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Coexistence of unbounded and periodic solutions to perturbed damped isochronous oscillators at resonance

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In this paper, we are concerned with the existence of unbounded orbits of the mapping

$$\theta_1 = \theta + 2\pi + \frac{1}{\rho}\mu(\theta) + o(\rho^{-1}),$$
$$\rho_1 = \rho + c - \mu'(\theta) + o(1), \quad \rho \to \infty,$$

where c is a constant and $\mu(\theta)$ is 2π-periodic. Assume that $c \neq 0$, that $\mu(\theta)$ is non-negative (or non-positive) and that $\mu(\theta)$ has finitely many degenerate zeros in $[0, 2\pi]$. We prove that every orbit of the given mapping tends to infinity in the future or in the past for sufficiently large ρ. On the basis of this conclusion, we further prove that the equation $x'' + f(x)x' + V'(x) + \phi(x) = p(t)$ has unbounded solutions provided that V is an isochronous potential at resonance and $F(x)$ ($F(x) = \int_0^x f(s)\,\mathrm{d}s$) and $\phi(x)$ satisfy some limit conditions. Meanwhile, we also obtain the existence of 2π-periodic solutions of this equation.

1. Introduction

This paper deals with a Liénard equation of the form

$$x'' + f(x)x' + V'(x) + g(x) = p(t), \tag{1.1}$$

where $f, g \in C(\mathbb{R})$, $V \in C^2(\mathbb{R})$ and $p \in C(\mathbb{R})$ is 2π-periodic. We also assume that g is locally Lipschitz. The function V is a $2\pi/n$-isochronous potential, i.e. all the solutions of $x'' + V'(x) = 0$ are $2\pi/n$-periodic, with $n \in \mathbb{N}$; according to [3] (see also [12]), $V \in C^2$ is $2\pi/n$-isochronous, provided its graph is obtained by horizontally shearing the graph of $V(x) = \frac{1}{2}nx^2$. We also recall that (see [2]) the origin in $\mathbb{R}^2$ is not an isochronous centre for $z' = A(z)$ of period ω/k, $k \in \mathbb{N}$, if and only if a perturbed system of the form $z' = A(z) + \mu\alpha(t, z, \mu)$ has an ω-periodic solution, for $|\mu|$ sufficiently small. For more examples, together with comments and remarks, we refer the reader to [3].

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In the case when $f \equiv 0$, a result on the existence of 2π-periodic solutions for an equation of the form (1.1) has been proved in [3].

In the present paper, in the general case when f is not identically zero, we tackle a problem which has been widely studied in the last few years for some particular cases of equation (1.1), i.e. the question of the coexistence of 2π-periodic solutions and unbounded solutions. The situation we deal with is a generalization of the well-known resonant situation where

$$V'(x) = ax^+ - bx^-, \tag{1.2}$$

with $a, b \in \mathbb{R}$ such that $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in \mathbb{N}$.

Results on the existence of 2π-periodic solutions when V satisfies (1.2) and $f \equiv 0$ can be found, among others, in [6, 7, 11, 13, 15]. More recently, Fabry and Mawhin [9] generalized previous results and proved the coexistence of 2π-periodic and unbounded solutions in the case when $f \equiv 0$. On the other hand, a result on the existence of 2π-periodic solutions in the case when f is not identically zero is given in [4]. The results in [4] are based on a detailed study of the Poincaré map associated with a first-order system in $\mathbb{R}^2$ obtained from the given second-order equation via some suitable change of variables. More precisely, an asymptotic expression for the Poincaré map has the form

$$\left.\begin{aligned} \theta_1 &= \theta_0 + 2\pi + 2\pi a \Sigma_1(\theta_0)\rho_0^{-1} + o(\rho_0^{-1}), \\ \rho_1 &= \rho_0 - 2\pi a \Sigma_2(\theta_0) + o(1), \end{aligned}\right\} \tag{1.3}$$

for sufficiently large ρ_0.

In the above formula, setting $F(x) := \int_0^x f(s)\,\mathrm{d}s$, the functions Σ_1 and Σ_2 are defined by

$$\begin{aligned} \Sigma_1(\theta) &= \frac{n}{\pi}\left[\frac{g(+\infty)}{a} - \frac{g(-\infty)}{b}\right] - \frac{1}{2\pi}\int_0^{2\pi} p(t)\psi(t+\theta)\,\mathrm{d}t, \\ \Sigma_2(\theta) &= \frac{n}{\pi}[F(+\infty) - F(-\infty)] + \Sigma_1'(\theta), \end{aligned}$$

where $F(\pm\infty)$ and $g(\pm\infty)$ are the limits of F and g at infinity, respectively, and ψ is the 2π-periodic solution of $x'' + ax^+ - bx^- = 0$ satisfying $\psi(0) = 0$, $\psi'(0) = 1$. It was proved in [4] that $x'' + f(x)x' + ax^+ - bx^- + g(x) = p(t)$ has at least one 2π-periodic solution, provided that either Σ_1 or Σ_2 is of constant sign. In the case when the zeros of the function Σ_1 are non-degenerate and the zeros of the functions Σ_1, Σ_2 are different and the signs of Σ_2 at the zeros of Σ_1 in $[0, 2\pi/n)$ do not change or change more than twice, the same conclusion was also obtained in [4]. The proof is performed using the asymptotic expansion of the Poincaré mapping; indeed, it is possible to guarantee the applicability of the Brouwer fixed-point theorem and obtain the existence of at least one 2π-periodic solution. For more recent developments on this subject, we refer the reader to [7, 10].

Note that the technique based on the study of the asymptotic expansion of the Poincaré map has been successfully applied by Alonso and Ortega in [1], where (with minor changes) a development of the form (1.3) is examined. Alonso and Ortega, assuming that the zeros of Σ_1 are non-degenerate, construct a periodic function p such that all solutions of the equation

$$x'' + ax^+ - bx^- = p(t) \tag{1.4}$$

with large initial conditions are unbounded, provided that $1/\sqrt{a}+1/\sqrt{b}$ is a rational number. This result disproves a conjecture that the existence of periodic solutions of (1.4) may imply the boundedness of all solutions.

In the more general situation when f and g are not identically zero and when (1.2) holds, it was proved in [17] that all solutions with large initial values are unbounded provided that the function Σ_1 is of constant sign. When Σ_1 has non-degenerate zeros, we can prove (arguing as in [1]) the existence of unbounded solutions as well. For related results, we refer the reader to [5, 8, 16, 18] and the references therein.

In the present paper we present a two-fold generalization of the above quoted results. On the one hand, we consider an isochronous potential V which generalizes (1.2) (see assumptions (1)–(4) in § 3); on the other hand, we examine the situation when Σ_1 is non-negative (or non-positive) and all zeros of Σ_1 are degenerate. More precisely, we study the dynamics of a class of mappings defined on the plane, which have an asymptotic expression of the form

$$\left.\begin{aligned} \theta_1 &= \theta + 2\pi + \frac{2\pi}{a\rho}\Sigma_1(\theta) + o(\rho^{-1}), \\ \rho_1 &= \rho + c - \frac{2\pi}{a}\Sigma_1'(\theta) + o(1), \quad c \in \mathbb{R}, \end{aligned}\right\} \tag{1.5}$$

where $\rho \to +\infty$.

