Proceedings of the Royal Society of Edinburgh, **132A**, 181–219, *2002*

Non-classical Riemann solvers and kinetic relations. II An hyperbolic–elliptic model of phase-transition dynamics

Philippe G. LeFloch
Centre de Mathématiques Appliquées and Centre National de la Recherche Scientifique, UMR 7641, Ecole Polytechnique, 91128 Palaiseau Cedex, France (leflochcmap.polytechnique.fr)

Mai Duc Thanh
Hanoi Institute of Mathematics, PO Box 631 BoHo, 10.000 Hanoi, Vietnam (thanh@cmap.polytechnique.fr)

(MS received 26 May 2000; accepted 2 February 2001)

This paper deals with the Riemann problem for a partial differential equation's model arising in phase-transition dynamics and consisting of an hyperbolic–elliptic system of two conservation laws. First of all, we provide a complete description of all solutions of the Riemann problem that are consistent with the mathematical entropy inequality associated with the total energy of the system. Second, following Abeyaratne and Knowles, we impose a *kinetic relation* to determine the dynamics of subsonic phase boundaries. Based on the requirement that subsonic phase boundaries are preferred whenever available, we determine the corresponding wave curves associated with composite waves (shocks, rarefaction fans, phase boundaries). It turns out that even after the kinetic relation is imposed, the Riemann problem may admit up to two solutions. A *nucleation criterion* is necessary to select between a solution remaining in a single phase and a solution containing two phase boundaries. Alternatively, a *strong assumption* on the kinetic relation ensures that the Riemann solution is unique and depends continuously upon its initial data.

1. Introduction

This is the second paper of a series [10,11] devoted to constructing Riemann solvers for hyperbolic or hyperbolic–elliptic systems of partial differential equations in conservative form, modelling phase-transition dynamics. In [11] we treated a *hyperbolic model* of phase transitions, which was shown to admit under-compressive non-classical shock waves singled out by the so-called *kinetic relation*. In the present paper we extend the analysis to a hyperbolic–elliptic model, which exhibits under-compressive subsonic phase boundaries, also characterized via a kinetic relation. Our construction extends, to rather general equations of state, the construction obtained by Abeyaratne and Knowles [2] (in the piecewise linear case) and Shearer and Yang [16] (in the cubic case). Understanding the dynamics of phase boundaries in solids undergoing phase transformations is essential in many applications

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

of material science, for instance, in the study of smart materials. Active research by physicists and applied mathematicians is now under way to identify and study the relevant models. There is now a vast literature on the subject and we refer the reader to [1–5, 7–9, 13, 15–20] and the references cited therein.

In this paper we deal with an important model of continuum physics, consisting of two conservation laws for the velocity $v = v(x, t)$ and deformation gradient $w = w(x, t) > -1$ of some fluid or solid,

$$\left.\begin{aligned} \partial_t v - \partial_x \sigma(w) = 0,\\ \partial_t w - \partial_x v = 0.\end{aligned}\right\} \tag{1.1}$$

The stress σ is a twice-differentiable function of w satisfying (see figure 1 below)

$$\left.\begin{gathered} \sigma'(0) < 0, \quad w\sigma''(w) > 0 \quad \text{for } w \neq 0,\\ \lim_{w\to -1} \sigma'(w) = +\infty, \qquad \lim_{w\to+\infty} \sigma'(w) = +\infty.\end{gathered}\right\} \tag{1.2}$$

