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THE FINITISTIC DIMENSION AND CHAIN CONDITIONS ON IDEALS

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Abstract. Let Λ be an artin algebra and $0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ a chain of ideals of Λ such that $(I_{i+1}/I_i)\,\text{rad}(\Lambda/I_i) = 0$ for any $0 \leq i \leq n-1$ and Λ/I_n is semisimple. If either none or the direct sum of exactly two consecutive ideals has infinite projective dimension, then the finitistic dimension conjecture holds for Λ. As a consequence, we have that if either none or the direct sum of exactly two consecutive terms in the radical series of Λ has infinite projective dimension, then the finitistic dimension conjecture holds for Λ. Some known results are obtained as corollaries.

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1. Introduction. Throughout this paper, Λ is an artin algebra, $\text{rad}(\Lambda)$ is the Jacobson radical of Λ and $\text{mod}\,\Lambda$ is the category of finitely generated left Λ-modules. For a module M in $\text{mod}\,\Lambda$, we use $\text{pd}_\Lambda M$ to denote the projective dimension of M.

Recall that the *finitistic dimension* $\text{fin.dim}\,\Lambda$ of Λ is defined as

$$\sup\{\text{pd}_\Lambda M \mid \text{pd}_\Lambda M < \infty \text{ with } M \in \text{mod}\,\Lambda\}.$$

The famous finitistic dimension conjecture states that $\text{fin.dim}\,\Lambda < \infty$ for any artin algebra Λ. This conjecture was initially an open question posed by Rosenberg and Zelinsky, published by Bass in 1960 ([**1**]). The finitistic dimension conjecture is one of the main problems in the representations theory of artin algebras and has a close relation with some other homological conjectures, such as the (generalised) Nakayama conjecture, the Gorenstein symmetry conjecture and the Wakamatsu tilting conjecture, and so on ([**2**, **24**]). These conjectures are still open. See [**21**, **26**] for some progress on the finitistic dimension conjecture.

Igusa and Todorov introduced the ϕ-function and the ψ-function from $\text{mod}\,\Lambda$ to $\mathbb{N}$ (the natural numbers) in [**11**]. These two functions are powerful in studying the finitistic dimension conjecture, see [**3**, **4**], [**7**]–[**20**], [**22**, **23**, **25**] and references therein. In particular, in [**18**], the finitistic dimension conjecture was investigated in terms of some chain conditions of ideals. Following this philosophy, the aim of this paper is to prove the following

THEOREM 1.1. *Let*

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$$

be a chain of ideals of Λ *such that* $(I_{i+1}/I_i)\operatorname{rad}(\Lambda/I_i)=0$ *for any* $0\leq i\leq n-1$ *and* Λ/I_n *is semisimple. If either none or the direct sum of exactly two consecutive ideals has infinite projective dimension, then the finitistic dimension conjecture holds for* Λ.

Recall that the *Loewy length* $LL(\Lambda)$ of Λ is defined as $\min\{l \mid \operatorname{rad}^{l-1}(\Lambda)\neq 0$ and $\operatorname{rad}^{l}(\Lambda)=0\}$. Let $LL(\Lambda)=n$ and

$$0=\operatorname{rad}^n(\Lambda)\subseteq \operatorname{rad}^{n-1}(\Lambda)\subseteq \operatorname{rad}^{n-2}(\Lambda)\subseteq\cdots\subseteq \operatorname{rad}(\Lambda)\subseteq\Lambda$$

be the radical series of Λ. By putting $I_i=\operatorname{rad}^{n-i}(\Lambda)$ for any $0\leq i\leq n-1$ in Theorem 1.1, we immediately have the following

COROLLARY 1.2. *Let* $LL(\Lambda)=n$. *If either none or the direct sum of exactly two consecutive terms in the radical series of* Λ *has infinite projective dimension, then the finitistic dimension conjecture holds for* Λ.

The following three results are special cases of Corollary 1.2.

COROLLARY 1.3 ([**5**, Theorem 16]). *If* $LL(\Lambda)\leq 3$, *then the finitistic dimension conjecture holds for* Λ.