This class of maps includes the Poincaré maps of equations of the form (1.1) (cf. (1.3)).

For the study of the dynamics of (1.5), we observe that if $\Sigma_1(\theta) \geqslant 0$ (or $\Sigma_1(\theta) \leqslant 0$), $\theta \in [0, 2\pi]$ and the zeros of Σ_1 are degenerate, then the methods in [1, 17] cannot be applied. However, we can still prove the existence of orbits which tend to infinity in the future or in the past according to the sign of c. On the basis of this conclusion, we deal with the unboundedness of solutions of (1.1). Meanwhile, we can still prove the existence of periodic solutions of (1.1).

In § 2 we study the behaviour of a one-to-one continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ and give sufficient conditions (proposition 2.1) which guarantee the unboundedness of its orbits.

In § 3 we show that, under conditions (1)–(4), proposition 2.1 is applicable to the Poincaré map associated with (1.1). This enables us to prove our main result (theorem 3.5), where we obtain the coexistence of 2π-periodic solutions and unbounded solutions to (1.1).

2. Unbounded orbits of planar mappings

In this section we will study the behaviour of the iterates of a one-to-one continuous mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$; we assume that there exist $c \in \mathbb{R}$, $\mu \in C^2(S^1)$, $h_1, h_2 \in C((0, +\infty) \times S^1)$ such that the lift of P can be expressed in the form

$$P : \begin{cases} \theta_1 = \theta + 2\pi + \dfrac{1}{\rho}\mu(\theta) + h_1(\rho, \theta), \\ \rho_1 = \rho + c - \mu'(\theta) + h_2(\rho, \theta). \end{cases} \tag{2.1}$$

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Moreover, we suppose that

$$h_1(\rho,\theta) = o\left(\frac{1}{\rho}\right), \quad h_2(\rho,\theta) = o(1), \quad \rho \to +\infty. \tag{2.2}$$

Given a point (ρ_0, θ_0) we denote by $\{(\rho_j, \theta_j)\}$ the orbit of the mapping P through the point (ρ_0, θ_0), i.e.

$$P(\rho_j, \theta_j) = (\rho_{j+1}, \theta_{j+1}).$$

PROPOSITION 2.1. *Assume that condition (2.2) is fulfilled. Then the following conclusions hold:*

(a) *if $c > 0$ and μ is non-negative (or non-positive) and has finitely many zeros in $[0, 2\pi]$, all of which are degenerate, then there exists $R_0 > 0$ such that for $\rho_0 \geqslant R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies*

$$\lim_{j\to+\infty} \rho_j = +\infty;$$

(b) *if $c < 0$ and μ is non-negative (or non-positive) and has finitely many zeros in $[0, 2\pi]$, all of which are degenerate, then there exists $R_0 > 0$ such that for $\rho_0 \geqslant R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the past and satisfies*

$$\lim_{j\to-\infty} \rho_j = +\infty.$$

We observe that the assumption on the degeneracy of the zeros of μ implies that the methods in [1, 17] cannot be applied; to overcome this difficulty we develop an approximation method.

We deal only with the case $\mu(\theta) \geqslant 0$, $\theta \in [0, 2\pi]$; the case $\mu(\theta) \leqslant 0$, $\theta \in [0, 2\pi]$ can be handled similarly.

Let $\varepsilon > 0$ be a sufficiently small constant. Let

$$\nu(\theta) = \mu(\theta) + \varepsilon \quad \text{for all } \theta \in [0, 2\pi].$$

Obviously, $\nu(\theta) > 0$ and $\nu'(\theta) = \mu'(\theta)$, for every $\theta \in [0, 2\pi]$. Therefore, (2.1) can be written as

$$P: \begin{cases} \theta_1 = \theta + 2\pi + \dfrac{1}{\rho}\nu(\theta) + h_1(\rho,\theta) - \dfrac{\varepsilon}{\rho}, \\ \rho_1 = \rho + c - \nu'(\theta) + h_2(\rho,\theta). \end{cases} \tag{2.3}$$

Let

$$\frac{1}{\rho} = \delta r,$$

where $\delta > 0$ is a parameter to be determined later. Under this transformation, the mapping P becomes

$$\bar{P}: \begin{cases} \theta_1 = \theta + 2\pi + \delta r\nu(\theta) + h_{11}(r,\theta,\delta) - \varepsilon\delta r, \\ r_1 = r + \delta r^2(-c + \nu'(\theta)) + \delta r^2 h_{21}(r,\theta,\delta), \end{cases} \tag{2.4}$$

where

$$h_{11}(r,\theta,\delta) = h_1(\delta^{-1}r^{-1},\theta),$$
$$h_{21}(r,\theta,\delta) = -h_2(\delta^{-1}r^{-1},\theta) + \frac{[c-\nu'(\theta)+h_2(\delta^{-1}r^{-1},\theta)]^2}{\delta^{-1}r^{-1}+c-\nu'(\theta)+h_2(\delta^{-1}r^{-1},\theta)}.$$

From condition (2.2) we deduce that

$$\lim_{\delta\to 0^+}\delta^{-1}r^{-1}h_{11}(r,\theta,\delta) = \lim_{\delta\to 0^+}\delta^{-1}r^{-1}h_1(\delta^{-1}r^{-1},\theta) = 0, \qquad \lim_{\delta\to 0^+}h_{21}(r,\theta,\delta) = 0, \tag{2.5}$$

uniformly for $\theta \in [0,2\pi]$ and sufficiently small r.

We observe that in the asymptotic expression (2.4) for $\bar{P}$ the term $-c+\nu'(\theta)$ in general does not have constant sign; the next change of variables transforms this term to one with definite sign.

Towards this aim, consider the system

$$\theta' = r\nu(\theta), \quad r' = r^2\nu'(\theta), \quad r>0, \tag{2.6}$$

whose first integral is

$$I(r,\theta) = \frac{\nu(\theta)}{r}.$$

Therefore, the orbits of (2.6) can be expressed in the form

$$\mathit{\Gamma}_h : I(r,\theta) = \frac{\nu(\theta)}{r} = h,$$

where h is an arbitrary constant. Let $(r(t),\theta(t))$ be the solution of (2.6) lying on the curve $\mathit{\Gamma}_h$. Obviously, $(r(t),\theta(t))$ is a periodic solution. Denote by $T(h)$ the minimal period of $(r(t),\theta(t))$; from the first equation in (2.6) we deduce that

$$T(h) = h\int_0^{2\pi}\frac{\mathrm{d}\theta}{\nu^2(\theta)} = dh,$$

where

$$d = \int_0^{2\pi}\frac{\mathrm{d}\theta}{\nu^2(\theta)}.$$

Now let us introduce the frequency function

$$\omega(h) = \frac{2\pi}{T(h)} = \frac{2\pi}{dh}$$

and the function [14]

$$K(r,\theta) = \frac{\nu(\theta)}{r}\int_0^{\theta}\frac{\mathrm{d}s}{\nu^2(s)}.$$

Immediately we see that the quantity $K(r,\theta)$ denotes the time needed for a solution $(r(t),\theta(t))$ to go from the vertical axis $\theta=0$ to the point (r,θ).

