Proceedings of the Royal Society of Edinburgh, **140A**, 821–831, *2010*

On backward self-similar blow-up solutions to a supercritical semilinear heat equation

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(MS received 27 February 2009; accepted 24 September 2009)

We are concerned with a Cauchy problem for the semilinear heat equation

$$\left.\begin{aligned} u_t &= \Delta u + u^p && \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) &= u_0(x) \geqslant 0 && \text{in } \mathbb{R}^N. \end{aligned}\right\} \tag{P}$$

If $u(x,t) = (T-t)^{-1/(p-1)}\varphi((T-t)^{-1/2}x)$ for $x \in \mathbb{R}^N$ and $t \in [0,T)$ with a solution $\varphi \not\equiv 0$ of

$$\Delta\varphi - \tfrac{1}{2}y\nabla\varphi - \frac{1}{p-1}\varphi + \varphi^p = 0 \quad \text{in } \mathbb{R}^N,$$

then u is called a backward self-similar solution blowing up at $t = T$. Let p_{S} and p_{L} be the Sobolev and the Lepin exponents, respectively. It was shown by Mizoguchi (*J. Funct. Analysis* **257** (2009), 2911–2937) that $\kappa \equiv (p-1)^{-1/(p-1)}$ is a unique regular radial solution of (P) if $p > p_{\mathrm{L}}$. We prove that it remains valid for $p = p_{\mathrm{L}}$. We also show the uniqueness of singular radial solution of (P) for $p > p_{\mathrm{S}}$. These imply that the structure of radial backward self-similar blow-up solutions is quite simple for $p \geqslant p_{\mathrm{L}}$.

1. Introduction

We consider a Cauchy problem for a semilinear heat equation

$$\left.\begin{aligned} u_t &= \Delta u + u^p && \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) &= u_0(x) \geqslant 0 && \text{in } \mathbb{R}^N \end{aligned}\right\} \tag{1.1}$$

with $p > p_{\mathrm{S}}$, where p_{S} is the Sobolev exponent. A solution u of (1.1) is said to blow up at $t = T$ if $\limsup_{t \nearrow T} |u(t)|_\infty = \infty$ with the norm $|\cdot|_\infty$ of $L^\infty(\mathbb{R}^N)$. Set

$$w(y,s) = (T-t)^{1/(p-1)}u(x,t) \tag{1.2}$$

with $y = (T-t)^{-1/2}x$ and $s = -\log(T-t)$ for a solution u of (1.1) blowing up at $t = T$. Then w satisfies

$$\left.\begin{aligned} w_s &= \Delta w - \tfrac{1}{2}y \cdot \nabla w - \frac{1}{p-1}w + w^p && \text{in } \mathbb{R}^N \times (s^{\mathrm{T}}, \infty), \\ w(y, s^{\mathrm{T}}) &= T^{1/(p-1)}u_0(T^{1/2}y) && \text{in } \mathbb{R}^N, \end{aligned}\right\} \tag{1.3}$$

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where $s^{\mathrm{T}} = -\log T$. If a solution u of (1.1) defined in $\mathbb{R}^N \times (-\infty, 0)$ satisfies $\lambda^{2/(p-1)} u(\lambda x, \lambda^2 t) = u(x,t)$ in $\mathbb{R}^N \times (-\infty,0)$ for all $\lambda > 0$, then u is called backward self-similar. It is equivalent to $u(x,t) = (-t)^{-1/(p-1)} \varphi((-t)^{-1/2} x)$ for a positive solution of

$$\Delta\varphi - \tfrac{1}{2} y \nabla\varphi - \frac{1}{p-1}\varphi + \varphi^p = 0 \tag{1.4}$$

in $\mathbb{R}^N$. Here we say that a function f is positive if $f(x) > 0$ for all $x \in \mathbb{R}^N$. In the radial case, (1.4) is represented as

$$\varphi'' + \frac{N-1}{r}\varphi' - \tfrac{1}{2} r\varphi' - \frac{1}{p-1}\varphi + \varphi^p = 0 \tag{1.5}$$

with $r = |y|$.

It was shown in [5] that $\kappa \equiv (p-1)^{-1/(p-1)}$ is a unique regular solution of (1.4) if $1 < p \leqslant p_{\mathrm{S}}$. On the other hand, when $p_{\mathrm{S}} < p < p_{\mathrm{L}}$ there exist regular solutions of (1.5) which are spatially inhomogeneous by [2, 4, 6, 7, 15], where p_{L} is the Lepin exponent, i.e.

$$p_{\mathrm{L}} = \begin{cases} \infty & \text{if } N \leqslant 10, \\ 1 + \dfrac{6}{N-10} & \text{if } N \geqslant 11. \end{cases}$$

In the case of $p \geqslant p_{\mathrm{L}}$, the existence of such a solution has remained undiscovered for many years. Recently, a numerical experiment in [13] suggested the non-existence of a regular solution of (1.5) except κ for $p \geqslant p_{\mathrm{L}}$ with $N \geqslant 11$. In [11], a rigorous proof was given of the non-existence in the case of $p > 1 + 7/(N-11)$ and $N \geqslant 12$. The author improved the condition on p and N to the Lepin exponent in [12] as follows: if $p > p_{\mathrm{L}}$ and $N \geqslant 11$, then there exists no regular solution of (1.5) which is spatially inhomogeneous. We first extend the non-existence result to $p = p_{\mathrm{L}}$ in the following theorem.

