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Regularity results for the 2D critical Oldroyd-B model in the corotational case

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This paper studies the regularity results of classical solutions to the two-dimensional critical Oldroyd-B model in the corotational case. The critical case refers to the full Laplacian dissipation in the velocity or the full Laplacian dissipation in the non-Newtonian part of the stress tensor. Whether or not their classical solutions develop finite time singularities is a difficult problem and remains open. The object of this paper is two-fold. Firstly, we establish the global regularity result to the case when the critical case occurs in the velocity and a logarithmic dissipation occurs in the non-Newtonian part of the stress tensor. Secondly, when the critical case occurs in the non-Newtonian part of the stress tensor, we first present many interesting global *a priori* bounds, then establish a conditional global regularity in terms of the non-Newtonian part of the stress tensor. This criterion comes naturally from our approach to obtain a global L^∞-bound for the vorticity ω.

Keywords: Oldroyd-B model; global regularity; classical solutions

2010 *Mathematics subject classification:* Primary: 76A10; 76D03; 76A05; 35Q35

1. Introduction

The two-dimensional (2D) Oldroyd-B type model with diffusive stress in the whole space $\mathbb{R}^2$ takes the following form [10]

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla\pi = \kappa\nabla\cdot\tau, \\ \partial_t\tau + (u\cdot\nabla)\tau + \beta\tau - \mu\Delta\tau - Q(\nabla u,\tau) = \gamma\mathcal{D}u, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ \tau(x,0) = \tau_0(x), \end{cases} \tag{1.1}$$

where the unknown vector $u = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ is the velocity of the fluid, $\pi = \pi(x,t) \in \mathbb{R}$ is the scalar pressure and $\tau = \tau(x,t)$ which is a symmetric tensor is the non-Newtonian part of the stress tensor. The parameters ν, μ β, κ, γ are such that $\nu \geqslant 0$, $\mu \geqslant 0$ $\beta \geqslant 0$ and $\kappa > 0$, $\gamma > 0$. Here $\mathcal{D}u$ is the symmetric part of the velocity gradient, namely $\mathcal{D}u = 1/2(\nabla u + \nabla u^\top)$. Q is a given bilinear form and usually chooses the following form

$$Q(\nabla u, \tau) = \Omega\tau - \tau\Omega + b(\mathcal{D}u\tau + \tau\mathcal{D}u), \tag{1.2}$$

where $\Omega = 1/2(\nabla u - \nabla u^\top)$ is the skew symmetric part of ∇u and $b \in [-1, 1]$ is a parameter. If $b = 0$, we call the corresponding system as corotational case. The classical Oldroyd-B type model ($\mu = 0$) originally was introduced by Oldroyd [**36**] which is one of the basic macroscopic models for viscoelastic flows such as polymer flows; fluids of this type have both elastic properties and viscous properties. For more discussions and the derivation of Oldroyd-B model, we refer the readers to [**6**, **16**, **36**].

Due to the physical applications and mathematical significance, the Oldroyd-B model has recently attracted considerable attention. Let us first briefly review some existence theories of the Oldroyd-B model from various aspects, we will not attempt to address exhaustive reference in this paper. The study of the most interesting case $\nu > 0$ and $\mu = 0$ (which is the classical case) started by Guillopé and Saut in [**17**, **18**], where the existence of local strong solutions to the Oldroyd-B model was proved in Hilbert space. In the frame of critical Besov spaces, Chemin and Masmoudi [**6**] constructed global solutions to the incompressible Oldroyd-B model with small initial data (see Chen-Miao [**7**]). Very recently, the results of global existence of smooth solution with small initial data were significantly improved by many works, see [**11**, **14**, **15**, **19**, **39**, **40**] and references therein. Moreover, Chemin and Masmoudi [**6**], they also provided some interesting blowup criteria. An improvement of the Chemin-Masmoudi blow-up criterion was presented by Lei, Masmoudi and Zhou in [**31**]. For the Oldroyd-B fluids with diffusive stress (namely, the system (1.1) with $\nu > 0$, $\mu > 0$ and $b = 1$), the existence and uniqueness of global strong solutions in 2D case was established by Constantin and Kliegel [**10**]. Very recently, Elgindi and Rousset [**13**] proved the global regularity with general initial data to the system (1.1) in the case $\nu = 0$, $\mu > 0$ and $Q = 0$. Moreover, they also obtained the global existence of smooth solution with small initial data in the case of $\nu = 0$, $\mu > 0$ and the general Q given by (1.2). Many other interesting results on the Oldroyd-B and related models have been established (see, e.g., [**8**, **22**–**24**, **28**–**30**, **33**–**35**, **37**] and the references therein).

In 2000, Lions and Masmoudi [**34**] proved the global existence of weak solutions (without uniqueness) to the following 2D critical Oldroyd-B model in the corotational case, namely,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla\pi = \kappa\nabla \cdot \tau, \\ \partial_t \tau + (u \cdot \nabla)\tau + \beta\tau + \eta(\tau\Omega - \Omega\tau) = \gamma\mathcal{D}u, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x),\ \tau(x, 0) = \tau_0(x), \end{cases} \tag{1.3}$$

which was further generalized by Bejaoui and Mohamed Majdoub [**3**]. However, the global existence of smooth solutions is open and quite challenging (see [**13**] for more details). We also mention that global weak for the above system (1.3) with $\eta(\tau\Omega - \Omega\tau)$ replaced by general Q, namely (1.2), is still open up to now. As pointed out in [**13**], in the case where $\eta = 0$ and $\nu > 0$, the global existence of

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smooth solutions to the above system (1.3) is also open and quite challenging, not to speak of the case $\eta > 0$. Very recently, when one adds the fractional dissipation $(-\Delta)^{\gamma}\tau$ (the power $\gamma > 0$ can be arbitrarily small) to the τ equation, the author [**38**] obtained the global existence of classical solution for the general initial data for the corresponding system. Consequently, the above system (1.3) is the first object of the study.

On the other hand, as indicated in [**13**], the global regularity of the following 2D critical Oldroyd-B model in the corotational case (with $b = 0$) with diffusive stress is also an unsolved critical problem

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = \kappa\nabla\cdot\tau, \\ \partial_t\tau + (u\cdot\nabla)\tau + \beta\tau + \eta(\tau\Omega - \Omega\tau) - \mu\Delta\tau = \gamma\mathcal{D}u, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ \tau(x,0) = \tau_0(x). \end{cases} \tag{1.4}$$

As stated above, when the system (1.4) with $\mu > 0$ and $\eta = 0$, Elgindi and Rousset [**13**] proved the global regularity with general initial data. The major obstacle to get the global regularity for the system (1.4) with $\mu > 0$ and $\eta > 0$ is to show the global L^{∞}-bound for the vorticity ω. However, the corotational term $\tau\Omega - \Omega\tau$ prevents us to get the global $\|\omega\|_{L^{\infty}}$ bound (see lemma 4.5 for details). Actually, the above system (1.4) is also our object of the study.

Now we are in the position to state our main results. The first goal of this paper is to establish the global regularity for the case when the critical case occurs in the velocity and a logarithmic dissipation occurs in the non-Newtonian part of the stress tensor. More precisely, the first main result can be stated as follows.

Theorem 1.1. *Consider the following system*

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\pi = \nabla\cdot\tau, \\ \partial_t\tau + (u\cdot\nabla)\tau + \mathcal{L}\tau + \tau\Omega - \Omega\tau = \mathcal{D}u, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x),\ \tau(x,0) = \tau_0(x), \end{cases} \tag{1.5}$$

where the dissipation operator $\mathcal{L}$ is defined via

$$\mathcal{L}\tau(x) = P.V.\int_{\mathbb{R}^2} \frac{\tau(x) - \tau(x-y)}{|y|^2 m(|y|)}\,\mathrm{d}y \tag{1.6}$$

for some non-decreasing, smooth function $m : [0,\infty) \mapsto [0,\infty)$ satisfying the following conditions:

(I) *There exists a positive constant γ such that*

$$\limsup_{r\to 0^+} \frac{r^{\delta}}{m(r)} \leqslant \gamma \tag{1.7}$$

for some $\delta \in (0, 1)$.

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(II) *It satisfies the assumption*

$$\int_1^\infty \frac{\mathrm{d}r}{rm(r)} < \infty. \tag{1.8}$$

(III) *It satisfies the assumption*

$$\int_0^1 \frac{m(r)}{r}\,\mathrm{d}r < \infty. \tag{1.9}$$

If $(u_0, \tau_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla \cdot u_0 = 0$*, then the system (1.5) admits a unique global solution such that for any given* $T > 0$

$$u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^2)), \quad \tau \in C([0,T]; H^s(\mathbb{R}^2)).$$

REMARK 1.2. The classical example of m satisfying the conditions (I)–(III) is

$$m(r) = r^\alpha \quad \text{with } 0 < \alpha \leqslant \delta,$$

which corresponds to the fractional Laplacian operator. But it is not the purpose of this paper. What we care is the case that m behaves like $1/(-\ln r)^{\alpha_1}$ for sufficiently small r with some $\alpha_1 > 1$, and behaves like $(\ln r)^{\alpha_2}$ for sufficiently large r with some $\alpha_2 > 1$.

REMARK 1.3. It should be noted that the condition (1.7) can be replaced by the following doubling condition

$$m(2r) \leqslant C_0 m(r)$$

with $C_0 \in (1,\, 2)$. As a matter of fact, it directly gives

$$m(1) \leqslant C_0 m\left(\frac{1}{2}\right) < \cdots < C_0^k m\left(\frac{1}{2^k}\right), \quad \forall\, k \in \mathbb{N}.$$

Then for any $r \in (0,\, 1)$, there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2^k} \leqslant r < \frac{1}{2^{k-1}}.$$

This implies for any $r \in (0,\, 1)$ that

$$m(r) \geqslant m\left(\frac{1}{2^k}\right) \geqslant m(1) C_0^{-k} \geqslant m(1) C_0^{-(1-\ln r/\ln 2)} \geqslant \frac{m(1)}{C_0} e^{\ln C_0/\ln 2 \ln r} \geqslant \frac{r^\delta}{\widetilde{\gamma}},$$

where $\delta = \ln C_0/\ln 2 \in (0,\, 1)$ and $\widetilde{\gamma} = C_0/m(1)$. The above estimate immediately implies (1.7).

REMARK 1.4. Obviously, the dissipation $\mathcal{L}$ given in theorem 1.1 is weaker than any power of the fractional Laplacian. Consequently, theorem 1.1 is a further improvement of [**38**]. However, it remains an open problem whether there exists a global smooth solution when the operator $\mathcal{L}$ is absent from the system (1.5).

The next goal of this paper is to gain further understanding of the global regularity problem for the case when the operator $\mathcal{L}$ is absent from the system (1.5), namely (1.3). Several global *a priori* bounds are also presented, which will be useful on the eventual resolution of this difficult global regularity problem. More precisely, we have the following results for the system (1.3).

THEOREM 1.5. *Assume that* $(u_0, \tau_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla \cdot u_0 = 0$. *Then, the solution* (u, τ) *of the system (1.3) admits the following global bounds, for any* $T > 0$ *and* $t \leqslant T$,

(d) *Global bound for* u,

$$\|\nabla u\|_{L^p(0,T;L^q)} \leqslant C, \quad \forall\, 2 \leqslant p,\, q < \infty, \tag{1.10}$$

where $C = C(p, q, T, u_0, \tau_0)$;

(e) *Global bound for* τ,

$$\|\tau(t)\|_{L^p} \leqslant C, \quad \forall\, 2 \leqslant p < \infty, \tag{1.11}$$

where $C = C(p, T, u_0, \tau_0)$;

(f) *Global bounds for* G,

$$\|G\|_{L^p(0,T;W^{1,q})} \leqslant C, \quad \forall\, 2 \leqslant p,\, q < \infty, \tag{1.12}$$

where $C = C(p, q, T, u_0, \tau_0)$;

$$\|\Lambda^\gamma G(t)\|_{L^p} \leqslant C, \quad \forall\, 0 \leqslant \gamma < 1, \quad \forall\, 2 \leqslant p < \infty, \tag{1.13}$$

where $C = C(\gamma, p, T, u_0, \tau_0)$ *and* $\Lambda := (-\Delta)^{1/2}$.

Here the G *is given by*

$$G = \omega - (-\Delta)^{-1} \mathit{curl}\ \mathit{div}(\tau), \quad \omega = \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

REMARK 1.6. The desired estimates (1.10), (1.11), (1.12) and (1.13) can be obtained by the same arguments adopted in proving lemmas 3.4 and 3.5. In order to avoid the redundancy, we thus omit the details.

The third goal of this paper is to gain further understanding of the global regularity problem for the 2D critical Oldroyd-B model in the corotational case with diffusive stress, namely (1.4). We first present many global *a priori* bounds and then establish a conditional global regularity in terms of the non-Newtonian part of the stress tensor. We hope that these results will shed light on the eventual resolution of this difficult global regularity problem.

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THEOREM 1.7. *Assume that* $(u_0, \tau_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla \cdot u_0 = 0$. *Then, the solution* (u, τ) *of the system (1.4) admits the following global bounds, for any* $T > 0$ *and* $t \leqslant T$,

- *Global a priori bounds:*
  (a) *Global* $W^{1,q}$*-bound for* u,

$$\|u(t)\|_{W^{1,q}} \leqslant C, \quad \forall\, 2 \leqslant q < \infty, \tag{1.14}$$

  where $C = C(q, T, u_0, \tau_0)$;

  (b) *Global bounds for* τ,

$$\|\tau(t)\|_{W^{1,p}} \leqslant C, \quad \forall\, 2 \leqslant p < \infty, \tag{1.15}$$

  where $C = C(p, T, u_0, \tau_0)$;

$$\|\nabla^2 \tau\|_{L^p(0,T;L^q)} \leqslant C, \quad \forall\, 2 \leqslant p,\, q < \infty, \tag{1.16}$$

  where $C = C(p, q, T, u_0, \tau_0)$;

$$\|\Lambda^\delta \tau(t)\|_{L^\infty} \leqslant C, \quad \forall\, 0 \leqslant \delta < 2, \tag{1.17}$$

  where $C = C(\delta, T, u_0, \tau_0)$.

- *Regularity criterion:*
  Let (u, τ) *be the local (in time) smooth solution of the system* (1.4) *on* $[0, T_0)$ *associated with initial value* (u_0, τ_0). *Let* $T > T_0$. *If there is an integer* $j_0 > 0$ *such that* τ *satisfies*

$$\int_0^T \sum_{j \geqslant j_0} \|S_{j-1}\tau(\sigma)\|_{L^\infty}\, \mathrm{d}\sigma < \infty, \tag{1.18}$$

  then the local solution can be extended to $[0, T]$, *where* S_j *denotes the low-frequency cut-off operator defined through the Littlewood–Paley decomposition (see* § 2 *for details).*

REMARK 1.8. Although we can obtain many global *a priori* bounds (1.15)–(1.17) for τ, however it is not clear whether (1.18) can be guaranteed by these *a priori* estimates.

If we add the dissipation term $(-\Delta)^\alpha u$ (the power $\alpha > 0$ can be arbitrarily small) to the velocity equation of the system (1.4), then we have the global regularity result for the corresponding system.

THEOREM 1.9. *Consider the following system*

$$\begin{cases}\partial_t u + (u\cdot\nabla)u + (-\Delta)^\alpha u + \nabla\pi = \nabla\cdot\tau,\\ \partial_t \tau + (u\cdot\nabla)\tau + \tau + \tau\Omega - \Omega\tau - \Delta\tau = \mathcal{D}u,\\ \nabla\cdot u = 0,\\ u(x,0) = u_0(x),\ \ \tau(x,0) = \tau_0(x).\end{cases} \tag{1.19}$$

If $(u_0,\tau_0)\in H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ *with* $s>2$ *and* $\nabla\cdot u_0 = 0$*, then for any* $\alpha>0$ *the system (1.19) admits a unique global solution such that for any given* $T>0$

$$u \in C([0,T];H^s(\mathbb{R}^2))\cap L^2([0,T];H^{s+\alpha}(\mathbb{R}^2)),$$
$$\tau \in C([0,T];H^s(\mathbb{R}^2))\cap L^2([0,T];H^{s+1}(\mathbb{R}^2)).$$

The rest part of this paper is organized as follows. Section 2 makes several preparations including presenting the Littlewood–Paley decomposition, functional spaces and some related inequalities. Section 3 is devoted to the proof of theorem 1.1. Section 4 provides the proof of theorem 1.7. A brief of the proof of theorem 1.9 is carried out in § 5. In appendix, we provide the proof of several facts.

