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THE LIMIT DISTRIBUTION OF THE MAXIMUM PROBABILITY NEAREST-NEIGHBOUR BALL

LÁSZLÓ GYÖRFI,* *Budapest University of Technology and Economics*
NORBERT HENZE,** *Karlsruhe Institute of Technology (KIT)*
HARRO WALK,*** *University of Stuttgart*

Abstract

Let $X_1, \ldots, X_n$ be independent random points drawn from an absolutely continuous probability measure with density f in $\mathbb{R}^d$. Under mild conditions on f, we derive a Poisson limit theorem for the number of large probability nearest-neighbour balls. Denoting by P_n the maximum probability measure of nearest-neighbour balls, this limit theorem implies a Gumbel extreme value distribution for $nP_n - \ln n$ as $n \to \infty$. Moreover, we derive a tight upper bound on the upper tail of the distribution of $nP_n - \ln n$, which does not depend on f.

Keywords: Nearest neighbour; Gumbel extreme value distribution; Poisson limit theorem; exchangeable event

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1. Introduction

Let $X, X_1, \ldots, X_n$ be independent, identically distributed (i.i.d.) random vectors taking values in $\mathbb{R}^d$. We assume throughout the paper that the distribution of X, which is denoted by μ, has a density f with respect to Lebesgue measure λ.

Writing $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^d$, set

$$R_{i,n} := \min_{j \neq i, j \leq n} \|X_i - X_j\|,$$

and let

$$P_n := \max_{1 \leq i \leq n} \mu\{S(X_i, R_{i,n})\}$$

denote the maximum probability of the nearest-neighbour balls, where $S(x, r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ stands for the closed ball with centre x and radius r. In this paper we deal with both the finite sample and the asymptotic distribution of

$$nP_n - \ln n \quad \text{as } n \to \infty.$$

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* Postal address: Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Magyar Tudósok krt. 2., Budapest, H-1117, Hungary.
** Postal address: Institute of Stochastics, Karlsruhe Institute of Technology (KIT), Englerstr. 2, D-76133 Karlsruhe, Germany.
*** Postal address: Institute of Stochastics and Applications, University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany.

There is a large related literature on the Poisson sample size. Let N be a random variable that is independent of $X_1, X_2, \ldots$ and has a Poisson distribution with $\mathbb{E}(N) = n$. Then

$$X_1, \ldots, X_N \tag{1.1}$$

is a nonhomogeneous Poisson process with intensity function nf. For the nuclei $X_1, \ldots, X_N$,

$$\widetilde{A}_n(X_j) := \Big\{ y \in \mathbb{R}^d \colon \|X_j - y\| \le \min_{1\le i\le n, i\ne j} \|X_i - y\| \Big\}$$

denotes the Voronoi cell around X_j, and $\widehat{r}_j$ and $\widehat{R}_j$ denote the inscribed and circumscribed radii of $\widetilde{A}_n(X_j)$, respectively, i.e. we have

$$\widehat{r}_j = \sup\{r > 0 \colon S(X_j, r) \subset \widetilde{A}_n(X_j)\}$$

and

$$\widehat{R}_j = \inf\{r > 0 \colon \widetilde{A}_n(X_j) \subset S(X_j, r)\}.$$

If $X_1, X_2, \ldots$ are i.i.d. uniformly distributed on a convex set $W \subset \mathbb{R}^d$ with volume 1, then (2a) and (2c) of Theorem 1 of [5] read

$$\lim_{n\to\infty} \mathbb{P}\Big(2^d n\lambda \Big\{ S\Big(0, \max_{1\le j\le N} \widehat{r}_j\Big) \Big\} - \ln n \le y \Big) = G(y)$$

and

$$\lim_{n\to\infty} \mathbb{P}\Big(n\lambda \Big\{ S\Big(0, \max_{1\le j\le N} \widehat{R}_j)\Big) \Big\} - \ln\left(\alpha_d n(\ln n)^{d-1}\right) \le y \Big) = G(y)$$

for $y \in \mathbb{R}$. Here $\alpha_d > 0$ is a universal constant, and

$$G(y) = \exp\left(-\exp\left(-y\right)\right)$$

denotes the distribution function of the Gumbel extreme value distribution. In the sequel we consider a related problem, namely that of finding the limit distribution of the largest probability content of nearest-neighbour spheres in a nonhomogeneous i.i.d. setting such that the support W can be arbitrary. Such a generality is important for designing and analysing wireless networks; see [1] and [2].

The paper is organized as follows. In Section 2 we study the distribution of $nP_n - \ln n$. Theorem 2.1 is on a universal and tight bound on the upper tail of $nP_n - \ln n$. Under mild conditions on the density, Theorem 2.2 shows that the number of exceedances of nearest-neighbour ball probabilities over a certain sequence of thresholds has an asymptotic Poisson distribution as $n \to \infty$. As a consequence, the limit distribution of $nP_n - \ln n$ is the Gumbel extreme value distribution. Theorem 3.1 is the extension of Theorem 2.1 for a Poisson sample size. All proofs are presented in Section 4. The main tool for proving Theorem 2.2 is a novel Poisson limit theorem for sums of indicators of exchangeable events, which is formulated as Proposition 4.1. The final section sheds some light on a technical condition on f that is used in the proof of the main result. We conjecture that our main result holds without any condition on the density f.

Although there is a weak dependence between the probabilities of nearest-neighbour balls, a main message of this paper is that one can neglect this dependence when looking for the limit distribution of the maximum probability.

2. The maximum nearest-neighbour ball

Under the assumption that the density f is sufficiently smooth and bounded away from 0, Henze ([13], [12]) derived the limit distribution of the maximum *approximate* probability measure

$$\max_{1\le i\le n} f(X_i)R_{i,n}^d v_d \tag{2.1}$$

of nearest-neighbour balls. Here $v_d = \pi^{d/2}/\Gamma(1+d/2)$ denotes the volume of the unit ball in $\mathbb{R}^d$.