COROLLARY 1.4 ([**18**, Corollary 0.3]). *If* $\operatorname{pd}_\Lambda \operatorname{rad}^i(\Lambda)<\infty$ *for all* $i\geq 3$, *then the finitistic dimension conjecture holds for* Λ.

COROLLARY 1.5 ([**18**, Corollary 3.8]). *Let* $LL(\Lambda)\leq 4$. *If either* $\operatorname{pd}_\Lambda \operatorname{rad}^2(\Lambda)<\infty$ *or* $\operatorname{pd}_\Lambda \operatorname{rad}^3(\Lambda)<\infty$, *then the finitistic dimension conjecture holds for* Λ.

2. Preliminaries. In this section, we give some terminology and some preliminary results.

For a module M in $\operatorname{mod}\Lambda$, we use $\operatorname{rad}_\Lambda(M)$ and $\Omega^i_\Lambda(M)$ to denote the radical and the i-th syzygy of M (in particular, $\Omega^0_\Lambda(M):=M$), respectively, and use $\operatorname{add}_\Lambda M$ to denote the subcategory of $\operatorname{mod}\Lambda$ consisting of all direct summands of finite direct sums of copies of M.

Let K_0 be the abelian group generated by all $[M]$, where $M\in\operatorname{mod}\Lambda$, subject to the relations $[C]=[A]+[B]$ if $C\cong A\oplus B$ and $[P]=0$ if P is projective. Define a homomorphism $L:K_0\to K_0$ via $L[M]=[\Omega(M)]$. Let $M\in\operatorname{mod}\Lambda$. Denote by $\langle\operatorname{add}_\Lambda M\rangle$ the subgroup of K_0 generated by all indecomposable direct summands of M. Let f be an endomorphism of M and X a submodule of M. By the Fitting lemma, there exists a smallest integer $\eta_f(X)$ such that $f|_{f^m(X)}:f^m(X)\to f^{m+1}(X)$ is an isomorphism for any $m\geq\eta_f(X)$. Moreover, if Y is a submodule of X, then $\eta_f(Y)\leq\eta_f(X)$. In [**11**], Igusa and Todorov defined

$$\phi(M):=\eta_L(\langle\operatorname{add}_\Lambda M\rangle),$$

$$\psi(M):=\phi(M)+\sup\{\operatorname{pd}_\Lambda X\mid \operatorname{pd}_\Lambda X<\infty \text{ with } X \text{ a direct summand of } \Omega^{\phi(M)}_\Lambda(M)\}.$$

LEMMA 2.1 ([**11**]). *Let* $M\in\operatorname{mod}\Lambda$. *Then the function* $\psi:\operatorname{mod}\Lambda\to\mathbb{N}$ *satisfies the following properties.*

(1) *If* $\operatorname{pd}_\Lambda M<\infty$, *then* $\psi(M)=\phi(M)=\operatorname{pd}_\Lambda M$. *If* M *is indecomposable and* $\operatorname{pd}_\Lambda M=\infty$, *then* $\psi(M)=0$.
(2) $\psi(M^{(n)})=\psi(M)$ *for any* $n\geq 1$.
(3) $\psi(M)\leq\psi(M\oplus N)$ *for any* $N\in\operatorname{mod}\Lambda$.

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(4) $\psi(M) = \psi(M \oplus P)$ *for any projective module* P *in* $\operatorname{mod}\Lambda$.

(5) Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence in $\operatorname{mod}\Lambda$ *with* $\operatorname{pd}_\Lambda C < \infty$*, then* $\operatorname{pd}_\Lambda C \leq \psi(A \oplus B) + 1$.

3. Proof of Theorem 1.1. In this section, we give the proof of Theorem 1.1. We need some lemmas. The first assertion of the following lemma is essentially contained in the proof of [18, Theorem 3.6].

LEMMA 3.1. *Let*

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$$

be a chain of ideals of Λ *such that* $(I_{i+1}/I_i)\operatorname{rad}(\Lambda/I_i) = 0$ *for any* $0 \leq i \leq n-1$ *and* Λ/I_n *is semisimple. Then for any* $M \in \operatorname{mod}\Lambda$*, there exists an exact sequence*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0$$

in $\operatorname{mod}\Lambda$ *such that* $M_i \in \operatorname{add}_\Lambda \Lambda/I_i$ *for any* $0 \leq i \leq n$.

