Ergod. Th. & Dynam. Sys., (2021), **41**, 3780–3806

doi:10.1017/etds.2020.125

Value distribution of derivatives in polynomial dynamics

YÛSUKE OKUYAMA† and GABRIEL VIGNY‡

† *Division of Mathematics, Kyoto Institute of Technology, Sakyo-ku, Kyoto 606-8585, Japan*
(e-mail: okuyama@kit.ac.jp)
‡ *LAMFA, UPJV, 33 rue Saint-Leu, 80039 Amiens Cedex 1, France*
(e-mail: gabriel.vigny@u-picardie.fr)

(*Received* 24 *April* 2019 *and accepted in revised form* 28 *October* 2020)

Abstract. For every $m \in \mathbb{N}$, we establish the equidistribution of the sequence of the averaged pullbacks of a Dirac measure at any given value in $\mathbb{C} \setminus \{0\}$ under the mth order derivatives of the iterates of a polynomials $f \in \mathbb{C}[z]$ of degree $d > 1$ towards the harmonic measure of the filled-in Julia set of f with pole at ∞. We also establish non-archimedean and arithmetic counterparts using the potential theory on the Berkovich projective line and the adelic equidistribution theory over a number field k for a sequence of effective divisors on $\mathbb{P}^1(\overline{k})$ having small diagonals and small heights. We show a similar result on the equidistribution of the analytic sets where the derivative of each iterate of a Hénon-type polynomial automorphism of $\mathbb{C}^2$ has a given eigenvalue.

Key words: complex dynamics, Hénon map, higher derivative, non-archimedean dynamics, value distribution
2020 Mathematics Subject Classification: 37F10 (Primary); 37P30, 32H50 (Secondary)

1. *Introduction*

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$. The *filled-in Julia set*

$$K(f) := \left\{ z \in \mathbb{C} : \limsup_{n\to\infty} |f^n(z)| < \infty \right\}$$

of f is a *non-polar* compact subset in $\mathbb{C}$. Let g_f be the *Green function* of $K(f)$ with pole at ∞, regarding $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$ (see, for example, [**25**, §4.4]). We extend g_f equal to $= 0$ on $K(f)$. For every $n \in \mathbb{N}$, the difference $g_f - (\log \max\{1, |f^n|\})/d^n$ on $\mathbb{C}$ is harmonic and bounded near ∞ so it admits a harmonic extension across ∞, and we have the estimate

$$g_f - \frac{\log \max\{1, |f^n|\}}{d^n} = O(d^{-n}) \quad \text{as } n \to \infty \tag{1.1}$$

on $\mathbb{P}^1$ uniformly.

Let us denote by δ_a the Dirac measure on $\mathbb{P}^1$ at each $a \in \mathbb{P}^1$. The *harmonic measure* of $K(f)$ with pole at ∞ is the probability measure

$$\mu_f := \Delta g_f + \delta_\infty \quad \text{on } \mathbb{P}^1,$$

which has no atoms on $\mathbb{P}^1$ and is supported by $\partial K(f)$. The exceptional set of f is defined as

$$E(f) := \left\{ a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(a) < \infty \right\},$$

which consists of ∞ ($f^{-1}(\infty) = \{\infty\}$) and at most one point $b \in \mathbb{C}$ ($f^{-1}(b) = \{b\}$). For every $h \in \mathbb{C}(z)$ of $\deg h > 0$ and every $a \in \mathbb{P}^1$, by the definition of the pullback operator h^*, we have $h^*\delta_a = \sum_{w \in h^{-1}(a)} (\deg_w h)\delta_w$ on $\mathbb{P}^1$, where $\deg_w h$ is the local degree of h at w.

Brolin [8] studied the value distribution of the iteration sequence ($f^n : \mathbb{P}^1 \to \mathbb{P}^1$) of f and established that *for every $a \in \mathbb{C} \setminus E(f)$,*

$$\lim_{n \to \infty} \frac{(f^n)^* \delta_a}{d^n} = \mu_f \quad \text{weakly on } \mathbb{P}^1.$$

This equidistribution of pullbacks of points under iterations initiated the study of value distribution of complex dynamics (see, for example, [25, §6.5], [7, §VIII], [10, 27]). In [17, §2] and [23, Theorem 1], a similar equidistribution statement replacing f^n with the first order derivative $(f^n)'$ of f^n has been proved first for $a \in \mathbb{C}$ outside a polar set and then for any $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, respectively.

Our aim is to contribute to the study of the parallelism between the value distribution of the sequence of higher derivatives (or jets) of the iterations of f and the value distribution of higher derivatives (or jets) of meromorphic mappings (cf. [29]), extending the results mentioned above to several different settings: higher derivatives of polynomials over various valued fields and Hénon-type polynomial automorphisms of $\mathbb{C}^2$.

1.1. *Over the field $\mathbb{C}$ of complex numbers.*

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$. For every $h \in \mathbb{C}[z]$ and every $m \in \mathbb{N}$, we write the mth order derivative $(\mathrm{d}^m/\mathrm{d}z^m)h(z)$ of h as $h^{(m)}$.

Our first principal result is the following theorem.

THEOREM 1. *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$, and $m \in \mathbb{N}$. Then, for every $a \in \mathbb{C}^*$,*

$$\lim_{n \to \infty} \frac{((f^n)^{(m)})^* \delta_a}{d^n - m} = \mu_f \quad \textit{weakly on } \mathbb{P}^1. \tag{1.2}$$

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

In Theorem 1, in general, the values $a = 0, \infty$ need to be excluded as, for every $n \in \mathbb{N}$, $((f^n)^{(m)})^*\delta_\infty/(d^n - m) = \delta_\infty \neq \mu_f$ and, if there is $b \in E(f) \cap \mathbb{C}$, then for every $n \in \mathbb{N}$, $((f^n)^{(m)})^*\delta_0/(d^n - m) = \delta_b \neq \mu_f$ (see also Remark 2.4 below). An affine coordinate on $\mathbb{C}$ is fixed in Theorem 1, but note that $A^*(((f^n)^{(m)})^*\delta_a - (d^n - m)\cdot\mu_f) = (((A \circ f \circ A^{-1})^n)^{(m)})^*\delta_{(A')^{m-1}(a)} - (d^n - m)\cdot\mu_{A\circ f\circ A^{-1}}$ on $\mathbb{P}^1$ for any affine transformation A on $\mathbb{C}$.

The equidistribution (1.2) for $m > 1$ was expected in [17, §2.4], at least when f has no Siegel disks. As seen in the proof below, (1.2) follows only by an analysis of $(f^n)^{(m)}$ on $\mathbb{P}^1 \setminus K(f)$ in this case. This analysis is not difficult for $m = 1$ by the chain rule, but for $m > 1$ it requires care with the higher order derivatives of the Böttcher coordinates of f near ∞. An extra and more involved effort is required to treat the situation on $K(f)$ under the presence of Siegel disks of f in general.

1.2. *Over a non-archimedean complete valued field K.*

Let K be an algebraically closed field. We say that an absolute value $|\cdot|$ on K is *non-trivial* if $|K| \not\subset \{0, 1\}$ and that it is *non-archimedean* if the *strong triangle inequality* $|z + w| \leq \max\{|z|, |w|\}$ holds for any $z, w \in K$. For the details on the Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$, the canonical action of f on P^1, and the equilibrium (or canonical) measure μ_f of f on P^1, see §3.1 below. By convention, we say f *has no potentially good reductions* if $\mu_f(\{\mathcal{S}\}) = 0$ for any $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$; this definition coincides with the usual algebraic one (cf. [3, Corollary 10.33]).

Our second principal result is a non-archimedean counterpart of Theorem 1.

THEOREM 2. *Let K be an algebraically closed field of characteristic 0 that is complete with respect to a non-trivial and non-archimedean absolute value. Let $m \in \mathbb{N}$ and $f \in K[z]$ be a polynomial of degree $d > 1$ having no potentially good reductions. Then, for every $a \in K$,*

$$\lim_{n\to\infty} \frac{((f^n)^{(m)})^*\delta_a}{d^n - m} = \mu_f \quad \textit{weakly on } \mathsf{P}^1. \tag{1.3}$$

The assumption of no potentially good reductions allows us to deal with the Berkovich filled-in Julia set $\mathsf{K}(f)$ of f. The analysis on $\mathsf{P}^1 \setminus \mathsf{K}(f)$ in the proof is similar to that in the archimedean case, using the (non-archimedean) Böttcher coordinate near ∞ and a non-archimedean potential theory instead (see [24]).

1.3. *Over a product formula field k.*

Let k be a field. We denote by $\overline{k}$ an algebraic closure of k. An effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$ is the scheme-theoretic vanishing of some $P \in \bigcup_{d\in\mathbb{N}} k[z_0, z_1]_d$. Then, $\mathcal{Z}$ is supported by $\overline{k}$ (regarding $\mathbb{P}^1(\overline{k})$ as $\overline{k} \cup \{\infty\}$) if and only if $P(z_0, z_1) = z_0^{\deg p} p(z_1/z_0)$ for some $p(z) \in k[z]$ of degree greater than 0 (identifying $[z_0 : z_1]$ with z_1/z_0 when $z_0 \neq 0$, that is, $\infty = [0 : 1]$ as the convention in [16]), which is unique up to multiplication in $k^* = k \setminus \{0\}$ and is called a *representative* of $\mathcal{Z}$.

A field k is a *product formula field* if k is equipped with a (possibly uncountable) family M_k of (not necessarily all) places of k, a family $(|\cdot|_v)_{v\in M_k}$ of non-trivial absolute values

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

$|\cdot|_v$ representing v, and a family $(N_v)_{v\in M_k}$ in $\mathbb{N}$ satisfying the *product formula property* in that, for every $z\in k^*$,

$$|z|_v=1 \text{ for all but finitely many } v\in M_k, \text{ and } \prod_{v\in M_k}|z|_v^{N_v}=1.$$

A place $v\in M_k$ is said to be finite (respectively, infinite) if $|\cdot|_v$ is non-archimedean (respectively, archimedean). If M_k contains an infinite place of v, then k is (isomorphic to) a number field (so there are at most finitely many infinite places of a product formula field). For each $v\in M_k$, let k_v be the completion of k with respect to $|\cdot|_v$. Then $|\cdot|_v$ extends to $\overline{k_v}$. Let $\mathbb{C}_v$ be the completion of $\overline{k_v}$ with respect to $|\cdot|_v$ (so $|\cdot|_v$ extends to $\mathbb{C}_v$) and fix an embedding of $\overline{k}$ to $\mathbb{C}_v$ extending that of k to k_v. By convention, the dependence of a *local* quantity induced by $|\cdot|_v$ on each $v\in M_k$ is emphasized by adding the suffix to it, like k_v and $\mathbb{C}_v$.

Let $\hat{h}_f(\mathcal{Z})$ be the *Call–Silverman canonical height* of an effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$ (see §3.2 below for the definition). The following theorem is our third principal result.

THEOREM 3. *Let k be a product formula field of characteristic* 0*, and let $f\in k[z]$ be a polynomial of degree $d>1$ and $m\in\mathbb{N}$. Then, for every $a\in k$, denoting by $[(f^n)^{(m)}=a]$ the effective k-divisor on $\mathbb{P}^1(\overline{k})$ whose representative is $(f^n)^{(m)}-a\in k[z]$, we have the* $(g_{f,v})_{v\in M_k}$-small heights property

$$\lim_{n\to\infty}\hat{h}_f([(f^n)^{(m)}=a])=0 \tag{1.4}$$

of the sequence $([(f^n)^{(m)}=a])_n$ of effective k-divisors on $\mathbb{P}^1(\overline{k})$.

Assume, in addition, that k is a number field and $a\in k^*$. Then the uniform asymptotically $(g_{f,v})_{v\in M_k}$-Fekete configuration property

$$\lim_{n\to\infty}\sup_{v\in M_k}N_v\int_{\mathsf{P}^1(\mathbb{C}_v)\times\mathsf{P}^1(\mathbb{C}_v)\setminus\mathrm{diag}_{\mathbb{P}^1(\mathbb{C}_v)}}(\log|\mathcal{S}-\mathcal{S}'|_v-g_{f,v}(\mathcal{S})-g_{f,v}(\mathcal{S}'))\\ \left(\left(\frac{((f^n)^{(m)})^*\delta_a}{d^n-m}-\mu_{f,v}\right)\times\left(\frac{((f^n)^{(m)})^*\delta_a}{d^n-m}-\mu_{f,v}\right)\right)(\mathcal{S},\mathcal{S}')=0 \tag{1.5}$$

of $([(f^n)^{(m)}=a])$ holds, so in particular, for every $v\in M_k$,

$$\lim_{n\to\infty}\frac{((f^n)^{(m)})^*\delta_a}{d^n-m}=\mu_{f,v}\quad\textit{weakly on }\mathsf{P}^1(\mathbb{C}_v). \tag{1.6}$$

The proof is based on an adelic equidistribution result for effective divisors on $\mathbb{P}^1(\overline{k})$ having *small diagonals and* small heights [21].

1.4. *The derivatives of the iterates of a Hénon-type polynomial automorphism of* $\mathbb{C}^2$. Let $[t:z:w]$ be the homogeneous coordinate on $\mathbb{P}^2$, endowed with the Fubini–Study form. Identifying $\mathbb{C}^2$ with $\{t=1\}$, we let

$$L_\infty:=\{t=0\}=\mathbb{P}^2\setminus\mathbb{C}^2$$

be the *line at infinity* in $\mathbb{P}^2$. We fix the orthonormal frame (∂_z,∂_w) of the tangent space $T\mathbb{C}^2$ of $\mathbb{C}^2$, so that for a polynomial endomorphism f of $\mathbb{C}^2$, the derivative $\mathrm{d}f$ of f

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

is identified with the $M(2, \mathbb{C})$-valued function $(z, w) \mapsto (Df)_{(z,w)}$. Here, a polynomial automorphism of $\mathbb{C}^2$ is a polynomial endomorphism of $\mathbb{C}^2$ whose inverse exists and is a polynomial endomorphism of $\mathbb{C}^2$.