In the following, we consider the number of points among $X_1, \ldots, X_n$ for which the probability content of the nearest-neighbour ball exceeds some (large) threshold. To be more specific, we fix $y \in \mathbb{R}$ and consider the random variable

$$C_n := \sum_{i=1}^n \mathbf{1}\{n\mu\{S(X_i, R_{i,n})\} > y + \ln n\},$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Writing '$\xrightarrow{\text{D}}$' to denote convergence in distribution, we will show that, under some conditions on the density f,

$$C_n \xrightarrow{\text{D}} Z \quad \text{as } n \to \infty,$$

where Z is a random variable with the Poisson distribution $\text{Po}(\exp(-y))$. Now $C_n = 0$ if and only if $nP_n - \ln n \le y$, and it follows that

$$\lim_{n\to\infty} \mathbb{P}(nP_n - \ln n \le y) = \mathbb{P}(Z = 0) = G(y), \qquad y \in \mathbb{R}. \tag{2.2}$$

Since $1 - G(y) \le \exp(-y)$ if $y \ge 0$, (2.2) implies that

$$\limsup_{n\to\infty} \mathbb{P}(nP_n - \ln n \ge y) \le \mathrm{e}^{-y}, \qquad y \ge 0. \tag{2.3}$$

Our first result is a nonasymptotic upper bound on the upper tail of the distribution of $nP_n - \ln n$. This bound holds without any condition on the density and thus entails (2.3) universally.

Theorem 2.1. *Without any restriction on the density f, we have*

$$\mathbb{P}(nP_n - \ln n \ge y) \le \exp\left(-\frac{n-1}{n}y + \frac{\ln n}{n}\right)\mathbf{1}\{y \le n - \ln n\}, \qquad y \in \mathbb{R}. \tag{2.4}$$

Theorem 2.1 implies a nonasymptotic upper bound on the mean of $nP_n - \ln n$ since

$$\begin{aligned}
\mathbb{E}[nP_n - \ln n] &\le \mathbb{E}[(nP_n - \ln n)^+] \\
&= \int_0^\infty \mathbb{P}(nP_n - \ln n \ge y)\,\mathrm{d}y \\
&\le \int_0^\infty \exp\left(-\frac{n-1}{n}y + \frac{\ln n}{n}\right)\mathrm{d}y \\
&= \frac{n}{n-1}\exp\left(\frac{\ln n}{n}\right).
\end{aligned}$$

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Note that this upper bound approaches 1 for large n, and that the mean of the standard Gumbel distribution is the Euler–Mascheroni constant, which is $-\int_0^\infty e^{-y} \ln y \, dy = 0.5772\ldots$.

Recall that the support of μ is defined by

$$\mathrm{supp}(\mu) := \{x \in \mathbb{R}^d : \mu\{S(x, r)\} > 0 \text{ for each } r > 0\},$$

i.e. the support of μ is the smallest closed set in $\mathbb{R}^d$ having μ-measure one.

Theorem 2.2. *Assume that* $\beta \in (0, 1)$, $c_{\max} < \infty$, *and* $\delta > 0$ *such that, for any* $r, s > 0$ *and any* $x, z \in \mathrm{supp}(\mu)$ *with* $\|x - z\| \geq \max\{r, s\}$ *and* $\mu(S(x, r)) = \mu(S(z, s)) \leq \delta$, *we have*

$$\frac{\mu(S(x, r) \cap S(z, s))}{\mu(S(z, s))} \leq \beta \tag{2.5}$$

and

$$\mu(S(z, 2s)) \leq c_{\max} \mu(S(z, s)). \tag{2.6}$$

Then

$$\sum_{i=1}^{n} \mathbf{1}\{n\mu\{S(X_i, R_{i,n})\} > y + \ln n\} \xrightarrow{\mathrm{D}} \mathrm{Po}(\exp(-y)), \qquad y \in \mathbb{R}, \tag{2.7}$$

and, hence,

$$\lim_{n\to\infty} \mathbb{P}(nP_n - \ln n \leq y) = G(y), \qquad y \in \mathbb{R}. \tag{2.8}$$

Remark 2.1. It is easy to see that (2.5) and (2.6) hold if the density is both bounded from above by $f_{\max}$ and bounded away from 0 by $f_{\min} > 0$. Indeed, setting

$$\beta := 1 - \frac{1}{2}\frac{f_{\min}}{f_{\max}}, \qquad c_{\max} := 2^d \frac{f_{\max}}{f_{\min}},$$

we have

$$\begin{aligned}
\frac{\mu(S(x, r) \cap S(z, s))}{\mu(S(z, s))} &= 1 - \frac{\mu(S(z, s) \setminus S(x, r))}{\mu(S(z, s))} \\
&\leq 1 - \frac{f_{\min}\, \lambda(S(z, s) \setminus S(x, r))}{f_{\max} \lambda(S(z, s))} \\
&\leq \beta,
\end{aligned}$$

because $\|x - z\| \geq \max\{r, s\}$. Moreover,

$$\mu(S(z, 2s)) \leq f_{\max} \lambda(S(z, 2s)) = f_{\max} 2^d \lambda(S(z, s)) \leq c_{\max} \mu(S(z, s)).$$

A challenging problem left is to weaken the conditions of Theorem 2.2 or to prove that (2.7) and (2.8) hold without any conditions on the density. We believe that such universal limit results are possible because the summands in (2.7) are identically distributed, and their distribution does not depend on the actual density. Further discussion of condition (2.5) is given in Section 5.

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3. The maximum nearest-neighbour ball for a nonhomogeneous Poisson process

In this section we consider the nonhomogeneous Poisson process $X_1, \ldots, X_N$ defined by (1.1). Setting

$$\widetilde{R}_{i,n} := \min_{j\neq i, j\leq N} \|X_i - X_j\| \quad \text{and} \quad \widetilde{P}_n = \max_{1\leq i\leq N} \mu\{S(X_i, \widetilde{R}_{i,n})\},$$

the following result is the Poisson analogue to Theorem 2.1.

Theorem 3.1. *Without any restriction on the density f, we have*

$$\mathbb{P}(n\widetilde{P}_n - \ln n \geq y) \leq \mathrm{e}^{-y} \exp\left(\frac{(y+\ln n)^2}{n}\right), \qquad y \in \mathbb{R}.$$