2. Preliminaries

This section presents the Besov spaces as well as several inequalities to be extensively used in the subsequent section. In this paper, all constants will be denoted by C that is a generic constant depending only on the quantities specified in the context. We shall write $C(\lambda_1,\lambda_2,\ldots,\lambda_k)$ as the constant C depends on the quantities $\lambda_1,\lambda_2,\ldots,\lambda_k$. We denote the fractional operator Λ by $(-\Delta)^{1/2}$. For a quasi-Banach space X and for any $0<T\leqslant\infty$, we use standard notation $L^p(0,T;X)$ or $L^p_T(X)$ for the quasi-Banach space of Bochner measurable functions f from $(0,T)$ to X endowed with the norm

$$\|f\|_{L^p_T(X)} := \begin{cases}\left(\displaystyle\int_0^T \|f(.,t)\|_X^p\,\mathrm{d}t\right)^{1/p}, & 1\leqslant p<\infty,\\ \displaystyle\sup_{0\leqslant t\leqslant T}\|f(.,t)\|_X, & p=\infty.\end{cases}$$

Now we recall the so-called Littlewood–Paley operators and their elementary properties which allow us to define the Besov spaces (see e.g,. [1]). Let (χ,φ) be a couple of smooth functions with values in $[0,1]$ such that $\chi\in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B} := \{\xi\in\mathbb{R}^n, |\xi|\leqslant 4/3\}, \varphi\in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C} := \{\xi\in\mathbb{R}^n, 3/4\leqslant|\xi|\leqslant 8/3\}$ and satisfy

$$\chi(\xi)+\sum_{j\in\mathbb{N}}\varphi(2^{-j}\xi) = 1,\quad \forall\xi\in\mathbb{R}^n;\quad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\quad \forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

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Let $h=\mathcal{F}^{-1}\varphi$ and $g=\mathcal{F}^{-1}\chi$ and then we define the non-homogeneous Littlewood–Paley operators as follows

$$\Delta_j u=\varphi(2^{-j}D)u=2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)\,\mathrm{d}y,\quad \forall j\geqslant 0;$$
$$S_j u=\chi(2^{-j}D)u=\sum_{k=-1}^{j-1}\Delta_k u=2^{jn}\int_{\mathbb{R}^n} g(2^j y)u(x-y)\,\mathrm{d}y,$$

and

$$\Delta_j u=0,\ j\leqslant -2;\quad \Delta_{-1}u=\chi(D)u=S_0 u.$$

Also, we denote

$$\widetilde{\Delta}_j u:=\Delta_{j-1}u+\Delta_j u+\Delta_{j+1}u.$$

We now point out several simple facts concerning the operators Δ_j: By compactness of the supports of the series of Fourier transform, we have

$$\Delta_j\Delta_l u\equiv 0,\quad |j-l|\geqslant 2\quad \text{and}\quad \Delta_k(S_l u\Delta_l v)\equiv 0,\quad |k-l|\geqslant 5.$$

Moreover, it is easy to check that

$$\operatorname{supp}\mathcal{F}(S_{j-1}u\Delta_j v)\approx\left\{\xi\,|\,\frac{1}{12}2^j\leqslant|\xi|\leqslant\frac{10}{3}2^j\right\}$$

and

$$\operatorname{supp}\mathcal{F}(\widetilde{\Delta}_j u\Delta_j v)\subset\{\xi\,|\,|\xi|\leqslant 8\times 2^j\},$$

where $\mathcal{F}$ denotes the Fourier transform. Let us recall the definition of inhomogeneous Besov spaces through the dyadic decomposition.

Definition 2.1. Let $s\in\mathbb{R},(p,r)\in[1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f\in S'(\mathbb{R}^n)$ such that

$$B^s_{p,r}=\{f\in S'(\mathbb{R}^n);\|f\|_{B^s_{p,r}}<\infty\},$$

where

$$\|f\|_{B^s_{p,r}}=\begin{cases}\left(\sum_{j\geqslant -1}2^{jrs}\|\Delta_j f\|^r_{L^p}\right)^{1/r}, & \forall\ r<\infty,\\ \sup_{j\geqslant -1}2^{js}\|\Delta_j f\|_{L^p}, & \forall\ r=\infty.\end{cases}$$

In addition, for two tempered distributions f and g, we also recall the notion of paraproducts

$$T_f g = \sum_j S_{j-1} f \Delta_j g, \quad R(f,g) = \sum_j \widetilde{\Delta}_j f \Delta_j g$$

and Bony's decomposition

$$fg = T_f g + T_g f + R(f,g).$$

Bernstein inequalities are fundamental in the analysis involving Besov spaces and these inequalities trade integrability for derivatives.

LEMMA 2.2 (see [1]). *Let $k \geqslant 0, 1 \leqslant a \leqslant b \leqslant \infty$. Assume that*

$$supp\, \widehat{f} \subseteq \{\xi \in \mathbb{R}^n : \ |\xi| \leqslant A_0 2^j\},$$

for some integer j, then there exists a constant C_1 such that

$$\|\Lambda^k f\|_{L^b} \leqslant C_1\, 2^{jk+jn(1/a-1/b)} \|f\|_{L^a}.$$

If f satisfies

$$supp\, \widehat{f} \subseteq \{\xi \in \mathbb{R}^n : \ A_1 2^j \leqslant |\xi| \leqslant A_2 2^j\}$$

for some integer j, then

$$C_1\, 2^{jk} \|f\|_{L^b} \leqslant \|\Lambda^k f\|_{L^b} \leqslant C_2\, 2^{jk+jn(1/a-1/b)} \|f\|_{L^a},$$

where C_1 and C_2 are constants depending on k, a, b and dimension n only.

Now we recall some properties involving the heat operator. We set (n is the space dimension)

$$e^{t\Delta} f = H(t,x) * f, \quad H(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

The following estimate is a direct consequence of the Young inequality.

LEMMA 2.3. *Let $1 \leqslant p \leqslant q \leqslant \infty$ and $k \geqslant 0$, then it holds*

$$\|\Lambda^k e^{t\Delta} f\|_{L^q} \leqslant C t^{-k/2-n/2(1/p-1/q)} \|f\|_{L^p}.$$

Finally, we recall the following Maximal $L^q_t L^p_x$ regularity for the heat kernel (see e.g,. [32]).

LEMMA 2.4. *The operator A defined by*

$$Af(x,t) := \int_0^t \mathrm{e}^{(t-s)\Delta} \Delta f(s,x)\, \mathrm{d}s$$

is bounded from $L^q(0,T; L^p(\mathbb{R}^n))$ to $L^q(0,T; L^p(\mathbb{R}^n))$ for very $(q,p) \in (1,\infty) \times (1,\infty)$ and $T \in (0,\infty]$.

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3. The proof of theorem 1.1

This section is devoted to the proof of theorem 1.1. For two n order matrices $\mathbb{A}$ and $\mathbb{B}$, we denote $\mathbb{A}:\mathbb{B}=\sum_{i,j=1}^{n}a_{ij}b_{ij}$ where a_{ij} and b_{ij} are the components of matrices $\mathbb{A}$ and $\mathbb{B}$, respectively. The existence and uniqueness of local smooth solutions in H^{s} $(s>2)$ can be done without any difficulty, thus it is sufficient to establish *a priori* estimates for (u,τ) at the interval $[0,\,T]$. We first establish some properties of the operator $\mathcal{L}$, namely, lemmas 3.1 and 3.2.

Lemma 3.1. *Under the assumptions (1.7) and (1.8), it holds that for any* $1<p<\infty$

$$\|\mathcal{L}h\|_{L^{p}}\leqslant C\|h\|_{W^{\widetilde{\delta},\,p}}, \tag{3.1}$$

where $\delta<\widetilde{\delta}<1$.

Proof of lemma 3.1. Recalling the operator $\mathcal{L}$, namely,

$$\mathcal{L}h(x)=\mathrm{P.V.}\int_{\mathbb{R}^{2}}\frac{h(x)-h(x-y)}{|y|^{2}m(|y|)}\,\mathrm{d}y,$$

we immediately have

$$\|\mathcal{L}h(t)\|_{L^{p}}=\left(\int_{\mathbb{R}^{2}}\left|\mathrm{P.V.}\int_{\mathbb{R}^{2}}\frac{h(x)-h(x-y)}{|y|^{2}m(|y|)}\,\mathrm{d}y\right|^{p}\,\mathrm{d}x\right)^{1/p}\leqslant I+J,$$

where I and J are given by

$$I=\left(\int_{\mathbb{R}^{2}}\left|\mathrm{P.V.}\int_{|y|\leqslant 1}\frac{h(x)-h(x-y)}{|y|^{2}m(|y|)}\,\mathrm{d}y\right|^{p}\,\mathrm{d}x\right)^{1/p},$$

$$J=\left(\int_{\mathbb{R}^{2}}\left|\mathrm{P.V.}\int_{|y|\geqslant 1}\frac{h(x)-h(x-y)}{|y|^{2}m(|y|)}\,\mathrm{d}y\right|^{p}\,\mathrm{d}x\right)^{1/p}.$$

By the Minkowski inequality and the Hölder inequality, one has

$$\begin{aligned}
I&\leqslant\int_{|y|\leqslant 1}\left(\int_{\mathbb{R}^{2}}\frac{|h(x)-h(x-y)|^{p}}{|y|^{2p}m^{p}(|y|)}\,\mathrm{d}x\right)^{1/p}\,\mathrm{d}y\\
&=\int_{|y|\leqslant 1}\left(\int_{\mathbb{R}^{2}}\frac{|h(x)-h(x-y)|^{p}}{|y|^{2+\widetilde{\delta}p}|y|^{(2-\widetilde{\delta})p-2}m^{p}(|y|)}\,\mathrm{d}x\right)^{1/p}\,\mathrm{d}y\\
&=\int_{|y|\leqslant 1}\left(\int_{\mathbb{R}^{2}}\frac{|h(x)-h(x-y)|^{p}}{|y|^{2+\widetilde{\delta}p}}\,\mathrm{d}x\right)^{1/p}\frac{1}{|y|^{2-\widetilde{\delta}-2/p}m(|y|)}\,\mathrm{d}y\\
&\leqslant\left(\int_{|y|\leqslant 1}\int_{\mathbb{R}^{2}}\frac{|h(x)-h(x-y)|^{p}}{|y|^{2+\widetilde{\delta}p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\\
&\quad\times\left(\int_{|y|\leqslant 1}\frac{1}{|y|^{(2-\widetilde{\delta}-2/p)p/(p-1)}m^{p/(p-1)}(|y|)}\,\mathrm{d}y\right)^{(p-1)/p}
\end{aligned}$$

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$$\leqslant C\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|h(x)-h(x-y)|^p}{|y|^{2+\widetilde{\delta}p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$$
$$\times\left(\int_0^1\frac{r}{r^{(2-\widetilde{\delta}-2/p)p/(p-1)}m(r)^{p/(p-1)}}\,\mathrm{d}r\right)^{(p-1)/p}$$
$$\leqslant C\|h(t)\|_{W^{\widetilde{\delta},p}}\left(\int_0^1\frac{r}{r^{(2-\widetilde{\delta}-2/p)p/(p-1)}m(r)^{p/(p-1)}}\,\mathrm{d}r\right)^{(p-1)/p}.$$

By means of (1.7), there exists $r_0\in(0,\,1)$ such that for any $r\in(0,\,,r_0]$

$$\frac{r^\delta}{m(r)}\leqslant\gamma+1\quad\text{or}\quad m(r)\geqslant\frac{r^\delta}{\gamma+1}.$$

Consequently, one has

$$I\leqslant C\|h(t)\|_{W^{\widetilde{\delta},p}}\left(\int_0^{r_0}\frac{r}{r^{(2-\widetilde{\delta}-2/p)p/(p-1)}m(r)^{p/(p-1)}}\,\mathrm{d}r\right.$$
$$\left.+\int_{r_0}^1\frac{r}{r^{(2-\widetilde{\delta}-2/p)p/(p-1)}m(r)^{p/(p-1)}}\,\mathrm{d}r\right)^{(p-1)/p}$$
$$\leqslant C\|h(t)\|_{W^{\widetilde{\delta},p}}\left(\int_0^{r_0}\frac{1}{r^{1-(\widetilde{\delta}-\delta)p/(p-1)}}\,\mathrm{d}r+\int_{r_0}^1\frac{1}{r^{1-p\widetilde{\delta}/(p-1)}m(r_0)^{p/(p-1)}}\,\mathrm{d}r\right)^{(p-1)/p}$$
$$\leqslant C\|h(t)\|_{W^{\widetilde{\delta},p}},$$

where in the last line we have applied the following bound

$$\int_0^{r_0}\frac{1}{r^{1-(\widetilde{\delta}-\delta)p/(p-1)}}\,\mathrm{d}y\leqslant C$$

due to $\widetilde{\delta}>\delta$. Using the Minkowski inequality again, we obtain by (1.8) that

$$J\leqslant\int_{|y|\geqslant1}\left(\int_{\mathbb{R}^2}\frac{|h(x)-h(x-y)|^p}{|y|^{2p}m^p(|y|)}\,\mathrm{d}x\right)^{1/p}\mathrm{d}y$$
$$=\int_{|y|\geqslant1}\left(\int_{\mathbb{R}^2}|h(x)-h(x-y)|^p\,\mathrm{d}x\right)^{1/p}\frac{1}{|y|^2m(|y|)}\,\mathrm{d}y$$
$$\leqslant C\int_{|y|\geqslant1}\|h\|_{L^p}\frac{1}{|y|^2m(|y|)}\,\mathrm{d}y$$
$$\leqslant C\|h\|_{L^p}\int_1^\infty\frac{1}{rm(r)}\,\mathrm{d}r$$
$$\leqslant C\|h\|_{L^p}$$
$$\leqslant C\|h(t)\|_{W^{\widetilde{\delta},p}}.$$

Combining the above two estimates yields (3.1). □

Lemma 3.2. *For the operator $\mathcal{L}$ given by* (1.6)*, it holds that for any $2 \leqslant p < \infty$*

$$\int_{\mathbb{R}^2} |h|^{p-2} h\mathcal{L}h \,\mathrm{d}x \geqslant 0. \tag{3.2}$$

Proof of lemma 3.2. Let us first show the case $p = 2$. It is easy to see that $\mathcal{L}h(x)$ can be rewritten as

$$\mathcal{L}h(x) = \mathrm{P.V.} \int_{\mathbb{R}^2} \frac{h(x) - h(y)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y.$$

Therefore, we get

$$\begin{aligned}\int_{\mathbb{R}^2} h\mathcal{L}h \,\mathrm{d}x &= \int_{\mathbb{R}^2} h(x)\mathrm{P.V.} \int_{\mathbb{R}^2} \frac{h(x) - h(y)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y \,\mathrm{d}x \\ &= \mathrm{P.V.} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{h^2(x) - h(x)h(y)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y \,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{h^2(x)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y \,\mathrm{d}x \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{2h(x)h(y)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y \,\mathrm{d}x.\end{aligned}$$

One may check that by exchanging x and y

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{h^2(x)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y \,\mathrm{d}x = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{h^2(y)}{|y-x|^2 m(|y-x|)} \,\mathrm{d}y \,\mathrm{d}x.$$

Consequently,

$$\int_{\mathbb{R}^2} h\mathcal{L}h \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(h(x) - h(y))^2}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y \,\mathrm{d}x \geqslant 0. \tag{3.3}$$

This is the desired (3.2) with the case $p = 2$. For the general case, we first verify that for any $q > 1$

$$|h(x)|^{q-2} h(x)\mathcal{L}h(x) \geqslant \frac{1}{q}\mathcal{L}(|h(x)|^q). \tag{3.4}$$

Now it is obvious that

$$|h(x)|^{q-2} h(x)\mathcal{L}h(x) = \mathrm{P.V.} \int_{\mathbb{R}^2} \frac{|h(x)|^q - |h(x)|^{q-2}h(x)h(y)}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y.$$

According to the Young inequality, it implies

$$|h(x)|^{q-2} h(x)h(y) \leqslant |h(x)|^{q-1}|h(y)| \leqslant \frac{q-1}{q}|h(x)|^q + \frac{1}{q}|h(y)|^q.$$

We therefore deduce

$$|h(x)|^{q-2} h(x)\mathcal{L}h(x) \geqslant \frac{1}{q}\mathrm{P.V.} \int_{\mathbb{R}^2} \frac{|h(x)|^q - |h(y)|^q}{|x-y|^2 m(|x-y|)} \,\mathrm{d}y = \frac{1}{q}\mathcal{L}(|h(x)|^q).$$

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Now for $2 < p < \infty$, we appeal to (3.4) to conclude

$$\begin{aligned}\int_{\mathbb{R}^2} |h|^{p-2} h\mathcal{L}h \,\mathrm{d}x &= \int_{\mathbb{R}^2} |h|^{p/2} \underbrace{|h|^{p/2-2} h\mathcal{L}h}\,\mathrm{d}x \\ &\geqslant \frac{2}{p}\int_{\mathbb{R}^2} |h|^{p/2}\mathcal{L}|h|^{p/2}\,\mathrm{d}x \\ &\geqslant 0,\end{aligned}$$

where in the last line we have applied (3.3). Therefore, the proof of lemma 3.2 is completed. □

With lemmas 3.1 and 3.2 in hand, it is not difficult to show the following lemma.

