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ON H-ANTIMAGICNESS OF DISCONNECTED GRAPHS

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Abstract

A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to at least one subgraph of G isomorphic to a given graph H. Then the graph G is (a, d)-H-antimagic if there exists a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that, for all subgraphs H' of G isomorphic to H, the H'-weights, $wt_f(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$, form an arithmetic progression with the initial term a and the common difference d. When $f(V) = \{1, 2, \dots, |V|\}$, then G is said to be super (a, d)-H-antimagic. In this paper, we study super (a, d)-H-antimagic labellings of a disjoint union of graphs for $d = |E(H)| - |V(H)|$.

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1. Introduction

Let $G = (V, E)$ be a finite simple graph. An *edge covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of E belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an H-*covering*. A bijective function $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ is an (a, d)-H-*antimagic labelling* of a graph G admitting an H-covering whenever, for all subgraphs H' isomorphic to H, the H'-weights

$$wt_f(H') = \sum_{v\in V(H')} f(v) + \sum_{e\in E(H')} f(e)$$

form an arithmetic progression $a, a + d, a + 2d, \dots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs of G isomorphic to H. Such a labelling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a, d)-H-antimagic labelling is called *(super)* (a, d)-H-*antimagic*. For $d = 0$, it is called H-*magic* and H-*supermagic*, respectively.

The H-(super)magic labellings were first studied by Gutiérrez and Lladó [8] as an extension of the edge-magic and super edge-magic labellings introduced by Kotzig

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and Rosa [11] and Enomoto *et al.* [7], respectively. In [8], there are considered star-(super)magic and path-(super)magic labellings of some connected graphs and it is proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [13] studied the cycle-(super)magic behaviour of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h-magic for some h. Maryati *et al.* [16] and also Salman *et al.* [18] proved that certain families of trees are path-supermagic. Ngurah *et al.* [17] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati *et al.* [15] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h.

The (a, d)-H-antimagic labelling was introduced by Inayah *et al.* [9]. In [10], there are investigated the super (a, d)-H-antimagic labellings for some shackles of a connected graph H. In [4], wheels are proved to be cycle-antimagic. The (super) (a, d)-H-antimagic labelling is related to a super d-antimagic labelling of type $(1, 1, 0)$ of a plane graph that is the generalisation of a face-magic labelling introduced by Lih [12]. Further information on super d-antimagic labellings can be found in [2, 6].

For $H \cong K_2$, (super) (a, d)-H-antimagic labellings are also called (super) (a, d)-edge-antimagic total labellings [19]. More results on (a, d)-edge-antimagic total labellings can be found in [5, 14]. The vertex version of these labellings for generalised pyramid graphs is given in [1].

In this paper we mainly investigate the existence of super (a, d)-H-antimagic labellings for disconnected graphs. The main result of the paper is that if a graph G admits a (super) (a, d)-H-antimagic labelling, where $d = |E(H)| - |V(H)|$, then the disjoint union of m copies of the graph G, denoted by mG, admits a (super) (b, d)-H-antimagic labelling.

2. Super (a, d)-H-antimagic labelling

In this section we will study super (a, d)-H-antimagicness for the disjoint union of graphs. Since, for every simple connected graph H,

$$|V(H)| - 1 \le |E(H)| \le \frac{|V(H)|(|V(H)| - 1)}{2},$$

then $|E(H)| - |V(H)| \ge -1$. Thus, only for the purposes of the following theorem we allow (a, d)-H-antimagic labelling of G also for negative differences d. This amounts to $(a + (t - 1)d, -d)$-H-antimagic labelling of G, where t is the number of all subgraphs of G isomorphic to H.