Moreover, let us define

$$\tau(\theta) = \omega(I(r,\theta))K(r,\theta) = \frac{2\pi}{d}\int_0^{\theta}\frac{\mathrm{d}s}{\nu^2(s)};$$

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the function τ satisfies

$$\tau(0) = 0, \qquad \tau(\theta + 2\pi) = \tau(\theta) + 2\pi.$$

Now, let $\Psi : \mathbb{R}^+ \times S^1 \to \mathbb{R}^+ \times S^1$ be defined by

$$\Psi : (r, \theta) \to (I, \tau) = (I(r, \theta), \tau(\theta)).$$

It is easy to check that the mapping Ψ is a bijective mapping. The inverse, Ψ^{-1}, of Ψ satisfies the relations

$$\Psi^{-1}(I, \tau) = (r, \theta),$$
$$r(I, \tau) = \frac{\nu(\theta(\tau))}{I}, \qquad \frac{2\pi}{d} \int_0^{\theta(\tau)} \frac{\mathrm{d}s}{\nu^2(s)} = \tau.$$

Obviously, we have

$$\theta(0) = 0, \qquad \theta(\tau + 2\pi) = \theta(\tau) + 2\pi.$$

Finally, let us consider the map

$$\hat{P} = \Psi \circ \bar{P} \circ \Psi^{-1} : (I, \tau) \to (I_1, \tau_1) = \hat{P}(I, \tau).$$

We are able to prove the following lemma.

LEMMA 2.2. *For every $\varepsilon > 0$, the mapping $\hat{P}$ can be expressed in the form*

$$\hat{P} : \begin{cases} \tau_1 = \tau + 2\pi + \delta\omega(I) + \delta h_{12}(I, \tau, \delta, \varepsilon), \\ I_1 = I + \delta c \nu(\theta(\tau)) - \varepsilon\delta\nu'(\theta(\tau)) + \delta h_{22}(I, \tau, \delta, \varepsilon), \end{cases} \tag{2.7}$$

where h_{12} and h_{22} satisfy

$$\lim_{\delta \to 0^+} h_{12}(I, \tau, \delta, \varepsilon) = 0, \qquad \lim_{\delta \to 0^+} h_{22}(I, \tau, \delta, \varepsilon) = 0,$$

uniformly in $\tau \in \mathbb{R}$, for sufficiently large I.

Proof. Let us consider the asymptotic expansion of $\bar{P}$ given in (2.4); under the transformation $\Psi^{-1}(I, \tau) = (r(I, \tau), \theta(\tau))$, relations (2.4) become

$$\left.\begin{aligned} \theta_1 &= \theta(\tau) + 2\pi + \delta r(I, \tau)\nu(\theta(\tau)) + h_{11}(r(I, \tau), \theta(\tau), \delta) - \varepsilon\delta r(I, \tau), \\ r_1 &= r(I, \tau) + \delta r^2(I, \tau)(-c + \nu'(\theta(\tau))) + \delta r^2(I, \tau) h_{21}(r(I, \tau), \theta(\tau), \delta). \end{aligned}\right\} \tag{2.8}$$

Recalling that $r(I, \tau) = \nu(\theta(\tau))/I$, we may infer that

$$\left.\begin{aligned} \theta_1 &= \theta(\tau) + 2\pi + \frac{\delta\nu^2(\theta(\tau))}{I} + h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right) - \frac{\varepsilon\delta\nu(\theta(\tau))}{I} \\ r_1 &= \frac{\nu(\theta(\tau))}{I} + \frac{\delta\nu^2(\theta(\tau))(-c + \nu'(\theta(\tau)))}{I^2} + \frac{\delta\nu^2(\theta(\tau))}{I^2} h_{21}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right). \end{aligned}\right\} \tag{2.9}$$

From (2.9) we can deduce the asymptotic expression for (I_1, τ_1). Indeed, let us recall that

$$I_1 = \frac{\nu(\theta_1)}{r_1}, \qquad \tau_1 = \frac{2\pi}{d} \int_0^{\theta_1} \frac{\mathrm{d}s}{\nu^2(s)}.$$

Expanding $\nu(\theta_1)$, we get

$$\nu(\theta_1) = \nu(\theta(\tau)) + \frac{\delta\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I} - \frac{\varepsilon\delta\nu'(\theta(\tau))\nu(\theta(\tau))}{I} + \bar{h}_{11},$$

where $\bar{h}_{11} = \bar{h}_{11}(I, \theta, \delta, \varepsilon)$ is defined by

$$\begin{aligned}\bar{h}_{11} &= \nu'(\theta(\tau))h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right)\\ &\quad + \int_0^1 (1-s)\nu''\left[\theta(\tau) + s\frac{\delta\nu^2(\theta(\tau))}{I} + sh_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right) - \frac{s\varepsilon\delta\nu(\theta(\tau))}{I}\right]\\ &\quad \times \left[\frac{\delta\nu^2(\theta(\tau))}{I} + h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right) - \frac{\varepsilon\delta\nu(\theta(\tau))}{I}\right]^2 \mathrm{d}s.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{r_1} &= \frac{I}{\nu(\theta(\tau))(1 + \delta\nu(\theta(\tau))(-c + \nu'(\theta(\tau)))/I + (\delta\nu(\theta(\tau))/I)h_{21}(\nu(\theta(\tau))/I, \theta(\tau), \delta))}\\ &= \frac{I}{\nu(\theta(\tau))} + \delta(c - \nu'(\theta(\tau))) + \delta\bar{h}_{21},\end{aligned}$$

with $\bar{h}_{21} = \bar{h}_{21}(I, \tau, \delta, \varepsilon)$ defined by

$$\begin{aligned}\bar{h}_{21} &= -h_{21}(\nu(\theta(\tau))/I, \theta(\tau), \delta)\\ &\quad + \frac{\delta\nu(\theta(\tau))}{I}\frac{[-c + \nu'(\theta(\tau)) + h_{21}(\nu(\theta(\tau))/I, \theta(\tau), \delta)]^2}{1 + (\delta\nu(\theta(\tau))/I)[-c + \nu'(\theta(\tau)) + h_{21}(\nu(\theta(\tau))/I, \theta(\tau), \delta)]}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}I_1 &= I + \delta c\nu(\theta(\tau)) + \delta\nu(\theta(\tau))\bar{h}_{21}(I, \tau, \delta, \varepsilon)\\ &\quad + \frac{\delta^2(c - \nu'(\theta(\tau)))}{I}\nu'(\theta(\tau))\nu^2(\theta(\tau))\\ &\quad + \frac{\delta^2\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\bar{h}_{21}(I, \tau, \delta, \varepsilon) - \varepsilon\delta\nu'(\theta(\tau))\\ &\quad - \frac{\varepsilon\delta^2(c - \nu'(\theta(\tau)))}{I}\nu'(\theta(\tau))\nu(\theta(\tau))\\ &\quad - \frac{\varepsilon\delta^2\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\bar{h}_{21}(I, \tau, \delta, \varepsilon)\\ &\quad + \frac{I}{\nu(\theta(\tau))}\bar{h}_{11}(I, \tau, \delta, \varepsilon)\\ &\quad + \delta(c - \nu'(\theta(\tau)))\bar{h}_{11}(I, \tau, \delta, \varepsilon) + \delta\bar{h}_{11}(I, \tau, \delta, \varepsilon)\bar{h}_{21}(I, \tau, \delta, \varepsilon),\end{aligned}$$