Lemma 3.3. *Let (u, τ) be the corresponding solution of (1.5), then it holds true*

$$\|u(t)\|_{L^2}^2 + \|\tau(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s \leqslant \|u_0\|_{L^2}^2 + \|\tau_0\|_{L^2}^2. \tag{3.5}$$

Proof of lemma 3.3. Multiplying $(1.5)_1$ by u and $(1.5)_2$ by τ and adding them up yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|\tau(t)\|_{L^2}^2) + \int_{\mathbb{R}^2} \mathcal{L}\tau : \tau \,\mathrm{d}x + \|\nabla u\|_{L^2}^2 = 0, \tag{3.6}$$

where we have used the following identity

$$\int_{\mathbb{R}^2} (\nabla \cdot \tau) \cdot u \,\mathrm{d}x + \int_{\mathbb{R}^2} \mathcal{D}u : \tau \,\mathrm{d}x = 0$$

and the following cancellation property (see (3.17) below)

$$\int_{\mathbb{R}^2} (\tau\Omega - \Omega\tau) : \tau \,\mathrm{d}x = 0.$$

By (3.2), we have

$$\int_{\mathbb{R}^2} \mathcal{L}\tau : \tau \,\mathrm{d}x \geqslant 0.$$

Therefore, the desired estimate (3.5) follows by integrating (3.6) in time. This ends the proof of the lemma. □

Next, we will establish the following lemma, which concerns the global H^1-estimate of u and the L^r-estimate of τ for any $r < \infty$.

Lemma 3.4. *Let (u, τ) be the corresponding solution of (1.5). Then, for any $2 < r < \infty$, (u, τ) obeys the global bound*

$$\|\nabla u(t)\|_{L^2} + \|\tau(t)\|_{L^r} \leqslant C, \tag{3.7}$$

where C is a constant depending on T, r and the initial data.

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Proof of lemma 3.4. In order to get the above estimate (3.7), we first apply operator curl to the equation $(1.5)_1$ to obtain the vorticity ω equation as follows

$$\partial_t\omega + (u\cdot\nabla)\omega - \Delta\omega = \mathrm{curl}\,\mathrm{div}\ (\tau).$$

We denote $\mathcal{R}$ as the singular integral operator

$$\mathcal{R} = -(-\Delta)^{-1}\mathrm{curl}\,\mathrm{div}.$$

Applying the operator $\mathcal{R}$ to equation $(1.5)_2$, one has

$$\partial_t\mathcal{R}\tau + (u\cdot\nabla)\mathcal{R}\tau = \mathcal{R}\mathcal{D}u - [\mathcal{R},\, u\cdot\nabla]\tau - \mathcal{R}(\tau\Omega - \Omega\tau) - \mathcal{R}\mathcal{L}\tau.$$

In this case, the combined quantity $G := \omega + \mathcal{R}\tau$ satisfies the equation

$$\partial_t G + (u\cdot\nabla)G - \Delta G = \mathcal{R}\mathcal{D}u - [\mathcal{R},\, u\cdot\nabla]\tau - \mathcal{R}(\tau\Omega - \Omega\tau) - \mathcal{R}\mathcal{L}\tau. \tag{3.8}$$

Multiplying (3.8) by G, we obtain, after integration by parts

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|G\|_{L^2}^2 + \|\nabla G\|_{L^2}^2 = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4, \tag{3.9}$$

where

$$\mathcal{N}_1 = \int_{\mathbb{R}^2} \mathcal{R}\mathcal{D}uG\,\mathrm{d}x, \quad \mathcal{N}_2 = -\int_{\mathbb{R}^2} [\mathcal{R},\, u\cdot\nabla]\tau G\,\mathrm{d}x,$$
$$\mathcal{N}_3 = -\int_{\mathbb{R}^2} \mathcal{R}(\tau\Omega - \Omega\tau)G\,\mathrm{d}x, \quad \mathcal{N}_4 = -\int_{\mathbb{R}^2} \mathcal{R}\mathcal{L}\tau G\,\mathrm{d}x.$$

The Young inequality implies

$$\mathcal{N}_1 \leqslant C\|\nabla u\|_{L^2}\|G\|_{L^2}. \tag{3.10}$$

Thanks to the following commutator estimate (see [**20**, theorem 3.3] for its proof)

$$\|[\mathcal{R},\, u]f\|_{H^\sigma} \leqslant C(\sigma)(\|\nabla u\|_{L^2}\|f\|_{B^{\sigma-1}_{\infty,r}} + \|u\|_{L^2}\|f\|_{L^2})$$

for any $\sigma \in (0,\, 1)$, one concludes

$$\begin{aligned}
\mathcal{N}_2 &= \int_{\mathbb{R}^2} \nabla\cdot[\mathcal{R},\, u]\tau G\,\mathrm{d}x \\
&\leqslant C\|[\mathcal{R},\, u]\tau\|_{\dot{H}^{(r-2)/2r}}\|G\|_{\dot{H}^{(r+2)/2r}} \\
&\leqslant C\|[\mathcal{R},\, u]\tau\|_{H^{(r-2)/2r}}\|G\|_{\dot{H}^{(r+2)/2r}} \\
&\leqslant C(\|\nabla u\|_{L^2}\|\tau\|_{B^{(-r-2)/2r}_{\infty,2}} + \|u\|_{L^2}\|\tau\|_{L^2})\|G\|_{L^2}^{(r-2)/2r}\|\nabla G\|_{L^2}^{(r+2)/2r} \\
&\leqslant C(\|\nabla u\|_{L^2}\|\tau\|_{L^r} + \|u\|_{L^2}\|\tau\|_{L^2})\|G\|_{L^2}^{(r-2)/2r}\|\nabla G\|_{L^2}^{(r+2)/2r} \\
&\leqslant \frac{1}{8}\|\nabla G\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{4r/(3r-2)}\|\tau\|_{L^r}^{4r/(3r-2)}\|G\|_{L^2}^{2(r-2)/(3r-2)} \\
&\quad + C(\|u\|_{L^2}\|\tau\|_{L^2})^{4r/(3r-2)}\|G\|_{L^2}^{2(r-2)/(3r-2)} \\
&\leqslant \frac{1}{8}\|\nabla G\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2)(\|\tau\|_{L^r}^2 + \|G\|_{L^2}^2),
\end{aligned} \tag{3.11}$$

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where we have used the following fact $L^r(\mathbb{R}^2) \hookrightarrow B_{\infty,2}^{(-r-2)/2r}(\mathbb{R}^2)$ with $2 \leqslant r < \infty$. Finally, by the Young inequality and the Gagliardo-Nirenberg inequality, one can check that

$$\begin{aligned}
\mathcal{N}_3 &\leqslant \|\mathcal{R}(\tau\Omega - \Omega\tau)\|_{L^{2r/(r+2)}} \|G\|_{L^{2r/(r-2)}} \\
&\leqslant C\|\Omega\|_{L^2}\|\tau\|_{L^r}\|G\|_{L^{2r/(r-2)}} \\
&\leqslant C\|\nabla u\|_{L^2}\|\tau\|_{L^r}\|G\|_{L^2}^{(r-2)/r}\|\nabla G\|_{L^2}^{2/r} \\
&\leqslant \frac{1}{4}\|\nabla G\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{r/(r-1)}(\|\tau\|_{L^r}\|G\|_{L^2}^{(r-2)/r})^{r/(r-1)} \\
&\leqslant \frac{1}{4}\|\nabla G\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2)(\|\tau\|_{L^r}^2 + \|G\|_{L^2}^2). \qquad (3.12)
\end{aligned}$$

Finally, by (3.1), the term $\mathcal{N}_4$ admits the bound

$$\begin{aligned}
\mathcal{N}_4 &= -\int_{\mathbb{R}^2} \tau\mathcal{L}\mathcal{R}G \,\mathrm{d}x \\
&\leqslant C\|\tau\|_{L^2}\|\mathcal{L}\mathcal{R}G\|_{L^2} \\
&\leqslant C\|\tau\|_{L^2}(\|\mathcal{R}G\|_{L^2} + \|\nabla\mathcal{R}G\|_{L^2}) \\
&\leqslant C\|\tau\|_{L^2}(\|G\|_{L^2} + \|\nabla G\|_{L^2}) \\
&\leqslant \frac{1}{16}\|\nabla G\|_{L^2}^2 + C(\|\tau\|_{L^2}^2 + \|G\|_{L^2}^2), \qquad (3.13)
\end{aligned}$$

where we have used the fact (see appendix for its proof)

$$\int_{\mathbb{R}^2} \mathcal{L}fg \,\mathrm{d}x = \int_{\mathbb{R}^2} f\mathcal{L}g \,\mathrm{d}x. \qquad (3.14)$$

Putting (3.10), (3.11), (3.12) and (3.13) into (3.9), and absorbing the dissipative term, it gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G\|_{L^2}^2 + \|\nabla G\|_{L^2}^2 \leqslant C(1+\|\nabla u\|_{L^2}^2)(\|\tau\|_{L^r}^2 + \|G\|_{L^2}^2) + C\|\tau\|_{L^2}^2. \qquad (3.15)$$

In order to close the above inequality (3.15), we need to establish the differential inequality of estimate of $\|\tau\|_{L^r}$. Now multiplying the stress tensor τ equation of (1.5) by $|\tau|^{r-2}\tau$ and integrating over $\mathbb{R}^2$, it follows that

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{L^r}^r + \int_{\mathbb{R}^2} |\tau|^{r-2}\tau : \mathcal{L}\tau \,\mathrm{d}x = \int_{\mathbb{R}^2} \mathcal{D}u\tau|\tau|^{r-2} \,\mathrm{d}x, \qquad (3.16)$$

where the following simple fact has been used

$$\int_{\mathbb{R}^2} (\tau\Omega - \Omega\tau) : |\tau|^{r-2}\tau \,\mathrm{d}x = 0$$

for any $r \geqslant 2$ due to the symmetry of τ. As a matter of fact, we have

$$\begin{aligned}\int_{\mathbb{R}^2} (\tau\Omega - \Omega\tau) : |\tau|^{r-2}\tau \,\mathrm{d}x &= \int_{\mathbb{R}^2} \tau_{ik}\Omega_{kj}|\tau|^{r-2}\tau_{ij} \,\mathrm{d}x - \int_{\mathbb{R}^2} \Omega_{kj}\tau_{ji}|\tau|^{r-2}\tau_{ki} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \tau_{ik}\Omega_{kj}|\tau|^{r-2}\tau_{ij} \,\mathrm{d}x - \int_{\mathbb{R}^2} \Omega_{kj}\tau_{ij}|\tau|^{r-2}\tau_{ik} \,\mathrm{d}x \\ &= 0, \end{aligned} \tag{3.17}$$

where we used $\tau_{ji} = \tau_{ij}$ and $\tau_{ki} = \tau_{ik}$ due to the symmetry of τ. Thanks to (3.2), one gets

$$\int_{\mathbb{R}^2} |\tau|^{r-2}\tau : \mathcal{L}\tau \,\mathrm{d}x \geqslant 0.$$

We thus deduce from (3.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{L^r}^2 \leqslant C\|\mathcal{D}u\|_{L^r}\|\tau\|_{L^r}.$$

Now it is easy to see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{L^r}^2 &\leqslant C\|\mathcal{D}u\|_{L^r}\|\tau\|_{L^r} \\ &\leqslant C\|\omega\|_{L^r}\|\tau\|_{L^r} \\ &\leqslant C(\|G\|_{L^r} + \|\tau\|_{L^r})\|\tau\|_{L^r} \\ &\leqslant C\|G\|_{L^2}^{2/r}\|\nabla G\|_{L^2}^{(r-2)/r}\|\tau\|_{L^r} + \|\tau\|_{L^r}^2 \\ &\leqslant \frac{1}{4}\|\nabla G\|_{L^2}^2 + C(\|G\|_{L^2}^2 + \|\tau\|_{L^r}^2). \end{aligned} \tag{3.18}$$

Adding up the above estimates (3.15) and (3.18) altogether, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|G\|_{L^2}^2 + \|\tau\|_{L^r}^2) + \|\nabla G\|_{L^2}^2 \leqslant C(1 + \|\nabla u\|_{L^2}^2)(\|\tau\|_{L^r}^2 + \|G\|_{L^2}^2).$$

The Gronwall inequality gives

$$\|G(t)\|_{L^2}^2 + \|\tau\|_{L^r}^2 + \int_0^t \|\nabla G(s)\|_{L^2}^2 \,\mathrm{d}s < \infty,$$

which also shows

$$\|\nabla u\|_{L^2} \leqslant \|\omega\|_{L^2} \leqslant \|G\|_{L^2} + \|\tau\|_{L^2} < \infty.$$

Therefore, this concludes the proof of lemma 3.4. □

We are now in the position to prove the following lemma.

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LEMMA 3.5. *Let (u,τ) be the corresponding solution of (1.5). Then, for any $2 \leqslant p$, $q < \infty$, (u, G) obeys the global bound*

$$\|\nabla u\|_{L^q(0,T;L^p)} \leqslant C, \tag{3.19}$$

$$\|G\|_{L^q(0,T;W^{1,p})} \leqslant C, \tag{3.20}$$

where C is a constant depending on T, p, q and the initial data. For any $0 \leqslant \gamma < 1$

$$\|G\|_{L^\infty(0,T;W^{\gamma,q})} \leqslant C, \tag{3.21}$$

where C is a constant depending on T, γ, q and the initial data.

Proof of lemma 3.5. To avoid the pressure term, we resort to the incompressible condition to deduce from the first equation of (1.5) that

$$-\pi = \frac{\mathrm{divdiv}}{-\Delta}(\tau - u \otimes u) := \widetilde{\mathcal{R}}(\tau - u \otimes u).$$

Hence, the first equation of (1.5) can be rewritten as

$$\partial_t u - \Delta u = (\mathrm{div} + \nabla\widetilde{\mathcal{R}})(\tau - u \otimes u).$$

Applying operator ∇ to above equality yields

$$\partial_t \nabla u - \Delta\nabla u = \nabla(\mathrm{div} + \nabla\widetilde{\mathcal{R}})(\tau - u \otimes u).$$

Owing to the Duhamel Principle, we have

$$\begin{aligned}\nabla u(x,t) = \mathrm{e}^{t\Delta}\nabla u_0(x) - \int_0^t \mathrm{e}^{(t-s)\Delta} \\ \times \Delta(-\Delta)^{-1}\nabla(\mathrm{div} + \nabla\widetilde{\mathcal{R}})(\tau - u\otimes u)(x,s)\,\mathrm{d}s.\end{aligned} \tag{3.22}$$

Note that the following estimate

$$\|u\|_{L^p} \leqslant \|u\|_{L^2}^{2/p}\|\nabla u\|_{L^2}^{1-2/p} \leqslant C, \quad \forall\, 2 < p < \infty,$$

which together with lemma 2.4 allows us to deduce from (3.22) that

$$\begin{aligned}\|\nabla u\|_{L^q(0,T;L^p)} &\leqslant \|e^{t\Delta}\nabla u_0\|_{L^q(0,T;L^p)} \\ &\quad + \left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\Delta(-\Delta)^{-1}\nabla(\mathrm{div} + \nabla\widetilde{\mathcal{R}})(\tau - u\otimes u)\,\mathrm{d}s\right\|_{L^q(0,T;L^p)} \\ &\leqslant C\|H(t,x)\|_{L^q(0,T;L^1)}\|\nabla u_0\|_{L^p_x} \\ &\quad + C\|(-\Delta)^{-1}\nabla(\mathrm{div} + \nabla\widetilde{\mathcal{R}})(\tau - u\otimes u)\|_{L^q(0,T;L^p)} \\ &\leqslant C\|\nabla u_0\|_{L^p} + C\|\tau - u\otimes u\|_{L^q(0,T;L^p)} \\ &\leqslant C + CT^{1/q}\left(\|\tau\|_{L^\infty(0,T;L^p)} + \|u\|^2_{L^\infty(0,T;L^{2p})}\right) \\ &\leqslant C,\end{aligned}$$

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which is (3.19). Now making use of the incompressible condition and applying operator ∇, we deduce from (3.8) that

$$\partial_t \nabla G - \Delta \nabla G = \nabla(\mathcal{R}\mathcal{D}u - \nabla\cdot[\mathcal{R},\, u]\tau - \nabla\cdot(uG) - \mathcal{R}(\tau\Omega - \Omega\tau)) - \nabla\mathcal{R}\mathcal{L}\tau.$$

Again using the Duhamel Principle, we can rewrite ∇G as follows

$$\begin{aligned}\nabla G(x,t) &= e^{t\Delta}\nabla G_0(x)\\ &\quad - \int_0^t \mathrm{e}^{(t-s)\Delta}\nabla(\mathcal{R}\mathcal{D}u - \nabla\cdot[\mathcal{R},\, u]\tau - \nabla\cdot(uG) - \mathcal{R}(\tau\Omega-\Omega\tau))(x,s)\,\mathrm{d}s\\ &\quad - \int_0^t \mathrm{e}^{(t-s)\Delta}\nabla\mathcal{R}\mathcal{L}\tau(x,s)\,\mathrm{d}s := J_1 + J_2 + J_3.\end{aligned}$$