Proof. Set $\Lambda_i := \Lambda/I_i$ for any $0 \leq i \leq n$; in particular, $\Lambda_0 = \Lambda$. Taking $X_0 \in \operatorname{mod}\Lambda$ and writing $X_1 := \Omega^1_\Lambda(X_0)$, then we get an exact sequence

$$0 \longrightarrow X_1 \longrightarrow P_{X_0} \longrightarrow X_0 \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ with $P_{X_0} \to X_0$ the projective cover of X_0. Since $I_1\operatorname{rad}(\Lambda) = 0$, we have that $I_1X_1 = I_1\Omega^1_\Lambda(X_0) \subseteq I_1\operatorname{rad}_\Lambda(P_{X_0}) = 0$ and $X_1 \in \operatorname{mod}\Lambda_1$. Inductively, for $X_i \in \operatorname{mod}\Lambda_i$ with $0 \leq i \leq n-1$, we have an exact sequence

$$0 \longrightarrow X_{i+1} \longrightarrow P_{X_i} \longrightarrow X_i \longrightarrow 0,$$

in $\operatorname{mod}\Lambda_i$ with $P_{X_i} \to X_i$ the projective cover of X_i, such that $X_{i+1} \in \operatorname{mod}\Lambda_{i+1}$. Moreover, restricting these exact sequences to Λ-modules and combining them, we get the following exact sequence

$$0 \longrightarrow X_n \longrightarrow P_{X_{n-1}} \longrightarrow P_{X_{n-1}} \longrightarrow \cdots \longrightarrow P_{X_0} \longrightarrow X_0 \longrightarrow 0$$

in $\operatorname{mod}\Lambda$, where $X_n \in \operatorname{mod}\Lambda_n$ and each P_{X_i} is projective as a Λ_i-module. We have $P_{X_i} \in \operatorname{add}_{\Lambda_i}\Lambda_i$ for any $0 \leq i \leq n-1$. Because $\Lambda_n = \Lambda/I_n$ is semisimple, we have that X_n is projective as a Λ_n-module and $X_n \in \operatorname{add}_{\Lambda_n}\Lambda_n$. Thus, $P_{X_i} \in \operatorname{add}_\Lambda \Lambda_i$ for any $0 \leq i \leq n$. Now putting $M := X_0$, $M_n := X_n$ and $M_i := P_{X_i}$ for any $0 \leq i \leq n-1$, we get the required exact sequence. □

The following lemma is useful.

LEMMA 3.2. *Let*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0$$

be an exact sequence in $\operatorname{mod}\Lambda$.

(1) If $\operatorname{pd}_\Lambda M_i < \infty$ *for all* $0 \leq i \leq n-1$*, then for all* $t \geq 0$*, we have*

$$\Omega^{(n+m)+t}_\Lambda(M) \oplus P \cong \Omega^{m+t}_\Lambda(M_n) \oplus Q,$$

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where $m = \max\{\mathrm{pd}_\Lambda M_0, \mathrm{pd}_\Lambda M_1, \cdots, \mathrm{pd}_\Lambda M_{n-1}\}$ *and* P, Q *are projective in* $\mathrm{mod}\,\Lambda$.

(2) If $\mathrm{pd}_\Lambda M_i < \infty$ *for all* $1 \le i \le n$*, then for all* $t \ge 0$*, we have*

$$\Omega_\Lambda^{(n+m+1)+t}(M) \oplus P \cong \Omega_\Lambda^{(n+m+1)+t}(M_0) \oplus Q,$$

where $m = \max\{\mathrm{pd}_\Lambda M_1, \mathrm{pd}_\Lambda M_2, \cdots, \mathrm{pd}_\Lambda M_n\}$ *and* P, Q *are projective in* $\mathrm{mod}\,\Lambda$.