Under the hypotheses (1.2), there exist a and $b \in (-1, +\infty)$, with $a < 0 < b$, such that

$$\sigma'(w) < 0 \quad \text{if and only if } a < w < b. \tag{1.3}$$

The system (1.1) under consideration has the general form of a system of conservation laws,

$$\partial_t u + \partial_x f(u) = 0, \tag{1.4}$$

provided we set $u = (v, w)$ and $f(u) = -(\sigma(w), v)$. Under the assumptions (1.2), we have $\sigma'(w) > 0$ for $w < a$ or $w > b$, and the matrix $Df(u)$ admits two real and distinct eigenvalues depending only on w and denoted by

$$\lambda_1(w) := -\sqrt{\sigma'(w)} < 0 < \sqrt{\sigma'(w)} := \lambda_2(w).$$

Therefore, system (1.1) is strictly hyperbolic for $w \notin [a, b]$. Right-eigenvectors are chosen to be $r_i(u) = r_i(w) := (-\lambda_i(w), 1) = (\pm\sqrt{\sigma'(w)}, 1)$ for $i = 1, 2$. We also define the sound speed to be $c(w) := \sqrt{\sigma'(w)}$. Throughout this paper, we restrict the discussion to values in the hyperbolic regions only.

To exhibit fundamental features of the discontinuous solutions of (1.1), we solve in this paper the Riemann problem corresponding to the initial data

$$(v, w)(x, 0) = \begin{cases} (v_\mathrm{l}, w_\mathrm{l}), & x < 0,\\ (v_\mathrm{r}, w_\mathrm{r}), & x > 0,\end{cases} \tag{1.5}$$

where $(v_\mathrm{l}, w_\mathrm{l})$ and $(v_\mathrm{r}, w_\mathrm{r})$ are constants. As is customary, we constrain the weak solutions to satisfy the entropy inequality

$$\left.\begin{gathered} \partial_t U(u) + \partial_x F(u) \leqslant 0,\\ U(v, w) := \tfrac{1}{2}v^2 + \Sigma(w), \qquad F(v, w) = -v\sigma(w),\\ \Sigma(w) := \int_0^w \sigma(z)\,\mathrm{d}z.\end{gathered}\right\} \tag{1.6}$$

Since σ is increasing in both intervals $(-1, a]$ and $[b, +\infty)$, the internal energy function $\Sigma(w)$ is convex in each of these intervals, and actually strictly convex

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

away from the points $w = a, b$. However, Σ is not globally convex, as it admits two local minima at $w = a, b$. This is a typical shape in phase dynamics problems. The values $w < a$ will be referred to as the phase-1 region, and the values $w > b$ as the phase-2 region.

Recall that the Riemann problem just described was first solved by Shearer [15], using the Liu entropy criterion [12] (see § 2). Abeyaratne and Knowles proposed a more general construction for the Riemann problem, restricting attention to trilinear equations of state, that is, linear in each of the region $w < a$, $a < w < b$ and $w > b$. In [1,2,19,20], the concept of *kinetic relation* was introduced, and Abeyaratne and Knowles [2] successfully applied it to construct a Riemann solver for (1.1). The continuous dependence of these solutions was investigated by LeFloch [7]. Moreover, a Riemann solver is explicitly constructed in Shearer and Yang [16] for a cubic stress–strain function and an impact problem for general constitutive law is solved by Pence [13]. Indeed, there exists an extensive literature on the Riemann problem for models of the type (1.1) (for more details, we refer the reader to [2, 8, 13–18] and to the references cited therein). In particular, in Slemrod's pioneering work [17, 18], the mathematical properties of the capillarity were identified. The problem under consideration is also closely related to the recent activity on non-classical shock waves of systems of conservation laws generated by diffusive-dispersive limits (see [8, 9] for a review and references).

The principal aim of this paper is to generalize the Riemann solver in [2] to the nonlinear equation of state (1.2). Based on the concepts of the kinetic relation and nucleation criterion, we are going to derive here a unique Riemann solution for (1.1).

An outline of this paper is as follows.

To begin with, in § 2, we impose [15] that any stationary phase boundary is admissible, which allows one to construct a unique Riemann solver. In this case, the Riemann solutions depend L^1 continuously on their initial data (see theorem 2.7).