We resort to lemma 2.4 again to obtain that

$$\begin{aligned}&\|J_1\|_{L^q(0,T;L^p)} + \|J_2\|_{L^q(0,T;L^p)}\\ &\quad\leqslant \|\mathrm{e}^{t\Delta}\nabla G_0\|_{L^q(0,T;L^p)} + \left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla\nabla\cdot[\mathcal{R},\, u]\tau\,\mathrm{d}s\right\|_{L^q(0,T;L^p)}\\ &\qquad + \left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla\mathcal{R}\mathcal{D}u\,\mathrm{d}s\right\|_{L^q(0,T;L^p)} + \left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla\nabla\cdot(uG)\,\mathrm{d}s\right\|_{L^q(0,T;L^p)}\\ &\qquad + \left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla\mathcal{R}(\tau\Omega-\Omega\tau)\,\mathrm{d}s\right\|_{L^q(0,T;L^p)}\\ &\quad\leqslant C\|H(t,x)\|_{L^q(0,T;L^1)}\|\nabla G_0\|_{L^p} + C\,\|[\mathcal{R},\, u]\tau\|_{L^q(0,T;L^p)}\\ &\qquad + C\|u\|_{L^q(0,T;L^p)} + C\|uG\|_{L^q(0,T;L^p)} + C\|\Lambda^{-1}(\tau\Omega-\Omega\tau)\|_{L^q(0,T;L^p)}\\ &\quad\leqslant C + CT^{1/q}\|[\mathcal{R},\, u]\tau\|_{L^\infty(0,T;L^p)} + C\|u\|_{L^\infty(0,T;L^{2p})}\|G\|_{L^q(0,T;L^{2p})}\\ &\qquad + CT^{1/q}\|u\|_{L^\infty(0,T;L^p)} + C\|\tau\Omega\|_{L^q(0,T;L^{2p/(p+2)})}\\ &\quad\leqslant C + CT^{1/q}\|[\mathcal{R},\, u]\tau\|_{L^\infty(0,T;L^p)} + C\|u\|_{L^\infty(0,T;L^{2p})}\|G\|_{L^q(0,T;L^{2p})}\\ &\qquad + CT^{1/q}\|u\|_{L^\infty(0,T;L^p)} + C\|\Omega\|_{L^\infty(0,T;L^2)}\|\tau\|_{L^q(0,T;L^p)}\\ &\quad\leqslant C,\end{aligned}$$

where we have used the following estimate

$$\|[\mathcal{R},\, u]\tau\|_{L^p} \leqslant C\|u\|_{\mathrm{BMO}}\|\tau\|_{L^p} \leqslant C\|\nabla u\|_{L^2}\|\tau\|_{L^p} \leqslant C.$$

One deduces by using (3.1) that

$$\begin{aligned}\|J_3\|_{L^q(0,T;L^p)} &= \left\|\int_0^t e^{(t-s)\Delta}\nabla\mathcal{R}\Lambda^{\widetilde{\delta}}\mathcal{L}\Lambda^{-\widetilde{\delta}}\tau(x,s)\,\mathrm{d}s\right\|_{L^q(0,T;L^p)}\\ &\leqslant C\|\Lambda^{\widetilde{\delta}-1}\mathcal{L}\Lambda^{-\widetilde{\delta}}\tau(x,s)\|_{L^q(0,T;L^p)}\end{aligned}$$

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$$
\begin{aligned}
&\leqslant C\|\mathcal{L}\Lambda^{-\widetilde{\delta}}\tau(x,s)\|_{L^q(0,T;L^{\Upsilon})}\\
&\leqslant C\|\Lambda^{-\widetilde{\delta}}\tau(x,s)\|_{L^q(0,T;W^{\widetilde{\delta},\Upsilon})}\\
&\leqslant C\|\tau(x,s)\|_{L^q(0,T;L^{\Upsilon})}+C\|\tau(x,s)\|_{L^q(0,T;L^2)}\\
&\leqslant C,
\end{aligned}
$$

where $\Upsilon = 2p/2 + (1-\widetilde{\delta})p$. We point out that here and in what follows, we use the fact (see appendix for its proof)

$$
\Lambda^{-\delta}\mathcal{L}f(x) = \mathcal{L}\Lambda^{-\delta}f(x), \quad \delta \in (0,\, 2). \tag{3.23}
$$

We thus obtain

$$
\|\nabla G\|_{L^q(0,T;L^p)} \leqslant C,
$$

which along with the simple interpolation inequality implies (3.20). Applying Λ^{γ} to the equation (3.8) and using the incompressible condition yield

$$
\begin{aligned}
\partial_t\Lambda^{\gamma}G - \Delta\Lambda^{\gamma}G = \Lambda^{\gamma}\mathcal{R}\mathcal{D}u - \Lambda^{\gamma}\nabla\cdot[\mathcal{R},\, u]\tau - \Lambda^{\gamma}(u\cdot\nabla G)\\
- \Lambda^{\gamma}\mathcal{R}(\tau\Omega - \Omega\tau) - \Lambda^{\gamma}\mathcal{R}\mathcal{L}\tau.
\end{aligned}
$$

The Duhamel Principle entails

$$
\begin{aligned}
\Lambda^{\gamma}G(x,t) = \mathrm{e}^{t\Delta}\Lambda^{\gamma}G_0(x) - \int_0^t e^{(t-s)\Delta}(\Lambda^{\gamma}\nabla\cdot[\mathcal{R},\, u]\tau\\
- \Lambda^{\gamma}\mathcal{R}\mathcal{D}u + \Lambda^{\gamma}(u\cdot\nabla G))(s)\,\mathrm{d}s\\
- \int_0^t \mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}\mathcal{R}(\tau\Omega - \Omega\tau)(s)\,\mathrm{d}s - \int_0^t \mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}\mathcal{R}\mathcal{L}\tau(s)\,\mathrm{d}s.
\end{aligned}
$$

Now we have that

$$
\|\Lambda^{\gamma}G(t)\|_{L^q} \leqslant K_1 + K_2,
$$

where

$$
\begin{aligned}
K_1 &= C\|\Lambda^{\gamma}G_0\|_{L^q} + C\int_0^t \|\mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}\nabla\cdot[\mathcal{R},\, u]\tau\|_{L^q}\,\mathrm{d}s\\
&\quad + C\int_0^t \|\mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}\mathcal{R}\mathcal{D}u\|_{L^q}\,\mathrm{d}s + C\int_0^t \|\mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}(u\cdot\nabla G)\|_{L^q}\,\mathrm{d}s\\
&\quad + C\int_0^t \|\mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}\mathcal{R}(\tau\Omega - \Omega\tau)\|_{L^q}\,\mathrm{d}s,\\
K_2 &= C\int_0^t \|\mathrm{e}^{(t-s)\Delta}\Lambda^{\gamma}\mathcal{R}\mathcal{L}\tau(s)\|_{L^q}\,\mathrm{d}s.
\end{aligned}
$$

By means of lemma 2.3, it is not hard to check that

$$
\begin{aligned}
K_1 &\leqslant C\|\Lambda^\gamma G_0\|_{L^q} + C\int_0^t (t-s)^{-(\gamma+1)/2}\|[\mathcal{R},\, u]\tau\|_{L^q}\,\mathrm{d}s \\
&\quad + C\int_0^t (t-s)^{-\gamma/2}\|\nabla u\|_{L^q}\,\mathrm{d}s + C\int_0^t (t-s)^{-\gamma/2}\|(u\cdot\nabla G)\|_{L^q}\,\mathrm{d}s \\
&\quad + C\int_0^t (t-s)^{-\gamma/2}\|\tau\Omega\|_{L^q}\,\mathrm{d}s \\
&\leqslant C + C\int_0^t s^{-(\gamma+1)/2}\,\mathrm{d}s + C\|\nabla u\|_{L_t^{2/\gamma}L_x^q}\left(\int_0^t s^{-\gamma/(2-\gamma)}\,\mathrm{d}s\right)^{(2-\gamma)/2} \\
&\quad + C\int_0^t (t-s)^{-\gamma/2}\|u\|_{L^{2q}}\|\nabla G\|_{L^{2q}}\,\mathrm{d}s + C\int_0^t (t-s)^{-\gamma/2}\|\tau\|_{L^{2q}}\|\Omega\|_{L^{2q}}\,\mathrm{d}s \\
&\leqslant C + Ct^{(1-\gamma)/2} + C\|\nabla u\|_{L_t^{2/\gamma}L_x^q}t^{1-\gamma} \\
&\quad + C\|\tau\|_{L_t^\infty L_x^{2q}}\|\Omega\|_{L_t^{2/\gamma}L_x^{2q}}\left(\int_0^t s^{-\gamma/(2-\gamma)}\,\mathrm{d}s\right)^{(2-\gamma)/2} \\
&\quad + C\|u\|_{L_t^\infty L_x^2}^{1/q}\|\omega\|_{L_t^\infty L_x^2}^{(q-1)/q}\int_0^t (t-s)^{-\gamma/2}\|\nabla G\|_{L^{2q}}\,\mathrm{d}s \\
&\leqslant C + Ct^{(1-\gamma)/2} + C\|\nabla u\|_{L_t^{2/\gamma}L_x^q}t^{1-\gamma} + C\|\tau\|_{L_t^\infty L_x^{2q}}\|\nabla u\|_{L_t^{2/\gamma}L_x^{2q}}t^{1-\gamma} \\
&\quad + C\|u\|_{L_t^\infty L_x^2}^{1/q}\|\omega\|_{L_t^\infty L_x^2}^{(q-1)/q}\|\nabla G\|_{L_t^{2/\gamma}L_x^{2q}}t^{1-\gamma} \\
&\leqslant C(1+t)^{1-\gamma}.
\end{aligned}
$$

According to (3.1) and (3.23), it follows that

$$
\begin{aligned}
K_2 &\leqslant C\int_0^t \|\mathrm{e}^{(t-s)\Delta}\Lambda^\gamma\mathcal{R}\Lambda^{\widetilde{\delta}}\mathcal{L}\Lambda^{-\widetilde{\delta}}\tau(s)\|_{L^q}\,\mathrm{d}s \\
&\leqslant C\int_0^t (t-s)^{-(\gamma+\widetilde{\delta})/2}\|\mathcal{L}\Lambda^{-\widetilde{\delta}}\tau\|_{L^q}\,\mathrm{d}s \\
&\leqslant C + C\int_0^t (t-s)^{-(\gamma+\widetilde{\delta})/2}(\|\tau(s)\|_{L^q} + \|\tau(s)\|_{L^2})\,\mathrm{d}s \\
&\leqslant C(\|\tau\|_{L^\infty(0,T;L^q)} + \|\tau\|_{L^\infty(0,T;L^2)})\int_0^t s^{-(\gamma+\widetilde{\delta})/2}\,\mathrm{d}s \\
&\leqslant C.
\end{aligned}
$$

Therefore, we obtain the desired estimate (3.21). We thus complete the proof of the lemma. □

Next we will derive the bound for $\|\omega\|_{L^\infty}$ which will play a significant role in obtaining the global H^s-bound. More precisely, it reads as follows.

LEMMA 3.6. *Let (u, τ) be the corresponding solution of (1.5), then it holds*

$$\max_{0\leqslant t\leqslant T} \|\tau(t)\|_{L^\infty} \leqslant C, \tag{3.24}$$

where C is a constant depending on T and the initial data.

Proof of lemma 3.6. To begin with, it is easy to see that

$$\begin{aligned} u &= -\nabla^\perp(-\Delta)^{-1}\omega \\ &= -\nabla^\perp(-\Delta)^{-1}(G + (-\Delta)^{-1}\mathrm{curl}\,\mathrm{div}\tau) \\ &= -\nabla^\perp(-\Delta)^{-1}G - \nabla^\perp\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau, \end{aligned} \tag{3.25}$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})^{\mathrm{T}}$. Consequently, we have

$$\partial_t\tau + (u\cdot\nabla)\tau + \mathcal{L}\tau + \tau\Omega - \Omega\tau = -\mathcal{D}\nabla^\perp(-\Delta)^{-1}G - \mathcal{D}\nabla^\perp\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau. \tag{3.26}$$

Multiplying (3.26) by $\tau(x, t)$ implies

$$\begin{aligned} \frac{1}{2}(\partial_t + u\cdot\nabla)|\tau(x,t)|^2 + \tau(x,t)\mathcal{L}\tau(x,t) &= -\tau(x,t)\mathcal{D}\nabla^\perp(-\Delta)^{-1}G \\ &\quad - \tau(x,t)\mathcal{D}\nabla^\perp\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau, \end{aligned}$$

where we have used the following identity

$$\begin{aligned} (\tau\Omega - \Omega\tau) : |\tau|^{r-2}\tau &= \tau_{ik}\Omega_{kj}|\tau|^{r-2}\tau_{ij} - \Omega_{kj}\tau_{ji}|\tau|^{r-2}\tau_{ki} \\ &= \tau_{ik}\Omega_{kj}|\tau|^{r-2}\tau_{ij} - \Omega_{kj}\tau_{ij}|\tau|^{r-2}\tau_{ik} \\ &= 0. \end{aligned}$$

Notice the identity

$$\tau(x,t)\mathcal{L}\tau(x,t) = \frac{1}{2}\mathcal{L}(|\tau(x,t)|^2) + \frac{1}{2}D(x,t),$$

where

$$D(x,t) = \mathrm{P.V.}\int_{\mathbb{R}^2} \frac{(\tau(x,t) - \tau(x-y,t))^2}{|y|^2 m(|y|)}\,\mathrm{d}y.$$

This identity can be deduced by

$$\begin{aligned}\tau(x,t)\mathcal{L}\tau(x,t) &= \tau(x,t)\text{P.V.}\int_{\mathbb{R}^2}\frac{\tau(x,t)-\tau(x-y,t)}{|y|^2m(|y|)}\,\mathrm{d}y\\ &= \text{P.V.}\int_{\mathbb{R}^2}\frac{|\tau(x,t)|^2-\tau(x,t)\tau(x-y,t)}{|y|^2m(|y|)}\,\mathrm{d}y\\ &= \frac{1}{2}\text{P.V.}\int_{\mathbb{R}^2}\frac{|\tau(x,t)|^2-|\tau(x-y,t)|^2}{|y|^2m(|y|)}\,\mathrm{d}y\\ &\quad+\frac{1}{2}\text{P.V.}\int_{\mathbb{R}^2}\frac{|\tau(x,t)|^2-2\tau(x,t)\tau(x-y,t)+|\tau(x-y,t)|^2}{|y|^2m(|y|)}\,\mathrm{d}y\\ &= \frac{1}{2}\mathcal{L}(|\tau(x,t)|^2)+\frac{1}{2}\text{P.V.}\int_{\mathbb{R}^2}\frac{(\tau(x,t)-\tau(x-y,t))^2}{|y|^2m(|y|)}\,\mathrm{d}y\\ &= \frac{1}{2}\mathcal{L}(|\tau(x,t)|^2)+\frac{1}{2}D(x,t).\end{aligned}$$

We thus deduce

$$\begin{aligned}\frac{1}{2}(\partial_t+u\cdot\nabla+\mathcal{L})|\tau(x,t)|^2++\frac{1}{2}D(x,t) &= -\tau(x,t)\mathcal{D}\nabla^{\perp}(-\Delta)^{-1}G\\ &\quad-\tau(x,t)\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau.\end{aligned}$$

Thanks to (3.21), it gives that for $p > 2/\gamma$

$$\|\mathcal{D}\nabla^{\perp}(-\Delta)^{-1}G\|_{L^\infty} \leqslant C(\|G\|_{L^2}+\|\Lambda^\gamma G\|_{L^p}) \leqslant C.$$

This leads to

$$|-\tau(x,t)\mathcal{D}\nabla^{\perp}(-\Delta)^{-1}G| \leqslant C|\tau(x,t)|.$$

Noticing the fundamental solutions for biharmonic equation (see, e.g., [25])

$$\frac{r^2}{8\pi}\ln r,\qquad r=|x-y|,$$

it gives that

$$(-\Delta)^{-2}\tau(x) = \frac{1}{8\pi}\int_{\mathbb{R}^2}|y|^2\ln|y|\tau(x-y)\,\mathrm{d}y.$$

Let $\chi(x)\in C_c^\infty(\mathbb{R}^2)$ be the standard smooth cut-off function satisfying

$$\chi(x)=\begin{cases}1, & |x|\leqslant \frac{1}{2},\\ 0, & |x|>1.\end{cases}$$

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Therefore, it gives

$$
\begin{aligned}
&|\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau(x)|\\
&= \frac{1}{8\pi}\left|\int_{\mathbb{R}^2}|y|^2\ln|y|(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y\tau(x-y)\,\mathrm{d}y\right|\\
&= \frac{1}{8\pi}\left|\int_{\mathbb{R}^2}|y|^2\ln|y|(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y(\tau(x-y)-\tau(x))\,\mathrm{d}y\right|\\
&\leqslant \frac{1}{8\pi}\left|\int_{|y|\leqslant 1}\chi^4(y)|y|^2\ln|y|(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y(\tau(x-y)-\tau(x))\,\mathrm{d}y\right|\\
&\quad + \frac{1}{8\pi}\left|\int_{|y|\geqslant \frac{1}{2}}(1-\chi^4(y))|y|^2\ln|y|(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y\tau(x-y)\,\mathrm{d}y\right|\\
&= \frac{1}{8\pi}\left|\int_{|y|\leqslant 1}(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y(\chi^4(y)|y|^2\ln|y|)(\tau(x-y)-\tau(x))\,\mathrm{d}y\right|\\
&\quad + \frac{1}{8\pi}\left|\int_{|y|\geqslant 1/2}(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y((1-\chi^4(y))|y|^2\ln|y|)\tau(x-y)\,\mathrm{d}y\right|\\
&:= H_1 + H_2.
\end{aligned}
$$