Recall some basic facts on a *Hénon-type* polynomial automorphism f of $\mathbb{C}^2$ of degree $d > 1$ [4, 11]. The Jacobian determinant $J_f := \det(Df) \in \mathbb{C}[z, w]$ of f is a non-zero constant on $\mathbb{C}^2$, so for every $n \in \mathbb{N}$, the Jacobian determinant $J_{f^n} = \det(D(f^n)) \in \mathbb{C}[z, w]$ of f^n on $\mathbb{C}^2$ is equal to the non-zero constant J_f^n. This f extends to a birational self-map on $\mathbb{P}^2$, which is still denoted by f for simplicity, so that both the indeterminacy loci I^+, I^- of f, f^{-1} are singletons in L_∞, that $I^- \neq I^+$ (so often normalized as $I^+ = \{[0 : 0 : 1]\}$, $I^- = \{[0 : 1 : 0]\}$), and that $I^- = f(L_\infty \setminus I^+)$. Moreover, the unique point in I^- is a superattracting fixed point of $f|(\mathbb{P}^2 \setminus I^+)$, and the attractive basin B^+ of $f|(\mathbb{P}^2 \setminus I^+)$ associated to I^- satisfies $B^+ \setminus \mathbb{C}^2 = L_\infty \setminus I^+$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{C}^2$. The *filled-in Julia set* of f is defined by

$$K^+ := \Big\{(z, w) \in \mathbb{C}^2 : \limsup_{n\to\infty} \|f^n(z, w)\| < \infty\Big\}.$$

Then $\overline{K^+} = K^+ \cup I^+$ in $\mathbb{P}^2$ and $\mathbb{P}^2 = \overline{K^+} \cup B^+$ (see, for example, [11, Proposition 5.5]). The *Green function* g^+ of f is the locally uniform limit

$$g^+ := \lim_{n\to\infty} \frac{\log \max\{1, \|f^n\|\}}{d^n} \quad \text{on } \mathbb{C}^2.$$

It is continuous and plurisubharmonic on $\mathbb{C}^2$, it is > 0 and pluriharmonic on B^+, and it is $\equiv 0$ on K^+. The *Green current* T^+ of f is defined as the trivial extension of $dd^c g^+$ on $\mathbb{C}^2$ to $\mathbb{P}^2$. It is a positive closed $(1, 1)$-current on $\mathbb{P}^2$ and, moreover, of mass 1 [11, Lemma 6.3].

For a non-constant polynomial $P \in \mathbb{C}[z, w]$, let $[P]$ be the current of integration along the hypersurface in $\mathbb{P}^2$ defined by the zeros of (the homogenized) P in $\mathbb{P}^2$, taking into account their multiplicities. The mass of $[P]$ equals $\deg P$ by Bézout's theorem. Let $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the *identity matrix* in $M(2, \mathbb{C})$.

Our final principal result is the following theorem.

THEOREM 4. *Let f be a Hénon-type polynomial automorphism of $\mathbb{C}^2$ of degree $d > 1$ and $\lambda \in \mathbb{C}^*$. Then, for every $n \in \mathbb{N}$, $\det(D(f^n) - \lambda I_2) \in \mathbb{C}[z, w]$ is of degree $d^n - 1$, and*

$$\lim_{n\to\infty} \frac{[\det(D(f^n) - \lambda I_2)]}{d^n - 1} = T^+ \quad \text{on } \mathbb{P}^2 \tag{1.7}$$

as currents.

In the proof, we show the L^1_{loc}-convergence of a sequence of potentials of $[\det(D(f^n) - \lambda I_2)]/(d^n - 1)$ towards g^+ on B^+ as $n \to \infty$ using the first order partial derivatives of g^+. The pleasant *uniqueness* of T^+ among all positive closed $(1, 1)$-currents on $\mathbb{P}^2$ of mass 1 which are supported by $\overline{K^+}$ ([15]; see also [11, Theorem 6.5]) allows us to deal with K^+.

Organization of the paper. In §2, we treat the field $\mathbb{C}$ of complex numbers. In §2.1, we recall some notions and facts from complex dynamics. In §2.2, we give a proof of

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Theorem 1, and in §2.3, we give a simpler treatment for the cases $m = 1, 2$. In §3, we treat a non-archimedean field K and a product formula field k. In §3.1 and §3.2, we recall background material from non-archimedean and arithmetic dynamics, respectively, and in §3.3, we show Theorems 2 and 3. In §4, we show Theorem 4 in a slightly more general form.

2. *Proof of Theorem 1*

2.1. *Background from complex dynamics.* Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$. The superattractive basin

$$I_\infty(f) := \left\{ z \in \mathbb{P}^1 : \lim_{n\to\infty} f^n(z) = \infty \right\}$$

of f associated to the superattracting fixed point ∞ of f (regarding $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$) is a domain in $\mathbb{P}^1$ containing ∞, and coincides with $\mathbb{P}^1 \setminus K(f)$. Let $C(f)$ be the critical set of f (as a branched self-covering of $\mathbb{P}^1$), which consists of ∞ and all the zeros of f' on $\mathbb{C}$. The set $\bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f) \setminus \{\infty\})$ is bounded in $\mathbb{C}$.

The topology of $\mathbb{P}^1$ coincides with the induced one from the chordal metric on $\mathbb{P}^1$. The Julia set $J(f)$ of f is defined as the set of all $z \in \mathbb{P}^1$ at which the family $(f^n : \mathbb{P}^1 \to \mathbb{P}^1)_{n\in\mathbb{N}}$ is not normal. The Fatou set $F(f)$ of f is defined by $\mathbb{P}^1 \setminus J(f)$ and a component of $F(f)$ is called a Fatou component of f. Both $J(f)$ and $F(f)$ are totally invariant under f and

$$J(f) = \partial K(f) = \partial I_\infty(f).$$

Any Fatou component of f is either $I_\infty(f)$ or a component of the interior of $K(f)$ and is mapped properly to a Fatou component of f. Any Fatou component of f other than $I_\infty(f)$ is simply connected. A Fatou component W of f is said to be *cyclic* under f if there is $p \in \mathbb{N}$ such that $f^p(W) = W$. If in addition the restriction $f^p : W \to W$ is injective, W is called a *Siegel disk* of f and then there exists a holomorphic injection $h : W \to \mathbb{C}$ such that for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $h \circ f^p = e^{2i\pi\alpha} \cdot h$ on W. For more details on complex dynamics, see, for example, [**20**].

2.2. *Proof of Theorem 1.* Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$. Fix $m \in \mathbb{N}$.

LEMMA 2.1. *We have*

$$(f^n)^{(m)} = ((e^{O(1)} \cdot d^n)^m + O(d^{(m-1)n})) \cdot f^n \quad \textit{as } n \to \infty \tag{2.1}$$

on $I_\infty(f) \setminus \bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f))$ *locally uniformly. Moreover, for every* $a \in \mathbb{C}$, *the family* $((\log|(f^n)^{(m)} - a|)/(d^n - m))_n$ *of subharmonic functions on* $\mathbb{C}$ *is locally uniformly bounded from above on* $\mathbb{C}$ *and*

$$\lim_{n\to\infty} \frac{\log|(f^n)^{(m)} - a|}{d^n - m} = g_f \tag{2.2}$$

locally uniformly on $I_\infty(f) \setminus \bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f))$.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Proof. Fixing $r \gg 1$, there exists a biholomorphism $w = \psi(z)$ from $\mathbb{P}^1 \setminus \{g_f \le r\}$ to $\mathbb{P}^1 \setminus \{|w| \le e^r\}$, which is called a *Böttcher coordinate* near ∞ associated to f, such that $\psi(f(z)) = \psi(z)^d$ on $\mathbb{P}^1 \setminus \{g_f \le r\}$. Then $\psi(\infty) = \infty$, $\psi' \ne 0$ on $\mathbb{C} \setminus \{g_f \le r\}$, and letting $\iota : \mathbb{P}^1 \to \mathbb{P}^1$ be the involution $z \mapsto 1/z$ (regarding $1/0$ as ∞), $(\iota \circ \psi \circ \iota)'(0) = 1/(\iota \circ \psi^{-1} \circ \iota)'(0) \ne 0$.

We first claim that

$$\frac{(f^n)'}{f^n}(z) = d^n \cdot (1 + O(\psi(z)^{-d^n})) \cdot \frac{\psi'}{\psi}(z) \quad \text{as } n \to \infty \tag{2.3}$$

on $\mathbb{C} \setminus \{g_f \le r\}$ uniformly; indeed, for every $n \in \mathbb{N}$, since $\psi(f^n(z)) = \psi(z)^{d^n}$ on $\mathbb{C} \setminus \{g_f \le r\}$, we have $f^n(z) = \psi^{-1}(\psi(z)^{d^n})$ and $\psi'(f^n(z)) \cdot (f^n)'(z) = d^n \cdot \psi(z)^{d^n - 1} \cdot \psi'(z)$ on $\mathbb{C} \setminus \{g_f \le r\}$, so that

$$\frac{(f^n)'(z)}{f^n(z)} = \frac{d^n \cdot \psi(z)^{d^n-1} \cdot \psi'(z)}{\psi^{-1}(\psi(z)^{d^n}) \cdot \psi'(f^n(z))} = d^n \cdot \frac{\psi(z)^{d^n}/\psi^{-1}(\psi(z)^{d^n})}{\psi'(f^n(z))} \cdot \frac{\psi'(z)}{\psi(z)}$$

on $\mathbb{C} \setminus \{g_f \le r\}$. Moreover, we have

$$\begin{aligned}\frac{\psi(z)^{d^n}}{\psi^{-1}(\psi(z)^{d^n})} &= \frac{(\iota \circ \psi^{-1} \circ \iota)(1/\psi(z)^{d^n}) - (\iota \circ \psi^{-1} \circ \iota)(0)}{1/\psi(z)^{d^n} - 0} \\ &= (\iota \circ \psi^{-1} \circ \iota)'(0) + O(1/\psi(z)^{d^n}) \\ &= \frac{1}{(\iota \circ \psi \circ \iota)'(0)} + O(\psi(z)^{-d^n}) \quad \text{as } n \to \infty\end{aligned}$$

on $\mathbb{C} \setminus \{g_f \le r\}$ uniformly and, since $(\iota \circ \psi \circ \iota)'(1/f^n(z)) = -(\psi'(f^n(z)) \cdot \{-(f^n(z)^2)\})/\psi(f^n(z))^2$ on $\mathbb{C} \setminus \{g_f \le r\}$ by the chain rule, we also have

$$\begin{aligned}\psi'(f^n(z)) &= \frac{(\iota \circ \psi \circ \iota)'(1/f^n(z))}{((\iota \circ \psi \circ \iota)(1/f^n(z))/(1/f^n(z)))^2} \\ &= \frac{(\iota \circ \psi \circ \iota)'(0) + ((\iota \circ \psi \circ \iota)'(1/f^n(z)) - (\iota \circ \psi \circ \iota)'(0))}{(((\iota \circ \psi \circ \iota)(1/f^n(z)) - (\iota \circ \psi \circ \iota)(0))/(1/f^n(z) - 0))^2} \\ &= \frac{(\iota \circ \psi \circ \iota)'(0) + O(1/f^n(z))}{((\iota \circ \psi \circ \iota)'(0) + O(1/f^n(z)))^2} \\ &= \frac{1}{(\iota \circ \psi \circ \iota)'(0)} + O(1/f^n(z)) = \frac{1}{(\iota \circ \psi \circ \iota)'(0)} + O(\psi(z)^{-d^n}) \quad \text{as } n \to \infty\end{aligned}$$

on $\mathbb{C} \setminus \{g_f \le r\}$ uniformly. Hence the claim holds.

For any domain $D \Subset I_\infty(f) \cap \mathbb{C}$ and any $M \in \mathbb{N} \cup \{0\}$ so large that $f^M(D) \subset \mathbb{P}^1 \setminus \{g_f \le r\}$, by (2.3), we have

$$\frac{(f^n)'}{f^n} = \frac{((f^{n-M})' \circ f^M) \cdot (f^M)'}{f^{n-M} \circ f^M} = d^{n-M} \cdot \left(\frac{\psi'}{\psi} \circ f^M \cdot (f^M)'\right) + o(1) \quad \text{as } n \to \infty$$

on some open neighborhood of $\overline{D}$ uniformly. Let us show by induction *that for any* $m \in \mathbb{N}$,

$$\frac{(f^n)^{(m)}}{f^n} = \left(d^{n-M} \cdot \frac{\psi'}{\psi} \circ f^M \cdot (f^M)'\right)^m + O(d^{(m-1)n}) \quad \textit{as } n \to \infty \tag{2.4}$$

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

on some open neighborhood of $\overline{D}$ uniformly. We have just seen (2.4) for $m=1$ on some open neighborhood of $\overline{D}$ uniformly, so assume that $m>1$ and that (2.4) for $m-1$ holds on some open neighborhood of $\overline{D}$ uniformly. Then, using Cauchy's estimate, we have

$$\frac{(f^n)^{(m)}}{f^n}-\frac{(f^n)^{(m-1)}\cdot(f^n)'}{f^n\cdot f^n}=\bigg(\frac{(f^n)^{(m-1)}}{f^n}\bigg)'=O(d^{n(m-1)})\quad\text{as } n\to\infty$$

on some open neighborhood of $\overline{D}$ uniformly, which with (2.4) for both 1 and $m-1$ on some open neighborhood of $\overline{D}$ uniformly yields

$$\begin{aligned}\frac{(f^n)^{(m)}}{f^n}&=\frac{(f^n)^{(m-1)}\cdot(f^n)'}{f^n\cdot f^n}+O(d^{(m-1)n})\\&=\bigg(\bigg(d^{n-M}\cdot\frac{\psi'}{\psi}\circ f^M\cdot(f^M)'\bigg)^{m-1}+O(d^{(m-2)n})\bigg)\\&\quad\cdot\bigg(d^{n-M}\cdot\frac{\psi'}{\psi}\circ f^M\cdot(f^M)'+O(1)\bigg)+O(d^{(m-1)n})\quad\text{as } n\to\infty\end{aligned}$$

on some open neighborhood of $\overline{D}$ uniformly. This yields (2.4) for m on some open neighborhood of $\overline{D}$ uniformly and concludes the induction. Now, if in addition $D\Subset I_\infty(f)\setminus\bigcup_{n\in\mathbb{N}\cup\{0\}}f^{-n}(C(f))$, so $\inf_D|(\psi'/\psi)\circ f^M\cdot(f^M)'|>0$, then estimate (2.4) yields the asymptotic estimate (2.1).