4. Proofs

Proof of Theorem 2.1. Since the right-hand side of (2.4) is larger than 1 if $y < 0$, we take $y \geq 0$ in what follows. Moreover, in view of $P_n \leq 1$ the left-hand side of (2.4) vanishes if $y > n - \ln n$. We therefore assume without loss of generality that

$$\frac{y+\ln n}{n} \leq 1. \tag{4.1}$$

For a fixed $x \in \mathbb{R}^d$, let

$$H_x(r) := \mathbb{P}(\|x - X\| \leq r), \qquad r \geq 0, \tag{4.2}$$

be the distribution function of $\|x - X\|$. By the probability integral transform (cf. [3, p. 8]), the random variable

$$H_x(\|x - X\|) = \mu\{S(x, \|x - X\|)\}$$

is uniformly distributed on [0, 1]. We thus have

$$\mu\{S(x, H_x^{-1}(p))\} = p, \qquad 0 < p < 1, \tag{4.3}$$

where $H_x^{-1}(p) = \inf\{r \colon H_x(r) \geq p\}$. It follows that

$$\begin{aligned}
\mathbb{P}(nP_n - \ln n \geq y) &= \mathbb{P}(n \max_{1\leq i\leq n} \mu\{S(X_i, R_{i,n})\} - \ln n \geq y) \\
&\leq n\,\mathbb{P}(n\mu\{S(X_1, R_{1,n})\} - \ln n \geq y) \\
&= n\,\mathbb{P}\left(\mu\{S(X_1, R_{1,n})\} \geq \frac{y+\ln n}{n}\right) \\
&= n\,\mathbb{P}\left(\min_{2\leq j\leq n} \mu\{S(X_1, \|X_1 - X_j\|)\} \geq \frac{y+\ln n}{n}\right).
\end{aligned}$$

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Now (4.1) and (4.3) imply that

$$\begin{aligned}\mathbb{P}(nP_n - \ln n \geq y) &\leq n\,\mathbb{E}\bigg[\mathbb{P}\bigg(\min_{2\leq j\leq n} \mu\{S(X_1, \|X_1 - X_j\|)\} \geq \frac{y + \ln n}{n} \,\bigg|\, X_1\bigg)\bigg] \\ &= n\bigg(1 - \frac{y+\ln n}{n}\bigg)^{n-1} \\ &\leq n \exp\bigg(-\frac{(y + \ln n)(n-1)}{n}\bigg) \\ &= \exp\bigg(-\frac{n-1}{n}y + \frac{\ln n}{n}\bigg). \qquad \square\end{aligned}$$

Proof of Theorem 3.1. We again assume that (4.1) holds in what follows. By conditioning on N, we have

$$\begin{aligned}\mathbb{P}(n\widetilde{P}_n - \ln n \geq y) &= \sum_{k=1}^{\infty} \mathbb{P}(n\widetilde{P}_n - \ln n \geq y \mid N = k)\mathbb{P}(N = k) \\ &= \sum_{k=1}^{\infty} \mathbb{P}(nP_k - \ln n \geq y)\mathbb{P}(N = k).\end{aligned}$$

Setting $y_n := (y + \ln n)/n$, we obtain

$$\mathbb{P}(nP_k - \ln n \geq y) = \mathbb{P}(kP_k - \ln k \geq ky_n - \ln k),$$

and Theorem 2.1 implies that

$$\begin{aligned}\mathbb{P}(kP_k - \ln k \geq ky_n - \ln k) &\leq \exp\bigg(-\frac{k-1}{k}(ky_n - \ln k) + \frac{\ln k}{k}\bigg) \\ &= \exp(-(k-1)y_n + \ln k).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{P}(n\widetilde{P}_n - \ln n \geq y) &\leq \sum_{k=1}^{\infty} \exp(-(k-1)y_n + \ln k)\mathbb{P}(N = k) \\ &= \mathrm{e}^{y_n - n} \sum_{k=1}^{\infty} k(\mathrm{e}^{-y_n})^k \frac{n^k}{k!} \\ &= \mathrm{e}^{y_n - n} \sum_{k=1}^{\infty} \frac{(n\mathrm{e}^{-y_n})^k}{(k-1)!} \\ &= n\mathrm{e}^{y_n - n - y_n} \exp(n\mathrm{e}^{-y_n}) \\ &= n \exp(-n(1 - \mathrm{e}^{-y_n})).\end{aligned}$$

Since $z \geq 0$ entails $\mathrm{e}^{-z} \leq 1 - z + z^2$, we finally obtain

$$\mathbb{P}(n\widetilde{P}_n - \ln n \geq y) \leq n \exp(-n(y_n - y_n^2)) = \mathrm{e}^{-y} \exp\bigg(\frac{(y + \ln n)^2}{n}\bigg). \qquad \square$$

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The main tool in the proof of Theorem 2.2 is the following result. A similar result has been established in the context of random tessellations; see Lemma 4.1 of [6].

Proposition 4.1. *For each $n \geq 2$, let $A_{n,1}, \ldots, A_{n,n}$ be exchangeable events, and let*

$$Y_n := \sum_{j=1}^{n} \mathbf{1}\{A_{n,j}\}.$$

If, for some $\nu \in (0, \infty)$,

$$\lim_{n\to\infty} n^k \mathbb{P}(A_{n,1} \cap \cdots \cap A_{n,k}) = \nu^k \quad \text{for each } k \geq 1, \tag{4.4}$$

then

$$Y_n \xrightarrow{\mathrm{D}} Y \quad \text{as } n \to \infty,$$

where Y has the Poisson distribution $\mathrm{Po}(\nu)$.

Proof. The proof uses the method of moments; see, e.g. [4, Section 30]. Setting

$$S_{n,k} = \sum_{1\leq i_1<\cdots<i_k\leq n} \mathbb{P}(A_{n,i_1} \cap \cdots \cap A_{n,i_k}), \qquad k \in \{1, \ldots, n\},$$

and writing $Z^{(k)} = Z(Z-1)\cdots(Z-k+1)$ for the kth descending factorial of a random variable Z, we have

$$\mathbb{E}[Y_n^{(k)}] = k!\, S_{n,k}.$$

Since $A_{n,1}, \ldots, A_{n,n}$ are exchangeable, (4.4) implies that

$$\lim_{n\to\infty} \mathbb{E}[Y_n^{(k)}] = \nu^k, \qquad k \geq 1.$$

Now $\nu^k = \mathbb{E}[Y^{(k)}]$, where Y has the Poisson distribution $\mathrm{Po}(\nu)$. We thus have

$$\lim_{n\to\infty} \mathbb{E}[Y_n^{(k)}] = \mathbb{E}[Y^{(k)}], \qquad k \geq 1. \tag{4.5}$$

Since

$$Y_n^k = \sum_{j=0}^{k} \left\{ {k \atop j} \right\} Y_n^{(j)},$$

where $\left\{ {k \atop 0} \right\}, \ldots, \left\{ {k \atop k} \right\}$ denote Stirling numbers of the second kind (see, e.g. [10, p. 262]), (4.5) entails $\lim_{n\to\infty} \mathbb{E}[Y_n^k] = \mathbb{E}[Y^k]$ for each $k \geq 1$. Since the distribution of Y is uniquely determined by the sequence of moments $(\mathbb{E}[Y^k])$, $k \geq 1$, the assertion follows. □