Direct computations yield

$$
\begin{aligned}
\partial_{y_l}(|y|^2\ln|y|) &= y_l(2\ln|y|+1),\\
\partial_{y_k}\partial_{y_l}(|y|^2\ln|y|) &= \delta_{kl}(2\ln|y|+1) + 2\frac{y_k y_l}{|y|^2},\\
\partial_{y_j}\partial_{y_k}\partial_{y_l}(|y|^2\ln|y|) &= 2\frac{\delta_{kl}y_j+\delta_{jk}y_l+\delta_{jl}y_k}{|y|^2} - 4\frac{y_j y_k y_l}{|y|^4},\\
\partial_{y_i}\partial_{y_j}\partial_{y_k}\partial_{y_l}(|y|^2\ln|y|) &= 2\frac{\delta_{ij}\delta_{kl}+\delta_{kj}\delta_{il}+\delta_{jl}\delta_{ki}}{|y|^2} + 16\frac{y_i y_j y_k y_l}{|y|^6}\\
&\quad - 4\frac{\begin{array}{c}\delta_{kl}y_i y_j+\delta_{kj}y_i y_l+\delta_{jl}y_k y_i+\delta_{ik}y_l y_j\\ +\delta_{il}y_k y_j+\delta_{ij}y_k y_l\end{array}}{|y|^4},
\end{aligned}
$$

where

$$
\delta_{ij} = \begin{cases} 1, & i=j;\\ 0, & i\neq j.\end{cases}
$$

As a result, it is not hard to check for $|y|\leqslant 1$ that

$$
\begin{aligned}
&|\partial_{y_l}(|y|^2\ln|y|)| + |\partial_{y_k}\partial_{y_l}(|y|^2\ln|y|)| + |\partial_{y_j}\partial_{y_k}\partial_{y_l}(|y|^2\ln|y|)|\\
&\quad + |\partial_{y_i}\partial_{y_j}\partial_{y_k}\partial_{y_l}(|y|^2\ln|y|)|\\
&\qquad \leqslant \frac{C}{|y|^2},
\end{aligned}
$$

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which implies

$$H_1 \leqslant C \int_{|y|\leqslant 1} \frac{|\tau(x-y)-\tau(x)|}{|y|^2}\,\mathrm{d}y.$$

For the term H_2, we have

$$\begin{aligned}
H_2 &= \frac{1}{8\pi}\left|\int_{|y|\geqslant 1/2}(1-\chi^4(y))(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y(|y|^2\ln|y|)\tau(x-y)\,\mathrm{d}y\right| \\
&\quad + \frac{1}{8\pi}\left|\int_{|y|\geqslant 1/2}\sum_{m=1}^{4} C_4^m\nabla_y^m(1-\chi^4(y))\nabla_y^{4-m}(|y|^2\ln|y|)\tau(x-y)\,\mathrm{d}y\right| \\
&= \frac{1}{8\pi}\left|\int_{|y|\geqslant 1/2}(1-\chi^4(y))(\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div})_y(|y|^2\ln|y|)\tau(x-y)\,\mathrm{d}y\right| \\
&\quad + \frac{1}{8\pi}\left|\int_{1/2\leqslant|y|\leqslant 1}\sum_{m=1}^{4} C_4^m\nabla_y^m(1-\chi^4(y))\nabla_y^{4-m}(|y|^2\ln|y|)\tau(x-y)\,\mathrm{d}y\right| \\
&\leqslant C\int_{|y|\geqslant 1/2}\frac{|\tau(x-y)|}{|y|^2}\,\mathrm{d}y \\
&\quad + C\int_{1/2\leqslant|y|\leqslant 1}\left(|y|^2\ln|y| + |y|\ln|y| + |y| + \ln|y| + 1 + \frac{1}{|y|}\right)|\tau(x-y)|\,\mathrm{d}y \\
&\leqslant C\int_{|y|\geqslant 1/2}\frac{|\tau(x-y)|}{|y|^2}\,\mathrm{d}y + C\int_{1/2\leqslant|y|\leqslant 1}\frac{|\tau(x-y)|}{|y|^2}\,\mathrm{d}y \\
&\leqslant C\int_{|y|\geqslant 1/2}\frac{|\tau(x-y)|}{|y|^2}\,\mathrm{d}y.
\end{aligned}$$

Consequently, keeping in mind (1.9), we infer that

$$\begin{aligned}
&|\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau(x)| \\
&\quad \leqslant C\int_{|y|\leqslant 1}\frac{|\tau(x-y)-\tau(x)|}{|y|^2}\,\mathrm{d}y + C\int_{|y|\geqslant 1/2}|\tau(x-y)|/|y|^2\,\mathrm{d}y \\
&\quad \leqslant C\int_{|y|\leqslant 1}\frac{|\tau(x-y)-\tau(x)|}{|y|\sqrt{m(|y|)}}\,\frac{\sqrt{m(|y|)}}{|y|}\,\mathrm{d}y \\
&\qquad + C\|\tau\|_{L^2}\left(\int_{|y|\geqslant 1/2}\frac{1}{|y|^4}\,\mathrm{d}y\right)^{1/2} \\
&\quad \leqslant C\sqrt{D(x,t)}\left(\int_{|y|\leqslant 1}\frac{m(|y|)}{|y|^2}\,\mathrm{d}y\right)^{1/2} + C\|\tau\|_{L^2} \\
&\quad \leqslant C\sqrt{D(x,t)}\left(\int_0^1\frac{m(r)}{r}\,\mathrm{d}r\right)^{1/2} + C\|\tau\|_{L^2} \\
&\quad \leqslant C\sqrt{D(x,t)} + C.
\end{aligned}$$

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As a result, one obtains

$$|-\tau(x,t)\mathcal{D}\nabla^{\perp}\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau| \leqslant \frac{1}{4}D(x,t)+C(|\tau(x,t)|+|\tau(x,t)|^2).$$

The above estimates allow us to show

$$\frac{1}{2}(\partial_t+u\cdot\nabla+\mathcal{L})|\tau(x,t)|^2+\frac{1}{4}D(x,t)\leqslant C_1(|\tau(x,t)|+|\tau(x,t)|^2).$$

Resorting to [**12**, (5.18)], one has

$$D(x,t)\geqslant \frac{C_2}{m(1)}|\tau(x,t)|^2\ln\frac{1}{\rho}-\frac{C_3|\tau(x,t)|}{\rho m(\rho)},$$

where $\rho\in(0,\,1)$. We choose $\rho\in(0,\,1)$ such that

$$\frac{C_2}{16m(1)}\ln\frac{1}{\rho}\geqslant C_1,$$

which further implies

$$(\partial_t+u\cdot\nabla+\mathcal{L})|\tau(x,t)|^2+C_3|\tau(x,t)|^2\leqslant C_4|\tau(x,t)|. \tag{3.27}$$

Let $\varphi(r)$ be a non-decreasing positive convex smooth function which is identically 0 on $0\leqslant r\leqslant \max\{\|\tau_0\|_{L^\infty}^2,\,(C_4/C_3)^2\}$, and positive otherwise. Multiplying (3.27) by $\varphi'(|\tau(x,t)|^2)$ and applying the lower bound

$$\varphi'(f)\mathcal{L}f\geqslant\mathcal{L}(\varphi(f)),$$

which can be deduced from [**9**, proposition 6.3] due to the convexity of function φ. Therefore, we obtain

$$(\partial_t+u\cdot\nabla+\mathcal{L})\varphi(|\tau(x,t)|^2)\leqslant 0. \tag{3.28}$$

Keeping in mind (3.2), namely,

$$\int_{\mathbb{R}^2}|h(x)|^{p-2}h(x)\mathcal{L}h(x)\,\mathrm{d}x\geqslant 0,$$

it follows from (3.28) that

$$\|\varphi(|\tau(x,t)|^2)\|_{L^\infty}\leqslant\|\varphi(|\tau_0|^2)\|_{L^\infty}=0.$$

This implies

$$\|\tau(x,t)\|_{L^\infty}\leqslant\max\left\{\|\tau_0\|_{L^\infty},\,\frac{C_4}{C_3}\right\},$$

which is the desired estimate (3.24). We therefore conclude the proof of lemma 3.6. □

Finally we are ready to prove the global H^s-estimate.

Proof of the global $\boldsymbol{H^s}$-estimate. Applying Λ^s to the system (1.5), taking the L^2 inner product with $\Lambda^s u$ and $\Lambda^s \tau$ respectively, it yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s \tau(t)\|_{L^2}^2) + \|\Lambda^{s+1}u\|_{L^2}^2 \\
&\quad \leqslant \int_{\mathbb{R}^2} \Lambda^s \nabla\cdot\tau\cdot\Lambda^s u\,\mathrm{d}x + \int_{\mathbb{R}^2}\Lambda^s \mathcal{D}u : \Lambda^s\tau\,\mathrm{d}x + \int_{\mathbb{R}^2}[\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\tau) : \Lambda^s\tau\,\mathrm{d}x + \int_{\mathbb{R}^2}\Lambda^s(\tau\Omega - \Omega\tau) : \Lambda^s\tau\,\mathrm{d}x \\
&:= H_1 + H_2 + H_3 + H_4 + H_5,
\end{aligned}$$

where we have used the following fact

$$\int_{\mathbb{R}^2}\Lambda^s\tau\mathcal{L}\Lambda^s\tau\,\mathrm{d}x \geqslant 0.$$

Integrating by parts and using the Young inequality, we deduce

$$H_1 + H_2 \leqslant \|\Lambda^{s+1}u(t)\|_{L^2}\|\Lambda^s\tau\|_{L^2} \leqslant \frac{1}{16}\|\Lambda^{s+1}u\|_{L^2}^2 + C\|\Lambda^s\tau\|_{L^2}^2.$$

Now we recall the following classical estimates (see [**26**, **27**])

$$\|[\Lambda^s, f]g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \tag{3.29}$$

$$\|\Lambda^s(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}) \tag{3.30}$$

with $s > 0$, $p_2, p_3 \in (1,\infty)$ such that $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. In some context, we also need the following variant version of (3.29), whose proof is the same one as for (3.29)

$$\|[\Lambda^{s-1}\partial_i, f]g\|_{L^r} \leqslant C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}). \tag{3.31}$$

It follows from (3.29) that

$$\begin{aligned}
H_3 &\leqslant C\|[\Lambda^s, u\cdot\nabla]u\|_{L^{4/3}}\|\Lambda^s u\|_{L^4} \\
&\leqslant C\|\nabla u\|_{L^2}\|\Lambda^s u\|_{L^4}^2 \\
&\leqslant C\|\nabla u\|_{L^2}\|\Lambda^s u\|_{L^2}\|\Lambda^{s+1}u\|_{L^2} \\
&\leqslant \frac{1}{16}\|\Lambda^{s+1}u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\Lambda^s u\|_{L^2}^2.
\end{aligned}$$

Invoking the divergence free property, (3.31) and the Young inequality, we obtain

$$
\begin{aligned}
H_4 &= \int_{\mathbb{R}^2} \Lambda^s \partial_k (u_k \tau_{i,j}) \Lambda^s \tau_{i,j}\, \mathrm{d}x \\
&= \int_{\mathbb{R}^2} [\Lambda^s \partial_k,\, u_k] \tau_{i,j} \Lambda^s \tau_{i,j}\, \mathrm{d}x \\
&\leqslant C \|[\Lambda^s \partial_k,\, u_k]\tau_{i,j}\|_{L^2} \|\Lambda^s \tau\|_{L^2} \\
&\leqslant C(\|\tau\|_{L^\infty} \|\Lambda^{s+1} u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^s \tau\|_{L^2}) \|\Lambda^s \tau\|_{L^2} \\
&\leqslant \frac{1}{16} \|\Lambda^{s+1} u\|_{L^2}^2 + C(\|\tau\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}) \|\Lambda^s \tau\|_{L^2}^2.
\end{aligned}
$$

By (3.30), one gets

$$
\begin{aligned}
H_5 &\leqslant \|\Lambda^s(\tau\Omega - \Omega\tau)\|_{L^2} \|\Lambda^s \tau\|_{L^2} \\
&\leqslant C(\|\tau\|_{L^\infty} \|\Lambda^s \Omega\|_{L^2} + \|\Omega\|_{L^\infty} \|\Lambda^s \tau\|_{L^2}) \|\Lambda^s \tau\|_{L^2} \\
&\leqslant C(\|\tau\|_{L^\infty} \|\Lambda^{s+1} u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^s \tau\|_{L^2}) \|\Lambda^s \tau\|_{L^2} \\
&\leqslant \frac{1}{16} \|\Lambda^{s+1} u\|_{L^2}^2 + C(\|\tau\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}) \|\Lambda^s \tau\|_{L^2}^2).
\end{aligned}
$$

Combining all the above estimates yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s \tau(t)\|_{L^2}^2) + \|\Lambda^{s+1} u\|_{L^2}^2 \\
&\quad \leqslant C(1 + \|\nabla u\|_{L^\infty} + \|\tau\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s \tau\|_{L^2}^2).
\end{aligned}
$$

Recalling (3.25), it leads to

$$
\nabla u = -\nabla\nabla^\perp(-\Delta)^{-1} G - \nabla\nabla^\perp \mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau.
$$

Now we may deduce that

$$
\|\nabla u\|_{L^\infty} \leqslant C(1 + \|\tau\|_{L^\infty} \ln(e + \|\Lambda^s \tau\|_{L^2})). \tag{3.32}
$$

As a matter of fact, (3.32) can be deduced by using (3.21) and the high-low frequency technique

$$
\begin{aligned}
\|\nabla u\|_{L^\infty} &\leqslant \|\Delta_{-1} \nabla u\|_{L^\infty} + \sum_{j=0}^{N} \|\Delta_j \nabla u\|_{L^\infty} + \sum_{j=N+1}^{\infty} \|\Delta_j \nabla u\|_{L^\infty} \\
&\leqslant C\|u\|_{L^2} + \sum_{j=0}^{N} \|\Delta_j \nabla\nabla^\perp(-\Delta)^{-1} G\|_{L^\infty} \\
&\quad + \sum_{j=0}^{N} \|\Delta_j \nabla\nabla^\perp \mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau\|_{L^\infty}
\end{aligned}
$$

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$$
\begin{aligned}
&+\sum_{j=N+1}^{\infty}\|\Delta_j\nabla\nabla^{\perp}(-\Delta)^{-1}G\|_{L^\infty}\\
&+\sum_{j=N+1}^{\infty}\|\Delta_j\nabla\nabla^{\perp}\mathrm{curl}\,\mathrm{div}(-\Delta)^{-2}\tau\|_{L^\infty}\\
\leqslant{}& C+C\sum_{j=0}^{\infty}2^{j(2/p-\gamma)}\|\Delta_j\Lambda^{\gamma}G\|_{L^p}+C\sum_{j=0}^{N}\|\Delta_j\tau\|_{L^\infty}\\
&+C\sum_{j=N+1}^{\infty}2^{j(1-s)}2^{js}\|\Delta_j\tau\|_{L^2}\\
\leqslant{}& C+C\|\Lambda^{\gamma}G\|_{L^p}+CN\|\tau\|_{L^\infty}+C2^{N(1-s)}\|\Lambda^{s}\tau\|_{L^2}\\
\leqslant{}& C+CN\|\tau\|_{L^\infty}+C2^{N(1-s)}\|\Lambda^{s}\tau\|_{L^2}\\
\leqslant{}& C(1+\|\tau\|_{L^\infty}\ln(e+\|\Lambda^{s}\tau\|_{L^2})),
\end{aligned}
$$

where we have selected $p>2/\gamma$ and N satisfying

$$
N=\left[\frac{\ln(e+\|\Lambda^{s}\tau\|_{L^2})}{(s-1)\ln 2}\right]+1.
$$

Taking advantage of the above inequality, we easily get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{s}u(t)\|_{L^2}^2+\|\Lambda^{s}\tau(t)\|_{L^2}^2)+\|\Lambda^{s+1}u\|_{L^2}^2\\
&\quad\leqslant C(1+\|\tau\|_{L^\infty}^2)\ln(e+\|\Lambda^{s}u\|_{L^2}^2+\|\Lambda^{s}\tau\|_{L^2}^2)(\|\Lambda^{s}u\|_{L^2}^2+\|\Lambda^{s}\tau\|_{L^2}^2).
\end{aligned}
$$

Applying the Gronwall inequality leads to

$$
\max_{0\leqslant t\leqslant T}(\|\Lambda^{s}u(t)\|_{L^2}^2+\|\Lambda^{s}\tau(t)\|_{L^2}^2)+\int_0^T\|\Lambda^{s+1}u(t)\|_{L^2}^2\,\mathrm{d}t<\infty.
$$

This completely finishes the proof of theorem 1.1. □

4. The proof of theorem 1.7

This section is devoted to the proof of theorem 1.7. For simplicity, without loss of generality, we assume $\mu=\eta=\kappa=\gamma=1$ and $\beta=0$ in this section.