which can be written as

$$I_1 = I + \delta c\nu(\theta(\tau)) - \varepsilon\delta\nu'(\theta(\tau)) + \delta h_{22}(I, \tau, \delta, \varepsilon),$$

where

$$\begin{aligned} h_{22}(I,\tau,\delta,\varepsilon) &= \nu(\theta(\tau))\bar{h}_{21}(I,\tau,\delta,\varepsilon) + \frac{\delta(c-\nu'(\theta(\tau)))}{I}\nu'(\theta(\tau))\nu^2(\theta(\tau)) \\ &\quad + \frac{\delta\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\bar{h}_{21}(I,\tau,\delta,\varepsilon) \\ &\quad - \frac{\varepsilon\delta(c-\nu'(\theta(\tau)))}{I}\nu'(\theta(\tau))\nu(\theta(\tau)) \\ &\quad - \frac{\varepsilon\delta\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\bar{h}_{21}(I,\tau,\delta,\varepsilon) + \frac{I}{\delta\nu(\theta(\tau))}\bar{h}_{11}(I,\tau,\delta,\varepsilon) \\ &\quad + (c-\nu'(\theta(\tau)))\bar{h}_{11}(I,\tau,\delta,\varepsilon) + \bar{h}_{11}(I,\tau,\delta,\varepsilon)\bar{h}_{21}(I,\tau,\delta,\varepsilon). \end{aligned}$$

In what follows, we shall prove that, for every $\varepsilon > 0$, we have

$$\lim_{\delta\to 0^+} h_{22}(I,\tau,\delta,\varepsilon) = 0, \tag{2.10}$$

uniformly in $\tau \in [0, 2\pi]$ and sufficiently large I.

Indeed, from (2.5) and the fact that $\nu(\theta) > 0$, for $\theta \in [0, 2\pi]$, it follows that

$$\lim_{\delta\to 0^+} \delta^{-1} I h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right) = 0, \qquad \lim_{\delta\to 0^+} h_{21}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right) = 0, \tag{2.11}$$

uniformly in $\tau \in [0, 2\pi]$ and sufficiently large I. Furthermore, we have

$$\lim_{\delta\to 0^+} \delta^{-1} I \bar{h}_{11}(I,\tau,\delta,\varepsilon) = 0, \qquad \lim_{\delta\to 0^+} \bar{h}_{21}(I,\tau,\delta,\varepsilon) = 0 \tag{2.12}$$

for $\tau \in [0, 2\pi]$ and sufficiently large I. Hence, we may infer that

$$\lim_{\delta\to 0^+} \bar{h}_{11}(I,\tau,\delta,\varepsilon) = 0, \qquad \lim_{\delta\to 0^+} \nu(\theta(\tau))\bar{h}_{21}(I,\tau,\delta,\varepsilon) = 0, \tag{2.13}$$

and

$$\lim_{\delta\to 0^+} \frac{I}{\delta\nu(\theta(\tau))}\bar{h}_{11}(I,\tau,\delta,\varepsilon) = 0, \quad \lim_{\delta\to 0^+} \frac{\delta\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\bar{h}_{21}(I,\tau,\delta,\varepsilon) = 0, \tag{2.14}$$

uniformly in $\tau \in [0, 2\pi]$ and for sufficiently large I. On the other hand, it is easy to see that

$$\lim_{\delta\to 0^+} \frac{\delta(c-\nu'(\theta(\tau)))}{I}\nu'(\theta(\tau))\nu^2(\theta(\tau)) = 0, \qquad \lim_{\delta\to 0^+} \bar{h}_{11}(I,\tau,\delta,\varepsilon)\bar{h}_{21}(I,\tau,\delta,\varepsilon) = 0. \tag{2.15}$$

From (2.13), (2.14) and (2.15) we deduce that (2.10) holds.

Now we are in a position to prove the estimate on τ_1. From the definition of τ_1 we have

$$\tau_1 = \frac{2\pi}{d}\int_0^{\theta_1} \frac{\mathrm{d}s}{\nu^2(s)} = \frac{2\pi}{d}\int_0^{\alpha} \frac{\mathrm{d}s}{\nu^2(s)},$$

where

$$\alpha = 2\pi + \theta(\tau) + \frac{\delta\nu^2(\theta(\tau))}{I} + h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), \delta\right) - \frac{\varepsilon\delta\nu(\theta(\tau))}{I}.$$

Hence,

$$\tau_1 = \frac{2\pi}{d}\int_0^{2\pi}\frac{\mathrm{d}s}{\nu^2(s)} + \frac{2\pi}{d}\int_{2\pi}^{2\pi+\theta(\tau)}\frac{\mathrm{d}s}{\nu^2(s)} + \frac{2\pi}{d}\int_{2\pi+\theta(\tau)}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}\frac{\mathrm{d}s}{\nu^2(s)} + \frac{2\pi}{d}\int_{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I+h_{11}(\nu(\theta(\tau))/I,\theta(\tau),\delta)-\varepsilon\delta\nu(\theta(\tau))/I}\frac{\mathrm{d}s}{\nu^2(s)}. \tag{2.16}$$

From the definition of d and of $\theta(\tau)$, we obtain

$$\frac{2\pi}{d}\int_0^{2\pi}\frac{\mathrm{d}s}{\nu^2(s)} = 2\pi, \qquad \frac{2\pi}{d}\int_{2\pi}^{2\pi+\theta(\tau)}\frac{\mathrm{d}s}{\nu^2(s)} = \tau. \tag{2.17}$$

Moreover, we have

$$\begin{aligned}\frac{2\pi}{d}\int_{2\pi+\theta(\tau)}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}\frac{\mathrm{d}s}{\nu^2(s)} &= \frac{2\pi}{d}\int_{2\pi+\theta(\tau)}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}\frac{\mathrm{d}s}{\nu^2(\theta(\tau))} \\ &\quad + \frac{2\pi}{d}\int_{2\pi+\theta(\tau)}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}\frac{(\nu(\theta(\tau))+\nu(s))(\nu(\theta(\tau))-\nu(s))}{\nu^2(s)\nu^2(\theta(\tau))}\,\mathrm{d}s \\ &= \frac{2\pi\delta}{dI} + \delta\bar{h}_{12}(I,\tau,\delta,\varepsilon),\end{aligned} \tag{2.18}$$

where

$$\bar{h}_{12}(I,\tau,\delta,\varepsilon) = \frac{2\pi}{d\delta}\int_{2\pi+\theta(\tau)}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}\frac{(\nu(\theta(\tau))+\nu(s))(\nu(\theta(\tau))-\nu(s))}{\nu^2(s)\nu^2(\theta(\tau))}\,\mathrm{d}s.$$

From the fact that $\nu(\theta) \neq 0$, for every $\theta \in [0, 2\pi]$ and the Lagrange mean-value theorem, we infer that there exists a constant $\gamma > 0$ such that

$$\left|\int_{2\pi+\theta(\tau)}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}\frac{(\nu(\theta(\tau))+\nu(s))(\nu(\theta(\tau))-\nu(s))}{\nu^2(s)\nu^2(\theta(\tau))}\,\mathrm{d}s\right| \leqslant \frac{\gamma\delta^2}{I^2}. \tag{2.19}$$