Proof. Let

$$\cdots \to P_i^j \to P_i^{j-1} \to \cdots \to P_i^1 \to P_i^0 \to M_i \to 0$$

be the minimal projective resolution of M_i in $\mathrm{mod}\,\Lambda$ for any $0 \le i \le n$. Then by [6, Corollary 3.7], we get an exact sequence

$$\cdots \to \oplus_{i=0}^r P_i^{r-i} \to \cdots \to P_0^1 \oplus P_1^0 \to P_0^0 \to M \to 0. \tag{3.1}$$

(1) By assumption, we have $P_i^j = 0$ for all $0 \le i \le n-1$ and $j \ge m+1$. So (3.1) in fact is the following exact sequence

$$\cdots \to P_n^{m+1} \to P_n^m \to \oplus_{i=0}^n P_i^{(n+m-1)-i} \to \cdots \to P_0^1 \oplus P_1^0 \to P_0^0 \to M \to 0,$$

and the assertion follows.

(2) By assumption, we have $P_i^j = 0$ for all $1 \le i \le n$ and $j \ge m+1$. So (3.1) in fact is the following exact sequence

$$\cdots \to P_0^{n+m+2} \to P_0^{n+m+1} \to \oplus_{i=0}^n P_i^{(n+m)-i} \to \cdots \to P_0^1 \oplus P_1^0 \to P_0^0 \to M \to 0,$$

and the assertion follows. □

REMARK 3.3.

(1) Under the assumption of (1) (respectively, (2)) in Lemma 3.2, we have that $\mathrm{pd}_\Lambda M < \infty$ if and only if $\mathrm{pd}_\Lambda M_n < \infty$ (respectively, $\mathrm{pd}_\Lambda M_0 < \infty$).
(2) If $\mathrm{pd}_\Lambda M = \infty$, then all syzygies appear in Lemma 3.2 are non-zero. If $\mathrm{pd}_\Lambda M < \infty$, then the syzygies appear there might be zero.

The following observation is standard.

LEMMA 3.4. *Let*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0$$

be an exact sequence in $\mathrm{mod}\,\Lambda$. *If* $\mathrm{pd}_\Lambda M_i < \infty$ *for all* $0 \le i \le n$*, then for any* $l \ge 0$*, we have*

$$\mathrm{pd}_\Lambda \Omega_\Lambda^l(M) \le \mathrm{pd}_\Lambda(\Omega_\Lambda^l(\oplus_{i=0}^n M_i)) + n.$$

Proof. By the proof of Lemma 3.2(1), we have $\mathrm{pd}_\Lambda(M) \le \mathrm{pd}_\Lambda(\oplus_{i=0}^n M_i) + n$. So the assertion follows. □

Now we are in a position to prove the main result.

THEOREM 3.5. *Let*

$$0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$$

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be a chain of ideals of Λ *such that* $(I_{i+1}/I_i)\operatorname{rad}(\Lambda/I_i)=0$ *for any* $0\le i\le n-1$ *and* Λ/I_n *is semisimple. If either none or the direct sum of exactly two consecutive ideals has infinite projective dimension, then the finitistic dimension conjecture holds for* Λ.

Proof. Let $M\in\operatorname{mod}\Lambda$ with $\operatorname{pd}_\Lambda M<\infty$. By Lemma 3.1, there exists an exact sequence

$$0\to M_n\to M_{n-1}\to\cdots\to M_1\to M_0\to M\to 0$$

in $\operatorname{mod}\Lambda$, where $M_i\in\operatorname{add}_\Lambda \Lambda/I_i$ for any $0\le i\le n$. Set $K_i:=\operatorname{Im}(M_i\to M_{i-1})$ for any $1\le i\le n-1$, $K_n:=M_n$ and $K_0:=M$. Then by the construction in the proof of Lemma 3.1 we have the following facts: (i) Each M_i is a projective Λ/I_i-module; (ii) each K_i is a Λ/I_i-module; (iii) if $\operatorname{pd}_\Lambda I_i=\infty$, then $\operatorname{pd}_\Lambda M_i$ could be ∞; (iv) if $\operatorname{pd}_\Lambda M_i=\infty$, then at least one of the following are true: $\operatorname{pd}_\Lambda K_i=\infty$ or $\operatorname{pd}_\Lambda K_{i+1}=\infty$.

We will discuss the situation separately.