Theorem 1.1. *If $p = p_{\mathrm{L}}$ and $N \geqslant 11$, then κ is a unique regular solution of (1.5).*

In the proof of [11], an identity of Pohožaev type played an important role. The method introduced in [12] was quite different from it. However, the strict inequality $p > p_{\mathrm{L}}$ was an essential assumption there, so we need to take a new approach in order to solve for $p = p_{\mathrm{L}}$.

If there exists a constant $C > 0$ such that

$$|u(t)|_\infty \leqslant C(T-t)^{-1/(p-1)} \quad \text{for } t \in [0, T)$$

for a solution u of (1.1) blowing up at $t = T$, then the blow-up of u is said to be of type I, and of type II otherwise. According to [9, 10], any radially symmetric solution w of (1.3) corresponding to a type-I blow-up solution converges to φ as $s \to \infty$ for some regular positive solution φ of (1.5). The study of (1.5) is also important from the viewpoint of the dynamics of radial global solutions of (1.3), that is, the asymptotic behaviour of radial type-I blow-up solutions of (1.1).

We next obtain the uniqueness of the singular solution of (1.5) for $p > p_{\mathrm{S}}$.

Theorem 1.2. *Let φ_∞ be a singular solution of (1.5) defined by*

$$\varphi_\infty(r) = c_\infty r^{-2/(p-1)} \quad \textit{for } r > 0, \tag{1.6}$$

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with

$$c_\infty = \left\{ \frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right\}^{1/(p-1)}.$$

If $p > p_{\mathrm{S}}$ and $N \geqslant 3$, then φ_∞ is a unique singular solution of (1.5).

The uniqueness of the singular solution of (1.5) was given in [8] under the additional assumption

$$|\varphi(r)| \leqslant C(1 + r^{-2/(p-1)}) \quad \text{for } r > 0$$

with some constant $C > 0$, while we need no assumption. The following is immediate from theorems 1.1 and 1.2.

Corollary 1.3. *If $p \geqslant p_{\mathrm{L}}$ and $N \geqslant 11$, then there exist exactly two backward self-similar radial solutions $(T-t)^{-1/(p-1)}\kappa$ and φ_∞ of (1.1) blowing up at $t = T < \infty$.*

For a function $f \not\equiv 0$ on $[a, b)$ with $0 \leqslant a < b \leqslant \infty$, let $z(f : [a, b))$ be the supremum over all j such that there exist $a \leqslant r_1 < r_2 < \cdots < r_{j+1} < b$ with $f(r_i) \cdot f(r_{i+1}) < 0$ for $i = 1, 2, \dots, j$. Denote $z(f : [0, \infty))$ by $z(f)$ for simplicity. We number zeros of a function on $[a, b)$ with $0 \leqslant a < b \leqslant \infty$ with sign change in order enumerated from 0. We denote by $0 < c \ll 1$ and $d \gg 1$ a sufficiently small $c > 0$ and a sufficiently large d, respectively.

The paper is organized as follows. In § 2 we prove theorem 1.1. When $p > p_{\mathrm{S}}$, let $\varphi(r; \alpha; p)$ be a solution of (1.5) with $\varphi'(0) = 0$, and with $\varphi(0) = \alpha$ for $\alpha > 0$. Set (Figure 1)

$$r(\alpha; p) = \sup\{r > 0 : \varphi(\tilde{r}; \alpha; p) > 0 \text{ for all } \tilde{r} \in [0, r)\}$$

and let

$$\mathcal{S}_p = \{\alpha > 0 : r(\alpha; p) = \infty\}.$$

Then $\mathcal{S}_p$ is the set of $\alpha > 0$ for which $\varphi(r; \alpha)$ is a regular solution of (1.5) in $(0, \infty)$. It was given in [12] that, for $p > p_{\mathrm{S}}$,

(i) $\mathcal{S}_p \subset [\kappa, \infty)$,

(ii) $r(\alpha; p) < \infty$ for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$,

(iii) $z(\varphi(r; \alpha; p) - \varphi_\infty(r) : [0, r(\alpha; p))) = 2$ for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$.

Define

$$\alpha_*(p) = \sup\{\alpha > \kappa : z(\varphi(r; \tilde{\alpha}; p) - \varphi_\infty(r) : [0, r(\tilde{\alpha}; p))) = 2 \text{ for all } \tilde{\alpha} \in (\kappa, \alpha)\}.$$

It is immediate that $\alpha_*(p) \in \mathcal{S}_p$ and $z(\varphi(r; \alpha_*(p); p) - \varphi_\infty(r)) = 2$ for $p > p_{\mathrm{S}}$ with $\mathcal{S}_p \setminus \{\kappa\} \neq \varnothing$. We show that, for such p, the linearized operator $L_{\alpha_*(p)}(p)$ of (1.5) at $\varphi(r; \alpha_*(p); p)$ does not have 0 as an eigenvalue in a suitable setting of function space. On the contrary to the conclusion of theorem 1.1, assume that $\mathcal{S}_p \setminus \{\kappa\}$ is non-empty for $p_{\mathrm{S}} < p \leqslant p_{\mathrm{L}}$. Then $\varphi(r; \alpha_*(p); p)$ can be extended to $p_{\mathrm{L}} \leqslant p < p_{\mathrm{L}} + \delta$ with some $\delta > 0$ by the implicit function theorem. This is a contradiction, since $\mathcal{S}_p = \{\kappa\}$ for $p > p_{\mathrm{L}}$ from [12], which completes the proof.