Fix $a\in\mathbb{C}$. The final locally uniform convergence (2.2) follows from (2.1) and (1.1). Then, for every $R>0$ so large that $\bigcup_{n\in\mathbb{N}\cup\{0\}}f^{-n}(C(f)\setminus\{\infty\})\subset\{|z|<R\}$, we also have

$$\frac{\log|(f^n)^{(m)}-a|}{d^n-m}\le\frac{\log(2\max\{|(f^n)^{(m)}|,|a|\})}{d^n-m}\le g_f+O(1)\quad\text{as } n\to\infty$$

on $\{|z|=R\}$ uniformly. Hence, by the maximum principle for subharmonic functions, we deduce that the family $((\log|(f^n)^{(m)}-a|)/(d^n-m))_n$ is locally uniformly bounded from above on $\mathbb{C}$. □

Remark 2.2. (The Schwarzian and pre-Schwarzian derivatives S_{f^n},T_{f^n} of f^n) The expression for $(f^n)^{(m)}$ given by (2.4) in the proof of Lemma 2.1 also quantifies Ye's results [**30**, Theorems 1.1 and 3.3] as

$$\begin{aligned}S_{f^n}&:=\frac{(f^n)'''}{(f^n)'}-\frac{3}{2}\bigg(\frac{(f^n)''}{(f^n)'}\bigg)^2=-2d^{2n}\cdot(\partial_z g_f)^2+O(d^n)\quad\text{and}\\T_{f^n}&:=\frac{(f^n)''}{(f^n)'}=2d^n\cdot\partial_z g_f+O(1)\quad\text{as } n\to\infty\end{aligned}$$

on $I_\infty(f)\setminus\bigcup_{n\in\mathbb{N}\cup\{0\}}f^{-n}(C(f))$ locally uniformly. Indeed, recall that $g_f=\log|\psi|$ so $\partial_z g_f=\psi'/(2\psi)$ on $\mathbb{C}\setminus\{g_f\le r\}$, and $g_f\circ f=d\cdot g_f$ so $(\partial_z g_f)\circ f^M\cdot(f^M)'=d^M\cdot\partial_z g_f$ on $I_\infty(f)$. Hence (2.4) is rewritten as

$$\begin{aligned}(f^n)^{(m)}&=((d^{n-M}\cdot(2\partial_z g_f)\circ f^M\cdot(f^M)')^m+O(d^{(m-1)n}))\cdot f^n\\&=((2d^n\cdot\partial_z g_f)^m+O(d^{(m-1)n}))\cdot f^n\quad\text{as } n\to\infty\end{aligned}$$

on $\overline{D}$ uniformly. For $m\in\{1,2,3\}$, this yields the above asymptotics of S_{f^n} and T_{f^n}.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Fix $a \in \mathbb{C}$, and let us continue the proof of Theorem 1. By the final two assertions in Lemma 2.1, applying to $((\log|(f^n)^{(m)} - a|)/(d^n - m))_n$ a *compactness principle* (see [18, Theorem 4.1.9(a)]) for a family of subharmonic functions on a domain in $\mathbb{R}^N$, there are a sequence (n_j) in $\mathbb{N}$ tending to $+\infty$ as $j \to \infty$ and a subharmonic function ϕ_a on $\mathbb{C}$ such that

$$\phi_a := \lim_{j\to\infty} \frac{\log|(f^{n_j})^{(m)} - a|}{d^{n_j} - m} \quad \text{in } L^1_{\text{loc}}(\mathbb{C}, m_2) \tag{2.5}$$

(m_2 denotes the (real two-dimensional) Lebesgue measure on $\mathbb{C}$). By (2.2), we have $\phi_a \equiv g_f$ m_2-almost everywhere on $I_\infty(f) \setminus \bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f))$, and in turn on $I_\infty(f)$ by the subharmonicity of $\phi_a - g_f$ on $I_\infty(f) \cap \mathbb{C}$. Then also by $I_\infty(f) = \{g_f > 0\}$, the subharmonicity of ϕ_a on $\mathbb{C}$, and the maximum principle for subharmonic functions, we have $\phi_a \le \max_{\{g_f=\epsilon\}} \phi_a = \max_{\{g_f=\epsilon\}} g_f = \epsilon$ on $K(f) = \{g_f = 0\} \subset \{g_f < \epsilon\}$ for every $\epsilon > 0$, and in turn $\phi_a \le 0$ on $K(f)$. By the upper semicontinuity of $\phi_a - g_f$ on $\mathbb{C}$, the subset $\{\phi_a < g_f\}$ is open in $\mathbb{C}$.

LEMMA 2.3. *If $a \ne 0$, then $\phi_a = g_f$ on $\mathbb{C}$.*

Proof. Suppose that $\{\phi_a < g_f\} \ne \emptyset$, and let us show $a = 0$ (see also Remark 2.4 below).

By $\phi_a \equiv g_f$ on $I_\infty(f)$, there is a Fatou component $U \subset K(f)$ of f containing a component W of $\{\phi_a < g_f\}$. Since $\phi_a \le g_f = 0$ on U, we in fact have $U = W$ by the maximum principle for subharmonic functions.

(I) Taking a subsequence of (n_j) if necessary, there is a locally uniform limit

$$g := \lim_{j\to\infty} f^{n_j} \quad \text{on } U.$$

We claim that

$$g^{(m)} \equiv a$$

on U, so in particular we can say $g \in \mathbb{C}[z]$ (of degree at most m); indeed, for any domain $D \Subset U = W$, by Hartogs's lemma for a sequence of subharmonic functions on a domain in $\mathbb{R}^N$ (see [18, Theorem 4.1.9(b)]), we have

$$\limsup_{j\to\infty} \sup_{\overline{D}} \frac{\log|(f^{n_j})^{(m)} - a|}{d^{n_j} - m} \le \sup_{\overline{D}} \phi_a < 0. \tag{2.6}$$

Then $g^{(m)} = (\lim_{j\to\infty}(f^{n_j}))^{(m)} = \lim_{j\to\infty}((f^{n_j})^{(m)}) \equiv a$ on D, so the claim holds.

Hence in the case where g is constant, we have $g^{(m)} \equiv 0 = a$, so we are done.

(II) Let us show that the g in (I) is constant, by contradiction. Suppose to the contrary that g is non-constant. Then, by Hurwitz's theorem and Fatou's classification of cyclic Fatou components of f (see, for example, [20, §16]), there is $N \in \mathbb{N}$ such that $V := f^{n_N}(U) = g(U) (\supset g(\overline{D}))$ is a Siegel disk of f.

Setting $p := \min\{n \in \mathbb{N} : f^n(V) = V\}$, for any $j \ge N$, we have $p|(n_j - n_N)$ and there is a holomorphic injection $h : V \to \mathbb{C}$ such that for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, setting $\lambda := e^{2i\pi\alpha} \in \partial\mathbb{D}$, we have $h \circ f^p = \lambda \cdot h$ on V. Hence, for every $j \ge N$,

$$h \circ f^{n_j} = \lambda^{(n_j - n_N)/p} \cdot (h \circ f^{n_N}) \quad \text{on } U. \tag{2.7}$$

Taking a subsequence of (n_j) if necessary, the limit

$$\lambda_0 := \lim_{j\to\infty} \lambda^{(n_j - n_N)/p} \in \partial\mathbb{D}$$

also exists and then

$$h \circ g = \lambda_0 \cdot (h \circ f^{n_N}) \quad \text{on } U. \tag{2.7$'$}$$

Set $v_0 := h^{-1}(0)$ and fix $z_0 \in U \cap f^{-n_N}(v_0)$, so that $f^p(v_0) = v_0 = g(z_0)$ and $(f^p)'(v_0) = \lambda$. For every $0 < r \ll 1$, $\{|w| < 2r\} \Subset h(V)$, and letting D_r be a component of $(h \circ f^{n_N})^{-1}(\{|w| < r\})$ containing z_0, the restriction $h \circ f^{n_N} : D_r \setminus \{z_0\} \to \{0 < |w| < r\}$ is an unramified covering of degree $\deg_{z_0}(f^{n_N}) = \deg_{z_0} g$. Hence, the restriction $h \circ g : D_r \setminus \{z_0\} \to \{0 < |w| < r\}$ is also an unramified covering of the same degree as that of $h \circ f^{n_N}|D_r$ by Hurwitz's theorem. Let us denote by h^{-1} the holomorphic inverse of the biholomorphism $h : V \to h(V) \subset \mathbb{C}$.

Let us see by induction the following key observation that, for any $\ell \in \mathbb{N}$,

$$((h^{-1})^{(\ell)}(\lambda_0 \cdot h \circ f^{n_N}(z)) \cdot (h \circ f^{n_N}(z))^{\ell})^{(m)} \equiv 0 \quad \text{on } D_r; \tag{2.8}$$

indeed, for every $j \geq N$, applying Cauchy's integration formula to $f^{n_j} - g$ on D_r, by $g^{(m)} \equiv a$, (2.7), and (2.7$'$), we have

$$\begin{aligned}
&\frac{(f^{n_j})^{(m)}(z) - a}{m!} \\
&= \int_{\partial D_r} \frac{f^{n_j}(\zeta) - g(\zeta)}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&= \int_{\partial D_r} \frac{h^{-1}(\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta)) - h^{-1}(\lambda_0 \cdot h \circ f^{n_N}(\zeta))}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&= (\lambda^{(n_j-n_N)/p} - \lambda_0) \cdot \int_{\partial D_r} \frac{\frac{h^{-1}(\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta)) - h^{-1}(\lambda_0 \cdot h \circ f^{n_N}(\zeta))}{\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta) - \lambda_0 \cdot h \circ f^{n_N}(\zeta)} \cdot (h \circ f^{n_N}(\zeta))}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&= (\lambda^{(n_j-n_N)/p} - \lambda_0) \\
&\quad \times \int_{\partial D_r} \frac{((h^{-1})'(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) + O(\lambda^{(n_j-n_N)/p} - \lambda_0)) \cdot (h \circ f^{n_N}(\zeta))}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \quad \text{as } j \to \infty
\end{aligned} \tag{2.9}$$

on D_r, where, recalling $h \circ f^{n_N}(\partial D_r) = \{|w| = r\}$ and $\{|w| < 2r\} \Subset h(V)$ and applying Cauchy's estimate to the holomorphic function $h^{-1}|\{w' \in \mathbb{C} : |w' - w| \leq r\}$ for each $|w| = r$, the $O(\lambda^{(n_j-n_N)/p} - \lambda_0)$ term is estimated as

$$\begin{aligned}
&|O(\lambda^{(n_j-n_N)/p} - \lambda_0)| \\
&\leq \sum_{k=2}^{\infty} \frac{|(h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta))|}{k!} |\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta) - \lambda_0 \cdot h \circ f^{n_N}(\zeta)|^{k-1} \\
&\leq \sum_{k=2}^{\infty} \frac{\max_{|w|=r} |(h^{-1})^{(k)}(w)|}{k!} (|\lambda^{(n_j-n_N)/p} - \lambda_0| \cdot r)^{k-1} \\
&\leq \sum_{k=2}^{\infty} \frac{\max_{|w|=2r} |h^{-1}(w)|}{r^k} (|\lambda^{(n_j-n_N)/p} - \lambda_0| \cdot r)^{k-1}
\end{aligned}$$

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

$$= \frac{\max_{|w|=2r} |h^{-1}(w)|}{r} \cdot \frac{|\lambda^{(n_j-n_N)/p} - \lambda_0|}{1 - |\lambda^{(n_j-n_N)/p} - \lambda_0|} \quad \text{on } \partial D_r$$

so the implicit constant of it is independent of $z \in D_r$ and $\zeta \in \partial D_r$. On the other hand, for every $z \in D_r$, by (2.6) and [23, (3.8)], we also have

$$\limsup_{j\to\infty} \frac{\log |(f^{n_j})^{(m)}(z) - a|}{d^{n_j} - m} < -\delta_z < 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{\log |\lambda^{(n_j-n_N)/p} - \lambda_0|}{d^{n_j} - m} = 0 \tag{2.10}$$

for some $\delta_z > 0$. Hence also by Cauchy's integration formula, we have

$$\begin{aligned}
&\left|\frac{1}{m!}((h^{-1})'(\lambda_0 \cdot h \circ f^{n_N}(z)) \cdot h \circ f^{n_N}(z))^{(m)}\right| \\
&\quad = \left| \int_{\partial D_r} \frac{(h^{-1})'(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) \cdot (h \circ f^{n_N}(\zeta))}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \right| \\
&\quad \leq \frac{e^{-\delta_z(d^{n_j}-m)}}{m! \cdot e^{-(\delta_z/2)(d^{n_j}-m)}} \\
&\qquad + |O(\lambda^{(n_j-n_N)/p} - \lambda_0)| \cdot \frac{\max_{\partial D_r} |h \circ f^{n_N}|}{(\min_{\partial D_r} |\cdot - z|)^{m+1}} \to 0 \quad \text{as } j \to \infty
\end{aligned}$$

for this $z \in D_r$, that is, (2.8) holds for $\ell = 1$.