As a consequence, we obtain

$$\lim_{\delta\to 0^+}\bar{h}_{12}(I,\tau,\delta,\varepsilon) = 0, \tag{2.20}$$

uniformly in $\tau \in [0, 2\pi]$ and for sufficiently large I. Similarly, there exists a constant $\bar{\gamma} > 0$ such that

$$\left|\int_{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I+h_{11}(\nu(\theta(\tau))/I,\theta(\tau),\delta)-\varepsilon\delta\nu(\theta(\tau))/I}\frac{\mathrm{d}s}{\nu^2(s)}\right| \leqslant \bar{\gamma}\left|h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),\delta,\varepsilon\right)\right|. \tag{2.21}$$

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From (2.11) and (2.21) we deduce that

$$\lim_{\delta\to 0^+}\hat{h}_{12}(I,\tau,\delta,\varepsilon)=0, \tag{2.22}$$

where

$$\hat{h}_{12}(I,\tau,\delta,\varepsilon)=\frac{2\pi}{d\delta}\int_{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I}^{2\pi+\theta(\tau)+\delta\nu^2(\theta(\tau))/I+h_{11}(\nu(\theta(\tau))/I,\theta(\tau),\delta)-\varepsilon\delta\nu(\theta(\tau))/I}\frac{\mathrm{d}s}{\nu^2(s)}.$$

From (2.16)–(2.18), (2.20) and (2.22) we deduce that the asymptotic expansion of τ_1 in (2.7) holds. □

Proof of proposition 2.1. Assume that $\mu(\theta)\geqslant 0$ for $\theta\in[0,2\pi]$. Given a point (I_0,τ_0), denote by $\{(I_j,\tau_j)\}$ the orbit of the mapping $\hat{P}$ through the point (I_0,τ_0). We will prove that $I_j\to+\infty$; this will imply that $\rho_j\to+\infty$, as $j\to+\infty$ (or $-\infty$, according to the sign of c).

Let $0\leqslant\vartheta_1<\vartheta_2<\cdots<\vartheta_m<2\pi$ be m degenerate zeros of $\mu(\theta)$ in $[0,2\pi)$, i.e.

$$\mu(\vartheta_i)=\mu'(\vartheta_i)=0,\quad i=1,2,\dots,m.$$

Since $\theta(\tau)$ is increasing and $\theta(0)=0$, there exist $0\leqslant\varsigma_1<\varsigma_2<\cdots<\varsigma_m<2\pi$ such that

$$\theta(\varsigma_i)=\vartheta_i,\quad i=1,2,\dots,m.$$

From the continuity of $\mu(\theta(\tau))$ we deduce that there exist constants $\eta>0$, $\beta>0$ such that if $|\tau-\varsigma_i|<\eta$ for some $i=1,2,\dots,m$, then

$$|\mu'(\theta(\tau))|=|\mu'(\theta(\tau))-\mu'(\theta(\varsigma_i))|<\tfrac{1}{3}|c|, \tag{2.23}$$

and, if $|\tau-\varsigma_i|\geqslant\eta$, $\tau\in[0,2\pi]$, then

$$\mu(\theta(\tau))\geqslant\beta. \tag{2.24}$$

Let $0<\varepsilon_0<\beta$ be a sufficiently small constant such that

$$\varepsilon_0|\mu'(\theta(\tau))|\leqslant\tfrac{1}{3}|c|\beta,\quad \tau\in[0,2\pi]. \tag{2.25}$$

From lemma 2.2 we know that there exist $\delta_0>0$ and $l_0>0$ such that

$$|h_{22}(I,\tau,\delta_0,\varepsilon_0)|<\tfrac{1}{3}|c|\varepsilon_0, \tag{2.26}$$

uniformly for $I_0\geqslant l_0$ and $\tau\in[0,2\pi]$.

In order to conclude the proof we distinguish two cases, according to the sign of c.

Case 1 ($c>0$). We shall prove that there exist positive constants α and l_0 such that

$$I_1\geqslant I_0+\alpha, \tag{2.27}$$

for $\tau\in R$ and $I_0\geqslant l_0$. Obviously, it suffices to prove (2.27) for $\tau\in[0,2\pi]$ and $I_0\geqslant l_0$.

If $|\tau-\varsigma_i|\leqslant\eta$ for some $i=1,2,\dots,m$, then it follows from (2.23) and (2.26) that

$$I_1\geqslant I_0+\delta_0c\varepsilon_0-\delta_0\varepsilon_0|\mu'(\theta(\tau))|-\delta_0|h_{22}(I,\tau,\delta_0,\varepsilon_0)|\geqslant I_0+\tfrac{1}{3}c\varepsilon_0\delta_0$$

for $I_0 \geqslant l_0$, $\tau \in [0, 2\pi]$; on the other hand if $|\tau - \varsigma_i| \geqslant \eta$ for some $i = 1, 2, \ldots, m$ and $\tau \in [0, 2\pi]$, then from (2.24)–(2.26) we may infer that

$$I_1 \geqslant I_0 + \delta_0 c\varepsilon_0 + \delta_0 c\beta - \varepsilon_0\delta_0|\mu'(\theta(\tau))| - \delta_0|h_{22}(I, \tau, \delta_0, \varepsilon_0)| \geqslant I_0 + \tfrac{1}{3}c\varepsilon_0\delta_0.$$

Taking $\alpha = \frac{1}{3}c\varepsilon_0\delta_0$, we prove that (2.27) holds.

From (2.27) we can plainly deduce that $\lim_{j\to+\infty} I_j = +\infty$; as a consequence, by observing that $\nu(\theta) = \mu(\theta) + \varepsilon_0 > 0$, for every $\theta \in [0, 2\pi]$, and $r_j = r(I_j, \theta_j) = \nu(\theta(\tau_j))/I_j$, we can deduce that

$$\lim_{j\to+\infty} r_j = 0.$$

Recalling that $1/\rho = \delta_0 r$, this implies that $\lim_{j\to+\infty} \rho_j = +\infty$.