Section 3 is devoted to the proof of theorem 1.2. We first assume that there exists a singular solution φ with $\varphi \not\equiv \varphi_\infty$. Set $h(\eta) = r^{2/(p-1)}\varphi(r)$ and $\eta = \log r$. Then h satisfies

$$h'' + \left(N - 2 - \frac{4}{p-1}\right)h' - \tfrac{1}{2}\mathrm{e}^{2\eta}h' - c_\infty^{p-1}h + h^p = 0 \quad \text{in } \mathbb{R}. \tag{1.7}$$

Define

$$E[h(\eta)] = \tfrac{1}{2}(h'(\eta))^2 + F(h(\eta)) \quad \text{for } \eta \in \mathbb{R},$$

where

$$F(\tau) = -\tfrac{1}{2}c_\infty^{p-1}\tau^2 + \frac{1}{p+1}\tau^{p+1} \quad \text{for } \tau \geqslant 0.$$

Multiplying (1.7) by h' yields

$$\frac{\mathrm{d}}{\mathrm{d}\eta}E[h(\eta)] = \left\{\frac{\mathrm{e}^{2\eta}}{2} - \left(N - 2 - \frac{4}{p-1}\right)\right\}h'(\eta)^2 \quad \text{for } \eta \in \mathbb{R}. \tag{1.8}$$

We divide into two cases:

(i) $z(h - c_\infty : (-\infty, 0]) = \infty$;

(ii) $z(h - c_\infty : (-\infty, 0]) < \infty$.

We obtain a contradiction in each case through estimates based on (1.8).

2. Proof of theorem 1.1

In this section we prove the uniqueness of the regular solution of (1.5), which is spatially inhomogeneous for $p = p_{\mathrm{L}}$ since it was solved in the case of $p > p_{\mathrm{L}}$ in [12]. Let $p > p_{\mathrm{S}}$. For $\alpha > 0$, let $\varphi(r;\alpha;p)$ be a solution of (1.5) with $\varphi'(0) = 0$ and $\varphi(0) = \alpha$. Set

$$r(\alpha;p) = \sup\{r > 0 : \varphi(\tilde{r};\alpha;p) > 0 \text{ for all } \tilde{r} \in [0, r)\}.$$

In order to avoid complicated notation, we denote $\varphi(r;\alpha;p)$ and $r(\alpha;p)$ by $\varphi(r;\alpha)$ and $r(\alpha)$, respectively, if there is no fear of confusion. We also denote simply by κ and φ_∞ for all $p > p_{\mathrm{S}}$, though they depend on p. Let

$$\mathcal{S}_p = \{\alpha > 0 : r(\alpha;p) = \infty\}.$$

For $q \geqslant 1$, let L^q_w be the class of Lebesgue measurable functions on $[0, \infty)$ such that

$$\int_0^\infty |f(r)|^q r^{N-1}\rho(r)\,\mathrm{d}r < \infty,$$

where $\rho(r) = \exp(-r^2/4)$ for $r \geqslant 0$. Let

$$H^1_w = \{f \in L^2_w : f' \in L^2_w\}.$$

The following result was shown in [10].

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PROPOSITION 2.1. *Let $p > p_{\mathrm{S}}$. For $\alpha \in \mathcal{S}_p$, there exists $c(\alpha) > 0$ such that*

$$\varphi(r;\alpha) = c(\alpha) r^{-2/(p-1)}(1 - d(\alpha) r^{-2} + o(r^{-2})) \quad \text{as } r \to \infty,$$

where $d(\alpha) = c(\alpha)^{p-1} - c_\infty^{p-1}$.

The following results were obtained in [12].

LEMMA 2.2. *Let $p > p_{\mathrm{S}}$. For $\alpha \in \mathcal{S}_p$, it holds that $\alpha \geqslant \kappa$ and $\varphi(r;\alpha)$ is non-increasing with respect to r.*

LEMMA 2.3. *Let $p > p_{\mathrm{S}}$. If $\alpha > \kappa$ is sufficiently close to κ, then $r(\alpha) < \infty$ and*

$$z(\varphi(r;\alpha) - \varphi_\infty(r) : [0, r(\alpha))) = 2.$$

For $p > p_{\mathrm{S}}$, define

$$\alpha_*(p) = \sup\{\alpha > \kappa : z(\varphi(r;\tilde{\alpha};p) - \varphi_\infty(r) : [0, r(\tilde{\alpha};p))) = 2 \text{ for all } \tilde{\alpha} \in (\kappa, \alpha)\}. \tag{2.1}$$

As stated in § 1, if $p_{\mathrm{S}} < p < p_{\mathrm{L}}$, then $\mathcal{S}_p \setminus \{\kappa\}$ is non-empty and hence

$$\kappa < \alpha_*(p) < \infty.$$

It is immediate that $\alpha_*(p) \in \mathcal{S}_p$. Since κ is a unique element of $\mathcal{S}_p$ to which the corresponding solution intersects φ_∞ exactly once by [1], we have

$$z(\varphi(r;\alpha_*(p);p) - \varphi_\infty(r)) \geqslant 2 \quad \text{for } p > p_{\mathrm{S}}$$

with $\mathcal{S}_p \setminus \{\kappa\} \neq \varnothing$. For $\alpha \in \mathcal{S}_p$, let $L_\alpha(p)$ be the linearized operator at $\varphi(\cdot;\alpha;p)$, i.e.