(1) If $\operatorname{pd}_\Lambda I_i<\infty$ for all $0\le i\le n$, then $\operatorname{pd}_\Lambda M_i\le\operatorname{pd}_\Lambda \Lambda/I_i<\infty$ for all $0\le i\le n$. It follows from Lemma 3.4 that

$$\operatorname{pd}_\Lambda M\le\operatorname{pd}_\Lambda(\oplus_{i=0}^n M_i)+n\le\operatorname{pd}_\Lambda(\oplus_{i=0}^n\Lambda/I_i)+n<\infty$$

and $\operatorname{fin.dim}\Lambda\le\operatorname{pd}_\Lambda(\oplus_{i=0}^n\Lambda/I_i)+n<\infty$.

(2) If there is some integer s with $1\le s\le n$ such that $\operatorname{pd}_\Lambda I_s=\infty$ and $\operatorname{pd}_\Lambda I_i<\infty$ for all $1\le i\le n$ but $i\ne s$, then $\operatorname{pd}_\Lambda M_i\le\operatorname{pd}_\Lambda \Lambda/I_i<\infty$ for all $0\le i\le n$ but $i\ne s$. Since $\operatorname{pd}_\Lambda M<\infty$, we have $\operatorname{pd}_\Lambda M_s<\infty$; that is, we have $\operatorname{pd}_\Lambda M_i<\infty$ for all $0\le i\le n$. Note that $M_i\in\operatorname{add}_\Lambda \Lambda/I_i$ for any $0\le i\le n$. Thus, we have

$$\begin{aligned}\operatorname{pd}_\Lambda M&\le\operatorname{pd}_\Lambda(\oplus_{i=0}^n M_i)+n \quad \text{(by Lemma 3.4)}\\&=\psi(\oplus_{i=0}^n M_i)+n \quad \text{(by Lemma 2.1(1))}\\&\le\psi(\oplus_{i=0}^n\Lambda/I_i)+n \quad \text{(by Lemma 2.1(2)(3))}\end{aligned}$$

and $\operatorname{fin.dim}\Lambda\le\psi(\oplus_{i=0}^n\Lambda/I_i)+n$.

(3) If there is some integer s with $1\le s<n$ such that $\operatorname{pd}_\Lambda I_s=\infty$, $\operatorname{pd}_\Lambda I_{s+1}=\infty$ and $\operatorname{pd}_\Lambda I_i<\infty$ for all $1\le i\le n$ but $i\ne s,\ s+1$, then $\operatorname{pd}_\Lambda M_i\le\operatorname{pd}_\Lambda \Lambda/I_i<\infty$ for all $0\le i\le n$ but $i\ne s,\ s+1$.

By Lemma 3.2(1) and the exactness of the following sequence

$$0\to K_s\to M_{s-1}\to\cdots\to M_1\to M_0\to M\to 0,$$

we have

$$\Omega_\Lambda^{s+m_1}(M)\oplus P_1\cong\Omega_\Lambda^{m_1}(K_s)\oplus Q_1, \tag{3.2}$$

where

$$\begin{aligned}m_1&:=\max\{\operatorname{pd}_\Lambda \Lambda/I_0,\operatorname{pd}_\Lambda \Lambda/I_1,\cdots,\operatorname{pd}_\Lambda \Lambda/I_{s-1}\}\\&\ge\max\{\operatorname{pd}_\Lambda M_0,\operatorname{pd}_\Lambda M_1,\cdots,\operatorname{pd}_\Lambda M_{s-1}\}\end{aligned}$$

and P_1, Q_1 are projective in $\operatorname{mod}\Lambda$.