Proof of Theorem 2.2. Fix $y \in \mathbb{R}$. In what follows, we will verify (4.4) for

$$A_{n,i} := \{n\mu\{S(X_i, R_{i,n})\} \geq y + \ln n\}, \qquad i \in \{1, \ldots, n\},$$

and $\nu = \exp(-y)$. Throughout the proof, we tacitly assume that

$$0 < y_n := \frac{y + \ln n}{n} < 1.$$

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This assumption entails no loss of generality since n tends to ∞. With $H_x(\cdot)$ given in (4.2), we set

$$R^*_{i,n} := H^{-1}_{X_i}\left(\frac{y+\ln n}{n}\right), \qquad i \in \{1, \dots, n\}.$$

For the special case $k=1$, conditioning on X_1 and (4.3) yield

$$\begin{aligned} n\mathbb{P}(A_{n,1}) &= n\mathbb{P}(\mu(S(X_1, R_{1,n})) \geq y_n) \\ &= n\mathbb{E}[\mathbb{P}(\mu(S(X_1, R_{1,n})) \geq y_n \mid X_1)] \\ &= n\mathbb{E}[(1-\mu(S(X_1, H^{-1}_{X_1}(y_n))))^{n-1}] \\ &= n\left(1-\frac{y+\ln n}{n}\right)^{n-1}. \end{aligned}$$

Using the inequalities $1-1/t \leq \ln t \leq t-1$ gives $\lim_{n\to\infty} n\,\mathbb{P}(A_{n,1}) = \mathrm{e}^{-y}$. Thus, (4.4) is proved for $k=1$, remarkably without any condition on the underlying density f. We now assume that $k \geq 2$. The proof of (4.4) for this case is quite technical, but the basic reasoning is clear-cut: the main idea for showing that $n^k\mathbb{P}(A_{n,1} \cap \cdots \cap A_{n,n}) \approx (n\mathbb{P}(A_{n,1}))^k \approx \exp(-ky)$ – which yields the Poisson convergence (2.7) and the Gumbel limit (2.8) – is to prove that the radii have the same behaviour as if they were independent. To this end, we consider two subcases: in the first subcase, which leads to independent events, the balls are sufficiently separated from one another, with the opposite subcase shown to be of asymptotically negligible probability. To proceed, set

$$\widetilde{R}_{i,k,n} := \min_{k+1\leq j\leq n} \|X_i - X_j\|, \qquad r_{i,k} := \min_{j\neq i, j\leq k} \|X_i - X_j\|.$$

Then $R_{i,n} = \min\{\widetilde{R}_{i,k,n}, r_{i,k}\}$. Note that, on a set of probability 1, we have $R_{i,n} = \widetilde{R}_{i,k,n}$ for each $i \in \{1, \dots, k\}$ if n is large enough.

Conditioning on $X_1, \dots, X_k$ we have

$$\begin{aligned} \mathbb{P}\left(\bigcap_{i=1}^k A_{n,i}\right) &= \mathbb{P}\left(\bigcap_{i=1}^k \{\mu(S(X_i, \min\{\widetilde{R}_{i,k,n}, r_{i,k}\})) \geq y_n\}\right) \\ &= \mathbb{P}\left(\bigcap_{i=1}^k \{\mu(S(X_i, \widetilde{R}_{i,k,n})) \geq y_n,\, \mu(S(X_i, r_{i,k})) \geq y_n\}\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(\bigcap_{i=1}^k \{\mu(S(X_i, \widetilde{R}_{i,k,n})) \geq y_n,\, \mu(S(X_i, r_{i,k})) \geq y_n\} \,\middle|\, X_1, \dots, X_k\right)\right] \\ &= \mathbb{E}\left[\mathbb{P}\left(\bigcap_{i=1}^k \{\mu(S(X_i, \widetilde{R}_{i,k,n})) \geq y_n\} \,\middle|\, X_1, \dots, X_k\right)\mathbf{1}_{n,k}\right], \end{aligned}$$

where

$$\mathbf{1}_{n,k} := \prod_{i=1}^k \mathbf{1}\{\mu(S(X_i, r_{i,k})) \geq y_n\}.$$

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Furthermore, we obtain

$$\mathbb{P}\bigg(\bigcap_{i=1}^{k}\{\mu(S(X_i, \widetilde{R}_{i,k,n})) \geq y_n\} \,\bigg|\, X_1, \ldots, X_k\bigg)$$
$$= \mathbb{P}\bigg(\bigcap_{i=1}^{k}\{\widetilde{R}_{i,k,n} \geq H_{X_i}^{-1}(y_n)\} \,\bigg|\, X_1, \ldots, X_k\bigg)$$
$$= \mathbb{P}\bigg(\bigcap_{i=1}^{k}\{\widetilde{R}_{i,k,n} \geq R^*_{i,n}\} \,\bigg|\, X_1, \ldots, X_k\bigg)$$
$$= \mathbb{P}\bigg(X_{k+1}, \ldots, X_n \notin \bigcup_{i=1}^{k} S(X_i, R^*_{i,n}) \,\bigg|\, X_1, \ldots, X_k\bigg)$$
$$= \bigg(1 - \mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)^{n-k}.$$

Note that we have the obvious lower bound

$$n^k\bigg(1 - \mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)^{n-k}\mathbf{1}_{n,k} \geq n^k\bigg(1 - \sum_{i=1}^{k}\mu(S(X_i, R^*_{i,n}))\bigg)^{n-k}\mathbf{1}_{n,k}$$
$$= n^k\bigg(1 - k\frac{y + \ln n}{n}\bigg)^{n-k}\mathbf{1}_{n,k}.$$

Since the latter converges almost surely to e^{-ky} as $n \to \infty$, Fatou's lemma implies that

$$\liminf_{n\to\infty} n^k\mathbb{P}\bigg(\bigcap_{i=1}^{k} A_{n,i}\bigg) = \liminf_{n\to\infty} \mathbb{E}\bigg[n^k\bigg(1 - \mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)^{n-k}\mathbf{1}_{n,k}\bigg]$$
$$\geq \mathbb{E}\bigg[\liminf_{n\to\infty} n^k\bigg(1 - \mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)^{n-k}\mathbf{1}_{n,k}\bigg]$$
$$\geq \mathrm{e}^{-ky}.$$