Next, suppose that (2.8) holds for $1, \ldots, \ell - 1$. Then applying Cauchy's integration formula to $((h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(z)) \cdot (h \circ f^{n_N}(z))^k)^{(m)} \equiv 0$ on D_r for $k \in \{1, \ldots, \ell - 1\}$, also by (2.9), we have

$$\begin{aligned}
&\frac{(f^{n_j})^{(m)}(z) - a}{m!} \\
&= \frac{(f^{n_j})^{(m)}(z) - a}{m!} - \sum_{k=1}^{\ell-1} (\lambda^{(n_j-n_N)/p} - \lambda_0)^k \cdot \frac{((h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(z)) \cdot (h \circ f^{n_N}(z))^k)^{(m)}}{m!\,k!} \\
&= \int_{\partial D_r} \frac{h^{-1}(\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta)) - h^{-1}(\lambda_0 \cdot h \circ f^{n_N}(\zeta))}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&\quad - \sum_{k=1}^{\ell-1} (\lambda^{(n_j-n_N)/p} - \lambda_0)^k \cdot \int_{\partial D_r} \frac{(1/k!)(h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) \cdot (h \circ f^{n_N}(\zeta))^k}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&= \int_{\partial D_r} \frac{\sum_{k=\ell}^{\infty} (1/k!)(h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) \cdot (\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta) - \lambda_0 \cdot h \circ f^{n_N}(\zeta))^k}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&= (\lambda^{(n_j-n_N)/p} - \lambda_0)^\ell \\
&\quad \times \int_{\partial D_r} \frac{\frac{\sum_{k=\ell}^{\infty} (1/k!)(h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) \cdot (\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta) - \lambda_0 \cdot h \circ f^{n_N}(\zeta))^k}{(\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta) - \lambda_0 \cdot h \circ f^{n_N}(\zeta))^\ell} \cdot (h \circ f^{n_N}(\zeta))^\ell}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \\
&= (\lambda^{(n_j-n_N)/p} - \lambda_0)^\ell \\
&\quad \times \int_{\partial D_r} \frac{(1/\ell!)((h^{-1})^{(\ell)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) + O(\lambda^{(n_j-n_N)/p} - \lambda_0)) \cdot (h \circ f^{n_N}(\zeta))^\ell}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \quad \text{as } j \to \infty
\end{aligned}$$

on D_r, where, recalling $h \circ f^{n_N}(\partial D_r) = \{|w| = r\}$ and $\{|w| < 2r\} \Subset h(V)$ and applying Cauchy's estimate to the holomorphic function $h^{-1}|\{w' \in \mathbb{C} : |w' - w| \le r\}$ for each $|w| = r$, the $O(\lambda^{(n_j - n_N)/p} - \lambda_0)$ term is estimated as

$$
\begin{aligned}
&|O(\lambda^{(n_j-n_N)/p} - \lambda_0)| \\
&\quad \le \sum_{k=\ell+1}^{\infty} \frac{|(h^{-1})^{(k)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta))|}{k!} |\lambda^{(n_j-n_N)/p} \cdot h \circ f^{n_N}(\zeta) - \lambda_0 \cdot h \circ f^{n_N}(\zeta)|^{k-\ell} \\
&\quad \le \sum_{k=\ell+1}^{\infty} \frac{\max_{|w|=r} |(h^{-1})^{(k)}(w)|}{k!} (|\lambda^{(n_j-n_N)/p} - \lambda_0| \cdot r)^{k-\ell} \\
&\quad \le \sum_{k=\ell+1}^{\infty} \frac{\max_{|w|=2r} |h^{-1}(w)|}{r^k} (|\lambda^{(n_j-n_N)/p} - \lambda_0| \cdot r)^{k-\ell} \\
&\quad = \frac{\max_{|w|=2r} |h^{-1}(w)|}{r^\ell} \cdot \frac{|\lambda^{(n_j-n_N)/p} - \lambda_0|}{1 - |\lambda^{(n_j-n_N)/p} - \lambda_0|} \quad \text{on } \partial D_r
\end{aligned}
$$

so the implicit constant of it is independent of $z \in D_r$ and $\zeta \in \partial D_r$. Hence, by (2.10) again, also using Cauchy's integration formula, we have

$$
\begin{aligned}
&((h^{-1})^{(\ell)}(\lambda_0 \cdot h \circ f^{n_N}(z)) \cdot (h \circ f^{n_N}(z))^\ell)^{(m)} \\
&\quad = m! \int_{\partial D_r} \frac{(h^{-1})^{(\ell)}(\lambda_0 \cdot h \circ f^{n_N}(\zeta)) \cdot (h \circ f^{n_N}(\zeta))^\ell}{(\zeta - z)^{m+1}} \frac{d\zeta}{2i\pi} \equiv 0 \quad \text{on } D_r,
\end{aligned}
$$

that is, (2.8) holds for ℓ and concludes the induction.

Once this claim (2.8) is at our disposal, for every $\ell \in \mathbb{N}$, there is $P_\ell \in \mathbb{C}[z]$ of degree strictly less than m such that

$$(h^{-1})^{(\ell)}(\lambda_0 \cdot h \circ f^{n_N}(z)) \cdot (h \circ f^{n_N}(z))^\ell \equiv P_\ell(z) \quad \text{on } D_r.$$

Then, recalling $(h \circ f^{n_N})(z_0) = 0$, for every $\ell \ge m$, we have $P_\ell \equiv P_\ell(z_0) = 0$; for, otherwise, we must have $m > \deg P_\ell \ge \deg_{z_0} P_\ell \ge \ell \ge m$, which is a contradiction. Consequently, also by (2.7′) and $(h \circ f^{n_N})(D_r \setminus \{z_0\}) = \{0 < |w| < r\}$, for every $\ell \ge m$,

$$(h^{-1})^{(\ell)}((h \circ g)(z)) = (h^{-1})^{(\ell)}(\lambda_0 \cdot h \circ f^{n_N}(z)) \equiv 0 \quad \text{on } D_r,$$

which implies that there is $Q \in \mathbb{C}[z]$ (of degree strictly less than m) such that $h^{-1} \equiv Q$ on $\{0 < |w| < r\}$ since $h \circ g : D_r \setminus \{z_0\} \to \{0 < |w| < r\}$ is an unramified covering. Then $\deg Q > 0$ since h^{-1} is non-constant on $\{0 < |w| < r\}$.

On the other hand, we also have

$$f^p(Q(w)) = f^p(h^{-1}(w)) = h^{-1}(\lambda w) = Q(\lambda w) \quad \text{on } \{0 < |w| < r\},$$

and in turn $f^p(Q(w)) = Q(\lambda w)$ in $\mathbb{C}[w]$ by the identity theorem for holomorphic functions. Then $Q \in \mathbb{C}[w]$ must be constant since $\deg(f^p) = d^p > 1$. This contradicts $\deg Q > 0$.

(III) Hence g is constant, and the proof of Lemma 2.3 is complete. □

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Using Lemma 2.3, the $L^1_{\mathrm{loc}}(\mathbb{C}, m_2)$-convergence (2.5), a continuity of the Laplacian Δ, and the equalities

$$\Delta \frac{\log |(f^{n_j})^{(m)} - a|}{d^{n_j} - m} = \frac{((f^{n_j})^{(m)})^* \delta_a}{d^{n_j} - m} \quad \text{on } \mathbb{C}$$

for each $j \in \mathbb{N}$ and $\Delta g_f = \mu_f$ on $\mathbb{C}$, whenever $a \in \mathbb{C}^*$, we conclude the desired weak convergence (1.2) on $\mathbb{C}$, and in turn on $\mathbb{P}^1$ since supp $\mu_f \subset \mathbb{C}$. Now the proof of Theorem 1 is complete. □

Remark 2.4. From the proof of Lemma 2.3, no matter whether $a \neq 0$, if all the bounded Fatou components of f are eventually mapped to a Siegel disk of f under the dynamics of f, then $\phi_a = g_f$ on $\mathbb{C}$, and the weak convergence (1.2) on $\mathbb{P}^1$ still holds.

2.3. *On the proof of Theorem 1 for the first and second order derivatives.* In step (II) of the proof of Lemma 2.3 in §2.1, it might be interesting to show that $a = 0$ by direct computations in the case where g is non-constant, instead of showing that g is constant by contradiction. We include herewith such proofs in (II)′ and (II)″ below for the first and second order derivative cases $m = 1, 2$, respectively.

(II)′ Here, assume that $m = 1$ and that g is non-constant. For any $j \geq N$, differentiating both sides in (2.7), by the chain rule, we have

$$(h' \circ f^{n_j}) \cdot (f^{n_j})' = \lambda^{(n_j - n_N)/p} \cdot (h' \circ f^{n_N}) \cdot (f^{n_N})' \quad \text{on } U,$$

so that evaluating them at $z = z_0$, also by $h'(v_0) \neq 0$, we have

$$(f^{n_j})'(z_0) = \lambda^{(n_j - n_N)/p} \cdot (f^{n_N})'(z_0)$$

and, letting $j \to \infty$,

$$g'(z_0) = a = \lambda_0 \cdot (f^{n_N})'(z_0)$$

(here $m = 1$). Hence for any $j \geq N$, we have

$$(\lambda^{(n_j - n_N)/p} - \lambda_0)(f^{n_N})'(z_0) = (f^{n_j})'(z_0) - a.$$

On the other hand, by (2.6) (here $m = 1$) and [23, (3.8)], we have

$$\limsup_{j \to \infty} \frac{\log |(f^{n_j})'(z_0) - a|}{d^{n_j} - 1} < 0 \quad \text{and} \quad \lim_{j \to \infty} \frac{\log |\lambda^{(n_j - n_N)/p} - \lambda_0|}{d^{n_j} - 1} = 0.$$

Hence, we have

$$(f^{n_N})'(z_0) = 0, \tag{2.11}$$

which with $a = \lambda_0 \cdot (f^{n_N})'(z_0)$ yields $a = 0$. □

(II)″ Now assume that $m = 2$ and that g is non-constant. For any $j \geq N$, differentiating both sides in (2.7) twice, by the chain rule, we have

$$(h' \circ f^{n_j}) \cdot (f^{n_j})' = \lambda^{(n_j - n_N)/p} \cdot (h' \circ f^{n_N}) \cdot (f^{n_N})'$$

and then

$$(h'' \circ f^{n_j}) \cdot ((f^{n_j})')^2 + (h' \circ f^{n_j}) \cdot (f^{n_j})'' \\ = \lambda^{(n_j - n_N)/p} \cdot ((h'' \circ f^{n_N}) \cdot ((f^{n_N})')^2 + (h' \circ f^{n_N}) \cdot (f^{n_N})'')$$

on U, so that evaluating them at $z = z_0$, also by $h'(v_0) \neq 0$, we have

$$(f^{n_j})'(z_0) = \lambda^{(n_j - n_N)/p} \cdot (f^{n_N})'(z_0) \tag{2.12}$$

and

$$h''(v_0)((f^{n_j})'(z_0))^2 + h'(v_0)(f^{n_j})''(z_0) \\ = \lambda^{(n_j - n_N)/p} \cdot (h''(v_0) \cdot ((f^{n_N})'(z_0))^2 + h'(v_0)(f^{n_N})''(z_0)), \tag{2.13}$$

and in turn letting $j \to \infty$,

$$g'(z_0) = \lambda_0 \cdot (f^{n_N})'(z_0) \tag{2.14}$$

and

$$h''(v_0)(g'(z_0))^2 + h'(v_0)a = \lambda_0 \cdot (h''(v_0)((f^{n_N})'(z_0))^2 + h'(v_0)(f^{n_N})''(z_0)) \tag{2.15}$$

(here $m = 2$ so $a = g''(z_0)$). Hence, for any $j \geq N$, subtracting (2.15) from (2.13) and then eliminating $(f^{n_j})'(z_0)$ and $g'(z_0)$ by (2.12) and (2.14), the above four equalities yield

$$h''(v_0) \cdot ((\lambda^{(n_j - n_N)/p})^2 - \lambda_0^2)((f^{n_N})'(z_0))^2 - h'(v_0)((f^{n_j})''(z_0) - a) \\ = (\lambda^{(n_j - n_N)/p} - \lambda_0) \cdot (h''(v_0) \cdot ((f^{n_N})'(z_0))^2 + h'(v_0) \cdot (f^{n_N})''(z_0)),$$

which is rewritten as

$$\frac{(f^{n_j})''(z_0) - a}{\lambda^{(n_j - n_N)/p} - \lambda_0} = \frac{(\lambda^{(n_j - n_N)/p} + \lambda_0 - 1)h''(v_0)((f^{n_N})'(z_0))^2 - h'(v_0) \cdot (f^{n_N})''(z_0)}{h'(v_0)} \\ = (\lambda^{(n_j - n_N)/p} - \lambda_0) \cdot \frac{h''(v_0)((f^{n_N})'(z_0))^2}{h'(v_0)} \\ + \frac{(2\lambda_0 - 1)h''(v_0)((f^{n_N})'(z_0))^2 - h'(v_0) \cdot (f^{n_N})''(z_0)}{h'(v_0)}. \tag{2.16}$$

On the other hand, by (2.6) (here $m = 2$) and [23, (3.8)], we have

$$\limsup_{j \to \infty} \frac{\log |(f^{n_j})''(z_0) - a|}{d^{n_j} - 2} < 0 \quad \text{and} \quad \lim_{j \to \infty} \frac{\log |\lambda^{(n_j - n_N)/p} - \lambda_0|}{d^{n_j} - 2} = 0. \tag{2.17}$$

Hence, letting $j \to \infty$ in (2.16), we must have

$$(2\lambda_0 - 1)h''(v_0)((f^{n_N})'(z_0))^2 - h'(v_0) \cdot (f^{n_N})''(z_0) = 0, \tag{2.18}$$

which with (2.16) in turn yields

$$\frac{(f^{n_j})''(z_0) - a}{(\lambda^{(n_j - n_N)/p} - \lambda_0)^2} = \frac{(f^{n_N})''(z_0)}{2\lambda_0 - 1} \tag{2.16$'$}$$

for any $j \geq N$. Then by (2.17) again, from (2.16′), we have

$$(f^{n_N})''(z_0) = 0, \tag{2.19}$$

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

which with (2.18) and (2.14) yields

$$h''(v_0)((f^{n_N})'(z_0))^2 = 0 \quad \text{and} \quad 0 = \lambda_0^2 \cdot h''(v_0)((f^{n_N})'(z_0))^2 = h''(v_0)(g'(z_0))^2. \tag{2.20}$$

Consequently, by (2.15), (2.19), (2.20), and $h'(v_0) \neq 0$, we have $a = 0$. □

3. *Proofs of Theorems 2 and 3*

3.1. *Non-archimedean dynamics of polynomials of degree at least* 2. Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$. The Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is a compact augmentation of the *classical* projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$ and is also locally compact, Hausdorff, and uniquely arcwise connected. Let us go into more detail. As a set, the Berkovich affine line $\mathsf{A}^1 = \mathsf{A}^1(K)$ is the set of all multiplicative seminorms $K[z]$ which restricts to $|\cdot|$ on K. We write an element of A^1 like $\mathcal{S}$ and denote it by $[\cdot]_{\mathcal{S}}$ as a multiplicative seminorm on $K[z]$. A K*-closed disk* is a subset in K written as $B(a, r) := \{z \in K : |z - a| \leq r\}$ for some $a \in K$ and $r \geq 0$; by the strong triangle inequality, for any $b \in B(a, r)$, we have $B(b, r) = B(a, r)$, and for any two K-closed disks B, B' having non-empty intersection, we have either $B \subset B'$ or $B \supset B'$. By Berkovich's representation [6], any element $\mathcal{S} \in \mathsf{A}^1$ is induced by a non-increasing and nesting sequence (B_n) of K-closed disks in that

$$[\phi]_{\mathcal{S}} = \inf_{n\in\mathbb{N}} \sup_{z\in B_n} |\phi(z)| \quad \text{for any } \phi \in K[z]. \tag{3.1}$$

In particular, each point $a \in K$ is regarded as an element of A^1 induced by the (constant sequence of the) K-closed disk $B(a, 0) = \{a\}$, and more generally, each K-closed disk B is regarded as an element of A^1 induced by (the constant sequence of) B. In particular, K is regarded as a subset of A^1. The relative topology of A^1 is the weakest topology such that for any $\phi \in K[z]$, $\mathsf{A}^1 \ni \mathcal{S} \mapsto [\phi]_{\mathcal{S}} \in \mathbb{R}_{\geq 0}$ is continuous, and then A^1 is a locally compact, uniquely arcwise connected, Hausdorff topological space. The action on K of a polynomial $h \in K[z]$ continuously extends to A^1 as

$$[\phi]_{h(\mathcal{S})} = [\phi \circ h]_{\mathcal{S}} \quad \text{for every } \mathcal{S} \in \mathsf{A}^1 \text{ and every } \phi \in K[z], \tag{3.2}$$

preserving K and $\mathsf{A}^1 \setminus K$ if, in addition, $\deg h > 0$.