Case 2 ($c < 0$). Let us set $S = \{(I, \tau) : I \geqslant l_0, \tau \in R\}$. From the expression for the mapping $\hat{P}$ we know that $\hat{P}(S)$ contains a neighbourhood of infinity. Hence, there exists a constant $l_0' > 0$ such that if $I_0 \geqslant l_0'$ and $\hat{P}^{-1}(I_0, \tau_0) = (I_{-1}, \tau_{-1})$, then $I_{-1} \geqslant l_0$. Since

$$\begin{aligned} \tau_0 &= \tau_{-1} + 2\pi + \delta_0\omega(I_{-1}) + \delta_0 h_{12}(I_{-1}, \tau_{-1}, \delta_0, \varepsilon_0), \\ I_0 &= I_{-1} + \delta_0 c\nu(\theta(\tau_{-1})) - \varepsilon_0\delta_0\nu'(\theta(\tau_{-1})) + \delta_0 h_{22}(I_{-1}, \tau_{-1}, \delta_0, \varepsilon_0), \end{aligned}$$

we have

$$\left.\begin{aligned} \tau_{-1} &= \tau_0 - 2\pi - \delta_0\omega(I_{-1}) - \delta_0 h_{12}(I_{-1}, \tau_{-1}, \delta_0, \varepsilon_0), \\ I_{-1} &= I_0 + \delta_0|c|\nu(\theta(\tau_{-1})) + \varepsilon_0\delta_0\nu'(\theta(\tau_{-1})) - \delta_0 h_{22}(I_{-1}, \tau_{-1}, \delta_0, \varepsilon_0). \end{aligned}\right\} \tag{2.28}$$

From the second equality of (2.28) we infer that, if $I_0 \geqslant l_0'$ and $|\tau_{-1} - \varsigma_i| \leqslant \eta$ for some $i = 1, 2, \ldots, m$, then

$$I_{-1} \geqslant I_0 + \delta_0|c|\varepsilon_0 - \varepsilon_0\delta_0|\mu'(\theta(\tau_{-1}))| - \delta_0|h_{22}(I_{-1}, \tau_{-1}, \delta_0, \varepsilon_0)| \geqslant I_0 + \tfrac{1}{3}\delta_0|c|\varepsilon_0.$$

If $|\tau_{-1} - \varsigma_i| \geqslant \eta$ and $\tau_{-1} \in [0, 2\pi]$ for some $i = 1, 2, \ldots, m$, we can obtain the same result. Inductively, we deduce that

$$I_j \geqslant I_{j+1} + \tfrac{1}{3}|c|\varepsilon_0\delta_0 \geqslant \cdots \geqslant I_0 + \tfrac{1}{3}|jc|\varepsilon_0\delta_0,$$

for every $j = -1, -2, \ldots$. As a consequence, we obtain

$$\lim_{j\to-\infty} I_j = +\infty.$$

Arguing as in the previous case, we can conclude that $\lim_{j\to-\infty} \rho_j = +\infty$. □

3. Unbounded solutions and periodic solutions

In this section we will deal with the unboundedness of solutions and the existence of 2π-periodic solutions to the equation

$$x'' + f(x)x' + V'(x) + g(x) = p(t), \tag{3.1}$$

where $f, g \in C(\mathbb{R})$, $V \in C^2(\mathbb{R})$ and $p \in C(\mathbb{R})$ is 2π-periodic. Moreover, we assume that g is locally Lipschitz.

We will suppose that the following assumptions are satisfied.

(1) $V''(x) > 0$, for every $x \in \mathbb{R}$, $V'(0) = 0$ and all the solutions of

$$x'' + V'(x) = 0 \tag{3.2}$$

are $2\pi/n$ periodic, for some $n \in \mathbb{N}$.

(2) There exist $a, b > 0$ such that

$$\lim_{x \to +\infty} \frac{V'(x)}{x} = a, \qquad \lim_{x \to -\infty} \frac{V'(x)}{x} = b.$$

Moreover,

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{n}.$$

(3) There exist $g_\pm \in \mathbb{R}$ such that

$$\lim_{x \to \pm\infty} g(x) = g_\pm.$$

(4) There exist $F_\pm \in \mathbb{R}$ such that

$$\lim_{x \to \pm\infty} F(x) = F_\pm,$$

where $F(x) = \int_0^x f(u)\,\mathrm{d}u$.

For every $\rho > 0$ let $\phi(\cdot, \rho)$ be the solution of (3.2) such that $\phi(0, \rho) = 0$, $\phi_t(0, \rho) = \rho$; moreover, let ψ be the solution of

$$\begin{gathered} x'' + ax^+ - bx^- = 0, \\ x(0) = 0, \quad x'(0) = 1. \end{gathered}$$

We immediately see that ψ is 2π-periodic and satisfies

$$\psi(t) = \begin{cases} \dfrac{1}{\sqrt{a}} \sin \sqrt{a} t & \text{if } 0 \leqslant t \leqslant \dfrac{\pi}{\sqrt{a}}, \\ -\dfrac{1}{\sqrt{b}} \sin \sqrt{b} \left(t - \dfrac{\pi}{\sqrt{a}} \right) & \text{if } \dfrac{\pi}{\sqrt{a}} < t \leqslant \dfrac{2\pi}{n}. \end{cases}$$

Moreover, for every $\theta_0 \in \mathbb{R}$, let

$$I_0^+ = \{t \in [0, 2\pi] : \psi(\theta_0 + t) > 0\}, \qquad I_0^- = \{t \in [0, 2\pi] : \psi(\theta_0 + t) < 0\}.$$

From an elementary computation we deduce the following lemma.

Lemma 3.1. *For every $\theta_0 \in \mathbb{R}$, we have*

$$\begin{gathered} \int_{I_0^+} \psi(\theta_0 + t)\,\mathrm{d}t = \frac{2n}{a}, \qquad \int_{I_0^-} \psi(\theta_0 + t)\,\mathrm{d}t = -\frac{2n}{b}, \\ \int_{I_0^+} \psi'(\theta_0 + t)\,\mathrm{d}t = \int_{I_0^-} \psi'(\theta_0 + t)\,\mathrm{d}t = 0. \end{gathered}$$

The following result has been proved in [3].

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Lemma 3.2. *The function ϕ satisfies the following properties:*

$$\frac{\partial\phi}{\partial t}(t,\rho)\frac{\partial^2\phi}{\partial t\partial\rho}(t,\rho)+V'(\phi(t,\rho))\frac{\partial\phi}{\partial\rho}(t,\lambda)=V'(\rho)\quad \textit{for all } (t,\rho)\in\mathbb{R}\times(0,+\infty);$$

$$\lim_{\rho\to+\infty}\frac{\phi(t,\rho)}{\rho}=\psi(t);\qquad \lim_{\rho\to+\infty}\phi_\rho(t,\rho)=\psi(t);$$

$$\lim_{\rho\to+\infty}\frac{\phi_t(t,\rho)}{\rho}=\psi'(t);\quad \lim_{\rho\to+\infty}\frac{\partial^2\phi}{\partial t\partial\rho}(t,\rho)=\psi'(t).$$

All the previous limits are uniform in $t\in\mathbb{R}$.

Equation (3.1) is equivalent to the system

$$x'=y-F(x),\qquad y'=-V'(x)-g(x)+p(t). \tag{3.3}$$

Now we introduce a transformation due to B. Liu (2005, personal communication). Let

$$\varPhi:(\theta,\rho)\in S^1\times(0,+\infty)\to(x,y)\in\mathbb{R}^2\setminus\{0\}$$

be defined by

$$x=\phi(\theta,\rho),\quad y=\frac{\partial\phi}{\partial t}(\theta,\rho).$$

This change of variables transforms (3.3) into the system

$$\left.\begin{aligned}\theta'&=1+\frac{\alpha(\theta,\rho,t)}{\rho},\\ \rho'&=\beta(\theta,\rho,t),\end{aligned}\right\} \tag{3.4}$$

where

$$\alpha(\theta,\rho,t)=\frac{\rho}{V'(\rho)}(\phi_\rho(\theta,\rho)g(\phi(\theta,\rho))-\phi_\rho(\theta,\rho)p(t)-\phi_{t\rho}(\theta,\rho)F(\phi(\theta,\rho))),$$

$$\beta(\theta,\rho,t)=-\frac{\rho}{V'(\rho)}\biggl(\frac{\phi_t(\theta,\rho)}{\rho}g(\phi(\theta,\rho))-\frac{\phi_t(\theta,\rho)}{\rho}p(t)+\frac{V'(\phi(\theta,\rho))}{\rho}F(\phi(\theta,\rho))\biggr).$$

We also denote by P the Poincaré map associated with (3.4), i.e.

$$P(\theta_0,\rho_0)=(\theta(2\pi),\rho(2\pi)),$$

where $(\theta(\cdot),\rho(\cdot))$ is the solution of (3.4) satisfying $(\theta(0),\rho(0))=(\theta_0,\rho_0)$. Our aim is to show that the asymptotic development of P, as $\rho_0\to+\infty$, fits the framework of § 2.