Proof of the a priori estimates of theorem 1.7. We start with the following global L^2 estimate for the system (1.4).

LEMMA 4.1. *Let* (u,τ) *be the corresponding solution of (1.4), then* (u,τ) *obeys the global bound*

$$
\|u(t)\|_{L^2}^2+\|\tau(t)\|_{L^2}^2+2\int_0^t\|\nabla\tau\|_{L^2}^2\,\mathrm{d}t=\|u_0\|_{L^2}^2+\|\tau_0\|_{L^2}^2. \tag{4.1}
$$

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The next lemma provides the global bounds for $\|\nabla u(t)\|_{L^q}$ and $\|\tau(t)\|_{W^{1,p}}$ for any $2 \leqslant q,\, p < \infty$.

Lemma 4.2. *Let (u, τ) be the corresponding solution of (1.4). Then, for any $2 \leqslant p,\ q < \infty$, (u, τ) obeys the global bound, for any $t \in [0, T]$*

$$\|\nabla u(t)\|_{L^q} + \|\tau(t)\|_{W^{1,p}} + \int_0^t \|\Delta \tau(s)\|_{L^2}^2 \,\mathrm{d}s \leqslant C, \tag{4.2}$$

where C is a constant depending on T, p, q and the initial data.

Proof of lemma 4.2. Testing the equations $(1.4)_1$, $(1.4)_2$ by Δu and $\Delta\tau$, respectively, and summing them up, one easily gets

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla \tau\|_{L^2}^2) + \|\Delta\tau\|_{L^2}^2 &= \int_{\mathbb{R}^2} (u\cdot\nabla\tau) : \Delta\tau \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2} (\tau\Omega - \Omega\tau) : \Delta\tau\,\mathrm{d}x, \end{aligned}\tag{4.3}$$

where we have applied the following simple facts

$$\int_{\mathbb{R}^2} (\nabla\cdot\tau)\cdot\Delta u \,\mathrm{d}x + \int_{\mathbb{R}^2} \mathcal{D}u : \Delta\tau\,\mathrm{d}x = 0 \quad\text{and}\quad \int_{\mathbb{R}^2} (u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x = 0.$$

Recalling the incompressibility condition, the first term can be bounded by

$$\begin{aligned}\int_{\mathbb{R}^2} (u\cdot\nabla\tau) : \Delta\tau\,\mathrm{d}x &= -\int_{\mathbb{R}^2} \partial_l(u_i\partial_i\tau_{kj})\partial_l\tau_{kj}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^2} \partial_l u_i \partial_i\tau_{kj}\partial_l\tau_{kj}\,\mathrm{d}x \\ &\leqslant C\|\nabla u\|_{L^2}\|\nabla\tau\|_{L^4}^2 \\ &\leqslant C\|\nabla u\|_{L^2}\|\nabla\tau\|_{L^2}\|\Delta\tau\|_{L^2} \\ &\leqslant \frac{1}{8}\|\Delta\tau\|_{L^2}^2 + C\|\nabla\tau\|_{L^2}^2\|\nabla u\|_{L^2}^2. \end{aligned}\tag{4.4}$$

By the Young inequality, one immediately has the following estimate

$$\begin{aligned}\int_{\mathbb{R}^2} (\tau\Omega - \Omega\tau) : \Delta\tau\,\mathrm{d}x &\leqslant C\|\tau\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta\tau\|_{L^2} \\ &\leqslant \frac{1}{8}\|\Delta\tau\|_{L^2}^2 + C\|\tau\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \end{aligned}\tag{4.5}$$

Inserting (4.4) and (4.5) into (4.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla\tau\|_{L^2}^2) + \|\Delta\tau\|_{L^2}^2 \leqslant C(\|\nabla\tau\|_{L^2}^2 + \|\tau\|_{L^\infty}^2)\|\nabla u\|_{L^2}^2,$$

which along the basic L^2-energy estimate yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{H^1}^2 + \|\tau(t)\|_{H^1}^2) + \frac{3}{2}\|\Delta\tau\|_{L^2}^2 \leqslant C(\|\nabla\tau\|_{L^2}^2 + \|\tau\|_{L^\infty}^2)\|u\|_{H^1}^2. \tag{4.6}$$

Using the Brezis–Gallouet inequality in the form (see [**4**, **5**])

$$\|\tau\|_{L^\infty}^2 \leqslant C\|\tau\|_{H^1}^2 \ln(1+\|\Delta\tau\|_{L^2}^2)$$

and the following inequality

$$px \leqslant \frac{1}{2}e^x + p\ln p, \quad p,\, x>0,$$

we conclude that

$$\begin{aligned} C\|\tau\|_{L^\infty}^2\|u\|_{H^1}^2 &\leqslant C\|u\|_{H^1}^2\|\tau\|_{H^1}^2 \ln(1+\|\Delta\tau\|_{L^2}^2) \\ &\leqslant \frac{1}{2}+\frac{1}{2}\|\Delta\tau\|_{L^2}^2 + C\|\tau\|_{H^1}^2\|u\|_{H^1}^2 \ln(1+\|\nabla u\|_{L^2}^2+\|\tau\|_{H^1}^2) \\ &\leqslant C+\frac{1}{2}\|\Delta\tau\|_{L^2}^2 + C\|\tau\|_{H^1}^2\|u\|_{H^1}^2 \ln(1+\|u\|_{H^1}^2+\|\tau\|_{H^1}^2). \end{aligned} \tag{4.7}$$

Inserting (4.7) into (4.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + \|\Delta\tau\|_{L^2}^2 \leqslant C + C(1+\|\tau\|_{H^1}^2)Y(t)\ln(1+Y(t)),$$

where

$$Y(t) := \|u(t)\|_{H^1}^2 + \|\tau(t)\|_{H^1}^2.$$

Thanks to (4.1), we obtain

$$Y(t) + \int_0^t \|\Delta\tau(s)\|_{L^2}^2\,\mathrm{d}s \leqslant C,$$

which is equal to

$$\|u(t)\|_{H^1}^2 + \|\tau(t)\|_{H^1}^2 + \int_0^t \|\Delta\tau(s)\|_{L^2}^2\,\mathrm{d}s \leqslant C. \tag{4.8}$$

This also implies

$$\int_0^t \|\tau(s)\|_{L^\infty}^2\,\mathrm{d}s \leqslant C. \tag{4.9}$$

Next we apply operator curl to the equation $(1.4)_1$ to obtain the vorticity ω equation as follows

$$\partial_t\omega + (u\cdot\nabla)\omega = \mathrm{curl}\ \mathrm{div}(\tau). \tag{4.10}$$

However, the 'vortex stretching' term curl div(τ) appears to prevent us from proving any global bound for ω, except $\|\omega\|_{L^2}$ due to (4.8). A natural idea would be to eliminate curl div (τ) from the vorticity equation, which is motivated by the two elegant works [20, 21]. To realize this idea, we take $\mathcal{R}$ as the singular integral operator

$$\mathcal{R} = -(-\Delta)^{-1}\mathrm{curl}\ \mathrm{div}.$$

By applying the operator $\mathcal{R}$ to the equation $(1.4)_2$, it gives directly

$$\partial_t \mathcal{R}\tau + (u\cdot\nabla)\mathcal{R}\tau - \Delta\mathcal{R}\tau = \omega - \mathcal{R}(\tau\Omega - \Omega\tau) - [\mathcal{R},\, u\cdot\nabla]\tau, \tag{4.11}$$

where we have used the simple fact

$$\mathcal{R}\mathcal{D}u = \omega.$$

Now we denote $\Gamma = \omega - \mathcal{R}\tau$, which satisfies by combining (4.10) and (4.11)

$$\partial_t \Gamma + (u\cdot\nabla)\Gamma = [\mathcal{R},\, u\cdot\nabla]\tau + \mathcal{R}(\tau\Omega - \Omega\tau) - \omega. \tag{4.12}$$

According to the definition of Ω, one may check

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\omega := \mathcal{A}\omega,$$

which leads to

$$\|\Omega\|_{L^m} \leqslant \|\omega\|_{L^m}, \qquad 1 \leqslant m \leqslant \infty.$$

Multiplying (4.12) by $|\Gamma|^{q-2}\Gamma$ and integrating by parts, it can be obtained that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)\|_{L^q}^q = K_1 + K_2 + K_3, \tag{4.13}$$

where

$$K_1 = -\int_{\mathbb{R}^2} \omega\ |\Gamma|^{q-2}\Gamma\,\mathrm{d}x, \quad K_2 = \int_{\mathbb{R}^2} [\mathcal{R},\, u\cdot\nabla]\tau\ |\Gamma|^{q-2}\Gamma\,\mathrm{d}x,$$
$$K_3 = \int_{\mathbb{R}^2} \mathcal{R}(\tau\Omega - \Omega\tau)\ |\Gamma|^{q-2}\Gamma\,\mathrm{d}x.$$

We start with the first term which follows from the Young inequality and the boundedness of the Riesz transform between the $L^q\,(1 < q < \infty)$ space

$$\begin{aligned} K_1 &\leqslant C\|\omega\|_{L^q}\|\Gamma\|_{L^q}^{q-1} \\ &\leqslant C(\|\Gamma\|_{L^q} + \|\mathcal{R}\tau\|_{L^q})\|\Gamma\|_{L^q}^{q-1} \\ &\leqslant C(\|\Gamma\|_{L^q}^q + \|\mathcal{R}\tau\|_{L^q}^q) \\ &\leqslant C\|\Gamma\|_{L^q}^q + C\|\tau\|_{H^1}^q, \end{aligned} \tag{4.14}$$

where we have used the fact

$$\|\mathcal{R}\tau\|_{L^q} \leqslant C(q)\|\tau\|_{L^q} \leqslant C(q)\|\tau\|_{H^1}.$$

Making use of the following commutator estimate (see [21, theorem 3.3] or [13, corollary 3.2] for details)

$$\|[\mathcal{R},\, u\cdot\nabla]f\|_{B^0_{p,r}} \leqslant C(r,p)\|\nabla u\|_{L^p}(\|f\|_{B^0_{\infty,r}} + \|f\|_{L^p}),$$

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for any smooth incompressible vector field u and any $(p, r) \in [2, \infty) \times [1, \infty]$, we thus obtain

$$\begin{aligned}
K_2 &= \int_{\mathbb{R}^2} [\mathcal{R},\, u\cdot\nabla]\tau\ |\Gamma|^{q-2}\Gamma\,\mathrm{d}x \\
&\leqslant C\|[\mathcal{R},\, u\cdot\nabla]\tau\|_{L^q}\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|[\mathcal{R},\, u\cdot\nabla]\tau\|_{B^0_{q,2}}\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|\nabla u\|_{L^q}(\|\tau\|_{B^0_{\infty,2}} + \|\tau\|_{L^2})\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|\omega\|_{L^q}(\|\tau\|_{B^0_{\infty,2}} + \|\tau\|_{L^2})\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)(\|\Gamma\|_{L^q} + \|\mathcal{R}\tau\|_{L^q})(\|\tau\|_{B^0_{\infty,2}} + \|\tau\|_{L^2})\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)(\|\tau\|_{B^0_{\infty,2}} + \|\tau\|_{L^2})(\|\Gamma\|_{L^q}^{q} + \|\mathcal{R}\tau\|_{L^q}^{q}) \\
&\leqslant C(q)(\|\tau\|_{B^0_{\infty,2}} + \|\tau\|_{L^2})(\|\Gamma\|_{L^q}^{q} + \|\tau\|_{H^1}^{q}).
\end{aligned} \tag{4.15}$$

The last term can be easily estimated by

$$\begin{aligned}
K_3 &\leqslant \|\mathcal{R}(\tau\mathcal{A}\omega - \mathcal{A}\omega\tau)\|_{L^q}\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|\tau\mathcal{A}\omega - \mathcal{A}\omega\tau\|_{L^q}\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|\tau\|_{L^\infty}\|\omega\|_{L^q}\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|\tau\|_{L^\infty}(\|\Gamma\|_{L^q} + \|\mathcal{R}\tau\|_{L^q})\|\Gamma\|_{L^q}^{q-1} \\
&\leqslant C(q)\|\tau\|_{L^\infty}(\|\Gamma\|_{L^q}^{q} + \|\tau\|_{H^1}^{q}).
\end{aligned} \tag{4.16}$$

Putting estimates (4.14), (4.15) and (4.16) into (4.13), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma\|_{L^q}^{q} \leqslant C(q)(1 + \|\tau\|_{H^1} + \|\tau\|_{L^\infty})\|\Gamma\|_{L^q}^{q} + C(q)(1 + \|\tau\|_{H^1} + \|\tau\|_{L^\infty})\|\tau\|_{H^1}^{q},$$

where we have used

$$\|\tau\|_{B^0_{\infty,2}} \leqslant C\|\tau\|_{H^1}.$$

Thanks to the Gronwall inequality and (4.8)–(4.9), we show

$$\|\Gamma(t)\|_{L^q} + \|\tau(t)\|_{L^q} \leqslant C(q),$$

which leads to

$$\|\nabla u(t)\|_{L^q} \leqslant \|\omega(t)\|_{L^q} \leqslant C(q). \tag{4.17}$$

Next, our main target is to estimate $\|\nabla\tau\|_{L^q}$ for any $2 < q < \infty$. Applying ∇ to the equation $(1.4)_2$ leads to

$$\partial_t\nabla\tau + (u\cdot\nabla)\nabla\tau - \Delta\nabla\tau = \nabla\mathcal{D}u - \nabla(\tau\Omega - \Omega\tau) - \nabla u\cdot\nabla\tau. \tag{4.18}$$

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Multiplying (4.18) by $|\nabla\tau|^{q-2}\nabla\tau$ and integrating over $\mathbb{R}^2$, it thus follows that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tau(t)\|_{L^q}^q + (q-1)\int_{\mathbb{R}^2}|\nabla^2\tau|^2|\nabla\tau|^{q-2}\,\mathrm{d}x \leqslant F_1+F_2+F_3,$$

where F_1, F_2 and F_3 are given by

$$F_1 = \int_{\mathbb{R}^2}\nabla\mathcal{D}u(|\nabla\tau|^{q-2}\nabla\tau)\,\mathrm{d}x, \quad F_2 = -\int_{\mathbb{R}^2}\nabla(\tau\Omega-\Omega\tau)(|\nabla\tau|^{q-2}\nabla\tau)\,\mathrm{d}x,$$
$$F_3 = -\int_{\mathbb{R}^2}\nabla u\cdot\nabla\tau(|\nabla\tau|^{q-2}\nabla\tau)\,\mathrm{d}x.$$

Owing to integrating by parts and using the Hölder inequality, one obtains

$$\begin{aligned}
F_1 &\leqslant (q-1)\int_{\mathbb{R}^2}|\mathcal{D}u|\ \ |\nabla\tau|^{q-2}|\nabla^2\tau|\,\mathrm{d}x\\
&\leqslant \frac{q-1}{8}\int_{\mathbb{R}^2}|\nabla^2\tau|^2|\nabla\tau|^{q-2}\,\mathrm{d}x + C(q)\int_{\mathbb{R}^2}|\mathcal{D}u|^2\ \ |\nabla\tau|^{q-2}\,\mathrm{d}x\\
&\leqslant \frac{q-1}{8}\int_{\mathbb{R}^2}|\nabla^2\tau|^2|\nabla\tau|^{q-2}\,\mathrm{d}x + C(q)\|\nabla u\|_{L^q}^2\|\nabla\tau\|_{L^q}^{q-2}\\
F_2 &\leqslant (q-1)\int_{\mathbb{R}^2}|(\tau\Omega-\Omega\tau)|\ \ |\nabla\tau|^{q-2}|\nabla^2\tau|\,\mathrm{d}x\\
&\leqslant \frac{q-1}{8}\int_{\mathbb{R}^2}|\nabla^2\tau|^2|\nabla\tau|^{q-2}\,\mathrm{d}x + C(q)\int_{\mathbb{R}^2}|\tau\Omega|^2\ \ |\nabla\tau|^{q-2}\,\mathrm{d}x\\
&\leqslant \frac{q-1}{8}\int_{\mathbb{R}^2}|\nabla^2\tau|^2|\nabla\tau|^{q-2}\,\mathrm{d}x + C(q)\|\tau\|_{L^\infty}^2\|\nabla u\|_{L^q}^2\|\nabla\tau\|_{L^q}^{q-2}.
\end{aligned}$$

The term F_3 admits the same bound as F_2. As a matter of fact, by the incompressible condition, we have

$$\begin{aligned}
F_3 &= -\int_{\mathbb{R}^2}\nabla\cdot(\nabla u\tau)(|\nabla\tau|^{q-2}\nabla\tau)\,\mathrm{d}x\\
&\leqslant \frac{q-1}{8}\int_{\mathbb{R}^2}|\nabla^2\tau|^2|\nabla\tau|^{q-2}\,\mathrm{d}x + C(q)\|\tau\|_{L^\infty}^2\|\nabla u\|_{L^q}^2\|\nabla\tau\|_{L^q}^{q-2}.
\end{aligned}$$

Collecting the above estimates of $F_1 - F_3$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tau(t)\|_{L^q}^2 \leqslant C(q)(1+\|\tau\|_{L^\infty}^2)\|\nabla u\|_{L^q}^2.$$

By (4.17) and (4.9), we have

$$\max_{0\leqslant t\leqslant T}\|\nabla\tau(t)\|_{L^q} \leqslant C(q), \quad 2<q<\infty.$$

Combining all the above estimates gives (4.2). Therefore, we finally conclude the proof of lemma 4.2. □

The estimates of lemma 4.2 allow us to show the following lemma.