It thus remains to show that

$$\limsup_{n\to\infty} n^k\mathbb{P}\bigg(\bigcap_{i=1}^{k} A_{n,i}\bigg) \leq \mathrm{e}^{-ky}. \tag{4.6}$$

Let D_n be the event that the balls $S(X_i, R^*_{i,n})$, $i = 1, \ldots, k$, are pairwise disjoint. We have

$$\limsup_{n\to\infty} n^k\mathbb{P}\bigg(\bigcap_{i=1}^{k} A_{n,i}\bigg)$$
$$= \limsup_{n\to\infty} n^k\mathbb{E}\bigg[\bigg(1 - \mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)^{n-k}\mathbf{1}_{n,k}\bigg]$$
$$\leq \limsup_{n\to\infty} n^k\mathbb{E}\bigg[\exp\bigg(-(n-k)\mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)\mathbf{1}_{n,k}\bigg]$$

$$\leq \limsup_{n\to\infty} n^k \mathbb{E}\bigg[\exp\bigg(-(n-k)k\frac{y+\ln n}{n}\bigg)\mathbf{1}\{D_n\}\bigg]$$
$$+\limsup_{n\to\infty} n^k \mathbb{E}\bigg[\exp\bigg(-(n-k)\mu\bigg(\bigcup_{i=1}^k S(X_i, R_{i,n}^*)\bigg)\bigg)\mathbf{1}\{D_n^c\}\mathbf{1}_{n,k}\bigg]$$
$$\leq \mathrm{e}^{-ky} + \limsup_{n\to\infty} n^k \mathbb{E}\bigg[\exp\bigg(-(n-k)\mu\bigg(\bigcup_{i=1}^k S(X_i, R_{i,n}^*)\bigg)\bigg)\mathbf{1}\{D_n^c\}\mathbf{1}_{n,k}\bigg].$$

It thus remains to show that

$$\lim_{n\to\infty} n^k \mathbb{E}\bigg[\exp\bigg(-(n-k)\mu\bigg(\bigcup_{i=1}^k S(X_i, R_{i,n}^*)\bigg)\bigg)\mathbf{1}\{D_n^c\}\mathbf{1}_{n,k}\bigg] = 0. \tag{4.7}$$

Under some additional smoothness conditions on the density, Henze [13] verified (4.7) for the related problem of finding the limit distribution of the random variable figuring in (2.1). By analogy with his proof technique, we introduce an equivalence relation on the set $\{1, \ldots, k\}$ as follows. An equivalence class consists of a singleton $\{i\}$ if

$$S(X_i, R_{i,n}^*) \cap S(X_j, R_{j,n}^*) = \varnothing$$

for each $j \neq i$. Otherwise, i and j are called equivalent if there is a subset $\{i_1, \ldots, i_\ell\}$ of $\{1, \ldots, k\}$ such that $i = i_1, j = i_\ell$, and

$$S(X_{i_m}, R_{i_m,n}^*) \cap S(X_{i_{m+1}}, R_{i_{m+1},n}^*) \neq \varnothing$$

for each $m \in \{1, \ldots, \ell - 1\}$. Let $\mathcal{P} = \{Q_1, \ldots, Q_q\}$ be a partition of $\{1, \ldots, k\}$, and denote by E_u the event that Q_u forms an equivalence class. For the event D_n, the partition $\mathcal{P}_0 := \{\{1\}, \ldots, \{k\}\}$ is the trivial partition, while on the complement D_n^c any partition $\mathcal{P}$ is nontrivial, which means that $q < k$. In order to prove (4.7), we have to show that

$$\limsup_{n\to\infty} n^k \mathbb{E}\bigg[\exp\bigg(-(n-k)\mu\bigg(\bigcup_{i=1}^k S(X_i, R_{i,n}^*)\bigg)\bigg)\mathbf{1}_{n,k}\prod_{u=1}^q \mathbf{1}\{E_u\}\bigg] = 0 \tag{4.8}$$

for each nontrivial partition $\mathcal{P}$. Since balls that belong to different equivalence classes are disjoint, we have

$$\mu\bigg(\bigcup_{i=1}^k S(X_i, R_{i,n}^*)\bigg)\prod_{u=1}^q \mathbf{1}\{E_u\} = \mu\bigg(\bigcup_{u=1}^q \bigcup_{i\in Q_u} S(X_i, R_{i,n}^*)\bigg)\prod_{u=1}^q \mathbf{1}\{E_u\}$$
$$= \sum_{u=1}^q \mu\bigg(\bigcup_{i\in Q_u} S(X_i, R_{i,n}^*)\bigg)\prod_{u=1}^q \mathbf{1}\{E_u\}.$$

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Writing $|B|$ for the number of elements of a finite set B, it follows that

$$n^k \exp\bigg(-(n-k)\mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)\prod_{i=1}^{k}\mathbf{1}\{\mu(S(X_i, r_{i,k}))\geq y_n\}\prod_{u=1}^{q}\mathbf{1}\{E_u\}$$
$$\leq e^k \prod_{u=1}^{q} n^{|Q_u|}\prod_{u=1}^{q} e^{-n\mu(\bigcup_{i\in Q_u} S(X_i,R^*_{i,n}))}\prod_{u=1}^{q}\prod_{i\in Q_u}\mathbf{1}\{\mu(S(X_i, r_{i,k}))\geq y_n\}\prod_{u=1}^{q}\mathbf{1}\{E_u\}$$
$$= e^k \prod_{u=1}^{q}\bigg(n^{|Q_u|}e^{-n\mu(\bigcup_{i\in Q_u} S(X_i,R^*_{i,n}))}\prod_{i\in Q_u}\mathbf{1}\{\mu(S(X_i, r_{i,k}))\geq y_n\}\mathbf{1}\{E_u\}\bigg).$$