In § 3, following a similar analysis as in the first part of this series [11], in theorem 3.3 we investigate the consequences of the single entropy inequality (1.6) on the weak solutions of (1.1). We naturally distinguish between two types (subsonic, supersonic) of phase boundaries. Subsonic phase boundaries turn out to be the main source of non-uniqueness. Theorems 3.9 and 3.10 provide a complete description of the corresponding 1- and 2-wave sets (in the terminology introduced by Hayes and LeFloch [4]).

In § 4, following Abeyaratne and Knowles, we impose a kinetic relation for the propagation of subsonic phase boundaries, and under the requirement that subsonic phase boundaries are preferred whenever available, in theorems 4.1 and 4.2 we arrive at uniquely defined wave curves for each of the two wave families. However, it turns out in theorem 4.4 that the Riemann problem may admit two solutions, since the wave curves may intersect twice. A nucleation criterion must be imposed to select between a solution remaining in a single phase and a solution containing two phase boundaries. Alternatively, in theorem 4.5, a strong condition is assumed on the kinetic relation, implying that the Riemann solution is unique and depends continuously upon its initial data. However, this last assumption is probably not very realistic in the applications. The non-uniqueness of the Riemann solutions seem to be the rule rather than being the exception.

2. A unique Riemann solver based on stationary phase boundaries

In this section we discuss some basic properties of the system of conservation laws (1.1). Following Shearer [15], we provide a first approach to the Riemann problem, relying here on a stronger entropy condition than (1.6). Precisely, we consider as admissible all the jump discontinuities satisfying the Liu entropy criterion or else being stationary. These conditions indeed select uniquely defined wave curves, denoted below by $\mathcal{W}_1^c(u_{\rm l})$ and $\mathcal{W}_2^c(u_{\rm r})$.

Consider a shock wave for (1.1), connecting a left-hand state (v_0, w_0) to a right-hand state (v_1, w_1) and propagating with the speed $s \in \mathbb{R}$. In other words, set

$$u(x,t) = \begin{cases} u_0, & x < st, \\ u_1, & x > st, \end{cases}$$

where $u_0 = (v_0, w_0)$ and $u_1 = (v_1, w_1)$. Since u is a weak solution, the Rankine–Hugoniot relations

$$s(v_1 - v_0) + \sigma(w_1) - \sigma(w_0) = 0, \qquad s(w_1 - w_0) + v_1 - v_0 = 0 \tag{2.1}$$

yield

$$s(u_0, u_1) = -\frac{\sigma(w_1) - \sigma(w_0)}{v_1 - v_0} = -\frac{v_1 - v_0}{w_1 - w_0}.$$

Therefore, provided $\sigma(w_1) - \sigma(w_0)$ and $w_1 - w_0$ have the same sign, the shock speed

$$s = \mp\bar{c}(w_0, w_1) := \mp\sqrt{\frac{\sigma(w_1) - \sigma(w_0)}{w_1 - w_0}} \tag{2.2}$$

is well defined and independent of v_0 and v_1. We will simply write $s = s(w_0, w_1)$. In (2.2), the 1- and 2-shocks correspond to the $\mp$ signs, respectively.

To select a unique solution to the Riemann problem, we attempt to apply the Liu entropy criterion (first introduced by Wendroff for systems of two conservation laws). By definition, a shock satisfies the *Liu entropy condition* if and only if

$$\mp\bar{c}(w_0, w) \geqslant \mp\bar{c}(w_0, w_1) \quad \text{for all } w \text{ between } w_0 \text{ and } w_1. \tag{2.3}$$

Note that, for an i-shock ($i = 1, 2$), the inequality (2.3) implies the *Lax shock inequalities*

$$\lambda_i(u_0) = \mp c(w_0) \geqslant s(w_0, w_1) = \mp\bar{c}(w_0, w_1) \geqslant \mp c(w_1) = \lambda_i(u_1). \tag{2.4}$$

The Liu condition (2.3) can be explained geometrically. For instance, when $i = 2$, equation (2.3) is equivalent to

$$\frac{\sigma(w) - \sigma(w_0)}{w - w_0} \geqslant \frac{\sigma(w_1) - \sigma(w_0)}{w_1 - w_0} \quad \text{for all } w \text{ between } w_0 \text{ and } w_1. \tag{2.5}$$

This means that the graph of σ is below (respectively, above) the line connecting w_0 to w_1 when $w_1 < w_0$ (respectively, $w_1 > w_0$). When $i = 1$, the inequalities (2.5) are reversed. The condition (2.5) shows that the characteristic lines impinge on the discontinuity and the shock is called *compressive*.