Lemma 4.3. *Let (u,τ) be the corresponding solution of (1.4). Then, for any $2\leqslant p,\ q<\infty$, τ obeys the global bound*

$$\|\nabla^2\tau\|_{L^p(0,T;L^q)}\leqslant C, \tag{4.19}$$

where C is a constant depending on T, p, q and the initial data. For any $0\leqslant\delta<2$

$$\|\Lambda^\delta\tau\|_{L^\infty(0,T;L^\infty)}\leqslant C, \tag{4.20}$$

where C is a constant depending on T, δ and the initial data.

Remark 4.4. Unfortunately, we are unable to get the following key estimate

$$\|\nabla^2\tau\|_{L^\infty(0,T;L^\infty)}\leqslant C. \tag{4.21}$$

If one could obtain the above estimate (4.21), then according to the equation (4.10), we have the key estimate $\|\omega\|_{L^\infty}<\infty$, which allows us to conclude the global regularity result for the system (1.4).

Proof of lemma 4.3. Applying ∇^2 to equation $(1.4)_2$ and making use of the Duhamel Principle, we immediately have

$$\nabla^2\tau(x,t)=\mathrm{e}^{t\Delta}\nabla^2\tau_0(x)-\int_0^t \mathrm{e}^{(t-s)\Delta}\Delta(-\Delta)^{-1}\nabla^2(\mathcal{D}u-(u\cdot\nabla)\tau - (\tau\Omega-\Omega\tau))(s)\,\mathrm{d}s.$$

By means of lemma 2.4, it is clear that

$$\begin{aligned}
&\|\nabla^2\tau\|_{L^p(0,T;L^q)}\\
&\quad\leqslant \|\mathrm{e}^{t\Delta}\nabla^2\tau_0\|_{L^p(0,T;L^q)}\\
&\qquad+\left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\Delta(-\Delta)^{-1}\nabla^2(\mathcal{D}u-(u\cdot\nabla)\tau-(\tau\Omega-\Omega\tau))(s)\,\mathrm{d}s\right\|_{L^p(0,T;L^q)}\\
&\quad\leqslant C(T,\|\tau_0\|_{H^s})+C\|(-\Delta)^{-1}\nabla^2(\mathcal{D}u-(u\cdot\nabla)\tau-(\tau\Omega-\Omega\tau))\|_{L^p(0,T;L^q)}\\
&\quad\leqslant C(T,\|\tau_0\|_{H^s})+C\|\mathcal{D}u-(u\cdot\nabla)\tau-(\tau\Omega-\Omega\tau)\|_{L^p(0,T;L^q)}\\
&\quad\leqslant C(T,\|\tau_0\|_{H^s})+C\|\mathcal{D}u\|_{L^p(0,T;L^q)}+C\|(u\cdot\nabla)\tau\|_{L^p(0,T;L^q)}+C\|\tau\Omega\|_{L^p(0,T;L^q)}\\
&\quad\leqslant C(T,\|\tau_0\|_{H^s})+CT^{1/p}\|\omega\|_{L^\infty(0,T;L^q)}+CT^{1/p}\|u\|_{L^\infty(0,T;L^\infty)}\|\nabla\tau\|_{L^\infty(0,T;L^q)}\\
&\qquad+CT^{1/p}\|\tau\|_{L^\infty(0,T;L^\infty)}\|\Omega\|_{L^\infty(0,T;L^q)}\\
&\quad\leqslant C(p,q,T,u_0,\tau_0),
\end{aligned}$$

where in the last line we have used the estimates of lemma 4.2. This yields (4.19). Applying Λ^δ to equation $(1.4)_2$ and using the Duhamel Principle, we obtain

$$\Lambda^\delta\tau(x,t)=\mathrm{e}^{t\Delta}\Lambda^\delta\tau_0(x)-\int_0^t \mathrm{e}^{(t-s)\Delta}\Lambda^\delta(\mathcal{D}u-(u\cdot\nabla)\tau-(\tau\Omega-\Omega\tau))(s)\,\mathrm{d}s.$$

By choosing $q > \frac{2}{l}(2-\delta)$, it follows from lemma 2.3 that

$$
\begin{aligned}
\|\Lambda^\delta \tau\|_{L^\infty(0,T;L^\infty)} &\leqslant \|e^{t\Delta}\Lambda^\delta \tau_0\|_{L^\infty(0,T;L^\infty)} + \left\|\int_0^t e^{(t-s)\Delta}\Lambda^\delta \mathcal{D}u(s)\,ds\right\|_{L^\infty(0,T;L^\infty)} \\
&\quad + \left\|\int_0^t e^{(t-s)\Delta}\Lambda^\delta((u\cdot\nabla)\tau)(s)\,ds\right\|_{L^\infty(0,T;L^\infty)} \\
&\quad + \left\|\int_0^t e^{(t-s)\Delta}\Lambda^\delta(\tau\Omega-\Omega\tau)(s)\,ds\right\|_{L^\infty(0,T;L^\infty)} \\
&\leqslant C + C\left\|\int_0^t (t-s)^{-\delta/2-1/q}\|\omega(s)\|_{L^q}\,ds\right\|_{L^\infty(0,T)} \\
&\quad + C\left\|\int_0^t (t-s)^{-\delta/2-1/q}\|u(s)\|_{L^\infty}\|\nabla\tau(s)\|_{L^q}\,ds\right\|_{L^\infty(0,T)} \\
&\quad + C\left\|\int_0^t (t-s)^{-\delta/2-1/q}\|\tau(s)\|_{L^\infty}\|\omega(s)\|_{L^q}\,ds\right\|_{L^\infty(0,T)} \\
&\leqslant C + CT^{1-\delta/2-1/q} \\
&\leqslant C(\delta, T, u_0, \tau_0),
\end{aligned}
$$

which is (4.20). Therefore, this concludes the proof of the lemma. □

By this time, we have obtained the desired results (1.14)–(1.17), thus we complete the proof of the *a priori* estimates of theorem 1.7. □

We remark that the high regularity of τ was obtained, however, for the velocity u we only have (4.17), which has no effect on bounding the terms $\|\omega\|_{L^\infty}$, not to speak of the term $\|\nabla u\|_{L^\infty}$. As a matter of fact, at present we are unable to control the key quantity $\|\omega\|_{L^\infty}$ of the system (1.4). But if one imposes a condition on τ, namely (1.18), then we would derive the bound for $\|\omega\|_{L^\infty}$ which will play a significant role in obtaining the global H^s-bound. More precisely, we have

LEMMA 4.5. *Let (u,τ) be the corresponding solution of (1.4). If the condition (1.18) holds, then there exist some constants C such that*

$$\max_{0\leqslant t\leqslant T}\|\omega(t)\|_{L^\infty} \leqslant C. \tag{4.22}$$

Proof of lemma 4.5. Applying the maximum principle to the equation (4.12), it has

$$\frac{d}{dt}\|\Gamma(t)\|_{L^\infty} \leqslant \|[\mathcal{R},\, u\cdot\nabla]\tau\|_{L^\infty} + \|\mathcal{R}(\tau\Omega-\Omega\tau)\|_{L^\infty} + \|\omega\|_{L^\infty}. \tag{4.23}$$

By the following commutator estimate (see [**21**, theorem 3.3] or [**13**, corollary 3.2])

$$\|[\mathcal{R},\, u\cdot\nabla]f\|_{B^0_{\infty,r}} \leqslant C(\|\omega\|_{L^p} + \|\omega\|_{L^\infty})(\|f\|_{B^\varepsilon_{\infty,r}} + \|f\|_{L^p}), \quad \varepsilon > 0,$$

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one easily concludes that for any $0 < \varepsilon < 1 - 2/p$

$$\begin{aligned}
\|[\mathcal{R},\, u\cdot\nabla]\tau\|_{L^\infty} &\leqslant \|[\mathcal{R},\, u\cdot\nabla]\tau\|_{B^0_{\infty,1}}\\
&\leqslant C(\|\omega\|_{L^p}+\|\omega\|_{L^\infty})(\|\tau\|_{B^\varepsilon_{\infty,1}}+\|\tau\|_{L^p})\\
&\leqslant C(\|\omega\|_{L^p}+\|\Gamma\|_{L^\infty}+\|\mathcal{R}\tau\|_{L^\infty})(\|\tau\|_{B^\varepsilon_{\infty,1}}+\|\tau\|_{L^p})\\
&\leqslant C(\|\omega\|_{L^p}+\|\Gamma\|_{L^\infty}+\|\tau\|_{L^2}+\|\tau\|_{B^\varepsilon_{\infty,1}})(\|\tau\|_{B^\varepsilon_{\infty,1}}+\|\tau\|_{L^p})\\
&\leqslant C(\|\omega\|_{L^p}+\|\Gamma\|_{L^\infty}+\|\tau\|_{L^2}+\|\tau\|_{W^{1,p}})(\|\tau\|_{W^{1,p}}+\|\tau\|_{L^p})\\
&\leqslant C(1+\|\Gamma\|_{L^\infty}),
\end{aligned} \tag{4.24}$$

where we have applied the estimates of lemma 4.2. It is also easy to see

$$\|\omega\|_{L^\infty} \leqslant C(1+\|\Gamma\|_{L^\infty}). \tag{4.25}$$

Now we will handle the second term at the right-hand side of (4.23). By the definition of the Besov space and the Bony decomposition, we get

$$\begin{aligned}
\|\mathcal{R}(\tau\Omega-\Omega\tau)\|_{L^\infty} &= \|\mathcal{R}(\tau\mathcal{A}\omega-\mathcal{A}\omega\tau)\|_{L^\infty}\\
&\leqslant \|\mathcal{R}(\tau\mathcal{A}\omega-\mathcal{A}\omega\tau)\|_{B^0_{\infty,1}}\\
&\leqslant C\|\tau\omega\|_{B^0_{\infty,1}}+C\|u\|_{L^2}\|\tau\|_{L^2}\\
&= C\left(\|T_\tau\omega\|_{B^0_{\infty,1}}+\|T_\omega\tau\|_{B^0_{\infty,1}}+\|R(\omega\tau)\|_{B^0_{\infty,1}}\right)\\
&\quad + C\|u\|_{L^2}\|\tau\|_{L^2}.
\end{aligned} \tag{4.26}$$

According to the definition of the Besov space, it is not hard to check that

$$\begin{aligned}
\|T_\omega\tau\|_{B^0_{\infty,1}} &\leqslant C\|\omega\|_{L^\infty}\|\tau\|_{B^0_{\infty,1}}\\
&\leqslant C\|\omega\|_{L^\infty}\|\tau\|_{W^{1,p}}\\
&\leqslant C(1+\|\Gamma\|_{L^\infty}),\\
\|R(\omega\tau)\|_{B^0_{\infty,1}} &\leqslant C\|R(\omega\tau)\|_{B^\varepsilon_{\infty,1}}\\
&\leqslant C\|\omega\|_{B^0_{\infty,\infty}}\|\tau\|_{B^\varepsilon_{\infty,1}}\\
&\leqslant C\|\omega\|_{L^\infty}\|\tau\|_{W^{1,p}}\\
&\leqslant C(1+\|\Gamma\|_{L^\infty}),
\end{aligned}$$

where $0 < \varepsilon < 1 - 2/p$. In order to bound the term $T_\tau\omega$, we need the condition (1.18). Actually, it allows us to deduce

$$\begin{aligned}
\|T_\tau\omega\|_{B^0_{\infty,1}} &\leqslant \sum_{j\geqslant 0}\|S_{j-1}\tau\Delta_j\omega\|_{L^\infty}\\
&= \sum_{0\leqslant j\leqslant j_0-1}\|S_{j-1}\tau\Delta_j\omega\|_{L^\infty}+C\sum_{j\geqslant j_0}\|S_{j-1}\tau\Delta_j\omega\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C \sum_{0\leqslant j\leqslant j_0-1} \|S_{j-1}\tau\|_{L^\infty}\|\Delta_j\omega\|_{L^\infty} + C\sum_{j\geqslant j_0}\|S_{j-1}\tau\|_{L^\infty}\|\Delta_j\omega\|_{L^\infty}\\
&\leqslant Cj_0\|\tau\|_{L^\infty}\|\omega\|_{L^\infty} + C\sum_{j\geqslant j_0}\|S_{j-1}\tau\|_{L^\infty}\|\omega\|_{L^\infty}\\
&\leqslant C\|\omega\|_{L^\infty} + C\|\omega\|_{L^\infty}\sum_{j\geqslant j_0}\|S_{j-1}\tau\|_{L^\infty}\\
&\leqslant C(1+\|\Gamma\|_{L^\infty}) + \sum_{j\geqslant j_0}\|S_{j-1}\tau\|_{L^\infty}(1+\|\Gamma\|_{L^\infty}).
\end{aligned}
$$

We would like to point out that this is the only place in the proof where we use the condition (1.18). Putting the above estimates into (4.26) yields

$$
\|\mathcal{R}(\tau\Omega-\Omega\tau)\|_{L^\infty} \leqslant C(1+\|\Gamma\|_{L^\infty}) + \sum_{j\geqslant j_0}\|S_{j-1}\tau\|_{L^\infty}(1+\|\Gamma\|_{L^\infty}). \tag{4.27}
$$

Combining (4.23), (4.24), (4.25) and (4.27) ensures

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)\|_{L^\infty} \leqslant C(1+\|\Gamma\|_{L^\infty}) + \sum_{j\geqslant j_0}\|S_{j-1}\tau\|_{L^\infty}(1+\|\Gamma\|_{L^\infty}).
$$

The Gronwall inequality and the condition (1.18) allow us to obtain

$$
\max_{0\leqslant t\leqslant T}\|\Gamma(t)\|_{L^\infty} \leqslant C,
$$

which further implies that

$$
\max_{0\leqslant t\leqslant T}\|\omega(t)\|_{L^\infty} \leqslant C.
$$

Therefore, this completes the proof of the lemma. □

Now we are ready to prove our main theorem 1.7 with the obtained estimates at our disposal.