Note that the inequality above comes from the trivial inequality

$$\exp\bigg(k\mu\bigg(\bigcup_{i=1}^{k} S(X_i, R^*_{i,n})\bigg)\bigg)\leq e^k$$

and the fact that $k=\sum_{u=1}^{q}|Q_u|$. Thus, (4.8) is proved if we can show that

$$\lim_{n\to\infty}\mathbb{E}\bigg[n^{|Q_u|}e^{-n\mu(\bigcup_{i\in Q_u} S(X_i,R^*_{i,n}))}\prod_{i\in Q_u}\mathbf{1}\{\mu(S(X_i, r_{i,k}))\geq y_n\}\mathbf{1}\{E_u\}\bigg]=0$$

for each u with $2\leq |Q_u|<k$. Without loss of generality, assume that $Q_u=\{1,\dots,|Q_u|\}$. Then

$$\bigcap_{i=1}^{|Q_u|}\{\mu(S(X_i, r_{i,k}))\geq y_n\}\subset\bigcap_{i=1}^{|Q_u|}\Big\{\mu(S(X_i, \min_{j\neq i, j\leq |Q_u|}\|X_i-X_j\|))\geq y_n\Big\}$$
$$=\bigcap_{i=1}^{|Q_u|}\Big\{\min_{j\neq i, j\leq |Q_u|}\mu(S(X_i, \|X_i-X_j\|))\geq y_n\Big\}$$
$$=\bigcap_{i=1}^{|Q_u|}\bigcap_{j\neq i, j\leq |Q_u|}\{\mu(S(X_i, \|X_i-X_j\|))\geq y_n\}$$
$$=\bigcap_{i=1}^{|Q_u|}\bigcap_{j\neq i, j\leq |Q_u|}\{\|X_i-X_j\|\geq H^{-1}_{X_i}(y_n)\}$$
$$=\bigcap_{i,j\leq |Q_u|, i\neq j}\{\|X_i-X_j\|\geq \max(R^*_{i,n}, R^*_{j,n})\}.$$

Note that the last equality follows from the fact that $\bigcap_{i,j}A_{i,j}=\bigcap_{i,j}(A_{i,j}\cap A_{j,i})$ for any family of events $(A_{i,j})_{i,j}$. We now obtain

$$n^{|Q_u|}e^{-n\mu(\bigcup_{i\in Q_u} S(X_i,R^*_{i,n}))}\prod_{i\in Q_u}\mathbf{1}\{\mu(S(X_i, r_{i,k}))\geq y_n\}\mathbf{1}\{E_u\}$$
$$\leq n^{|Q_u|}e^{-n\mu(\bigcup_{i=1}^{|Q_u|} S(X_i,R^*_{i,n}))}\mathbf{1}\bigg\{\bigcap_{i,j\leq |Q_u|, i\neq j}\{\|X_i-X_j\|\geq \max\{R^*_{i,n}, R^*_{j,n}\}\}\bigg\}\mathbf{1}\{E_u\}$$
$$\leq n^{|Q_u|}e^{-n\mu(\bigcup_{i=1}^{2} S(X_i,R^*_{i,n}))}\mathbf{1}\bigg\{\bigcap_{i,j\leq |Q_u|, i\neq j}\{\|X_i-X_j\|\geq \max\{R^*_{i,n},R^*_{j,n}\}\}\bigg\}\mathbf{1}\{E_u\}.$$

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We now use the fact that $\mu(S(X_i, R^*_{i,n})) = y_n$ for each i. Moreover, on the event $\{\|X_i - X_j\| \geq \max\{R^*_{i,n}, R^*_{j,n}\}\}$, condition (2.5) implies that

$$\begin{aligned} n\mu\Bigg(\bigcup_{i=1}^{2} S(X_i, R^*_{i,n})\Bigg) &= n\mu(S(X_1, R^*_{1,n})) + n\mu(S(X_2, R^*_{2,n})) - n\mu(S(X_2, R^*_{2,n}) \cap S(X_1, R^*_{1,n})) \\ &= n\frac{y + \ln n}{n}\bigg(2 - \frac{\mu(S(X_2, R^*_{2,n}) \cap S(X_1, R^*_{1,n}))}{\mu(S(X_2, R^*_{2,n}))}\bigg) \\ &\geq (y + \ln n)(2 - \beta) \\ &=: (y + \ln n)(1 + \varepsilon) \quad \text{(say)}. \end{aligned}$$

Note that $\varepsilon > 0$ since $0 < \beta < 1$. Thus,

$$\begin{aligned} & n^{|Q_u|}\mathbb{E}\bigg[\mathrm{e}^{-n\mu(\bigcup_{i\in Q_u} S(X_i, R^*_{i,n}))} \prod_{i\in Q_u} \mathbf{1}\{\mu(S(X_i, r_{i,k})) \geq y_n\}\mathbf{1}\{E_u\}\bigg] \\ &\leq n^{|Q_u|}\mathrm{e}^{-(y+\ln n)(1+\varepsilon)}\mathbb{E}\bigg[\mathbf{1}\bigg\{\bigcap_{i,j\leq|Q_u|, i\neq j} \{\|X_i - X_j\| \geq \max\{R^*_{i,n}, R^*_{j,n}\}\}\bigg\}\mathbf{1}\{E_u\}\bigg] \\ &= \mathrm{O}(n^{|Q_u|-1-\varepsilon})\mathbb{P}(E_u). \end{aligned}$$

In order to bound $\mathbb{P}(E_u)$, we need the following lemma (recall that $Q_u = \{1, \ldots, |Q_u|\}$).

Lemma 4.1. *On E_u there is a random integer $L \in \{1, \ldots, |Q_u|\}$ depending on $X_1, \ldots, X_{|Q_u|}$ such that $Q_u \setminus \{L\}$ forms an equivalence class.*

Proof. Let $m := |Q_u|$. Regard $X_1, \ldots, X_m$ as vertices of a graph in which any two vertices X_i and X_j are connected by a node if $S(X_i, R^*_{i,n}) \cap S(X_j, R^*_{j,n}) \neq \varnothing$. Since $Q_u = \{1, \ldots, m\}$ is an equivalence class, this graph is connected. If there is at least one vertex X_j (say) with degree 1, set $L := j$. Otherwise, the degree of each vertex is at least 2, and we have $m \geq 3$. If $m = 3$, the graph is a triangle, and we can choose L arbitrarily. Now suppose that the lemma holds for any graph having $m \geq 3$ vertices, in which each vertex degree is at least 2. If we have an additional $(m + 1)$th vertex X_{m+1}, this is connected to at least two other vertices X_i and X_j (say). Of the graph with vertices $X_1, \ldots, X_m$ we can delete one vertex, and the remaining graph is connected. But X_{m+1} is then connected to either X_i or X_j, and we may choose $L = i$ or $L = j$. Note that, for $d = 1$, the proof is trivial since $\bigcup_{i\in Q_u} S(X_i, R^*_{i,n})$ is an interval, and L can be chosen as the index of either the smallest or the largest random variable. □