The following terminology will be used.

DEFINITION 2.1. A jump discontinuity connecting u_0 to u_1 satisfying the Liu entropy criterion will be called:

(i) a *classical shock wave* if the states u_0 and u_1 belong to the same phase;

(ii) a *supersonic phase boundary* if the states belong to different phases.

Observe that supersonic phase boundaries satisfy the Lax shock inequalities (2.4). (This will no longer be true of the *subsonic phase boundaries* to be exhibited in § 3.) In passing, we note the following result. geometrically from the graph of σ, we obtain easily:

LEMMA 2.2. *Under the assumption (1.2), the Lax inequalities and the Liu criterion are equivalent.*

Fix a left-hand state $u_0 = (v_0, w_0)$. Leaving from u_0, it is not difficult to determine the sets of all right-hand states that can be arrived at by crossing one elementary wave (a shock or a rarefaction wave). On one hand, in view of (2.1), (2.2), the Hugoniot curves for the first and the second families are given by

$$\mathcal{H}_1(u_0) := \{(v,w)/v - v_0 = \bar{c}(w_0,w)(w - w_0)\} \tag{2.6}$$

and

$$\mathcal{H}_2(u_0) := \{(v,w)/v - v_0 = -\bar{c}(w_0,w)(w - w_0)\}. \tag{2.7}$$

On the other hand, the integral curves associated with the vector fields $r_i(w)$, $i = 1, 2$, are

$$\mathcal{O}_1(u_0) := \left\{\frac{(v,w)}{v} - v_0 = \int_{w_0}^{w} c(z)\,\mathrm{d}z\right\} \tag{2.8}$$

and

$$\mathcal{O}_2(u_0) := \left\{\frac{(v,w)}{v} - v_0 = -\int_{w_0}^{w} c(z)\,\mathrm{d}z\right\}. \tag{2.9}$$

Consider the graph of the function σ in the (w,σ)-plane. By (1.2), for any $w \neq 0$, there exists a unique line that passes through the point with coordinates $(w, \sigma(w))$ and is tangent to the graph at a point $(\varphi^\natural(w), \sigma(\varphi^\natural(w))$, with $\varphi^\natural(w) \neq w$. In other words,

$$\sigma'(\varphi^\natural(w)) = \frac{\sigma(w) - \sigma(\varphi^\natural(w))}{w - \varphi^\natural(w)} \quad \text{for } w \neq 0 \tag{2.10}$$

(see figure 1). Note that $w\varphi^\natural(w) < 0$, and also set $\varphi^\natural(0) = 0$. Under the hypotheses (1.2), the map $\varphi^\natural : (-1, +\infty) \to (-1, +\infty)$ is strictly monotone decreasing and onto, and so is invertible. Denote by $\varphi^{-\natural} : (-1, +\infty) \to (-1, +\infty)$ its inverse function. We have

$$w\varphi^{-\natural}(w) \leqslant w\varphi^\natural(w) \leqslant 0 \quad \text{for all } w \in (-1, +\infty)$$

and

$$\varphi^\natural(0) = \varphi^{-\natural}(0) = 0.$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