Following Liu (2005, personal communication) we deduce that α and β are bounded; as a consequence, from the second equation in (3.4) we obtain

$$\rho(t)=\rho_0+O(1),\qquad \rho_0\to+\infty. \tag{3.5}$$

This relation implies that

$$\frac{1}{\rho(t)}=\frac{1}{\rho_0}+o\biggl(\frac{1}{\rho_0}\biggr),\quad \rho_0\to+\infty; \tag{3.6}$$

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by replacing this equality in the first equation of (3.4) we may infer that

$$\theta(t) = \theta_0 + t + O\left(\frac{1}{\rho_0}\right), \quad \rho_0 \to +\infty. \tag{3.7}$$

For every (θ_0, ρ_0), let

$$I^{+(-)}(\theta_0, \rho_0) = \{t \in [0, 2\pi] : \phi(\theta(t), \rho(t)) > (<) 0\}.$$

From lemma 3.2 and (3.7) it is easy to prove the validity of the following lemma.

Lemma 3.3. *For every $\theta_0 \in \mathbb{R}$, we have*

$$\lim_{\rho_0 \to +\infty} I^{\pm}(\theta_0, \rho_0) = I_0^{\pm}.$$

We are in position to prove the following proposition.

Proposition 3.4. *For every $\theta_0 \in \mathbb{R}$, the Poincaré map P satisfies*

$$P : \begin{cases} \theta_1 = \theta(2\pi) = \theta_0 + 2\pi + \dfrac{1}{\rho_0}\mu(\theta_0) + o\left(\dfrac{1}{\rho_0}\right), \\ \rho_1 = \rho(2\pi) = \rho_0 + c - \mu'(\theta_0) + o(1), \qquad \rho_0 \to +\infty, \end{cases} \tag{3.8}$$

where

$$\mu(\theta_0) = \frac{2n}{a}\left(\frac{g_+}{a} - \frac{g_-}{b}\right) - \frac{1}{a}\int_0^{2\pi} \psi(t + \theta_0) p(t)\,\mathrm{d}t, \quad c = -\frac{2n}{a}(F_+ - F_-).$$

Proof. We first prove the asymptotic formula for θ_1; from (3.6) and the first equation in (3.4), recalling that α is bounded, we obtain

$$\theta' = 1 + \frac{1}{\rho_0}\alpha(\theta, \rho, t) + o\left(\frac{1}{\rho_0}\right), \quad \rho_0 \to +\infty.$$

By integrating on $[0, 2\pi]$ and using the expression for α we may infer that

$$\begin{aligned} \theta_1 &= \theta_0 + 2\pi + \frac{1}{\rho_0}\int_0^{2\pi} \alpha(\theta, \rho, t)\,\mathrm{d}t + o\left(\frac{1}{\rho_0}\right) \\ &= \theta_0 + 2\pi + \frac{1}{\rho_0}\int_0^{2\pi} \frac{\rho}{V'(\rho)}(\phi_\rho(\theta, \rho) g(\phi(\theta, \rho)) - \phi_\rho(\theta, \rho) p(t) \\ &\qquad - \phi_{t\rho}(\theta, \rho) F(\phi(\theta, \rho)))\,\mathrm{d}t + o\left(\frac{1}{\rho_0}\right). \end{aligned} \tag{3.9}$$

From (3.5) and assumption (2) on the asymptotic behaviour of V' we get

$$\lim_{\rho_0 \to +\infty} \frac{\rho}{V'(\rho)} = \frac{1}{a}. \tag{3.10}$$

Moreover, lemma 3.2 and (3.7) imply that

$$\lim_{\rho_0 \to +\infty} \phi_\rho(\theta, \rho) = \psi(\theta_0 + t). \tag{3.11}$$

As a direct consequence, we infer that

$$\lim_{\rho_0\to+\infty}\int_0^{2\pi}\frac{\rho}{V'(\rho)}\phi_\rho(\theta,\rho)p(t)\,\mathrm{d}t=\frac{1}{a}\int_0^{2\pi}\psi(\theta_0+t)p(t)\,\mathrm{d}t. \tag{3.12}$$

Moreover, using also lemma 3.3, we have

$$\begin{aligned}
\lim_{\rho_0\to+\infty}&\int_0^{2\pi}\frac{\rho}{V'(\rho)}\phi_\rho(\theta,\rho)g(\phi(\theta,\rho))\,\mathrm{d}t\\
&=\lim_{\rho_0\to+\infty}\int_{I^+(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\phi_\rho(\theta,\rho)g(\phi(\theta,\rho))\,\mathrm{d}t\\
&\quad+\lim_{\rho_0\to+\infty}\int_{I^-(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\phi_\rho(\theta,\rho)g(\phi(\theta,\rho))\,\mathrm{d}t\\
&=\frac{1}{a}\biggl(g_+\int_{I_0^+}\psi(\theta_0+t)\,\mathrm{d}t+g_-\int_{I_0^-}\psi(\theta_0+t)\,\mathrm{d}t\biggr)\\
&=\frac{2n}{a}\biggl(\frac{g_+}{a}-\frac{g_-}{b}\biggr).
\end{aligned} \tag{3.13}$$

Analogously, we have

$$\begin{aligned}
\lim_{\rho_0\to+\infty}&\int_0^{2\pi}\frac{\rho}{V'(\rho)}\phi_{t\rho}(\theta,\rho)F(\phi(\theta,\rho))\,\mathrm{d}t\\
&=\lim_{\rho_0\to+\infty}\int_{I^+(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\phi_{t\rho}(\theta,\rho)F(\phi(\theta,\rho))\,\mathrm{d}t\\
&\quad+\lim_{\rho_0\to+\infty}\int_{I^-(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\phi_{t\rho}(\theta,\rho)F(\phi(\theta,\rho))\,\mathrm{d}t\\
&=\frac{1}{a}\biggl(F_+\int_{I_0^+}\psi'(\theta_0+t)\,\mathrm{d}t+F_-\int_{I_0^-}\psi'(\theta_0+t)\,\mathrm{d}t\biggr)\\
&=0.
\end{aligned} \tag{3.14}$$

From (3.9) and (3.12)–(3.14) we deduce the asymptotic development of θ_1 given in (3.8).