$$L_\alpha(p)\phi = \phi'' + \frac{N-1}{r}\phi' - \frac{r\phi'}{2} - \frac{1}{p-1}\phi + p\varphi(r;\alpha;p)^{p-1}\phi.$$

For $j = 0, 1, 2, \dots$, denote by $\lambda_j^\alpha(p)$ and $\phi_j^\alpha(p)$ the jth eigenvalue of

$$-L_\alpha(p)\phi = \lambda\phi \quad \text{in } H_w^1$$

and the jth eigenfunction with $\phi'(0) = 0$ and $\phi(0) = 1$, respectively. For simplicity, we denote $L_\alpha(p)$, $\lambda_j^\alpha(p)$ and $\phi_j^\alpha(p)$ by L_α, λ_j^α and ϕ_j^α, respectively, if there is no fear of confusion.

LEMMA 2.4. *For $p > p_{\mathrm{S}}$ with $\mathcal{S}_p \setminus \{\kappa\} \neq \varnothing$, let $\alpha_*(p)$ be defined in (2.1). Then 0 is not an eigenvalue of $L_{\alpha_*(p)}(p)$.*

Proof. Differentiating (1.5) in α yields

$$(\varphi_\alpha)'' + \frac{N-1}{r}(\varphi_\alpha)' - \tfrac{1}{2}r(\varphi_\alpha)' - \frac{1}{p-1}\varphi_\alpha + p\varphi(r;\alpha)^{p-1}\varphi_\alpha = 0, \tag{2.2}$$

where $\varphi_\alpha(r;\alpha) = \partial\varphi(r;\alpha)/\partial\alpha$. It is trivial that $(\varphi_\alpha)'(0;\alpha) = 0$ and also that $\varphi_\alpha(0;\alpha) = 1$.

Write $\alpha_* = \alpha_*(p)$ for simplicity. Let $\alpha \in (\kappa, \alpha_*)$. Since $\varphi(r;\alpha) > \varphi_\infty(r)$ for r in some interval, it is immediate that

$$z(\varphi_\alpha(r;\alpha) : [0, r(\alpha))) \geqslant 1. \tag{2.3}$$

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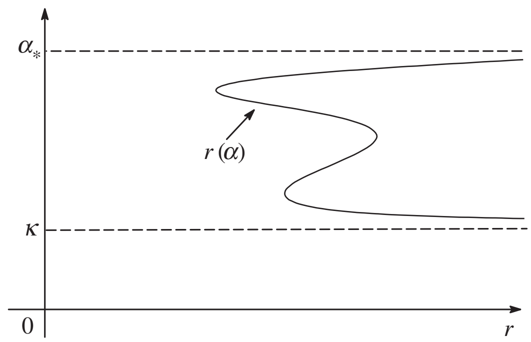

Figure 1. Rough sketch of $r(\alpha)$.

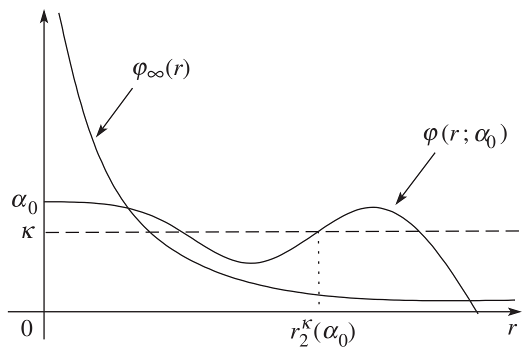

Figure 2. $\varphi(r;\alpha_0)$ is not non-increasing.

Since $\varphi(r(\alpha);\alpha)=0$, we have

$$r'(\alpha)=-\frac{\varphi_\alpha(r(\alpha);\alpha)}{\varphi_r(r(\alpha);\alpha)}, \tag{2.4}$$

where $\varphi_r(r;\alpha)=\partial/\partial r\varphi(r;\alpha)$. We first consider the case where $r(\alpha)$ is decreasing for $\alpha>\kappa$ with $\alpha-\kappa\ll 1$ and increasing for $\alpha<\alpha_*$ with $\alpha_*-\alpha\ll 1$. It follows from (2.4) that $\varphi_\alpha(r(\alpha);\alpha)<0$ for $\alpha>\kappa$ with $\alpha-\kappa\ll 1$ and $\varphi_\alpha(r(\alpha);\alpha)>0$ for $\alpha<\alpha_*$ with $\alpha_*-\alpha\ll 1$.

We show that $\varphi(r;\alpha)$ is non-increasing in $r\in[0,r(\alpha)]$ for any $\alpha\in(\kappa,\alpha_*)$. Assume that this is not valid for some $\alpha_0\in(\kappa,\alpha_*)$. Then we have $z(\varphi(r;\alpha_0)-\kappa:[0,r(\alpha_0)))\geqslant 3$. Let $r_i^\kappa(\alpha)$ be the ith zero of $\varphi(r;\alpha)-\kappa$ for a positive integer i.