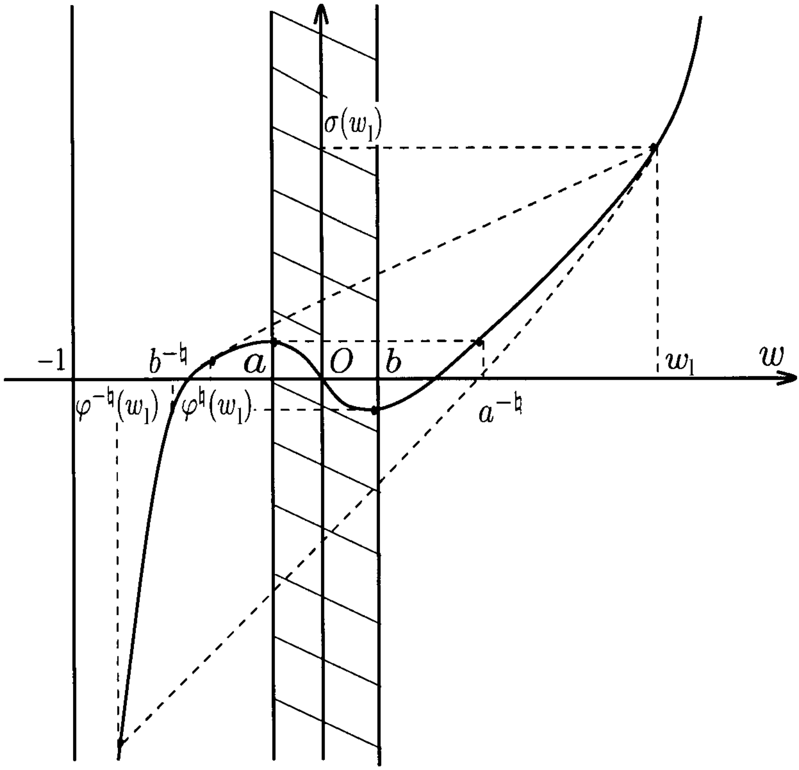

Figure 1. Stress–strain function.

In view of (1.2), (1.3), there exist unique points $a^{-\natural}$ and $b^{-\natural}$ satisfying $b^{-\natural} < a < 0 < b < a^{-\natural}$ and

$$\sigma(a) = \sigma(a^{-\natural}) \quad \text{and} \quad \sigma(b) = \sigma(b^{-\natural}). \tag{2.11}$$

Given a left-hand state $u_l = (v_l, w_l)$ and a right-hand state $u_r = (v_r, w_r)$, we will now determine explicitly the 1-wave curve $\mathcal{W}_1^c(u_l)$ and the 2-wave curve $\mathcal{W}_2^c(u_r)$. By definition, they consist of all states that can be reached through a combination of shocks, rarefactions and/or supersonic phase boundaries. These wave curves are the key to solving the Riemann problem with data u_l and u_r. We will check that the two wave curves intersect at a unique point, i.e.

$$\mathcal{W}_1^c(u_l) \cap \mathcal{W}_2^c(u_r) = \{u_m\},$$

where u_m represents the constant value taken by the Riemann solution in between the two wave fans.

First, the parts of the wave curves corresponding to Liu-admissible shocks are easily identified, as stated below.

LEMMA 2.3. *Consider shock waves satisfying the Liu entropy condition (2.3).*

(i) *The Liu-admissible* 1*-shock waves connecting the left-hand state* u_l *to some right-hand state cover the following section of the* 1*-wave curve,*

$$\mathcal{W}_1^c(u_l) \supset \{u = (v, w) \in \mathcal{H}_1(u_l) / w w_l \leqslant w_l \varphi^{-\natural}(w_l) \ \text{or}\ w w_l \geqslant w_l^2\}.$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(ii) *Similarly, using Liu-admissible* 2*-shocks connecting to the right-hand state* u_{r}*, we obtain*

$$\mathcal{W}_2^c(u_{\mathrm{r}}) \supset \{u = (v, w) \in \mathcal{H}_2(u_{\mathrm{r}}) / w w_{\mathrm{r}} \leqslant w_{\mathrm{r}} \varphi^{-\natural}(w_{\mathrm{r}}) \text{ or } w w_{\mathrm{r}} \geqslant w_{\mathrm{r}}^2\}.$$

Second, we consider rarefaction waves, relying on the geometrical constrain that the wave speed must be strictly increasing inside a rarefaction fan.