As a set, P^1 is nothing but $\mathsf{A}^1 \cup \{\infty\}$, regarding $\mathbb{P}^1$ as $K \cup \{\infty\}$, and as a topological space, P^1 is identified with the one-point compactification of A^1. An ordering $\leq_\infty$ on A^1 is defined so that for any $\mathcal{S}, \mathcal{S}' \in \mathsf{A}^1$, $\mathcal{S} \leq_\infty \mathcal{S}'$ if and only if $[\cdot]_{\mathcal{S}} \leq_\infty [\cdot]_{\mathcal{S}'}$ on $K[z]$, and this $\leq_\infty$ extends to the ordering on P^1 so that $\mathcal{S} \leq_\infty \infty$ for every $\mathcal{S} \in \mathsf{P}^1$. For any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, if $\mathcal{S} \leq_\infty \mathcal{S}'$, then set $[\mathcal{S}, \mathcal{S}'] = [\mathcal{S}', \mathcal{S}] := \{\mathcal{S}'' \in \mathsf{P}^1 : \mathcal{S} \leq_\infty \mathcal{S}'' \leq_\infty \mathcal{S}'\}$, and in general, we have $[\mathcal{S}, \infty] \cap [\mathcal{S}', \infty] = [\mathcal{S} \wedge_\infty \mathcal{S}', \infty]$, for some (unique) point $\mathcal{S} \wedge_\infty \mathcal{S}' \in \mathsf{P}^1$, and then set $[\mathcal{S}, \mathcal{S}'] := [\mathcal{S}, \mathcal{S} \wedge_\infty \mathcal{S}'] \cup [\mathcal{S} \wedge_\infty \mathcal{S}', \mathcal{S}']$. These *closed intervals* $[\mathcal{S}, \mathcal{S}'] \subset \mathsf{P}^1$ make P^1 an '$\mathbb{R}$'-tree in the sense of Jonsson [19, Definition 2.2]. For any $\mathcal{S} \in \mathsf{P}^1$, the equivalence class $T_{\mathcal{S}}\mathsf{P}^1 := (\mathsf{P}^1 \setminus \{\mathcal{S}\})/\sim$ is defined so that for any $\mathcal{S}', \mathcal{S}'' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$, $\mathcal{S}' \sim \mathcal{S}''$ if $[\mathcal{S}, \mathcal{S}'] \cap [\mathcal{S}, \mathcal{S}''] = [\mathcal{S}, \mathcal{S}' \wedge_{\mathcal{S}} \mathcal{S}'']$ for some (unique) point $\mathcal{S}' \wedge_{\mathcal{S}} \mathcal{S}'' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$. An element $\boldsymbol{v}$ of $T_{\mathcal{S}}\mathsf{P}^1$ is called a *direction* of P^1 at $\mathcal{S}$, which is denoted by $U(\boldsymbol{v})$ as a subset in $\mathsf{P}^1 \setminus \{\mathcal{S}\}$ and, if $\mathcal{S}' \in U(\boldsymbol{v})$, also by $\overrightarrow{\mathcal{S}\mathcal{S}'}$. A point $\mathcal{S} \in \mathsf{P}^1 \setminus \mathbb{P}^1$ is said to be of

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

type II, III, or IV, respectively if $\#T_{\mathcal{S}}\mathsf{P}^1 > 2, = 2$, or $= 1$, and let us denote by $\mathsf{H}^1_{\mathrm{II}}$, $\mathsf{H}^1_{\mathrm{III}}$, or $\mathsf{H}^1_{\mathrm{IV}}$ the set of all points in P^1 of type II, III, or IV, respectively. A non-empty subset in P^1 is called a *simple domain* (or a *Berkovich connected open affinoid*) if it is the intersection of some finitely many elements of $\{U(\boldsymbol{v}) : \mathcal{S} \in \mathsf{H}^1_{\mathrm{II}} \cup \mathsf{H}^1_{\mathrm{III}}, \boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1\}$. The topology of P^1 has an open basis consisting of all simple domains in P^1; in particular, a simple domain is nothing but a component of the complement in P^1 of a finite subset in $\mathsf{H}^1_{\mathrm{II}} \cup \mathsf{H}^1_{\mathrm{III}}$.

The point $[\cdot]_{\mathcal{O}_K}$ in P^1, where $\mathcal{O}_K := \{z \in K : |z| \le 1\}$ is the ring of K-integers, is called the *Gauss* or *canonical* point in P^1 and is denoted by $\mathcal{S}_{\mathrm{can}}$. Let us denote the continuous extension of $|\cdot|$ to A^1 by the same $|\cdot|$ for simplicity. More generally, let $|\mathcal{S} - \mathcal{S}'|$ be the *Hsia kernel* on A^1, which is the upper semicontinuous and separately continuous extension to $\mathsf{A}^1 \times \mathsf{A}^1$ of the function $|z - w|$ on $K \times K$ (although $\mathcal{S} - \mathcal{S}'$ itself is undefined unless $\mathcal{S}, \mathcal{S}' \in K$), and then, writing $|\mathcal{S}| = |\mathcal{S} - 0|$ for each $\mathcal{S} \in \mathsf{P}^1$, the function

$$[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} := \frac{|\mathcal{S} - \mathcal{S}'|}{\max\{1, |\mathcal{S}|\} \max\{1, |\mathcal{S}'|\}}$$

on $\mathsf{A}^1 \times \mathsf{A}^1$ extends to the *generalized* Hsia kernel on P^1 *with respect to* $\mathcal{S}_{\mathrm{can}}$, which is the upper semicontinuous and separately continuous extension to $\mathsf{P}^1 \times \mathsf{P}^1$ of the (normalized) chordal metric on $\mathbb{P}^1$ [3, §4.4].

A function $g : \mathsf{P}^1 \to \mathbb{R} \cup \{\pm\infty\}$ is said to be $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic if there is a probability Radon measure μ_g on P^1 such that

$$g = \int_{\mathsf{P}^1} \log[\cdot, \mathcal{S}']_{\mathrm{can}} \mu_g(\mathcal{S}') + \text{const.} \quad \text{on } \mathsf{P}^1; \tag{3.3}$$

then g belongs to the class $\mathrm{BDV}(\mathsf{P}^1)$, is not only upper semicontinuous on P^1 but also continuous on any closed interval in P^1, and satisfies

$$\Delta g = \mu_g - \delta_{\mathcal{S}_{\mathrm{can}}} \tag{3.4}$$

on P^1 (see [14, §2.4], and also [3, §5.8 and §6.3] for more details including specific information on $\mathrm{BDV}(\mathsf{P}^1)$). Here $\Delta = \Delta_{\mathsf{P}^1}$ is the Laplacian on P^1 (see [3, §5], [14, §2.4]; in [3] the opposite sign convention on Δ is adopted). For example, the function $\log \max\{1, |\cdot|\}$ on A^1 extends to a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1 so that

$$-\log \max\{1, |\cdot|\} = \log[\cdot, \infty]_{\mathrm{can}} = \int_{\mathsf{P}^1} \log[\cdot, \mathcal{S}']_{\mathrm{can}} \delta_\infty(\mathcal{S}') \quad \text{on } \mathsf{P}^1$$

and $\Delta(-\log \max\{1, |\cdot|\}) = \delta_\infty - \delta_{\mathcal{S}_{\mathrm{can}}}$ on P^1.

The continuous action on $\mathbb{P}^1$ of a rational function $h \in K(z)$ canonically extends to P^1. If in addition h is non-constant, then the action of h on P^1 preserves both $\mathbb{P}^1$ and $\mathsf{P}^1 \setminus \mathbb{P}^1$ and is open and surjective. The local degree function $w \mapsto \deg_w h$ on $\mathbb{P}^1$ also canonically extends to an upper semicontinuous function on P^1, satisfying $\sum_{\mathcal{S}' \in h^{-1}(\mathcal{S})} \deg_{\mathcal{S}'} h = \deg h$ for each $\mathcal{S} \in \mathsf{P}^1$. In particular, the action of h on P^1 induces the *pullback* action on the space of Radon measures on P^1 so that, letting $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at each $\mathcal{S} \in \mathsf{P}^1$, $h^*\delta_{\mathcal{S}} = \sum_{\mathcal{S}' \in h^{-1}(\mathcal{S})} (\deg_{\mathcal{S}'} h) \delta_{\mathcal{S}'}$ on P^1.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Let $f \in K[z]$ be a polynomial of degree $d > 1$. The *Berkovich* filled-in Julia set of f is

$$\mathsf{K}(f) := \Big\{\mathcal{S} \in \mathsf{A}^1 : \limsup_{n\to\infty} |f^n(\mathcal{S})| < \infty\Big\},$$

which is a compact subset in A^1, and the *escape rate function* of f on A^1 is the limit

$$g_f := \lim_{n\to\infty} \frac{\log\max\{1, |f^n|\}}{d^n} \quad \text{on } \mathsf{A}^1;$$

the function $T_f := (\log\max\{1, |f(\cdot)|\})/d - \log\max\{1, |\cdot|\}$ on P^1 is an $\mathbb{R}$-valued continuous and $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1 and satisfies $\Delta T_f = (f^*\delta_{\mathcal{S}_{\mathrm{can}}})/d - \delta_{\mathcal{S}_{\mathrm{can}}}$ on P^1, the difference $g_f - \log\max\{1, |\cdot|\}$ is the restriction to A^1 of the uniform limit $\sum_{j=0}^{\infty} d^{-j}(f^j)^* T_f$ on P^1, which is still an $\mathbb{R}$-valued continuous and $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1, and the function $g_f - (\log\max\{1, |f^n|\})/d^n$ for each $n \in \mathbb{N} \cup \{0\}$ extends continuously to P^1 so that

$$g_f - \frac{\log\max\{1, |f^n|\}}{d^n} = O(d^{-n}) \quad \text{as } n \to \infty \tag{3.5}$$

on P^1 uniformly. The function g_f is continuous, subharmonic, and non-negative on A^1, is harmonic and strictly positive on $\mathsf{A}^1 \setminus \mathsf{K}(f)$, and is zero on $\mathsf{K}(f)$ (for harmonic/subharmonic functions on an open subset in P^1, see [3, §7 and §8]). The *equilibrium* (or *canonical*) measure of f is the probability Radon measure

$$\mu_f := \Delta(g_f - \log\max\{1, |\cdot|\}) + \delta_{\mathcal{S}_{\mathrm{can}}} = \Delta g_f - \delta_\infty \quad \text{on } \mathsf{P}^1,$$

which is the weak limit $\lim_{n\to\infty}((f^n)^*\delta_{\mathcal{S}_{\mathrm{can}}})/d^n$ on P^1 and supported exactly by $\partial\mathsf{K}(f)$. The *Berkovich* superattractive basin

$$\mathsf{I}_\infty(f) := \Big\{z \in \mathsf{P}^1 : \lim_{n\to\infty} f^n(z) = \infty\Big\}$$

of f associated to the superattracting fixed point ∞ of f is a domain in P^1 containing ∞, and coincides with $\mathsf{P}^1 \setminus \mathsf{K}(f)$. Let $C(f)$ be the (classical) critical set of f; if K is of characteristic 0, then $C(f)$ consists of ∞ and all the (at most $d-1$) zeros of f' on K, and $\bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(C(f)\setminus\{\infty\})$ is bounded in K.

The *Berkovich Julia set* of f is defined as

$$\mathsf{J}(f) := \operatorname{supp}\mu_f = \partial\mathsf{K}(f).$$

The *Berkovich Fatou set* $\mathsf{F}(f)$ of f is defined by $\mathsf{P}^1 \setminus \mathsf{J}(f)$, and a component of $\mathsf{F}(f)$ is called a *Berkovich Fatou component* of f. Both $\mathsf{J}(f)$ and $\mathsf{F}(f)$ are totally invariant under f and any Berkovich Fatou component of f is either $\mathsf{I}_\infty(f)$ or a component of the interior of $\mathsf{K}(f)$.

Set $c_d := \lim_{K\ni z\to\infty} f(z)/z^d \in K^* = K \setminus \{0\}$. By the definition of μ_f and (3.3), the function $\mathcal{S} \mapsto \int_{\mathsf{P}^1} \log|\mathcal{S} - \mathcal{S}'|\mu_f(\mathcal{S}') - g_f(\mathcal{S})$ is constant on P^1. This with (3.5) and the strong triangle inequality yields the identity

$$\int_{\mathsf{P}^1} \log|\mathcal{S} - \mathcal{S}'|\mu_f(\mathcal{S}') \equiv g_f(\mathcal{S}) - \frac{\log|c_d|}{d-1} (\equiv \log|\mathcal{S}| \text{ if } |\mathcal{S}| \gg 1) \quad \text{on } \mathsf{P}^1. \tag{3.6}$$

For more details on the harmonic analysis and dynamics on P^1, see [3, 5, 12, 14, 19].