If some of the zeros of $\varphi(r;\alpha)-\kappa$ in $[0,r(\alpha))$ vanish as α varies from α_0 to α_*, then there exist $\alpha_1\in(\alpha_0,\alpha_*)$ and $\hat{r}\in(0,r(\alpha_1))$ such that $\varphi(\hat{r};\alpha_1)=\kappa$ and $\varphi'(\hat{r};\alpha_1)=0$. This is a contradiction by the uniqueness of solutions for (1.5) with the same initial condition at $r=\hat{r}$. Consequently, $z(\varphi(r;\alpha)-\kappa:[0,r(\alpha)))\geqslant 3$ for each $\alpha\in[\alpha_0,\alpha_*]$.

Suppose that there exists $\{\alpha_n\}$ with $\alpha_n\nearrow\alpha_*$ as $n\to\infty$ such that $r_2^\kappa(\alpha_n)\to\infty$ as $n\to\infty$. By the definition of $\alpha_*=\alpha_*(p)$, $\varphi(r;\alpha_n)$ does not intersect $\varphi_\infty(r)$ between the first zero of $\varphi(r;\alpha_n)-\varphi_\infty(r)$ and $r_2^\kappa(\alpha_n)$. Letting $n\to\infty$ yields $z(\varphi(r;\alpha_*)-\varphi_\infty(r))=1$. This is a contradiction, since κ is a unique regular solution which intersects φ_∞ exactly once [1]. Hence, there exists $C>0$ such that $r_2^\kappa(\alpha)\leqslant C$

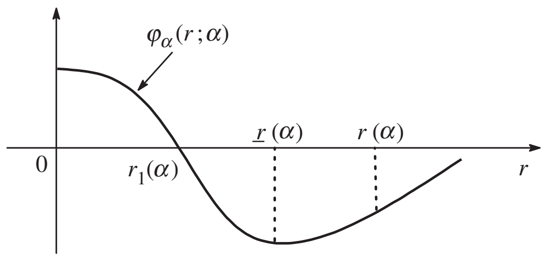

Figure 3. Rough sketch of $\varphi_\alpha(r(\alpha);\alpha)$.

for α with $0 < \alpha_* - \alpha \ll 1$. Then we have $z(\varphi(r;\alpha_*) - \kappa) \geqslant 2$ and hence $\varphi(r;\alpha_*)$ is not monotone. This contradicts lemma 2.2. Thus $\varphi(r;\alpha)$ is non-increasing in $r \in [0, r(\alpha)]$ for any $\alpha \in (\kappa, \alpha_*)$ (Figure 2).

If $\alpha > \kappa$ is sufficiently close to κ, then $z(\varphi_\alpha; [0, r(\alpha))) = 1$. In fact, assume that this is not true. Then there exist $r_i(\alpha) \in (0, r(\alpha))$ for $i = 1, 2, 3$, where $r_i(\alpha)$ is the ith zero of $\varphi_\alpha(r;\alpha)$ for positive integer i, since $\varphi_\alpha(r(\alpha);\alpha) < 0$ for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$. It was shown in [3] that $\lambda_0^\kappa < 0 = \lambda_1^\kappa < \lambda_2^\kappa < \cdots$. Therefore, ϕ_1^κ is an eigenfunction associated with $\lambda_1^\kappa = 0$, that is, ϕ_1^κ satisfies

$$\phi'' + \frac{N-1}{r}\phi' - \tfrac{1}{2}r\phi' - \frac{1}{p-1}\phi + p\kappa^{p-1}\phi = 0.$$

We take ϕ_1^κ again so that $\phi_1^\kappa(0) = 1$, which is denoted by ϕ_1^κ. Denote by R_1 the first zero of ϕ_1^κ. For any $0 < \varepsilon \ll 1$ there exists $\delta_1 > 0$ such that if $\kappa < \alpha < \kappa + \delta_1$, then $\varphi(r;\alpha) < \kappa$ for $r \in [R_1 + \varepsilon, r(\alpha)]$. For each $0 < \varepsilon \ll 1$, $R > 0$, there exists $\delta_2 > 0$ such that if $\kappa < \alpha < \kappa + \delta_2$, then

$$|\varphi_\alpha(r) - \phi_1^\kappa(r)| + |\varphi_\alpha'(r) - (\phi_1^\kappa)'(r)| < \varepsilon \quad \text{for } r \in [0, R].$$

Therefore, we have $r_2(\alpha), r_3(\alpha) \gg 1$ for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$. Since $\varphi(r;\alpha) < \kappa$ for $r \in [R_1 + \varepsilon, r(\alpha)]$ for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$, this is a contradiction by the standard comparison theorem on oscillation for elliptic equations [14].

We see that if $\alpha > \kappa$ is sufficiently close to κ, then $r_1(\alpha) < \underline{r}(\alpha) < r(\alpha)$, where $\underline{r}(\alpha)$ is the local minimizer of $\varphi_\alpha(r;\alpha)$ closest to $r(\alpha)$.

In fact, suppose that $r(\alpha) \leqslant \underline{r}(\alpha)$ for some $\alpha > \kappa$ with $\alpha - \kappa \ll 1$. If $r(\tilde{\alpha}) = \underline{r}(\tilde{\alpha})$ for some $\tilde{\alpha} \in (\kappa, \alpha_*)$, then substituting $\alpha = \tilde{\alpha}$ and $r = \underline{r}(\tilde{\alpha}) = r(\tilde{\alpha})$ into (2.2) yields a contradiction. This implies that $r(\alpha) < \underline{r}(\alpha)$ for all $\alpha \in (\kappa, \alpha_*)$. On the other hand, if $\alpha < \alpha_*$ is sufficiently close to α_*, then $\varphi_\alpha(r(\alpha);\alpha) > 0$ and hence $z(\varphi_\alpha(r;\alpha) : [0, r(\alpha))) = 0$ (Figure 3). This contradicts (2.3), which implies that $\underline{r}(\alpha) < r(\alpha)$ for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$.