Theorem 2.1. *Let $m, t \ge 1$ and $d \ge -1$ be integers. For $i = 1, 2, \ldots, m$, let G_i with an $(H_i^1, H_i^2, \ldots, H_i^t)$-covering be a super (a, d)-H-antimagic graph of order p and size q, where every graph H_i^j, $j = 1, 2, \ldots, t$, is isomorphic to the graph H and $d = |E(H)| - |V(H)|$. Then the disjoint union $\bigcup_{i=1}^m G_i$ is a super (b, d)-H-antimagic graph.*

Proof. Let $m \geq 1$, $t \geq 1$ be positive integers. Let $d \geq -1$ be an integer and, for a graph H, let $|E(H)| - |V(H)| = d$. Let G_i, $i = 1, 2, \ldots, m$, be a graph with p vertices and q edges that admits an $(H_i^1, H_i^2, \ldots, H_i^t)$-covering, where every graph H_i^j, $j = 1, 2, \ldots, t$, is isomorphic to the given graph H. Note that G_i is not necessarily isomorphic to G_j for $i \neq j$. Assume that every G_i, $i = 1, 2, \ldots, m$, has a super (a, d)-H-antimagic labelling $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, p + q\}$. Thus, the set of the corresponding H_i^j-weights forms an arithmetic sequence with difference d:

$$\{wt_{f_i}(H_i^j) : j = 1, 2, \ldots, t\} = \{a, a + d, \ldots, a + (t - 1)d\}. \tag{2.1}$$

We define the labelling f for the vertices and edges of $\bigcup_{i=1}^m G_i$ in the following way:

$$f(x) = \begin{cases} m(f_i(x) - 1) + i & \text{if } x \in V(G_i), \\ mf_i(x) + 1 - i & \text{if } x \in E(G_i). \end{cases}$$

First we prove that f is a bijection and that the vertices of $\bigcup_{i=1}^m G_i$ under the labelling f are labelled with the smallest possible numbers. As f_i, $i = 1, 2, \ldots, m$, is a super labelling, then

$$\{f_i(v) : v \in V(G_i)\} = \{1, 2, \ldots, p\},$$
$$\{f_i(e) : e \in E(G_i)\} = \{p + 1, p + 2, \ldots, p + q\}.$$

Thus, for $i = 1, 2, \ldots, m$,

$$\{f(v) : v \in V(G_i)\} = \{i, m + i, \ldots, m(p - 1) + i\},$$
$$\{f(e) : e \in E(G_i)\} = \{mp + 1 + m - i, mp + 1 + 2m - i, \ldots, mp + 1 + qm - i\}.$$

This means that

$$\Big\{f(v) : v \in V\Big(\bigcup_{i=1}^m G_i\Big)\Big\} = \{1, 2, \ldots, mp\}$$

and

$$\Big\{f(e) : e \in E\Big(\bigcup_{i=1}^m G_i\Big)\Big\} = \{mp + 1, mp + 2, \ldots, (p + q)m\}.$$

For the weight of every subgraph H_i^j isomorphic to the graph H under the labelling f,

$$\begin{aligned} wt_f(H_i^j) &= \sum_{v \in V(H_i^j)} f(v) + \sum_{e \in E(H_i^j)} f(e) \\ &= \sum_{v \in V(H_i^j)} (m(f_i(v) - 1) + i) + \sum_{e \in E(H_i^j)} (mf_i(e) + 1 - i) \\ &= m \sum_{v \in V(H_i^j)} f_i(v) - m|V(H_i^j)| + i|V(H_i^j)| + m \sum_{e \in E(H_i^j)} f_i(e) + |E(H_i^j)| - i|E(H_i^j)| \\ &= m\Big(\sum_{v \in V(H_i^j)} f_i(v) + \sum_{e \in E(H_i^j)} f_i(e)\Big) - m|V(H_i^j)| + |E(H_i^j)| + i|V(H_i^j)| - i|E(H_i^j)| \\ &= mwt_{f_i}(H_i^j) - m|V(H_i^j)| + |E(H_i^j)| + i|V(H_i^j)| - i|E(H_i^j)|. \end{aligned}$$

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As every H_i^j, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, is isomorphic to the graph H and as $|E(H)| - |V(H)| = d$,

$$|V(H_i^j)| = |V(H)| = k,$$
$$|E(H_i^j)| = |E(H)| = k + d.$$

Thus, for the H-weights,

$$wt_f(H_i^j) = mwt_{f_i}(H_i^j) + k(1 - m) + d(1 - i).$$

According to (2.1), for $i = 1, 2, \ldots, m$, the H-weights in the components are

$$\{wt_f(H_i^j) : j = 1, 2, \ldots, t\} = \{ma + k(1 - m) + d(1 - i),$$
$$m(a + d) + k(1 - m) + d(1 - i), \ldots, m(a + (t - 1)d) + k(1 - m) + d(1 - i)\}.$$