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

3.2. *Arithmetic dynamics of polynomials of degree at least* 2. Let k be a product formula field as in §1.3. Let $f \in k[z]$ be a polynomial of degree $d > 1$. For each $v \in M_k$, we obtain $g_{f,v}$ and $\mu_{f,v}$ on $\mathsf{P}^1(\mathbb{C}_v)$ from the action of f on $\mathbb{P}^1(\mathbb{C}_v)$. Writing $f(z)$ as $\sum_{j=0}^{d} c_j z^j \in k[z]$, so $c_d \in k^*$, there is a finite set E_f containing all the infinite places of k such that for every $v \in M_k \setminus E_f$, $|c_d|_v = 1, |c_0|_v, \ldots, |c_{d-1}|_v \leq 1$ and, moreover, $g_{f,v} = \log \max\{1, |\cdot|_v\}$ and $\mu_{f,v} = \delta_{\mathcal{S}_{\mathrm{can},v}}$ on $\mathsf{P}^1(\mathbb{C}_v)$.

Recall that an embedding of $\overline{k}$ in $\mathbb{C}_v$ is fixed for each $v \in M_k$. The Call–Silverman *f-canonical height* of an effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$ supported by $\overline{k}$ is

$$0 \leq \hat{h}_f(\mathcal{Z}) := \sum_{v \in M_k} N_v \frac{\sum_{z \in \overline{k}: p(z)=0} (\deg_z p) g_{f,v}(z)}{\deg p} \tag{3.7}$$

$$= h_{\mathrm{nv}}(\mathcal{Z}) + \sum_{v \in E_f} N_v \frac{\sum_{z \in \overline{k}: p(z)=0} (\deg_z p)(g_{f,v}(z) - \log \max\{1, |z|_v\})}{\deg p},$$

where $p \in k[z]$ is a representative of $\mathcal{Z}$ (so $\deg p > 0$) and the *naive* height

$$h_{\mathrm{nv}}(\mathcal{Z}) := \sum_{v \in M_k} N_v \frac{\sum_{z \in \overline{k}: p(z)=0} (\deg_z p) \log \max\{1, |z|_v\}}{\deg p}$$

of $\mathcal{Z}$ is in fact a *finite* sum by a standard argument involving the ramification theory of valuations (or [21, Lemma 2.3]). For every $v \in M_k$, setting $a_p := p^{(\deg p)}/(\deg p)! \in k^*$ (i.e., a_p is the coefficient of the monomial of p having the maximal degree $\deg p$), we have $\log |p(\cdot)|_v = \sum_{z \in \overline{k}: p(z)=0} (\deg_z p) \log |\cdot - z|_v + \log |a_p|_v$ on $\mathsf{A}^1(\mathbb{C}_v)$, integrating both sides of which against $\mu_{f,v}$ over $\mathsf{P}^1(\mathbb{C}_v)$, also by (3.6), we have

$$\int_{\mathsf{P}^1(\mathbb{C}_v)} \log |p|_v \mu_{f,v} = \sum_{z \in \overline{k}: p(z)=0} (\deg_z p) \int_{\mathsf{P}^1(\mathbb{C}_v)} \log |z - \mathcal{S}'|_v \mu_{f,v}(\mathcal{S}') + \log |a_p|_v$$

$$= \sum_{z \in \overline{k}: p(z)=0} (\deg_z p) g_{f,v}(z) - (\deg p) \cdot \frac{\log |c_d|_v}{d-1} + \log |a_p|_v.$$

Consequently, also by the product formula property of k, the defining equality (3.7) of $\hat{h}_f(\mathcal{Z})$ is rewritten as the *Mahler-type formula*

$$\hat{h}_f(\mathcal{Z}) = \sum_{v \in M_k} N_v \frac{\int_{\mathsf{P}^1(\mathbb{C}_v)} \log |p|_v \mu_{f,v}}{\deg p} \tag{3.7$'$}$$

(cf. [21, (1.1)]). For more details on canonical heights on P^1, see [1, 2, 9, 13]. For the treatment of effective divisors rather than Galois conjugacy classes, which are effective divisors represented by *irreducible* polynomials, see [21].

3.3. *Proofs of Theorems 2 and 3.* Let K be an algebraically closed field of characteristic 0 that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$. Let $f \in K[z]$ be a polynomial of degree $d > 1$, and fix $m \in \mathbb{N}$.

The following is a non-archimedean counterpart to Lemma 2.1.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

LEMMA 3.1. *We have*

$$(f^n)^{(m)} = ((e^{O(1)} \cdot d^n)^m + O(d^{(m-1)n})) \cdot f^n \quad \text{as } n \to \infty \tag{2.1'}$$

on $\mathsf{I}_\infty(f) \setminus \bigcup_{n \in \mathbb{N}\cup\{0\}} f^{-n}(C(f))$ *locally uniformly. Moreover, for every* $a \in K$*, the family* $((\log|(f^n)^{(m)} - a|)/(d^n - m) - \log\max\{1, |\cdot|\})_n$ *of* $\delta_{\mathcal{S}_{\text{can}}}$*-subharmonic functions on* P^1 *is locally uniformly bounded from above on* P^1 *and*

$$\lim_{n\to\infty}\left(\frac{\log|(f^n)^{(m)} - a|}{d^n - m} - g_f\right) = 0 \tag{2.2'}$$

on $\mathsf{I}_\infty(f) \setminus \bigcup_{n \in \mathbb{N}\cup\{0\}} f^{-n}(C(f))$ *locally uniformly.*

Proof. Fixing $r \gg 1$, there is a (rigid) *biholomorphism* $w = \psi(z)$ from $\mathbb{P}^1 \setminus \{g_f \le r\}$ to $\mathbb{P}^1 \setminus \{|w| \le e^r\}$, which is called a (non-archimedean) Böttcher coordinate near ∞ associated to f, such that $\psi(f(z)) = \psi(z)^d$ on $\mathbb{P}^1 \setminus \{g_f \le r\}$ (see Rivera-Letelier [26, the proof of Proposition 3.3(ii)]). Then $\psi(\infty) = \infty$ and $\psi' \neq 0$ on $\mathbb{P}^1 \setminus \{g_f \le r\}$. By a computation similar to that in the proof of Lemma 2.1, we have

$$\frac{(f^n)'}{f^n}(z) = d^n \cdot (1 + O(\psi(z)^{-d^n}))\frac{\psi'}{\psi}(z) \quad \text{as } n \to \infty \tag{2.3'}$$

on $K \setminus \{g_f \le r\}$ uniformly.

For any simple domain $D \Subset I_\infty(f) \cap \mathsf{A}^1$ and any $M \in \mathbb{N} \cup \{0\}$ so large that $f^M(D) \subset \mathsf{P}^1 \setminus \{g_f \le r\}$, from (2.3′), we also have

$$\frac{(f^n)'}{f^n} = d^{n-M} \cdot \frac{\psi'}{\psi} \circ f^M \cdot (f^M)' + o(1) \quad \text{as } n \to \infty$$

on $D \cap \mathbb{P}^1$ uniformly. Now fix $m \in \mathbb{N}$. Then noting that, by the definition of a simple domain, there is $0 < \epsilon \ll 1$ such that $B(z, \epsilon) \subset D \cap \mathbb{P}^1$ for any $z \in D \cap \mathbb{P}^1$, an induction which is similar to that in the proof of Lemma 2.1 and involves the (non-archimedean) Cauchy estimate for (rigid) analytic functions on those disks $B(z, \epsilon)$ yields

$$\frac{(f^n)^{(m)}}{f^n} = \left(d^{n-M} \cdot \frac{\psi'}{\psi} \circ f^M \cdot (f^M)'\right)^m + O(d^{(m-1)n}) \quad \text{as } n \to \infty \tag{2.4'}$$

on $D \cap \mathbb{P}^1$ uniformly. If in addition $D \Subset I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}\cup\{0\}} f^{-n}(C(f))$, so $\inf_D |(\psi'/\psi) \circ f^M \cdot (f^M)'| > 0$, then this (2.4′) yields the asymptotic estimate (2.1′) on $D \cap \mathbb{P}^1$ uniformly, and in turn on D uniformly by the continuity of $|(f^n)^{(m)}/f^n|$ on D and the density of $\mathbb{P}^1$ in P^1.

Fix also $a \in K$. The locally uniform convergence (2.2′) on $\mathsf{I}_\infty(f) \setminus \bigcup_{n \in \mathbb{N}\cup\{0\}} f^{-n}(C(f))$ follows from the estimate (2.1′). In particular, for $R \gg 1$, letting $\mathcal{S}_R \in [0, \infty] \setminus \mathbb{P}^1$ be the point in $\mathsf{P}^1 \setminus \mathbb{P}^1$ induced by the (constant sequence of the) K-closed disk $B(0, R)$, we have the convergence (2.2′) at $\mathcal{S} = \mathcal{S}_R$, and in turn, by the maximum principle for subharmonic functions (cf. [3, Proposition 8.14]), the family $(\log|(f^n)^{(m)} - a|/(d^n - m))_n$ is uniformly bounded from above on $\mathsf{P}^1 \setminus U(\overrightarrow{\mathcal{S}_R\infty})$ (whose boundary is $\{\mathcal{S}_R\}$). Similarly, for $R \gg 1$, noting that $\log|(f^n)^{(m)}/f^n|$ is a subharmonic function on $U(\overrightarrow{\mathcal{S}_R\infty})$ (whose boundary is $\{\mathcal{S}_R\}$), by the maximum principle for subharmonic functions (and (3.5)),

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

we have

$$\frac{\log|(f^n)^{(m)}|}{d^n-m}-\log\max\{1,|\cdot|\}\le\left(\frac{\log|f^n|}{d^n-m}+O(nd^{-n})\right)-\log\max\{1,|\cdot|\}$$
$$=g_f-\log\max\{1,|\cdot|\}+O(nd^{-n})=O(nd^{-n})\quad\text{as } n\to\infty$$

on $U(\overrightarrow{\mathcal{S}_R\infty})$ uniformly. Hence the family

$$((\log|(f^n)^{(m)}-a|)/(d^n-m)-\log\max\{1,|\cdot|\})_n$$

is locally uniformly bounded from above on P^1. □

Fix also $a\in K$. By the second and the last assertions in Lemma 3.1, a *compactness principle* for a family of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 (cf. [3, Proposition 8.57], [14, Proposition 2.18]) yields a sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j\to\infty$ and a function $\phi=\phi_a:\mathsf{P}^1\to\mathbb{R}\cup\{-\infty\}$ such that $\phi+(g_f-\log\max\{1,|\cdot|\})$ is a $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic function on P^1 (so, in particular, $\phi+g_f$ is subharmonic on A^1) and that

$$\phi=\lim_{j\to\infty}\left(\frac{\log|(f^{n_j})^{(m)}-a|}{d^{n_j}-m}-g_f\right)$$
$$\left(=\lim_{j\to\infty}\left(\frac{\log|(f^{n_j})^{(m)}-a|}{d^{n_j}-m}-\log\max\{1,|\cdot|\}\right)-(g_f-\log\max\{1,|\cdot|\})\right)\text{ on }\mathsf{P}^1\setminus\mathbb{P}^1.$$

Then, by (2.2′), we have $\phi\equiv0$ on $\mathsf{I}_\infty(f)\setminus\mathbb{P}^1$, and in turn

$$\phi\equiv0\quad\text{on }\mathsf{I}_\infty(f)\cup\mathsf{J}(f)$$

by $\mathsf{J}(f)=\partial\mathsf{I}_\infty(f)$ and the continuity of ϕ on any closed interval in P^1, and then $\phi(=\phi+g_f)\le0$ on $\mathsf{K}(f)$ by the maximum principle for subharmonic functions.

Let us also show that

$$\limsup_{n\to\infty}\frac{\int_{\mathsf{P}^1}\log|(f^n)^{(m)}-a|\mu_f}{d^n-m}\le0,\tag{3.8}$$

which will be used in the proof of Theorem 3 (but not in that of Theorem 2); indeed,

$$\limsup_{j\to\infty}\frac{\int_{\mathsf{P}^1}\log|(f^{n_j})^{(m)}-a|\mu_f}{d^{n_j}-m}\le\limsup_{j\to\infty}\sup_{\mathsf{J}(f)}\frac{\log|(f^{n_j})^{(m)}-a|}{d^{n_j}-m}$$
$$=\limsup_{j\to\infty}\sup_{\mathsf{J}(f)}\left(\frac{\log|(f^{n_j})^{(m)}-a|}{d^{n_j}-m}-\log\max\{1,|\cdot|\}+\log\max\{1,|\cdot|\}\right)$$
$$\le\sup_{\mathsf{J}(f)}((\phi+g_f-\log\max\{1,|\cdot|\})+\log\max\{1,|\cdot|\})=\sup_{\mathsf{J}(f)}(\phi+g_f)=0,$$

where the first inequality is by $\operatorname{supp}\mu_f=:\mathsf{J}(f)$, and the second one is by the continuity of $\log\max\{1,|\cdot|\}$ on $\mathsf{J}(f)$ and a version of Hartogs's lemma for a sequence of $\delta_{\mathcal{S}_{\mathrm{can}}}$-subharmonic functions on P^1 ([3, Proposition 8.57], [14, Proposition 2.18]).