We similarly obtain that $r(\alpha)$ cannot pass positive local maximizers of $\varphi_\alpha(r;\alpha)$ as α varies from κ to α_*.

Take

$$k > \max\left\{\frac{1}{2N(p-1)}, \frac{1}{4}\right\}$$

and set $\psi(r;\alpha) = \varphi_\alpha(r;\alpha)\exp(-kr^2)$ for $r \in [0, r(\alpha)]$. A straightforward calculation yields

$$\psi'' + \left\{\left(4k - \frac{1}{2}\right)r + \frac{N-1}{r}\right\}\psi' + \left\{p\varphi^{p-1} - \frac{1}{p-1} + 2kN + k(4k-1)r^2\right\}\psi = 0. \quad (2.5)$$

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There exists $\bar{\alpha} \in (\kappa, \alpha_*)$ with $r(\bar{\alpha}) = r_2(\bar{\alpha})$ such that $r(\alpha) > r_2(\alpha)$ for all $\alpha > \bar{\alpha}$. Then $\psi(r;\alpha)$ is non-decreasing for $r \in (r_2(\alpha), r(\alpha))$ for any $\alpha \in (\bar{\alpha}, \alpha_*)$. In fact, it is immediate that $\psi(r;\alpha)$ is increasing for $r \in (r_2(\alpha), r(\alpha))$ for any $\alpha \in (\bar{\alpha}, \alpha_*)$ with $\alpha - \bar{\alpha} \ll 1$. Assume that $\psi(r;\alpha_0)$ is not non-decreasing for $r \in (r_2(\alpha_0), r(\alpha_0))$ with some $\alpha_0 \in (\bar{\alpha}, \alpha_*)$. Then there exist $\alpha_1 \in (\bar{\alpha}, \alpha_0)$ and $\tilde{R} \in (r_2(\alpha_1), r(\alpha_1))$ such that $\psi'(\tilde{R};\alpha_1) = \psi''(\tilde{R};\alpha_1) = 0$. However, it is impossible from (2.5) and the choice of k. This contradiction implies that $\psi(r;\alpha)$ is non-decreasing for $r \in (r_2(\alpha), r(\alpha))$ for any $\alpha \in (\bar{\alpha}, \alpha_*)$.

Consequently, $\psi(r;\alpha_*)$ is non-decreasing for $r \in (r_2(\alpha_*), \infty)$. Taking $r_* > r_2(\alpha_*)$, we see that

$$\varphi_\alpha(r;\alpha_*)\exp(-kr^2) \geqslant \varphi_\alpha(r_*;\alpha_*)\exp(-kr_*^2) \qquad \text{for } r \geqslant r_*$$

and hence

$$\varphi_\alpha(r;\alpha_*) \geqslant \varphi_\alpha(r_*;\alpha_*)\exp(-kr_*^2)\exp(kr^2) \quad \text{for } r \geqslant r_*.$$

This implies that $\varphi_\alpha(r;\alpha_*) \notin H_w^1$.

When $r(\alpha)$ oscillates for $\alpha > \kappa$ with $\alpha - \kappa \ll 1$ or for $\alpha < \alpha_*$ with $\alpha_* - \alpha \ll 1$ we can modify the above argument to obtain the same conclusion. □

We are ready to prove theorem 1.1.

Proof of theorem 1.1. Arguing by contradiction, we assume that $\mathcal{S}_{p_{\mathrm{L}}} \setminus \{\kappa\}$ is non-empty.

We now introduce a standard formulation to apply the implicit function theorem. Define an operator F from $(p_{\mathrm{S}}, \infty) \times (H_w^1 \cap L^\infty)$ to H_w^1 by $F(p, f) = f - g(f)$, where $g(f)$ is a solution of

$$g'' + \frac{N-1}{r}g' - \tfrac{1}{2}rg' - \frac{1}{p-1}g = -|f|^{p-1}f$$

with $g'(0) = 0$ and $g(0) = f(0)$. It is trivial that $F(p, \varphi(\cdot;\alpha;p)) = 0$ for $\alpha \in \mathcal{S}_p$ and $p_S < p \leqslant p_{\mathrm{L}}$. Let F_f be the Fréchet derivative of F with respect to f. It is immediate that $F_f(p, \varphi(\cdot;\alpha;p))h = 0$ with $h \not\equiv 0$ if and only if $h \not\equiv 0$ satisfies

$$h'' + \frac{N-1}{r}h' - \tfrac{1}{2}rh' - \frac{1}{p-1}h + p\varphi(r;\alpha;p)^{p-1}h = 0,$$

that is, h is an eigenfunction of $L_\alpha(p)$ associated with 0. By lemma 2.4, 0 is not an eigenvalue of $L_{\alpha_*(p)}(p)$ for $p_{\mathrm{S}} < p \leqslant p_{\mathrm{L}}$. This implies that $F_f(p, \varphi(\cdot;\alpha_*(p);p))$ is invertible for $p_{\mathrm{S}} < p \leqslant p_{\mathrm{L}}$. Therefore, there exists $\delta > 0$ such that $\varphi(r;\alpha_*(p);p)$ can be extended to $p_{\mathrm{L}} < p < p_{\mathrm{L}} + \delta$ by the implicit function theorem.