By induction, we now show that

$$\mathbb{P}(E_u) = \mathrm{O}\bigg(\bigg(\frac{\ln n}{n}\bigg)^{|Q_u|-1}\bigg) \tag{4.9}$$

as $n \to \infty$ for each $m := |Q_u| \in \{2, \ldots, k - 1\}$. We start with the base case $m = 2$. Note that $\mathbb{P}(E_u) \leq \mathbb{P}(\|X_2 - X_1\| \leq R^*_{2,n} + R^*_{1,n})$ and

$$\begin{aligned} & \mathbb{P}(\|X_2 - X_1\| \leq R^*_{2,n} + R^*_{1,n} \mid X_1) \\ &\quad = \mathbb{P}(\|X_2 - X_1\| \leq R^*_{2,n} + R^*_{1,n}, R^*_{2,n} \leq R^*_{1,n} \mid X_1) \\ &\qquad + \mathbb{P}(\|X_2 - X_1\| \leq R^*_{2,n} + R^*_{1,n}, R^*_{2,n} > R^*_{1,n} \mid X_1) \\ &\quad \leq \mathbb{P}(\|X_2 - X_1\| \leq 2R^*_{1,n} \mid X_1) + \mathbb{P}(\|X_2 - X_1\| \leq 2R^*_{2,n} \mid X_1). \end{aligned}$$

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Now condition (2.6) entails

$$\begin{aligned}\mathbb{P}(\|X_2 - X_1\| \le 2R^*_{1,n} \mid X_1) &= \mu(S(X_1, 2R^*_{1,n}))\\ &\le c_{\max}\mu(S(X_1, R^*_{1,n}))\\ &= c_{\max}\frac{y+\ln n}{n}.\end{aligned}$$

Setting $\widetilde{R}_{2,n} := H^{-1}_{X_2}(c_{\max}(y+\ln n)/n)$, a second appeal to (2.6) yields

$$\mu(S(X_2, 2R^*_{2,n})) \le c_{\max}\mu(S(X_2, R^*_{2,n})) = c_{\max}\frac{y+\ln n}{n} = \mu(S(X_2, \widetilde{R}_{2,n})),$$

and, thus, $2R^*_{2,n} \le \widetilde{R}_{2,n}$. Consequently,

$$\mathbb{P}(\|X_2 - X_1\| \le 2R^*_{2,n} \mid X_1) \le \mathbb{P}(\|X_2 - X_1\| \le \widetilde{R}_{2,n} \mid X_1).$$

Let γ_d be the minimum number of cones of angle $\pi/3$ centred at 0 such that their union covers $\mathbb{R}^d$. Fritz [9] mentioned that γ_d is the minimal number of spheres with radius less than $\frac{1}{2}$ whose union covers the surface of the unit sphere. This constant is roughly $2^d d \log d$, while, according to Lemma 5.5 of [8], we have $\gamma_d \le 4.9^d$. Further upper and lower bounds on γ_d are given in [14]. We refer the reader to Section 20.7 of [3] for more information and further bounds. The cone covering lemma (cf. Lemma 10.1 of [7] and Lemma 6.2 of [11]) says that, for any $0 \le a \le 1$ and any x_1, we have

$$\mu(\{x_2 \in \mathbb{R}^d : \mu(S(x_2, \|x_2 - x_1\|)) \le a\}) \le \gamma_d a. \tag{4.10}$$

Now (4.10) implies that

$$\mu(\{x_2 \in \mathbb{R}^d : \|x_2 - x_1\| \le H^{-1}_{x_2}(a)\}) \le \gamma_d\, a;$$

whence,

$$\mathbb{P}(\|X_2 - X_1\| \le \widetilde{R}_{2,n} \mid X_1) \le \gamma_d c_{\max}\,\frac{y+\ln n}{n}.$$

We thus obtain

$$\mathbb{P}(\|X_2 - X_1\| \le R^*_{2,n} + R^*_{1,n} \mid X_1) = \mathrm{O}\left(\frac{\ln n}{n}\right), \tag{4.11}$$

and so (4.9) is proved for $m = 2$. For the induction step, assume that (4.9) holds for $|Q_u| = m \in \{2, \dots, k-2\}$. If Q_u with $|Q_u| = m+1$ is an equivalence class then, by Lemma 4.1, there are random integers L_1 and L_2 taking values in $\{1, \dots, m+1\}$ such that $Q_u \setminus \{L_1\}$ forms an equivalence class, and

$$\|X_{L_1} - X_{L_2}\| \le R^*_{L_1,n} + R^*_{L_2,n}.$$

It follows that

$$\begin{aligned}\mathbb{P}(E_u) &\le (m+1)m\mathbb{P}(E_u \cap \{L_1 = m+1, L_2 = 1\})\\ &\le k(k-1)\mathbb{P}(\{Q_u \setminus \{m+1\} \text{ forms an equivalence class}\}\\ &\qquad\qquad \cap \{\|X_{m+1} - X_1\| \le R^*_{m+1,n} + R^*_{1,n}\})\\ &= k(k-1)\mathbb{E}[\mathbf{1}\{Q_u \setminus \{m+1\} \text{ forms an equivalence class}\}\\ &\qquad\qquad \times \mathbb{P}(\|X_{m+1} - X_1\| \le R^*_{m+1,n} + R^*_{1,n} \mid X_1, \dots, X_m)]\end{aligned}$$

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$$
\begin{aligned}
&= k(k-1)\mathbb{E}[\mathbf{1}\{Q_u \setminus \{m+1\} \text{ forms an equivalence class}\} \\
&\qquad \times \mathbb{P}(\|X_{m+1} - X_1\| \le R^*_{m+1,n} + R^*_{1,n} \mid X_1)] \\
&\le \mathrm{O}\bigg(\frac{\ln n}{n}\bigg)\mathbb{P}(Q_u \setminus \{m+1\} \text{ forms an equivalence class}) \\
&= \mathrm{O}\bigg(\frac{\ln n}{n}\bigg)\mathrm{O}\bigg(\bigg(\frac{\ln n}{n}\bigg)^{m-1}\bigg) \\
&= \mathrm{O}\bigg(\bigg(\frac{\ln n}{n}\bigg)^{m}\bigg).
\end{aligned}
$$