Proof of the regularity criterion of theorem 1.7*.* Applying the operator Λ^s with $s>2$ to the system (1.4), taking the L^2 inner product with $\Lambda^s u$ and $\Lambda^s\tau$, respectively, and adding them up, we can get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\tau(t)\|_{L^2}^2) + \|\Lambda^{s+1}\tau\|_{L^2}^2\\
&\quad = \int_{\mathbb{R}^2}\Lambda^s\nabla\cdot\tau\cdot\Lambda^s u\,\mathrm{d}x + \int_{\mathbb{R}^2}\Lambda^s\mathcal{D}u:\Lambda^s\tau\,\mathrm{d}x + \int_{\mathbb{R}^2}[\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\,\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\tau):\Lambda^s\tau\,\mathrm{d}x + \int_{\mathbb{R}^2}\Lambda^s(\tau\Omega-\Omega\tau):\Lambda^s\tau\,\mathrm{d}x\\
&\quad := H_1+H_2+H_3+H_4+H_5.
\end{aligned} \tag{4.28}
$$

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In what follows, we will handle each term at the right-hand side of (4.28) separately. We get by integrating by parts and using the Young inequality that

$$H_1 + H_2 \leqslant \|\Lambda^s u(t)\|_{L^2}\|\Lambda^{s+1}\tau\|_{L^2} \leqslant \frac{1}{4}\|\Lambda^{s+1}\tau\|_{L^2}^2 + C\|\Lambda^s u\|_{L^2}^2. \tag{4.29}$$

As a consequence of (3.29), we have

$$H_3 \leqslant C\|[\Lambda^s, u\cdot\nabla]u\|_{L^2}\|\Lambda^s u\|_{L^2} \leqslant C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2. \tag{4.30}$$

By use of the divergence free property and (3.30), it follows that

$$\begin{aligned} H_4 &= \int_{\mathbb{R}^2} \Lambda^s \partial_k(u_k\tau_{i,j}) : \Lambda^s\tau_{i,j}\,\mathrm{d}x \\ &\leqslant C\|\Lambda^s(u\tau)\|_{L^2}\|\Lambda^{s+1}\tau\|_{L^2} \\ &\leqslant C(\|\tau\|_{L^\infty}\|\Lambda^s u\|_{L^2} + \|u\|_{L^\infty}\|\Lambda^s\tau\|_{L^2})\|\Lambda^{s+1}\tau\|_{L^2} \\ &\leqslant \frac{1}{4}\|\Lambda^{s+1}\tau\|_{L^2}^2 + C(\|\tau\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\Lambda^s\tau\|_{L^2}^2). \end{aligned} \tag{4.31}$$

Similarly, the last term can be estimated as follows

$$\begin{aligned} H_5 &\leqslant \|\Lambda^{s-1}(\tau\Omega - \Omega\tau)\|_{L^2}\|\Lambda^{s+1}\tau\|_{L^2} \\ &\leqslant C(\|\tau\|_{L^\infty}\|\Lambda^{s-1}\Omega\|_{L^2} + \|\Omega\|_{L^q}\|\Lambda^{s-1}\tau\|_{L^{2q/(q-2)}})\|\Lambda^{s+1}\tau\|_{L^2} \quad (q>2) \\ &\leqslant C(\|\tau\|_{L^\infty}\|\Lambda^{s-1}\Omega\|_{L^2} + \|\Omega\|_{L^q}\|\tau\|_{L^2}^{(q-2)/sq}\|\Lambda^s\tau\|_{L^2}^{((s-1)q+2)/sq})\|\Lambda^{s+1}\tau\|_{L^2} \\ &\leqslant \frac{1}{4}\|\Lambda^{s+1}\tau\|_{L^2}^2 + C(\|\tau\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2 + \|w\|_{L^q}^2\|\tau\|_{L^2}^{2(q-2)/sq}\|\Lambda^s\tau\|_{L^2}^{(2(s-1)q+4)/sq}) \\ &\leqslant \frac{1}{4}\|\Lambda^{s+1}\tau\|_{L^2}^2 + C(\|\tau\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2 + \|w\|_{L^q}^2 + \|w\|_{L^q}^2\|\Lambda^s\tau\|_{L^2}^2). \end{aligned} \tag{4.32}$$

Plugging together the preceding estimates (4.29), (4.30), (4.31) and (4.32) into (4.29), we find the following differential type inequality

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\tau(t)\|_{L^2}^2) + \|\Lambda^{s+1}\tau\|_{L^2}^2 \\ &\quad\leqslant C(1 + \|\nabla u\|_{L^\infty} + \|w\|_{L^q}^2 + \|u\|_{L^\infty}^2 + \|\tau\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\tau\|_{L^2}^2) \\ &\quad\leqslant C(1 + \|\nabla u\|_{L^\infty})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\tau\|_{L^2}^2). \end{aligned} \tag{4.33}$$

To control the term $\|\nabla u\|_{L^\infty}$, we will appeal to the following logarithmic Sobolev embedding inequality (see [**2**])

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leqslant C(1 + \|u\|_{L^2(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)}\ln(e + \|\Lambda^s u\|_{L^2(\mathbb{R}^2)})), \quad s>2. \tag{4.34}$$

Applying (4.34) to (4.33) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s\tau(t)\|_{L^2}^2) + \|\Lambda^{s+1}\tau\|_{L^2}^2 \\ &\quad\leqslant C(1 + \|\omega\|_{L^\infty})\ln(e + \|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\tau\|_{L^2}^2)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\tau\|_{L^2}^2). \end{aligned} \tag{4.35}$$

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Applying the Gronwall inequality to (4.35) gives that

$$\max_{0\leqslant t\leqslant T}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s\tau(t)\|_{L^2}^2)<\infty.$$

This bound implies that the local solution can be extended to $[0,T]$. Thus, we completely finish the proof of theorem 1.7. □

5. The proof of theorem 1.9

If we add the dissipation term $(-\Delta)^\alpha u$ to the velocity equation of the system (1.4), then with suitable modifications, it is not difficult to deduce that the estimates (1.14)–(1.17) are still true for the corresponding system. In order to avoid much of the repetition, we omit the details. In this case, the estimates (1.14)–(1.17) are sufficient for us to get the global H^s $(s>2)$ estimate without (4.22). Actually, it suffices to bound H_3 as follows which is different from (4.30)

$$\begin{aligned}
H_3&\leqslant C\|[\Lambda^s,u\cdot\nabla]u\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\leqslant C\|\nabla u\|_{L^q}\|\Lambda^s u\|_{L^{2q/(q-2)}}\|\Lambda^s u\|_{L^2}\\
&\leqslant C\|\nabla u\|_{L^q}\|\Lambda^s u\|_{L^2}^{(\alpha q-2)/\alpha q}\|\Lambda^{s+\alpha}u\|_{L^2}^{2/\alpha q}\|\Lambda^s u\|_{L^2}\\
&\leqslant \frac14\|\Lambda^{s+\alpha}u\|_{L^2}^2+C\|\nabla u\|_{L^q}^{\alpha q/(\alpha q-1)}\|\Lambda^s u\|_{L^2}^2,
\end{aligned}$$

where $q\geqslant 2/\alpha$. We would like to point out that this is the only place in the proof where we use the main assumption of the theorem, namely $\alpha>0$. The other four terms H_1, H_2, H_4, H_5 can be bounded as the same as in proving theorem 1.7. We thus complete the proof of theorem 1.9.

Appendix A. The proof of (3.14) and (3.23)

Let us start with the proof of (3.14). By the definition of operator $\mathcal{L}$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathcal{L}fg\,\mathrm{d}x\\
&=\mathrm{P.V.}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(f(x)-f(x-y))g(x)}{|y|^2m(|y|)}\,\mathrm{d}y\mathrm{d}x\\
&=\mathrm{P.V.}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{f(x)(g(x)-g(x-y))+f(x)g(x-y)-f(x-y)g(x)}{|y|^2m(|y|)}\,\mathrm{d}y\mathrm{d}x\\
&=\int_{\mathbb{R}^2}f\mathcal{L}g\,\mathrm{d}x+\mathrm{P.V.}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{f(x)g(x-y)-f(x-y)g(x)}{|y|^2m(|y|)}\,\mathrm{d}y\mathrm{d}x. \qquad (\mathrm{A}.1)
\end{aligned}$$

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By using the variable substitution $x = \widetilde{x} - \widetilde{y},\ y = -\widetilde{y}$, we have

$$\begin{aligned}
I :=& \text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(x)g(x-y) - f(x-y)g(x)}{|y|^2 m(|y|)} \, \mathrm{d}y\mathrm{d}x \\
=& \text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(\widetilde{x}-\widetilde{y})g(\widetilde{x}) - f(\widetilde{x})g(\widetilde{x}-\widetilde{y})}{|-\widetilde{y}|^2 m(|-\widetilde{y}|)} \, \mathrm{d}\widetilde{y}\mathrm{d}\widetilde{x} \\
=& -\text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(\widetilde{x})g(\widetilde{x}-\widetilde{y}) - f(\widetilde{x}-\widetilde{y})g(\widetilde{x})}{|\widetilde{y}|^2 m(|\widetilde{y}|)} \, \mathrm{d}\widetilde{y}\mathrm{d}\widetilde{x} \\
=& -I,
\end{aligned}$$

which implies

$$\text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(x)g(x-y) - f(x-y)g(x)}{|y|^2 m(|y|)} \, \mathrm{d}y\mathrm{d}x = 0.$$

This along with (A.1) gives

$$\int_{\mathbb{R}^2} \mathcal{L}f g \, \mathrm{d}x = \int_{\mathbb{R}^2} f \mathcal{L}g \, \mathrm{d}x.$$

This concludes the proof of (3.14).

The proof of (3.23) can be performed as follows. The well-known Riesz potential operator $\Lambda^{-\delta}$ with $\delta \in (0, 2)$ reads

$$\Lambda^{-\delta} f(x) = C(\delta) \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{2-\delta}} \, \mathrm{d}y,$$

where $C(\delta)$ is a positive constant depending only on δ. For the sake of simplicity, in what follows we ignore the constant $C(\delta)$. Recalling the definition of the operator $\mathcal{L}$, we have

$$\begin{aligned}
\Lambda^{-\delta} \mathcal{L} f(x) =& \int_{\mathbb{R}^2} \frac{\mathcal{L}f(y)}{|x-y|^{2-\delta}} \, \mathrm{d}y \\
=& \text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(y) - f(y-z)}{|x-y|^{2-\delta}|z|^2 m(|z|)} \, \mathrm{d}z\mathrm{d}y.
\end{aligned} \tag{A.2}$$

On the other hand, it also implies

$$\begin{aligned}
\mathcal{L}\Lambda^{-\delta} f(x) =& \text{P.V.} \int_{\mathbb{R}^2} \frac{\Lambda^{-\delta} f(x) - \Lambda^{-\delta} f(x-y)}{|y|^2 m(|y|)} \, \mathrm{d}y \\
=& \text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(z)}{|x-z|^{2-\delta}|y|^2 m(|y|)} \, \mathrm{d}z\mathrm{d}y \\
& - \text{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(z)}{|x-y-z|^{2-\delta}|y|^2 m(|y|)} \, \mathrm{d}z\mathrm{d}y.
\end{aligned} \tag{A.3}$$

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Thanks to the variable substitution $z = \widetilde{z} - \widetilde{y},\ y = \widetilde{y}$, we infer

$$
\begin{aligned}
&\mathrm{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(z)}{|x-y-z|^{2-\delta}|y|^2 m(|y|)}\,\mathrm{d}z\mathrm{d}y \\
&= \mathrm{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(\widetilde{z}-\widetilde{y})}{|x-\widetilde{z}|^{2-\delta}|\widetilde{y}|^2 m(|\widetilde{y}|)}\,\mathrm{d}\widetilde{z}\mathrm{d}\widetilde{y} \\
&= \mathrm{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(z-y)}{|x-z|^{2-\delta}|y|^2 m(|y|)}\,\mathrm{d}z\mathrm{d}y,
\end{aligned}
$$

which along with (A.3) further shows by the variable substitution $z = \widetilde{y},\ y = \widetilde{z}$ that

$$
\begin{aligned}
\mathcal{L}\Lambda^{-\delta} f(x) &= \mathrm{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(z)-f(z-y)}{|x-z|^{2-\delta}|y|^2 m(|y|)}\,\mathrm{d}z\mathrm{d}y \\
&= \mathrm{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(\widetilde{y})-f(\widetilde{y}-\widetilde{z})}{|x-\widetilde{y}|^{2-\delta}|\widetilde{z}|^2 m(|\widetilde{z}|)}\,\mathrm{d}\widetilde{z}\mathrm{d}\widetilde{y} \\
&= \mathrm{P.V.} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{f(y)-f(y-z)}{|x-y|^{2-\delta}|z|^2 m(|z|)}\,\mathrm{d}z\mathrm{d}y. \qquad \text{(A.4)}
\end{aligned}
$$

Combining (A.2) and (A.4), we finally get

$$
\Lambda^{-\delta}\mathcal{L} f(x) = \mathcal{L}\Lambda^{-\delta} f(x).
$$

This completes the proof of (3.23).

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References

1. H. Bahouri, J.-Y. Chemin and R. Danchin. Fourier Analysis and Nonlinear Partial Differential Equations, In *Grundlehren der Mathematischen Wissenschaften*, vol. 343 (Heidelberg: Springer, 2011.
2. J. T. Beale, T. Kato and A. Majda. Remarks on breakdown of smooth solutions for the 3D Euler equations. *Comm. Math. Phys.* **94** (1984), 61–66.
3. O. Bejaoui and M. Majdoub. Global weak solutions for some Oldroyd models. *J. Differ. Equ.* **254** (2013), 660–685.
4. H. Brezis and T. Gallouet. Nonlinear Schrödinger evolution equations. *Nonlinear Anal.* **4** (1980), 677–681.
5. H. Brezis and S. Wainger. A note on limiting cases of Sobolev embedding and convolution inequalities. *Comm. Partial Differ. Equ.* **5** (1980), 773–789.
6. J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.* **33** (2001), 84–112.
7. Q. Chen and C. Miao. Global well-posedness of viscoelastic fluids of Oldroyd type in Besov spaces. *NonlinearAnal.* **68** (2008), 1928–1939.
8. Y. Chen and P. Zhang. The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions. *Comm. Partial Differ. Equ.* **31** (2006), 1793–1810.

9 P. Constantin. Euler equations, Navier–Stokes equations and turbulence, In *Mathematical foundation of turbulent viscous flows*, Lecture Notes in Math., vol. 1871, pp. 1–43 (Berlin: Springer, 2006).

10 P. Constantin and M. Kliegl. Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress. *Arch. Ration. Mech. Anal* **206** (2012), 725–740.

11 P. Constantin and W. Sun. Remarks on Oldroyd-B and related complex fluid models. *Commun. Math. Sci.* **10** (2012), 33–73.

12 P. Constantin and V. Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *Geom. Funct. Anal.* **22** (2012), 1289–1321.

13 T. Elgindi and F. Rousset. Global regularity for some Oldroyd-B type models. *Comm. Pure Appl. Math.* **68** (2015), 2005–2021.

14 D. Fang and R. Zi. Global solutions to the Oldroyd-B model with a class of large initial data. *SIAM J. Math. Anal.* **48** (2016), 1054–1084.

15 D. Fang, M. Hieber and R. Zi. Global existence results for Oldroyd-B fluids in exterior domains: the case of non-small coupling parameters. *Math. Ann.* **357** (2013), 687–709.

16 E. Fernández-Cara, F. Guillén and R. Ortega. Mathematical modeling and analysis of viscoelastic fluids of the Oldroyd kind, *Handbook of numerical analysis*, vol. 8, pp. 543–661 (NorthHolland, Amsterdam: Elsevier, 2002).

17 C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.* **15** (1990), 849–869.

18 C. Guillopé and J.-C. Saut. Global existence and one-dimensional nonlinear stability of shearingmotions of viscoelastic fluids of Oldroyd type. *RAIROModél. Math. Anal. Numér.* **24** (1990), 369–401.

19 M. Hieber, Y. Naito and Y. Shibata. Global existence results for Oldroyd-B fluids in exterior domains. *J. Differ. Equ.* **252** (2012), 2617–2629.

20 T. Hmidi, S. Keraani and F. Rousset. Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation. *J. Differ. Equ.* **249** (2010), 2147–2174.

21 T. Hmidi, S. Keraani and F. Rousset. Global well-posedness for Euler-Boussinesq system with critical dissipation. *Comm. Partial Differ. Equ.* **36** (2011), 420–445.

22 D. Hu and T. Lelièvre. New entropy estimates for Oldroyd-B and related models. *Commun. Math. Sci.* **5** (2007), 909–916.

23 X. Hu and D. Wang. Local strong solution to the compressible viscoelastic flow with large data. *J. Differ. Equ.* **249** (2010), 1179–1198.

24 X. Hu and D. Wang. Global existence for the multi-dimensional compressible viscoelastic flows. *J. Differ. Equ.* **250** (2011), 1200–1231.

25 A. Karageorghis and G. Fairweather. The method of fundamental solutions for the numerical solution of the biharmonic equation. *J. Comput. Phys.* **69** (1987), 434–459.

26 T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.* **41** (1988), 891–907.

27 C. Kenig, G. Ponce and L. Vega. Well-posedness of the initial value problem for the Korteweg-de-Vries equation. *J. Amer. Math. Soc.* **4** (1991), 323–347.

28 Z. Lei. On 2D viscoelasticity with small strain. *Arch. Ration. Mech. Anal.* **198** (2010), 13–37.

29 Z. Lei and Y. Zhou. Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. *SIAM J. Math. Anal.* **37** (2005), 797–814.

30 Z. Lei, C. Liu and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal* **188** (2008), 371–398.

31 Z. Lei, N. Masmoudi and Y. Zhou. Remarks on the blowup criteria for Oldroyd models. *J. Differ. Equ.* **248** (2010), 328–341.

32 P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*, Chapman Hall/CRC Research Notes in Mathematics, vol. 431 (Boca Raton, FL: Chapman Hall/CRC, 2002).

33 F. Lin, C. Liu and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.* **58** (2005), 1437–1471.

34 P. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B* **21** (2000), 131–146.

35 N. Masmoudi. Global existence of weak solutions to macroscopic models of polymeric flows. *J. Math. Pures Appl.* **96** (2011), 502–520.

36 J. G. Oldroyd. Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. Roy. Soc. London Ser. A* **245** (1958), 278–297.

37 Z. Ye. *On the global regularity of the 2D Oldroyd-B-type model*, Annali di Matematica (2018). https://doi.org/10.1007/s10231-018-0784-2.

38 Z. Ye and X. Xu. Global regularity for the 2D Oldroyd-B model in the corotational case. *Math. Methods Appl. Sci.* **39** (2016), 3866–3879.

39 R. Zi. Global solution to the incompressible Oldroyd-B model in hybrid Besov spaces. *Filomat* **30** (2016), 3627–3639.

40 R. Zi, D. Fang and T. Zhang. Global solution to the incompressible Oldroyd-B model in the critical L^p framework: the case of the non-small coupling parameter. *Arch. Rational Mech. Anal.* **213** (2014), 651–687.