On the other hand, $\mathcal{S}_p = \{\kappa\}$ for $p > p_{\mathrm{L}}$ from [12]. This contradiction completes the proof. □

Remark 2.5. The proof of theorem 1.1 implies that $\varphi(0;\alpha_*(p);p) = \alpha_*(p) \to \infty$ as $p \nearrow p_{\mathrm{L}}$.

3. Proof of theorem 1.2

This section is devoted to the proof of the uniqueness of the singular solution of (1.5). The following result was proved by lemma 2.5 of [12].

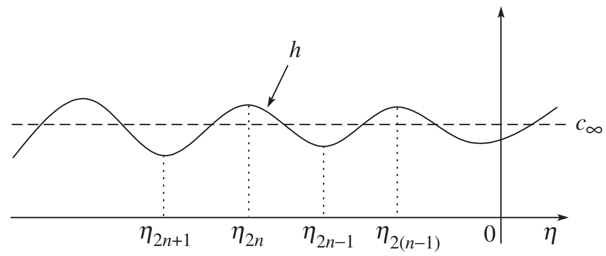

Figure 4. h in case (i).

PROPOSITION 3.1. *Let $p > p_S$. For $0 < \varepsilon \ll 1$ and $K, M \gg 1$ there exist $C_0, \alpha_0 > 0$ such that if $\alpha \geqslant \alpha_0$, then*

$$|\varphi(r;\alpha) - \varphi_\infty| \leqslant C_0 \varepsilon r^{-2/(p-1)} \quad \text{for } r \in [\alpha^{-(p-1)/2}K, M].$$

We now prove theorem 1.2.

Proof of theorem 1.2. Arguing by contradiction, we assume that there exists a singular solution φ of (1.5) which is different from φ_∞. Set

$$h(\eta) = r^{2/(p-1)}\varphi(r) \quad \text{and} \quad \eta = \log r. \tag{3.1}$$

Then h satisfies

$$h'' + \left(N - 2 - \frac{4}{p-1}\right)h' - \tfrac{1}{2}\mathrm{e}^{2\eta}h' - c_\infty^{p-1}h + h^p = 0 \quad \text{in } \mathbb{R}. \tag{3.2}$$

Define

$$E[h(\eta)] = \tfrac{1}{2}(h'(\eta))^2 + F(h(\eta)) \quad \text{for } \eta \in \mathbb{R}, \tag{3.3}$$

where

$$F(\tau) = -\tfrac{1}{2}c_\infty^{p-1}\tau^2 + \frac{1}{p+1}\tau^{p+1} \quad \text{for } \tau \geqslant 0.$$

Multiplying (3.2) by h' yields

$$\frac{\mathrm{d}}{\mathrm{d}\eta}E[h(\eta)] = \left\{\tfrac{1}{2}\mathrm{e}^{2\eta} - \left(N - 2 - \frac{4}{p-1}\right)\right\}h'(\eta)^2 \quad \text{for } \eta \in \mathbb{R}. \tag{3.4}$$

We divide into two cases:

(i) $z(h - c_\infty : (-\infty, 0]) = \infty$;

(ii) $z(h - c_\infty : (-\infty, 0]) < \infty$.

In case (i), there exists a sequence $\{\eta_n\}$ with $\eta_n \to -\infty$ as $n \to \infty$ such that $\{\eta_{2n}\}$ and $\{\eta_{2n+1}\}$ are sequences of consecutive local maximizers and minimizers of h, respectively, with $\eta_{2n+1} < \eta_{2n}$ for $n = 1, 2, \dots$ (see Figure 4).

It follows from (3.4) that

$$E[h(\eta_{2(n-1)})] \leqslant E[h(\eta_{2n-1}] \leqslant E[h(\eta_{2n})] \leqslant E[h(\eta_{2n+1}] \leqslant E[0] \quad \text{for all } n,$$

which implies that $\{h(\eta_{2n})\}$ is bounded and increasing. Therefore, $h(\eta_{2n}) \to a_0$ as $n \to \infty$ for some a_0 with $a_0 > c_\infty$. Taking $\hat{\eta}_n \in (\eta_{n+1}, \eta_n)$ with $h(\hat{\eta}_n) = c_\infty$, we have

$$E[h(\hat{\eta}_{2n})] \geqslant E[h(\eta_{2n})] \quad \text{for all } n. \tag{3.5}$$

Since $h(\eta_{2n}) \to a_0$ as $n \to \infty$, for $0 < \varepsilon \ll 1$ there exists $n_\varepsilon \gg 1$ such that $h(\eta_{2n}) \geqslant a_0 - \varepsilon \geqslant c_\infty + \varepsilon$ for $n \geqslant n_\varepsilon$. Then there exists $d_0 > 0$ such that $h'(\hat{\eta}_{2n}) \geqslant 2d_0$ for $n \geqslant n_\varepsilon$ from (3.5). Thus, there exists $0 < \delta_0 \ll 1$ such that $h'(\eta) \geqslant d_0$ for $\eta \in [\hat{\eta}_{2n} - \delta_0, \hat{\eta}_{2n} + \delta_0]$ and $n \geqslant n_\varepsilon$. It follows from (3.4) that