Proof of Theorem 2. We continue the above argument. Suppose that the open subset $\{\phi<0\}$ is non-empty. Then since $\phi\equiv0$ on $\mathsf{I}_\infty(f)$, there is a Berkovich Fatou component U of f other than $\mathsf{I}_\infty(f)$ (so $U\Subset\mathsf{A}^1$) such that $U\cap\{\phi<0\}\ne\emptyset$, and then ∂U is a

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

singleton, say $\{\mathcal{S}_0\}$, in $\mathsf{P}^1 \setminus \mathbb{P}^1$ (see [24, Lemma 2.1]). Moreover,

$$\phi \equiv 0 \quad \text{on } \partial U \subset \mathsf{J}(f).$$

Now set

$$\psi := \begin{cases} \phi & \text{on } U \\ 0 & \text{on } \mathsf{P}^1 \setminus U \end{cases} : \mathsf{P}^1 \to \mathbb{R}_{\le 0} \cup \{-\infty\},$$

so in particular that $\phi \le \psi$ on P^1, and we claim that the function $\psi + g_f$ is domination subharmonic on A^1, that is, $\psi + g_f$ is upper semicontinuous and $\not\equiv -\infty$ on A^1 and, for every harmonic function h on a simple domain $W \Subset \mathsf{A}^1$, if $\psi + g_f \le h$ on ∂W, then $\psi + g_f \le h$ on W (for the domination subharmonicity, which is in fact equivalent to the subharmonicity, of a function on an open subset in P^1, see [3, §8.2]); indeed, the function $\psi + g_f$ is not only upper semicontinuous on A^1 (since so is $\phi + g_f$ on A^1 and $\phi \equiv 0$ on ∂U) but also subharmonic on $\mathsf{A}^1 \setminus \partial U$ (since so are $\phi + g_f$ and g_f on U and $\mathsf{A}^1 \setminus \overline{U}$, respectively). Pick a harmonic function h on a simple domain $W \Subset \mathsf{A}^1$ and suppose that $\psi + g_f \le h$ on ∂W, or equivalently, that $\psi + g_f - h \le 0$ on ∂W. Then, noting that h extends continuously on $\overline{W}$, $\psi + g_f - h$ is upper semicontinuous on $\overline{W}$ so attains the maximum, say M, at some point $\mathcal{S} \in \overline{W}$. If $\mathcal{S} \in W \setminus \partial U$, then for any simple domain $W' \Subset W \setminus \partial U$ containing $\mathcal{S}$, the (domination) subharmonicity of $\psi + g_f - h$ on $W \setminus \partial U$ yields $(\psi + g_f - h)(\mathcal{S}) \le \int_{\partial(W')} (\psi + g_f - h)\mu_{\mathcal{S},W'}$, where $\mu_{\cdot,W'}$ is the Poisson–Jensen (or harmonic) measure associated to $\overline{W'}$ (for details on Poisson's integrals and Poisson–Jensen (or harmonic) measures, see [2, §7.3], [28, §3]). Then $\psi + g_f - h$ attains the maximum M at any point in $\partial(W')$, and in turn at some point in $(\partial W) \cup (\partial U)$ (increasing W' to the component of $W \setminus \partial U$ containing $\mathcal{S}$ and recalling the upper semicontinuity of $\psi + g_f - h$ on A^1). If $\mathcal{S} \in W \cap \partial U$, then we still have $(\psi + g_f - h)(\mathcal{S}) = (\phi + g_f - h)(\mathcal{S}) \le \int_{\partial W} (\phi + g_f - h)\mu_{\mathcal{S},W} \le \int_{\partial W} (\psi + g_f - h)\mu_{\mathcal{S},W}$ by $\psi = \phi (= 0)$ on ∂U, the (domination) subharmonicity of $\phi + g_f - h$ on A^1, and $\phi \le \psi$ on P^1 (so on ∂W). Then $\psi + g_f - h$ attains the maximum M at some (in fact any) point in ∂W. Hence $M \le 0$, that is, $\psi + g_f \le h$ on $\overline{W}$, and the claim holds. Once the claim is at our disposal, also noting that $\psi + g_f \equiv g_f$ near ∞, we obtain the probability Radon measure

$$\Delta\psi + \mu_f = \Delta(\psi + g_f) + \delta_\infty \quad \text{on } \mathsf{P}^1.$$

Suppose now that f has no potentially good reductions. Then $\mu_f(\partial U)(= \mu_f(\{\mathcal{S}_0\})) = 0$. We claim that $\Delta\psi = 0$ on P^1; for, by the definition of ψ, we have $\Delta\psi = 0$ on $\mathsf{P}^1 \setminus \overline{U}$ (or equivalently $\Delta\psi + \mu_f = \mu_f$ on $\mathsf{P}^1 \setminus \overline{U}$). This with $U \subset \mathsf{P}^1 \setminus \operatorname{supp} \mu_f$ also yields $(\Delta\psi + \mu_f)(\overline{U}) = 1 - (\Delta\psi + \mu_f)(\mathsf{P}^1 \setminus \overline{U}) = 1 - \mu_f(\mathsf{P}^1 \setminus \overline{U}) = \mu_f(\overline{U}) = \mu_f(U) + \mu_f(\partial U) = 0 + 0 = 0$. Hence, recalling that $\Delta\psi + \mu_f$ is a probability Radon measure on P^1, we conclude that $\Delta\psi + \mu_f = \mu_f$ on P^1, that is, the claim holds. Once the claim is at our disposal, we must have $\psi \equiv 0$ on $\mathsf{P}^1 \setminus \mathbb{P}^1$, which contradicts $U \cap \{\phi < 0\}$ being non-empty and open in P^1.

We have seen that $\phi \equiv 0$ on P^1 under the no potentially good reductions condition on f. Then the convergence (1.3) follows from the equality

$$\Delta\left(\frac{\log|(f^n)^{(m)}-a|}{d^n-m}-g_f\right)=\frac{((f^n)^{(m)})^*\delta_a}{d^n-m}-\mu_f \quad \text{on } \mathsf{P}^1$$

and continuity of the Laplacian Δ. □

Proof of Theorem 3. Let k be a product formula field of characteristic 0 and let $f \in k[z]$ be a polynomial of degree $d > 1$. Recall that, writing $f(z)$ as $\sum_{j=0}^{d} c_j z^j \in k[z]$, so $c_d \in k^*$, there is a finite subset E_f in M_k containing all the infinite places of k such that for every $v \in M_k \setminus E_f$,

$$|c_d|_v = 1, \quad |c_0|_v, |c_1|_v, \ldots, |c_{d-1}|_v \le 1$$

and, moreover, $g_{f,v} = \log\max\{1, |\cdot|_v\}$ and $\mu_{f,v} = \delta_{\mathcal{S}_{\mathrm{can},v}}$ on $\mathsf{P}^1(\mathbb{C}_v)$, regarding $f \in \mathbb{C}_v[z]$.

Fix $m \in \mathbb{N}$ and $a \in k$. For every $n \in \mathbb{N}$, $(f^n)^{(m)} \in (\mathbb{Z}[c_0, \ldots, c_d])[z]$ by induction. By the product formula property of k, there is an at most finite (and possibly empty) subset E_a in M_k such that, for every $v \in M_k \setminus E_a$, $|a|_v \in \{0, 1\}$. Then, for every $n \in \mathbb{N}$ and every $v \in M_k \setminus (E_f \cup E_a)$, we have

$$\begin{aligned}\int_{\mathsf{P}^1(\mathbb{C}_v)} \log|(f^n)^{(m)}-a|_v \mu_{f,v} &\le \int_{\mathsf{P}^1(\mathbb{C}_v)} \log\max\{|(f^n)^{(m)}|_v, |a|_v\}\delta_{\mathcal{S}_{\mathrm{can},v}}\\ &= \log\max\Big\{\sup_{z\in\mathcal{O}_{\mathbb{C}_v}} |(f^n)^{(m)}(z)|_v, |a|_v\Big\} \le \log\max\{|c_0|_v, \ldots, |c_d|_v, |a|_v\} = \log 1 = 0\end{aligned}$$

(see (3.1) and (3.2) for the first equality), which with the second assertions in Lemmas 3.1 and 2.1 (for finite and infinite $v \in M_k$, respectively) implies that

$$\sup_{v\in M_k}\sup_{n\in\mathbb{N}} N_v \frac{\int_{\mathsf{P}^1(\mathbb{C}_v)} \log|(f^n)^{(m)}-a|_v \mu_{f,v}}{d^n-m} < \infty.$$

Now by the Mahler-type formula (3.7′), Fatou's lemma, and (3.8), we have

$$\limsup_{n\to\infty} \hat{h}_f([(f^n)^{(m)}=a]) \le \sum_{v\in M_k} \limsup_{n\to\infty} N_v \frac{\int_{\mathsf{P}^1(\mathbb{C}_v)} \log|(f^n)^{(m)}-a|_v \mu_{f,v}}{d^n-m} \le 0,$$

which with the non-negativity (3.7) of $\hat{h}_f$ yields the *small* $(g_{f,v})_{v\in M_k}$*-heights property* (1.4) of the sequence $([(f^n)^{(m)}=a])_n$ of effective k-divisors on $\mathbb{P}^1(\bar{k})$.

We note that $\deg[(f^n)^{(m)}=a] = d^n - m \to \infty$ as $n \to \infty$ and that, whenever $v \in M_k$ is infinite, we have $\mathbb{C}_v \cong \mathbb{C}$. Suppose now that k is a number field and that $a \in k^*$, and choose an infinite place $v \in M_k$ of k. Then from the equidistribution (1.2) of $(((f^n)^{(m)})^*\delta_a/(d^n-m))_n$ towards $\mu_{f,v}$, which has no atoms, on $\mathsf{P}^1(\mathbb{C}_v) \cong \mathbb{P}^1(\mathbb{C})$, we have $\sup_{w\in\mathbb{P}^1(\bar{k}):(f^n)^{(m)}(w)=a} \deg_w((f^n)^{(m)}) = o((\deg[(f^n)^{(m)}=a]))$ as $n \to \infty$, so in particular the *small diagonal property*

$$\sum_{w\in\mathbb{P}^1(\bar{k}):(f^n)^{(m)}(w)=a} (\deg_w((f^n)^{(m)}))^2 = o((\deg[(f^n)^{(m)}=a])^2) \quad \text{as } n \to \infty$$

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

of $([(f^n)^{(m)} = a])_n$. Now the uniform *asymptotically* $(g_{f,v})_{v\in M_k}$*-Fekete configuration property* (1.5) of $([(f^n)^{(m)} = a])_n$ holds (see [22, Theorem 1]), so in particular the adelic equidistribution (1.6) holds. □

4. *Proof of Theorem 4*

Let us first show a slightly more general equidistribution statement (1.7′) under the normalization (4.1) below. Let f be a Hénon-type polynomial automorphism of $\mathbb{C}^2$ of degree $d > 1$ normalized as

$$I^+ = \{[0:0:1]\} \quad \text{and} \quad I^- = \{[0:1:0]\}. \tag{4.1}$$

Then the function

$$(z, w) \mapsto g^+(z, w) - \log \max\{1, |z|\} \quad \text{on } \mathbb{C}^2$$

extends pluriharmonically to an open neighborhood of $L_\infty \setminus I^+$ in $\mathbb{P}^2$ [11, Theorem 6.1]. Moreover, for every $n \in \mathbb{N}$, writing f^n as

$$f^n = (P_n, Q_n) \in (\mathbb{C}[z, w])^2,$$

we have $\deg P_n = \deg_z P_n = d^n > \deg Q_n$ [11, Proposition 5.11], and then

$$0 < g^+ = d^{-n} \log |P_n| + O(d^{-n}) \quad \text{and} \quad Q_n = o(P_n) \quad \text{as } n \to \infty \tag{4.2}$$

on $B^+ \cap \mathbb{C}^2$ locally uniformly, recalling also that $\lim_{n\to\infty} f^n = [0:1:0]$ on B^+ locally uniformly.

Fix a 2×2 matrix $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in \mathrm{M}(2, \mathbb{C})$ satisfying the condition

$$a_4 \neq 0, \tag{4.3}$$

so that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \det(D(f^n) - A) &= J_{f^n} - a_1 \partial_w Q_n - a_4 \partial_z P_n + a_3 \partial_w P_n + a_2 \partial_z Q_n + \det A \\ &= -a_1 \partial_w Q_n - a_4 \partial_z P_n + a_3 \partial_w P_n + a_2 \partial_z Q_n + J_f^n + \det A \in \mathbb{C}[z, w] \end{aligned} \tag{4.4}$$

is indeed of degree $d^n - 1$.

LEMMA 4.1. *For each $j \in \{z, w\}$,*

$$\partial_j P_n = 2d^n P_n \partial_j g^+ + O(1) \quad \textit{and} \quad \partial_j Q_n = o(d^n P_n) \quad \textit{as } n \to \infty \tag{4.5}$$

on $B^+ \cap \mathbb{C}^2$ locally uniformly.

Proof. Pick any open concentric bidisks $D \Subset D' \Subset B^+ \cap \mathbb{C}^2$, and fix $j \in \{z, w\}$. Let us write D, D' as $D_1 \times D_2$, $D'_1 \times D'_2$, respectively.

By the first half of (4.2), we have $\inf_{D'} |P_n| > 0$ if $n \gg 1$. We claim that

$$\partial_j g^+ = d^{-n} \partial_j \log |P_n| + O(d^{-n}) = \frac{1}{d^n} \frac{\partial_j P_n}{2P_n} + O(d^{-n}) \quad \text{as } n \to \infty \tag{4.6}$$

on $\overline{D}$ uniformly; indeed, for every $z \in \overline{D_1}$, using Poisson's integral of the function $w \mapsto g^+(z, w) - d^{-n} \log |P_n(z, w)|$ on $\partial D'_2$, the first half of (4.2) yields the asymptotic estimate (4.6) on $\{z\} \times \overline{D_2}$ uniformly, and moreover, the implicit constant in O depends only on D. Hence the claim holds. In particular, the first half of (4.5) holds.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Similarly, using the second half of (4.2) twice and Cauchy's integral of the function Q_n/P_n on $\partial D_1' \times \partial D_2'$, we also have

$$\frac{\partial_j Q_n}{P_n} = \frac{Q_n \partial_j P_n}{P_n^2} + \partial_j\left(\frac{Q_n}{P_n}\right) = o(1) \cdot \frac{\partial_j P_n}{P_n} + o(1) \quad \text{as } n \to \infty$$

on $\overline{D}$ uniformly, which together with (4.6) and $\sup_D |\partial_j g^+| < \infty$ yields

$$\frac{\partial_j Q_n}{P_n} = o(d^n) + o(1) = o(d^n) \quad \text{as } n \to \infty$$

on $\overline{D}$ uniformly. Hence the second half of (4.5) also holds. □

By the pluriharmonicity of g^+ on B^+, the function $a_4\partial_z g^+ - a_3\partial_w g^+$ is holomorphic on $B^+ \cap \mathbb{C}^2$. Set

$$Y := \{(z, w) \in B^+ \cap \mathbb{C}^2 : (a_4\partial_z g^+ - a_3\partial_w g^+)(z, w) = 0\}.$$

Recall the assumption that $a_4 \neq 0$.