Lemma 2.4.

(i) *The* 1*-rarefaction waves connecting the left-hand state* u_{l} *to some right-hand state cover the following section of the* 1*-wave curve. If* $w_{\mathrm{l}} > b$*, then*

$$\mathcal{W}_1^c(u_{\mathrm{l}}) \supset \{(v, w) \in \mathcal{O}_1(u_{\mathrm{l}}) / b \leqslant w \leqslant w_{\mathrm{l}}\}.$$

If $w_{\mathrm{l}} < a$*, then*

$$\mathcal{W}_1^c(u_{\mathrm{l}}) \supset \{(v, w) \in \mathcal{O}_1(u_{\mathrm{l}}) / w_0 \leqslant w \leqslant a\}.$$

(ii) *Similarly, using* 2*-rarefaction waves connecting to the right-hand state* u_{r}*, we obtain the following. If* $w_{\mathrm{r}} > b$*, then*

$$\mathcal{W}_2^c(u_{\mathrm{r}}) \supset \{(v, w) \in \mathcal{O}_2(u_{\mathrm{r}}) / b \leqslant w \leqslant w_{\mathrm{r}}\}.$$

If $w_{\mathrm{r}} < a$*, then*

$$\mathcal{W}_2^c(u_{\mathrm{r}}) \supset \{(v, w) \in \mathcal{O}_2(u_{\mathrm{r}}) / w_{\mathrm{r}} \leqslant w \leqslant a\}.$$

To complete the construction of the wave curves, we now combine shock and rarefaction waves. We assume that $w_{\mathrm{l}} > b$ and $w_{\mathrm{r}} > b$, as the other cases $w_{\mathrm{l}} < a$ and/or $w_{\mathrm{r}} < a$ can be handled similarly. To construct the 1-wave curve $\mathcal{W}_1^c(u_{\mathrm{l}})$, we proceed as follows. If $-1 < w \leqslant \varphi^{-\natural}(w_{\mathrm{l}})$ or if $w \geqslant w_{\mathrm{l}}$, then we use a supersonic phase boundary or a 1-classical shock, respectively, that is a part of the Hugoniot curve $\mathcal{H}_1(u_{\mathrm{l}})$ (see lemma 2.3 (i)). If $b \leqslant w < w_{\mathrm{l}}$, in view of lemma 2.3, we use a 1-rarefaction wave that is a part of the curve $\mathcal{O}_1(u_{\mathrm{l}})$. If $w \in (\varphi^{-\natural}(w_{\mathrm{l}}), b^{-\natural})$, then there exists a unique point $w^* \in (b, w_{\mathrm{l}})$ such that $-\bar{c}(w, w^*) = -c(w^*) = \lambda_1(w^*)$. Namely, $w^* = \varphi^{\natural}(w)$. In this case, the solution curve moves along $\mathcal{O}_1(u_{\mathrm{l}})$ until it reaches the state $u^* = (v^*, w^*) \in \mathcal{O}_1(u_{\mathrm{l}})$, and then jumps on $\mathcal{H}_1(u^*)$ to eventually reach (v, w). Hence we define the corresponding composite curve by

$$\mathcal{K}_1(u_{\mathrm{l}}) := \{(v, w) \in \mathcal{H}_1(u^*) / b \leqslant \varphi^{\natural}(w) \leqslant w_{\mathrm{l}},\ u^* = (v^*, \varphi^{\natural}(w)) \in \mathcal{O}_1(u_{\mathrm{l}})\}. \quad (2.12)$$

On the other hand, the values $w \in [b^{-\natural}, a]$ cannot be reached by combining Liu-admissible shocks and rarefactions. Additional phase boundaries must therefore must allowed in our construction, in order to ensure the existence of a Riemann solution. Following Shearer [15], we now postulate that *stationary phase boundaries* are admissible. These phase boundaries are called *subsonic*, as both characteristic families are transverse to the discontinuity. Note that non-stationary subsonic phase boundaries will be used later too (§ 3).