$$\begin{aligned} E[h(\eta_{2(n+1)})] - E[h(\eta_{2n})] &\geqslant \frac{1}{2}\left(N - 2 - \frac{4}{p-1}\right) \int_{\eta_{2n}}^{\eta_{2(n+1)}} h'(\eta)^2 \, \mathrm{d}\eta \\ &\geqslant \frac{1}{2}\left(N - 2 - \frac{4}{p-1}\right) \int_{\hat{\eta}_{2n}-\delta_0}^{\hat{\eta}_{2n}+\delta_0} h'(\eta)^2 \, \mathrm{d}\eta \\ &\geqslant \left(N - 2 - \frac{4}{p-1}\right) d_0^2 \delta_0 \end{aligned}$$

for $n \geqslant n_\varepsilon$. Since $E[h(\eta_{2(n+1)})] - E[h(\eta_{2n})] \to 0$ as $n \to \infty$, this is a contradiction.

In case (ii), $h(\eta)$ is monotone for η near $-\infty$. Indeed, if not, there exist infinitely many local maximizers and local minimizers of $h(\eta)$. From (3.2), any local maximum is larger than c_∞ and any local minimum is smaller than c_∞. Therefore, we have $z(h - c_\infty : (-\infty, 0]) = \infty$, which is case (i).

Consequently, the following two cases are possible:

(a) $h(\eta) \to a_1 > 0$ as $\eta \to -\infty$ for some $a_1 > 0$;

(b) $h(\eta) \to 0$ as $\eta \to -\infty$.

In fact, suppose that $\limsup_{\eta \to -\infty} h(\eta) = \infty$. Set $\tilde{h}(\eta) = h(-\eta)$ for $\eta \in \mathbb{R}$. Then $\tilde{h}$ satisfies

$$\tilde{h}'' - \left(N - 2 - \frac{4}{p-1}\right)\tilde{h}' + \tfrac{1}{2}\mathrm{e}^{-2\eta}\tilde{h}' - c_\infty^{p-1}\tilde{h} + \tilde{h}^p = 0 \quad \text{in } \mathbb{R}. \tag{3.6}$$

Since $h(\eta)$ is monotone for η near $-\infty$, there exists $\eta_0 \gg 1$ such that $\tilde{h}(\eta) \gg 1$ and $\tilde{h}'(\eta) > 0$ for $\eta \geqslant \eta_0$. We show that $\tilde{h}''(\eta) \geqslant 0$ for $\eta \geqslant \eta_0$. On the contrary, assume that $\tilde{h}''(\eta_1) < 0$ for some $\eta_1 \geqslant \eta_0$. Let

$$\eta^* = \sup\{\eta \geqslant \eta_1 : \tilde{h}''(\tilde{\eta}) < 0 \text{ for all } \tilde{\eta} \in [\eta_1, \eta)\}.$$

If $\eta^* < \infty$, then $\tilde{h}''(\eta^*) = 0$. Differentiating (3.6) in r yields

$$\tilde{h}''' - \left(N - 2 - \frac{4}{p-1}\right)\tilde{h}'' + \tfrac{1}{2}\mathrm{e}^{-2\eta}\tilde{h}'' - \mathrm{e}^{-2\eta}\tilde{h}' - c_\infty^{p-1}\tilde{h}' + p\tilde{h}^{p-1}\tilde{h}' = 0 \quad \text{in } \mathbb{R}.$$

Then we have $\tilde{h}'''(\eta) < 0$ for $\eta \in [\eta_1, \eta^*]$ and hence $\tilde{h}''(\eta^*) < \tilde{h}''(\eta_1) < 0$. This contradiction implies that $\eta^* = \infty$, that is, $\tilde{h}''(\eta) < 0$ for $\eta \geqslant \eta_1$. Since $\tilde{h}'(\eta)$ is decreasing for $\eta \geqslant \eta_1$, there exists $b_0 \geqslant 0$ such that $\tilde{h}'(\eta) \to b_0$ as $\eta \to \infty$. Then $\tilde{h}''(\eta) \to 0$ as $\eta \to \infty$. Letting $\eta \to \infty$ in (3.6) yields a contradiction. Thus, $\tilde{h}''(\eta) \geqslant 0$ for $\eta \geqslant \eta_0$. It then follows from (3.6) that

$$\left(N - 2 - \frac{4}{p-1}\right)\tilde{h}' \geqslant \tfrac{1}{2}\tilde{h}^p \quad \text{for } \eta \gg 1,$$

which implies that $\tilde{h}$ blows up at some finite η. This is a contradiction. Consequently, we obtain $\limsup_{\eta \to -\infty} h(\eta) < \infty$.

In the case of (a), we have

$$-c_{\infty}^{p-1}a_1 + a_1^p = 0,$$

letting $\eta \to -\infty$ in (3.2) since $h'(\eta) \to 0$ and $h''(\eta) \to 0$ as $\eta \to -\infty$. This is a contradiction since $a_1 \neq c_\infty$ by (3.4). In the case of (b), there exist $\alpha \gg 1$, $R > 0$, such that $\varphi(R;\alpha) = \varphi(R)$ and $\varphi(r;\alpha) < \varphi(r) < \varphi_\infty(r)$ for $r \in (0,R)$ by proposition 3.1. This is a contradiction by the standard comparison theorem on oscillation for elliptic equations. This completes the proof. □

Acknowledgements

The author is supported by the JST PRESTO programme.

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(*Issued 6 August 2010*)