It is easy to see that the set of all H-weights in $\bigcup_{i=1}^m G_i$ forms an arithmetic sequence with difference d,

$$\begin{aligned}\{wt_f(H_i^j) &: i = 1, 2, \ldots, m, j = 1, 2, \ldots, t\} \\ &= \{ma + k(1 - m) + d(1 - m), ma + k(1 - m) + d(2 - m), \ldots, \\ &\quad ma + k(1 - m) - d, ma + k(1 - m), ma + k(1 - m) + d, \ldots, \\ &\quad m(a + d) + k(1 - m), \ldots, m(a + (t - 1)d) + k(1 - m)\}.\end{aligned}$$

Thus, the graph $\bigcup_{i=1}^m G_i$ is a super $(ma + (k + d)(1 - m), d)$-H-antimagic graph. □

Theorem 2.1 has many interesting corollaries. First we present the result for H-antimagicness of an arbitrary number of copies of a super (a, d)-H-antimagic graph G, where $d = |E(H)| - |V(H)|$.

Corollary 2.2. *Let G be a super (a, d)-H-antimagic graph, where $d = |E(H)| - |V(H)|$. Then the disjoint union of an arbitrary number of copies of G, that is, mG with $m \geq 1$, admits a super (b, d)-H-antimagic labelling.*

We can extend the previous result also for the nonsuper case.

Theorem 2.3. *Let G be an (a, d)-H-antimagic graph, where $d = |E(H)| - |V(H)|$. Then mG, $m \geq 1$, is a (b, d)-H-antimagic graph.*

Proof. Let H be a graph of order k and size $k + d$, $k \geq 2$, $d \geq -1$. Let G be an (a, d)-H-antimagic graph of order p and size q. Let f be an (a, d)-H-antimagic labelling of G, that is, $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ and the corresponding H-weights are $a, a + d, a + 2d, \ldots, a + (t - 1)d$, where t is the number of subgraphs of G isomorphic to H.

For $i = 1, 2, \ldots, m$, let v_i denote the vertex corresponding to the vertex v in the ith copy of G in mG. Analogously, let u_iv_i be the edge corresponding to the edge uv in the ith copy of G in mG.

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Define a labelling g of mG, $m \geq 1$, in the following way:

$$g(v_i) = m(f(v) - 1) + i \quad \text{for } v \in V(G), i = 1, 2, \ldots, m,$$
$$g(u_iv_i) = mf(uv) + 1 - i \quad \text{for } uv \in E(G), i = 1, 2, \ldots, m.$$

According to the proof of Theorem 2.1, we only need to show that g is a bijection. Let $r \in \{1, 2, \ldots, p+q\}$. If the number r is assigned by the labelling f to a vertex v of G, then the labels of the corresponding vertices v_i, $i = 1, 2, \ldots, m$, in the copies G_i in mG are

$$\{g(v_i) : g(v_i) = m(r-1) + i, i = 1, 2, \ldots, m\} = \{m(r-1)+1, m(r-1)+2, \ldots, mr\}.$$

If the number r is assigned by the labelling f to an edge uv of G, then the labels of the corresponding edges u_iv_i, $i = 1, 2, \ldots, m$, in the copies G_i in mG are

$$\{g(u_iv_i) : g(u_iv_i) = mr + 1 - i, i = 1, 2, \ldots, m\} = \{mr, mr-1, \ldots, m(r-1)+1\}.$$

Thus, neither the vertex labels nor the edge labels in mG are overlapping. Under the labelling g, the minimum label is 1 and the maximum label is $m(p+q)$. Thus, g is a bijection. □

3. Cycle-antimagicness and tree-antimagicness of graphs

Immediately from Theorem 2.1, we can obtain many interesting corollaries if we consider special H-coverings of a given H-antimagic graph G.

If H is a graph isomorphic to a cycle C_n, then we get the following results for cycle-supermagicness of a disjoint union of graphs.