Note that the penultimate equation follows from the induction hypothesis, and the last '$\le$' is a consequence of (4.11). Note further that these limit relations imply (4.9); whence,

$$
\begin{aligned}
&n^{|Q_u|}\mathbb{E}\bigg[\mathrm{e}^{-n\mu(\bigcup_{i\in Q_u} S(X_i, R^*_{i,n}))} \prod_{i\in Q_u} \mathbf{1}\{\mu(S(X_i, r_{i,k})) \ge y_n\}\mathbf{1}\{E_u\}\bigg] \\
&\quad = \mathrm{O}(n^{|Q_u|-1-\varepsilon})\mathbb{P}(E_u) \\
&\quad = \mathrm{O}(n^{|Q_u|-1-\varepsilon})\mathrm{O}\bigg(\bigg(\frac{\ln n}{n}\bigg)^{|Q_u|-1}\bigg) \\
&\quad = \mathrm{O}(n^{-\varepsilon} \ln n).
\end{aligned}
$$

Summarizing, we have shown (4.8) and thus (4.6). Hence, (4.4) is verified with $\nu = \exp(-y)$, completing the proof of Theorem 2.2. □

5. Discussion of condition (2.5)

In this final section we comment on condition (2.5). For $d = 1$, we verify (2.5) if on $S(x, r) \cup S(z, s)$ the distribution function F of μ is either convex or concave. If $\|x - z\| \ge r + s$ then $S(x, r)$ and $S(z, s)$ are disjoint; therefore, suppose that $r + s \ge \|x - z\| \ge \max(r, s)$. Assume that F is convex; the proof for concave F is similar. If $x < z$, the convexity of F and

$$
\mu(S(z, s)) = F(z + s) - F(z - s) =: p \quad \text{(say)}
$$

imply that $F(z) - F(z - s) \le p/2$. Thus,

$$
\begin{aligned}
\mu(S(x, r) \cap S(z, s)) &= \mu([z - s, x + r]) \\
&\le \min\{\mu([z - s, z]), \mu([x, x + r])\} \\
&= \min\{F(z) - F(z - s), F(x + r) - F(x)\} \\
&\le F(z) - F(z - s) \\
&\le \frac{1}{2}p,
\end{aligned}
$$

and, hence,

$$
\frac{\mu(S(x, r) \cap S(z, s))}{\mu(S(z, s))} \le \frac{1}{2}.
$$

Thus, (2.5) is satisfied with $\beta = \frac{1}{2}$.

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For $d > 1$, the problem is more involved. Again, suppose that $r + s \geq \|x - z\| \geq \max(r, s)$. Writing $\langle\cdot, \cdot\rangle$ for the inner product in $\mathbb{R}^d$, introduce the half-spaces

$$H_1 := \{u \in \mathbb{R}^d \colon \langle u - x, z - x\rangle \geq 0\}, \qquad H_2 := \{u \in \mathbb{R}^d \colon \langle u - z, x - z\rangle \geq 0\}.$$

Then

$$\begin{aligned}\mu(S(x, r) \cap S(z, s)) &= \mu((S(z, s) \cap H_2) \cap (S(x, r) \cap H_1)) \\ &\leq \frac{\mu(S(z, s) \cap H_2) + \mu(S(x, r) \cap H_1)}{2}.\end{aligned}$$

We introduce another implicit condition as follows. Assume that $\alpha \in (1, 2)$ and $\delta > 0$ such that, for any $r, s > 0$ and any $x, z \in \operatorname{supp}(\mu)$ with $r + s \geq \|x - z\| \geq \max(r, s)$ and $\mu(S(x, r)) = \mu(S(z, s)) \leq \delta$, we have either

$$\mu(S(z, s) \cap H_2) \leq \alpha\mu(S(x, r) \cap H_1^c) \tag{5.1}$$

or

$$\mu(S(x, r) \cap H_1) \leq \alpha\mu(S(z, s) \cap H_2^c). \tag{5.2}$$

In the case of (5.1), we have

$$\begin{aligned}\frac{\mu(S(z, s) \cap H_2) + \mu(S(x, r) \cap H_1)}{2} &\leq \frac{\alpha\mu(S(x, r) \cap H_1^c) + \mu(S(x, r) \cap H_1)}{2} \\ &= \frac{\alpha}{2}\mu(S(x, r)),\end{aligned}$$

and (2.5) is verified with $\beta = \alpha/2$. The case of (5.2) is similar.

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References

[1] Baccelli, F. and Blaszczyszyn, B. (2009). *Stochastic Geometry and Wireless Networks: Theory* (Foundations Trends Networking **3**), Vol. I. NoW Publishers.

[2] Baccelli, F. and Blaszczyszyn, B. (2010). *Stochastic Geometry and Wireless Networks: Applications* (Foundations Trends Networking **4**), Vol. II. NoW Publishers.

[3] Biau, G. and Devroye, L. (2015). *Lectures on the Nearest Neighbor Method*. Springer, Cham.

[4] Billingsley, P. (1986). *Probability and Measure*. John Wiley, New York.

[5] Calka P. and Chenavier, N. (2014). Extreme values for characteristic radii of a Poisson-Voronoi tessellation. *Extremes* **17**, 359–385.

[6] Chenavier, N. and Hemsley, R. (2016). Extremes for the inradius in the Poisson line tessellation. *Adv. Appl. Prob.* **48**, 544–573.

[7] Devroye, L. and Györfi, L. (1985). *Nonparametric Density Estimation: The L_1 View*. John Wiley, New York.

[8] Devroye, L., Györfi, L. and Lugosi, G. (1996). *A Probabilistic Theory of Pattern Recognition*. Springer, New York.

[9] Fritz, J. (1975). Distribution-free exponential error bound for nearest neighbor pattern classification. *IEEE Trans. Inf. Theory* **IT-21**, 552–557.

[10] Graham, R. L., Knuth, D. E. and Patashnik, O. (1994). *Concrete Mathematics*. Addison-Wesley, Reading, MA.

[11] GYÖRFI, L., KOHLER, M., KRZYŻAK, A. AND WALK, H. (2002). *A Distribution-Free Theory of Nonparametric Regression*. Springer, New York.
[12] HENZE, N. (1982). The limit distribution for maxima of 'weighted' *r*th-nearest-neighbour distances. *J. Appl. Prob.* **19**, 344–354.
[13] HENZE, N. (1983). Ein asymptotischer Satz über den maximalen Minimalabstand von unabhängigen Zufallsvektoren mit Anwendungen auf einen Anpassungstest im $\mathbb{R}^p$ und auf der Kugel. [An asymptotic theorem on the maximum minimum distance of independent random vectors, with application to a goodness-of-fit test in $\mathbb{R}^p$ and on the sphere]. *Metrika* **30**, 245–259 (in German).
[14] ROGERS, C. A. (1963). Covering a sphere with spheres. *Mathematika* **10**, 157–164.

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