LEMMA 4.2. *Y is an analytic hypersurface in $B^+ \cap \mathbb{C}^2$, no irreducible component of which is horizontal, that is, $\{w = w_0\}$ for some $w_0 \in \mathbb{C}$.*

Proof. Let us first show that Y is not equal to $B^+ \cap \mathbb{C}^2$. Suppose to the contrary that $a_4\partial_z g^+ - a_3\partial_w g^+ \equiv 0$ on $B^+ \cap \mathbb{C}^2$. Then, letting L be the complex affine line $w = -(a_3/a_4)z$ in $\mathbb{C}^2$, there is $c \in \mathbb{R}$ such that $g^+ \equiv c$ on $L \cap B^+$. On the other hand, since the projective line $\overline{L}$ in $\mathbb{P}^2$ intersects L_∞ at $[0 : 1 : -a_3/a_4] \in L_\infty \setminus I^+$, near which $g^+(z, w) - \log\max\{1, |z|\}$ extends pluriharmonically, we must have $c = g^+(z, w) = \log\max\{1, |z|\} + O(1) \to \infty$ as $L \cap B^+ \ni (z, w) \to [0 : 1 : -a_3/a_4]$. This is a contradiction. Hence the former assertion holds.

The latter assertion is shown similarly, noting that the closure of any horizontal line intersects L_∞ at $[0 : 1 : 0] \in L_\infty \setminus I^+$. □

Recall the computation (4.4) of the polynomial $\det(D(f^n) - A) \in \mathbb{C}[z, w]$ of degree $d^n - 1$. For every $n \in \mathbb{N}$, set

$$\phi_n = \phi_n[A] := \frac{\log|\det(D(f^n) - A)|}{d^n - 1},$$

which is a plurisubharmonic function on $\mathbb{C}^2$ and satisfies $dd^c\phi_n = [\det(D(f^n) - A)]/(d^n - 1)$ as currents on $\mathbb{C}^2$ by the Poincaré–Lelong formula.

LEMMA 4.3. *We have $\phi_n = g^+ + O(nd^{-n})$ as $n \to \infty$ on $B^+ \cap (\mathbb{C}^2 \setminus Y)$ locally uniformly. Moreover, the family $(\phi_n)_n$ is locally uniformly bounded from above on $\mathbb{C}^2$.*

Proof. First, pick any open bidisk $D \Subset B^+ \cap (\mathbb{C}^2 \setminus Y)$. Then by (4.5) and the first half of (4.2), we have

$$\begin{aligned} &a_1\partial_w Q_n + a_4\partial_z P_n - a_3\partial_w P_n - a_2\partial_z Q_n \\ &\quad = 2d^n P_n \cdot (a_4\partial_z g^+ - a_3\partial_w g^+ + o(1)) \quad \text{as } n \to \infty \end{aligned}$$

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

on $\overline{D}$ uniformly, and then using the first half of (4.2) again and $D \Subset B^+ \cap (\mathbb{C}^2 \setminus Y)$, we have

$$\phi_n = \frac{1}{d^n - 1}\left(\log |P_n| + \log \left| 2d^n (a_4 \partial_z g^+ - a_3 \partial_w g^+ + o(1)) - \frac{J_f^n + \det A}{P_n} \right| \right)$$
$$= \frac{1}{d^n - 1} \log |P_n| + O(nd^{-n}) = g^+ + O(nd^{-n}) \quad \text{as } n \to \infty$$

on $\overline{D}$ uniformly. Hence the former assertion holds.

Fix $(z_0, w_0) \in \mathbb{C}^2$. By $L_\infty \setminus I^+ \subset B_+$ and the second half of Lemma 4.2, we have $\{|z - z_0| = r\} \times \{|w - w_0| = \epsilon\} \subset B^+ \cap (\mathbb{C}^2 \setminus Y)$ for $r \gg 1$ and $0 < \epsilon \ll 1$, so that by the former assertion and the maximum principle for the plurisubharmonic function ϕ_n on $\mathbb{C}^2$, we have

$$\sup_{\{|z-z_0|\le r\}\times\{|w-w_0|\le\epsilon\}} \phi_n \le \left(\sup_{\{|z-z_0|=r\}\times\{|w-w_0|=\epsilon\}} g^+ \right) + O(nd^{-n}) \quad \text{as } n \to \infty.$$

Hence the latter assertion also holds. □

Let us see

$$\lim_{n\to\infty} \frac{[\det(D(f^n) - A)]}{d^n - 1} = T^+ \quad \text{on } \mathbb{P}^2 \tag{1.7'}$$

as currents. First, let $\tilde{S} = \lim_{j\to\infty}[\det(D(f^{n_j}) - A)]/(d^{n_j} - 1)$ be any limit point, which is also a positive closed (1, 1)-current on $\mathbb{P}^2$ of mass 1, of the sequence $([\det(D(f^n) - A)]/(d^n - 1))_n$ of positive closed (1, 1)-currents on $\mathbb{P}^2$ of mass 1. On the other hand, by Lemma 4.3 and the *compactness principle* for plurisubharmonic functions on a domain in $\mathbb{C}^N$, taking a subsequence of (n_j) if necessary, there is a plurisubharmonic function ϕ on $\mathbb{C}^2$ such that $\phi = \lim_{j\to\infty} \phi_{n_j}$ in $L^1_{\text{loc}}(\mathbb{C}^2, m_4)$, where m_4 is the Lebesgue measure on $\mathbb{C}^2$. Then we have $\tilde{S}|\mathbb{C}^2 = dd^c\phi$ on $\mathbb{C}^2$ and, by the first half of Lemma 4.3, the plurisubharmonicity of ϕ on $\mathbb{C}^2$, and the pluriharmonicity of g^+ on B^+, we also have $\phi \equiv g^+$ on $B^+ \cap \mathbb{C}^2$. Hence $\text{supp}(\tilde{S}|\mathbb{C}^2) \subset K^+$. Next, let S be the *trivial extension* of $dd^c\phi$ to $\mathbb{P}^2$ across L_∞. It is a positive closed (1, 1)-current on $\mathbb{P}^2$ (cf. [11, Theorem 2.7]) and supported by $\overline{K^+} = K^+ \cup I^+$. Then, by the uniqueness of T^+ mentioned above among such currents, there is $c \ge 0$ such that $S = c \cdot T^+$ on $\mathbb{P}^2$. Moreover, for the current of integration $[L]$ along any projective line $L \subset \mathbb{P}^2 \setminus I^+$ other than L_∞ and passing through I^-, if $R \gg 1$, then we have $\phi \equiv g^+$ on $\{(z, w) \in \mathbb{C}^2 : \|(z, w)\| > R - 1\} \cap L \subset B^+$, and in turn, recalling the definition of S, T^+ and using Stokes's formula, we have

$$c - 1 = \int_{\mathbb{P}^2} (S - T^+) \wedge [L] = \int_{\{\|(z,w)\|\le R\}} dd^c(\phi - g^+) \wedge [L]$$
$$= \int_{\{\|(z,w)\|\le R\}\cap L} dd^c(\phi - g^+) = 0$$

(cf. [11, Proof of Lemma 6.3]). Hence $S = T^+$ on $\mathbb{P}^2$. Consequently, $S|\mathbb{C}^2 = T^+|\mathbb{C}^2 = dd^c\phi = \tilde{S}|\mathbb{C}^2$ on $\mathbb{C}^2$, and then $\tilde{S} \ge S$ on $\mathbb{P}^2$ by their construction. Since both $\tilde{S}$, S are of mass 1, we conclude that $\tilde{S} = S = T^+$ on $\mathbb{P}^2$. Hence (1.7′) holds.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

Proof of Theorem 4. Let f be a Hénon-type polynomial automorphism of $\mathbb{C}^2$ of degree $d > 1$. Fix $\lambda \in \mathbb{C}^*$, and set $A = \lambda I_2 \in \mathrm{M}(2, \mathbb{C})$. Then using the chain rule and the equivariance of T^+ under affine coordinate changes on $\mathbb{C}^2$, we can assume that f satisfies the normalization (4.1), without loss of generality. Noting also that $A = \lambda I_2$ satisfies condition (4.3), the desired (1.7) as currents on $\mathbb{P}^2$ is nothing but (1.7′) as currents on $\mathbb{P}^2$ for this $A = \lambda I_2$. □

Acknowledgement. The authors thank the referee for their very careful scrutiny. The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 15K04924 and (C) 19K03541. The authors were partially supported by Invitational Fellowships for Research in Japan (Short-term S18024) JSPS BRIDGE Fellowship 2018, and are grateful for the hospitality of Université de Picardie Jules Verne and Kyoto Institute of Technology, where they visited each other in 2018 and this work was developed.

REFERENCES

[1] M. H. Baker and L.-C. Hsia. Canonical heights, transfinite diameters, and polynomial dynamics. *J. Reine Angew. Math.* **585** (2005), 61–92.
[2] M. H. Baker and R. Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)* **56**(3) (2006), 625–688.
[3] M. H. Baker and R. Rumely. *Potential Theory and Dynamics on the Berkovich Projective Line (Mathematical Surveys and Monographs, 159)*. American Mathematical Society, Providence, RI, 2010.
[4] E. Bedford and J. Smillie. Polynomial diffeomorphisms of $\mathbf{C}^2$. V. Critical points and Lyapunov exponents. *J. Geom. Anal.* **8**(3) (1998), 349–383.
[5] R. L. Benedetto. *Dynamics in One Non-Archimedean Variable (Graduate Studies in Mathematics, 198)*. American Mathematical Society, Providence, RI, 2019.
[6] V. G. Berkovich. *Spectral Theory and Analytic Geometry over Non-Archimedean Fields (Mathematical Surveys and Monographs, 33)*. American Mathematical Society, Providence, RI, 1990.
[7] F. Berteloot and V. Mayer. *Rudiments de Dynamique Holomorphe (Cours Spécialisés, 7)*. Société Mathématique de France, Paris, 2001.
[8] H. Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.* **6** (1965), 103–144.
[9] A. Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.* **595** (2006), 215–235.
[10] T.-C. Dinh and N. Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. *Holomorphic Dynamical Systems (Lecture Notes in Mathematics, 1998)*. Springer, Berlin, 2010, pp. 165–294.
[11] T.-C. Dinh and N. Sibony. Rigidity of Julia sets for Hénon type maps. *J. Mod. Dyn.* **8**(3–4) (2014), 499–548.
[12] C. Favre and M. Jonsson. *The Valuative Tree (Lecture Notes in Mathematics, 1853)*. Springer, Berlin, 2004.
[13] C. Favre and J. Rivera-Letelier. Équidistribution quantitative des points de petite hauteur sur la droite projective. *Math. Ann.* **335**(2) (2006), 311–361.
[14] C. Favre and J. Rivera-Letelier. Théorie ergodique des fractions rationnelles sur un corps ultramétrique. *Proc. Lond. Math. Soc. (3)* **100**(1) (2010), 116–154.
[15] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimensions. *Complex Potential Theory (Montreal, PQ, 1993) (NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 439)*. Kluwer Academic Publishers, Dordrecht, 1994, pp. 131–186. Notes partially written by Estela A. Gavosto.
[16] J. Fresnel and M. van der Put. *Rigid Analytic Geometry and its Applications (Progress in Mathematics, 218)*. Birkhäuser, Boston, 2004.
[17] T. Gauthier and G. Vigny. Distribution of points with prescribed derivative in polynomial dynamics. *Riv. Math. Univ. Parma (N.S.)* **8**(2) (2017), 247–270.
[18] L. Hörmander. *The Analysis of Linear Partial Differential Operators. I: Distribution Theory and Fourier Analysis (Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 256)*. Springer, Berlin, 1983.

https://doi.org/10.1017/etds.2020.125 Published online by Cambridge University Press

[19] M. Jonsson. Dynamics on Berkovich spaces in low dimensions. *Berkovich Spaces and Applications*. Springer, Cham, 2015, pp. 205–366.

[20] J. Milnor. *Dynamics in One Complex Variable (Annals of Mathematics Studies, 160)*, 3rd edn. Princeton University Press, Princeton, NJ, 2006.

[21] Y. Okuyama. Effective divisors on the projective line having small diagonals and small heights and their application to adelic dynamics. *Pacific J. Math.* **280**(1) (2016), 141–175.

[22] Y. Okuyama. Adelically summable normalized weights and adelic equidistribution of effective divisors having small diagonals and small heights on the Berkovich projective lines. *Algebraic Number Theory and Related Topics 2014 (RIMS Kôkyûroku Bessatsu, B64)*. Research Institute for Mathematical Sciences (RIMS), Kyoto, 2017, pp. 55–66.

[23] Y. Okuyama. Value distribution of the sequences of the derivatives of iterated polynomials. *Ann. Acad. Sci. Fenn. Math.* **42**(2) (2017), 563–574.

[24] Y. Okuyama. An a priori bound of rational functions on the Berkovich projective line. *Preprint*, 2018, arXiv:1805.07668.

[25] T. Ransford. *Potential Theory in the Complex Plane (London Mathematical Society Student Texts, 28)*. Cambridge University Press, Cambridge, 1995.

[26] J. Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. *Geometric Methods in Dynamics. II (Astérisque, 287)*. Société Mathématique de France, Paris, 2003, pp. 147–230.

[27] A. Russakovskii and B. Shiffman. Value distribution for sequences of rational mappings and complex dynamics. *Indiana Univ. Math. J.* **46**(3) (1997), 897–932.

[28] A. Thuillier. Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov. *PhD Thesis*, Université Rennes 1, 2005.

[29] K. Yamanoi. Zeros of higher derivatives of meromorphic functions in the complex plane. *Proc. Lond. Math. Soc. (3)* **106**(4) (2013), 703–780.

[30] H. Ye. The Schwarzian derivative and polynomial iteration. *Conform. Geom. Dyn.* **15** (2011), 113–132.