Theorem 3.1. *Let $m, t \geq 1$ be integers. Let G_i with an $(H_i^1, H_i^2, \ldots, H_i^t)$-covering for $i = 1, 2, \ldots, m$ be a C_n-supermagic graph of order p and size q, where every graph H_i^j, $j = 1, 2, \ldots, t$, is isomorphic to the cycle C_n, $n \geq 3$. Then the disjoint union $\bigcup_{i=1}^m G_i$ is also a C_n-supermagic graph.*

Proof. The proof follows from the proof of Theorem 2.1 as $|E(C_n)| - |V(C_n)| = 0$ for every cycle C_n, $n \geq 3$. □

Theorem 3.2. *Let G be a C_n-supermagic (C_n-magic) graph with $n \geq 3$. Then the disjoint union of an arbitrary number of copies of G, that is, mG, $m \geq 1$, is also a C_n-supermagic (C_n-magic) graph.*

Note that it is possible to generalise the result not only for cycle-(super)magicness but also for general unicyclic graphs, providing the size and the order of the unicyclic graphs are the same.

Theorem 3.3. *Let $m, t \geq 1$ be integers. Let G_i with an $(H_i^1, H_i^2, \ldots, H_i^t)$-covering for $i = 1, 2, \ldots, m$ be a C-supermagic graph of order p and size q, where every graph H_i^j, $j = 1, 2, \ldots, t$, is isomorphic to the unicyclic graph C. Then the disjoint union $\bigcup_{i=1}^m G_i$ is also a C-supermagic graph.*

Theorem 3.4. *Let G be a C-supermagic (C-magic) graph, where C is a unicyclic graph. Then the disjoint union of an arbitrary number of copies of G, that is, mG, $m \geq 1$, is also a C-supermagic (C-magic) graph.*

If H is a tree, then $|V(H)| - |E(H)| = 1$. Also, by adding an edge e to a unicyclic graph C, we obtain the graph $H \cong C + e$ with $|E(H)| - |V(H)| = 1$. Thus, we get the following result.

Theorem 3.5. *Let $m, t \geq 1$ be positive integers. Let G_i with a $(T_i^1, T_i^2, \ldots, T_i^t)$-covering for $i = 1, 2, \ldots, m$ be a super $(a, 1)$-T-antimagic graph of order p and size q, where T is a tree and every tree T_i^j, $j = 1, 2, \ldots, t$, is isomorphic to T. Then the disjoint union $\bigcup_{i=1}^m G_i$ is a super $(b, 1)$-T-antimagic graph.*

Theorem 3.6. *Let G be a (super) $(a, 1)$-T-antimagic graph, where T is a tree. Then mG, $m \geq 1$, is also a (super) $(b, 1)$-T-antimagic graph.*

Note that Theorems 3.5 and 3.6 are also proved in [3].

Theorem 3.7. *Let $m, t \geq 1$ be integers. Let G_i with an $(H_i^1, H_i^2, \ldots, H_i^t)$-covering for $i = 1, 2, \ldots, m$ be an $(a, 1)$-$(C + e)$-antimagic graph of order p and size q, where every graph H_i^j, $j = 1, 2, \ldots, t$, is isomorphic to the graph $C + e$, where C is a unicyclic graph. Then the disjoint union $\bigcup_{i=1}^m G_i$ is a $(b, 1)$-$(C + e)$-antimagic graph.*

Theorem 3.8. *Let G be a (super) $(a, 1)$-$(C + e)$-antimagic graph, where C is a unicyclic graph. Then the disjoint union of an arbitrary number of copies of G, that is, mG, $m \geq 1$, is a (super) $(b, 1)$-$(C + e)$-antimagic graph.*

4. Conclusion

We have shown that the disjoint union of multiple copies of a (super) (a, d)-H-antimagic graph is also a (super) (b, d)-H-antimagic graph for $d = |E(H)| - |V(H)|$. It is a natural question whether a similar result holds also for other differences. For further investigation, we propose the following open problem.

Problem 4.1. Let G be a (super) (a, d)-H-antimagic graph. For the graph mG determine if there is a (super) (a, d)-H-antimagic total labelling, for certain values of $d \neq |E(H)| - |V(H)|$ and for all $m \geq 1$.

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