Now let us consider the asymptotic expansion of ρ_1; by integrating the second equation in (3.4) and recalling the expression for β, we may infer that

$$\begin{aligned}
\rho_1&=\rho_0+\int_0^{2\pi}\beta(\theta,\rho,t)\,\mathrm{d}t\\
&=\rho_0-\int_0^{2\pi}\frac{\rho}{V'(\rho)}\frac{\phi_t(\theta,\rho)}{\rho}g(\phi(\theta,\rho))\,\mathrm{d}t\\
&\quad+\int_0^{2\pi}\frac{\rho}{V'(\rho)}\biggl(\frac{\phi_t(\theta,\rho)}{\rho}p(t)-\frac{V'(\phi(\theta,\rho))}{\rho}F(\phi(\theta,\rho))\biggr)\,\mathrm{d}t.
\end{aligned} \tag{3.15}$$

Lemma 3.2 and (3.7) imply that

$$\lim_{\rho_0\to+\infty}\frac{\phi_t(\theta,\rho)}{\rho}=\psi'(\theta_0+t). \tag{3.16}$$

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As a direct consequence, we may infer that

$$\lim_{\rho_0\to+\infty}\int_0^{2\pi}\frac{\rho}{V'(\rho)}\frac{\phi_t(\theta,\rho)}{\rho}p(t)\,\mathrm{d}t=\frac{1}{a}\int_0^{2\pi}\psi'(\theta_0+t)p(t)\,\mathrm{d}t. \tag{3.17}$$

Moreover, by also using lemma 3.3, we have

$$\begin{aligned}\lim_{\rho_0\to+\infty}&\int_0^{2\pi}\frac{\rho}{V'(\rho)}\frac{\phi_t(\theta,\rho)}{\rho}g(\phi(\theta,\rho))\,\mathrm{d}t\\&=\lim_{\rho_0\to+\infty}\int_{I^+(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\frac{\phi_t(\theta,\rho)}{\rho}g(\phi(\theta,\rho))\,\mathrm{d}t\\&\quad+\lim_{\rho_0\to+\infty}\int_{I^-(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\frac{\phi_t(\theta,\rho)}{\rho}g(\phi(\theta,\rho))\,\mathrm{d}t\\&=\frac{1}{a}\biggl(g_+\int_{I_0^+}\psi'(\theta_0+t)\,\mathrm{d}t+g_-\int_{I_0^-}\psi'(\theta_0+t)\,\mathrm{d}t\biggr)\\&=0.\end{aligned}\tag{3.18}$$

Analogously, we have

$$\begin{aligned}\lim_{\rho_0\to+\infty}&\int_0^{2\pi}\frac{\rho}{V'(\rho)}\frac{V'(\phi(\theta,\rho))}{\rho}F(\phi(\theta,\rho))\,\mathrm{d}t\\&=\lim_{\rho_0\to+\infty}\int_{I^+(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\frac{V'(\phi(\theta,\rho))}{\phi(\theta,\rho)}\frac{\phi(\theta,\rho)}{\rho}F(\phi(\theta,\rho))\,\mathrm{d}t\\&\quad+\lim_{\rho_0\to+\infty}\int_{I^-(\theta_0,\rho_0)}\frac{\rho}{V'(\rho)}\frac{V'(\phi(\theta,\rho))}{\phi(\theta,\rho)}\frac{\phi(\theta,\rho)}{\rho}F(\phi(\theta,\rho))\,\mathrm{d}t\\&=\frac{1}{a}\biggl(aF_+\int_{I_0^+}\psi(\theta_0+t)\,\mathrm{d}t+bF_-\int_{I_0^-}\psi(\theta_0+t)\,\mathrm{d}t\biggr)\\&=\frac{2n}{a}(F_+-F_-).\end{aligned}\tag{3.19}$$

From (3.15) and (3.17)–(3.19) we deduce the asymptotic development of ρ_1 given in (3.8). This concludes the proof. □

We are now in a position to state the main theorem of this section.

THEOREM 3.5. *Suppose that assumptions (1)–(4) hold; moreover, assume that the function μ is non-negative (or non-positive) and has finitely many zeros in $[0,2\pi]$, all of which are degenerate.*

If $F_+\neq F_-$, then (3.1) has at least one 2π-periodic solution.

Moreover, the following conclusions hold:

(a) *if $F_+<F_-$, then there exists $R_0>0$ such that all the solutions $x(t)$ of (3.1) with $x(0)^2+x'(0)^2\geqslant R_0^2$ satisfy*

$$\lim_{t\to+\infty}(x^2(t)+x'(t)^2)=+\infty;$$

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(b) *if $F_+ > F_-$, then there exists $R_0 > 0$ such that all the solutions $x(t)$ of (3.1) with $x(0)^2 + x'(0)^2 \geqslant R_0^2$ satisfy*

$$\lim_{t\to-\infty}(x^2(t) + x'(t)^2) = +\infty.$$

Proof. We first show the existence of a 2π-periodic solution to (3.1). Let

$$\xi(\theta) = 2n(F_+ - F_-) - \mu'(\theta);$$

since all the zeros of μ are degenerate, the sign of ξ at every zero of μ is the same. This, together with the fact that μ does not change sign, ensures that the map P fits the framework of [4]; as a consequence, arguing as in [4], we deduce that P possesses at least one fixed point. Consequently, (3.1) has at least one 2π-periodic solution.

We prove the result on the unboundedness of solutions with large initial energy in the case when $F_+ < F_-$; the other case can be handled similarly.

By the assumption on μ and proposition 3.4, it follows that we can apply proposition 2.1 to the Poincaré map P. Hence, there exists $R_0 > 0$ such that, if $\rho_0 \geqslant R_0$, then the orbit $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies $\lim_{j\to+\infty}\rho_j = +\infty$.

On the other hand, it follows from the second equality of (3.4) and the fact that β is bounded that there exists a constant $d_0 > 0$ such that $|\rho(t) - \rho(s)| \leqslant d_0$ for t and s satisfying $|t - s| \leqslant 2\pi$. Hence, we obtain

$$\lim_{t\to+\infty}\rho(t) = +\infty.$$

By the expression for the change of variable Φ and by lemma 3.2 we deduce that

$$\tfrac{1}{2}y(t)^2 + V(x(t)) = V(\rho(t)) \quad \text{for all } t \in \mathbb{R},$$

which implies that

$$\lim_{t\to+\infty}(x^2(t) + y^2(t)) = +\infty.$$

From the boundedness of F and since $x'(t) = y(t) - F(x)$ we obtain

$$\lim_{t\to+\infty}(x^2(t) + x'(t)^2) = +\infty.$$

□

Remark 3.6. Following the approach of [3], it is possible to prove an analogous result for equation (3.1) in the case when the potential V is singular and at resonance. Indeed, a result along the lines of theorem 3.5 can be given when assumption (2) on the asymptotic behaviour of V' is replaced by

$$\lim_{x\to+\infty}\frac{V'(x)}{x} = \frac{n^2}{4}, \qquad \lim_{x\to a^+}\frac{V'(x)}{x} = +\infty,$$

for some $a < 0$. In this situation it is possible to see (see [3]) that all the computations can be repeated by replacing the function ψ by the function

$$\psi^*(t) = |\cos\tfrac{1}{2}nt|.$$

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