Math. Struct. in Comp. Science (2004), *vol.* 14, *pp.* 185–187. © 2004 Cambridge University Press
DOI: 10.1017/S0960129503004080 Printed in the United Kingdom

Corrigenda for 'Connected limits, familial representability and Artin glueing'

AURELIO CARBONI[†] and PETER JOHNSTONE[‡]

[†]*Dipartimento di Scienze Chemiche, Fisiche e Matematiche, Università dell'Insubria, Italy*
[‡]*Department of Pure Mathematics, University of Cambridge, England*

Received 29 October 2002

Since the publication of the paper Carboni and Johnstone (1995), we have become aware of two independent errors in it. Although neither of them has any effect on the main results of the paper, concerning when a category obtained by Artin glueing is a topos, we feel it is appropriate to publish this correction in the hope that it may prevent future readers of Carboni and Johnstone (1995) from being misled. We are grateful to Tom Leinster and to Marek Zawadowski for drawing our attention to the two errors.

The first error, which was pointed out by Tom Leinster, affects Proposition 3.2. Unfortunately, it is not true in general that a monad on **Set** whose functor part is familially representable, and whose unit and multiplication are cartesian, has a strongly regular presentation. In our purported proof, we constructed from such a monad (T, η, μ) an algebraic theory having a strongly regular presentation, and we correctly observed that the free functor T' for this algebraic theory is naturally isomorphic to T; but we neglected to verify that the isomorphism $T \cong T'$ is an isomorphism of monads, and, unfortunately, it is not in general – it always commutes with the units, but does not necessarily commute with the multiplications. In the construction of the theory, we (implicitly) chose total orderings for each of the sets in the family $(X_i)_{i \in I}$ representing T, and in order to make the isomorphism commute with μ, it is necessary to choose these orderings in a way that is 'compatible' with the morphism $m : T(I) \to I$ that induces μ, in the sense that if (i, g) is an element of $T(I) = \coprod_{i \in I} I^{X_i}$, then the induced bijection

$$\coprod_{x \in X_i} X_{g(x)} \longrightarrow X_{m(i,g)}$$

is order-preserving (where its domain is ordered lexicographically).

An example of a monad for which this is not possible is the one corresponding to the theory of monoids equipped with an anti-involution: that is, the theory presented by operations e, u, m of arities $0, 1, 2$, respectively, satisfying the equations $m(x, m(y, z)) = m(m(x, y), z)$, $m(e, x) = x$, $m(x, e) = x$, $u(u(x)) = x$ and $u(m(x, y)) = m(u(y), u(x))$. It is easily verified that the free functor T for this theory is familially representable: it may be represented by the family $(X_i)_{i \in I}$ where I is the set of all finite sequences of $+$ and $-$ signs (which we may identify with the free algebra on one generator) and if i is a sequence of length n, then X_i has n elements. Moreover, the unit and multiplication of the monad structure on T are cartesian, and so may be induced by morphisms

$e : 1 \to I$ and $m : T(I) \to I$ as described on page 449 of Carboni and Johnstone (1995). But we cannot choose orderings of the X_i that are compatible with m; for we have $m((-), g) = (-,-) = m((+,+), h)$, where g sends the unique element of $X_{(-)}$ to $(+,+)$ and h sends both elements of $X_{(+,+)}$ to $(-)$, and these equations give us two bijections

$$X_{(+,+)} \cong \coprod_{x \in X_{(+,+)}} X_{(-)} \rightrightarrows X_{(-,-)},$$

which are different, and so cannot both be order-preserving. If we apply the construction of Carboni and Johnstone (1995) to the monad (T, η, μ) (for any choice of orderings of the X_i), we obtain a theory equivalent to the theory of monoids with an involution, that is, that obtained on replacing the last equation in the presentation above by $u(m(x,y)) = m(u(x), u(y))$. Although the free functor for this theory is indeed isomorphic to that for the theory of monoids with an anti-involution, the two monads are not isomorphic, as may be seen by observing that a three-element set has twelve more algebra structures for the second monad than it does for the first. (There are six non-commutative monoid structures on a three-element set, and each of them has two involutions and no anti-involutions.)

Whether it is possible to give a syntactic characterisation of the class of monads on **Set** considered in Section 3 of Carboni and Johnstone (1995) remains an open problem. However, we may note that such a characterisation, if it exists, must involve a 'global' condition on the set of all equations in a presentation, rather than a condition like strong regularity, which may be verified 'equation by equation'; since the presentation above for monoids with an anti-involution should satisfy it, but if we add the strongly regular equation $u(x) = x$ to this presentation, we obtain the theory of commutative monoids, which should not.

The second error in Carboni and Johnstone (1995), which was pointed out to us by Marek Zawadowski, occurs in the final sentence of Example 4.6. It is correct that the category **Comp** of 2-computads (that is, formal presentations of 2-categories) is a presheaf topos; but this result does not extend to the category of 3-computads (or indeed to n-computads for any $n \geqslant 3$), as we claimed. The reason is that the functor P : **Comp** $\to$ **Set** that sends a 2-computad to its set of 2-dimensional pasting diagrams is not familially representable, unlike the corresponding functor for 1-computads (directed graphs). Indeed, P does not even preserve pullbacks over the terminal object, so the main theorem of the paper implies that the category of 3-computads is not even cartesian closed.

Once again, the culprit is (the lack of) an ordering: a 1-dimensional pasting diagram (that is, a sequence of formally composable 1-cells) has a canonical ordering, but a 2-dimensional diagram does not. Indeed, if f and g are 2-cells whose domain and codomain are the identity 1-cell (that is, the empty pasting diagram) on the same object, then the pasting diagrams for the (vertical) composites fg and gf are identical (though it is hard to draw them in a way that makes this evident!). Thus, if we restrict to the full subcategory of **Comp** (which is evidently closed under pullbacks) whose objects are computads with one object and no 1-cells, the functor P becomes isomorphic to the one that sends such a computad to the free commutative monoid on its set of 2-cells; and this functor does not preserve pullbacks.

From another point of view, the problem here is caused by the presence of identities (that is, empty pasting diagrams). If we had considered 'computads without identities' in which the domain and codomain of every n-cell are required to be non-empty $(n-1)$-dimensional pasting diagrams, then the argument of Carboni and Johnstone (1995) would apply to show that, for every n, the category of n-dimensional computads without identities is a presheaf topos. However, this result seems unlikely to be of much use.

Reference

Carboni, A. and Johnstone, P. T. (1995) Connected limits, familial representability and Artin glueing. *Mathematical Structures in Computer Science* **5** 441–459.