Consider the following exact sequence

$$0\to M_n\to M_{n-1}\to\cdots\to M_{s+1}\to K_{s+1}\to 0.$$

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By Lemma 3.2(2), we have

$$\Omega_\Lambda^{n-(s+2)+m_2+1}(K_{s+1}) \oplus P_2 \cong \Omega_\Lambda^{n-(s+2)+m_2+1}(M_{s+1}) \oplus Q_2, \tag{3.3}$$

where

$$\begin{aligned} m_2 &:= \max\{\mathrm{pd}_\Lambda\ \Lambda/I_{s+2}, \mathrm{pd}_\Lambda\ \Lambda/I_{s+3}, \cdots, \mathrm{pd}_\Lambda\ \Lambda/I_n\} \\ &\geq \max\{\mathrm{pd}_\Lambda\ M_{s+2}, \mathrm{pd}_\Lambda\ M_{s+3}, \cdots, \mathrm{pd}_\Lambda\ M_n\} \end{aligned}$$

and P_2, Q_2 are projective in $\mathrm{mod}\,\Lambda$. Set $r_1 := \max\{s+m_1, n-(s+2)+m_2+1\}+1$. By (3.2) and (3.3), we have

$$\Omega_\Lambda^{r_1}(M) \cong \Omega_\Lambda^{r_1-s}(K_s) \text{ and } \Omega_\Lambda^{r_1}(K_{s+1}) \cong \Omega_\Lambda^{r_1}(M_{s+1}). \tag{3.4}$$

Consider the following exact sequence

$$0 \longrightarrow K_{s+1} \longrightarrow M_s \longrightarrow K_s \to 0.$$

By the horseshoe lemma, we have

$$0 \longrightarrow \Omega_\Lambda^{r_1}(K_{s+1}) \longrightarrow \Omega_\Lambda^{r_1}(M_s) \oplus P \longrightarrow \Omega_\Lambda^{r_1}(K_s) \longrightarrow 0, \tag{3.5}$$

where P is projective in $\mathrm{mod}\,\Lambda$. Moreover, from (3.4) and (3.5) we obtain the following exact sequence

$$0 \longrightarrow \Omega_\Lambda^{r_1}(M_{s+1}) \longrightarrow \Omega_\Lambda^{r_1}(M_s) \oplus P \longrightarrow \Omega_\Lambda^{r_1+s}(M) \longrightarrow 0.$$

Thus,

$$\begin{aligned} \mathrm{pd}_\Lambda\ M &\leq \mathrm{pd}_\Lambda\ \Omega_\Lambda^{r_1+s}(M) + r_1 + s \\ &\leq \psi(\Omega_\Lambda^{r_1}(M_{s+1}) \oplus \Omega_\Lambda^{r_1}(M_s) \oplus P) + 1 + r_1 + s \quad (\text{by Lemma 2.1(5)}) \\ &\leq \psi(\Omega_\Lambda^{r_1}(\Lambda/I_{s+1}) \oplus \Omega_\Lambda^{r_1}(\Lambda/I_s)) + 1 + r_1 + s, \end{aligned}$$

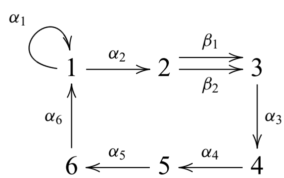
where the last inequality follows from Lemma 2.1(3)(4) and the fact that $M_s \in \mathrm{add}_\Lambda\ \Lambda/I_s$ and $M_{s+1} \in \mathrm{add}_\Lambda\ \Lambda/I_{s+1}$. Therefore,

$$\mathrm{fin.dim}\ \Lambda \leq \psi(\Omega_\Lambda^{r_1}(\Lambda/I_{s+1}) \oplus \Omega_\Lambda^{r_1}(\Lambda/I_s)) + 1 + r_1 + s < \infty.$$

The proof is finished. □

Finally, we give an example to illustrate Theorem 3.5.

EXAMPLE 3.6. Let k be an algebraically closed field and $\Lambda = kQ/I$, where Q the quiver



and I is generated by $\{\alpha_1^2, \alpha_2\alpha_1, \alpha_3\beta_1 - \alpha_3\beta_2, \alpha_6\alpha_5\}$. It is straightforward to verify that $LL(\Lambda) = 6$, $\mathrm{pd}_\Lambda\ \mathrm{rad}(\Lambda) = \infty$, $\mathrm{pd}_\Lambda\ \mathrm{rad}^2(\Lambda) = \infty$ and $\mathrm{pd}_\Lambda\ \mathrm{rad}^i(\Lambda) < \infty$ for any $3 \leq i \leq 5$. So $\mathrm{fin.dim}\ \Lambda < \infty$ by Theorem 3.5.

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