The function σ is not one-to-one and it is convenient to define two distinct inverse functions. The function $[b^{-\natural}, a] \ni w \mapsto \sigma(w) \in [\sigma(b^{-\natural}), \sigma(a)]$ is concave,

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

monotone increasing and onto. Therefore, it admits an inverse denoted here by $\tau_a : [\sigma(b^{-\natural}), \sigma(a)] \to [b^{-\natural}, a]$. The function τ_a is convex and monotone increasing. Similarly, σ is convex and monotone increasing on the interval $[b, a^{-\natural}]$, and its inverse function is denoted by $\tau_b : [\sigma(b), \sigma(a^{-\natural})] \to [b, a^{-\natural}]$. The function τ_b is concave and monotone increasing. Recall that $a^{-\natural}$ and $b^{-\natural}$ are defined by (2.11). The above functions allow us to define the mapping

$$[b^{-\natural}, a] \cup [b, a^{-\natural}] \ni w \mapsto \varphi_0(w) \in [b^{-\natural}, a] \cup [b, a^{-\natural}]$$

by

$$\varphi_0(w) := \begin{cases} \tau_a(\sigma(w)), & w \in (b, a^{-\natural}], \\ \tau_b(\sigma(w)), & w \in [b^{-\natural}, a), \\ a^{-\natural}, & w = a, \\ b^{-\natural}, & w = b. \end{cases} \tag{2.13}$$

On the interval $[b, a^{-\natural}]$, the function φ_0 is convex and increasing, since it is the composition of two convex and increasing functions. On the interval $[b^{-\natural}, a]$, φ_0 is concave and increasing, since it is the composition of two concave and increasing functions. Moreover, φ_0 is its own inverse, that is,

$$\varphi_0(\varphi_0(w)) = w \quad \text{for all } [b^{-\natural}, a] \cup [b, a^{-\natural}].$$

Now, using stationary phase boundaries, we are able to complete the construction of the 1-wave curve. Every value $w \in [b^{-\natural}, a]$ can be connected to the corresponding state $\varphi_0(w) \in [b, a^{-\natural}]$ by a stationary phase boundary. More generally, we can define a new section of the wave curve $\mathcal{W}_1^c(u_l)$ by using a shock or a rarefaction wave leaving from u_l and remaining in the phase-2 region, followed with a stationary phase boundary. This part of the wave curve is defined as follows.

(1) If $w_l \geqslant a^{-\natural}$,

$$\mathcal{Z}_1(u_l) := \{(v, w)/(v, \varphi_0(w)) \in \mathcal{O}_1(u_l), \text{ with } b \leqslant \varphi_0(w) \leqslant a^{-\natural}\}. \tag{2.14 a}$$

(2) If $w_l \in (b, a^{-\natural})$,

$$\begin{aligned} \mathcal{Z}_1(u_l) := \{(v, w)/(v, \varphi_0(w)) \in \mathcal{O}_1(u_l), \text{ with } b \leqslant \varphi_0(w) < w_l\} \\ \cup \{(v, w)/(v, \varphi_0(w)) \in \mathcal{H}_1(u_l), \text{ with } w_l \leqslant \varphi_0(w) \leqslant a^{-\natural}\}. \end{aligned} \tag{2.14 b}$$

The solution is made of two waves with distinct wave speeds. When the solution contains a rarefaction, the phase boundary is not attached to the rarefaction fan, except in the limiting case $w = b$ and along the curve $\mathcal{K}_1(u_l)$. This completes the construction of the wave curve $\mathcal{W}_1^c(u_l)$.

It is easy to see that, in each of the above cases, the v component is a monotone increasing function of w. So v varies continuously from $-\infty$ to $+\infty$ while w describes the union of intervals $(-1, b^{-\natural}) \cup [b, +\infty)$. The above results are summarized in the proposition below (see figure 2).

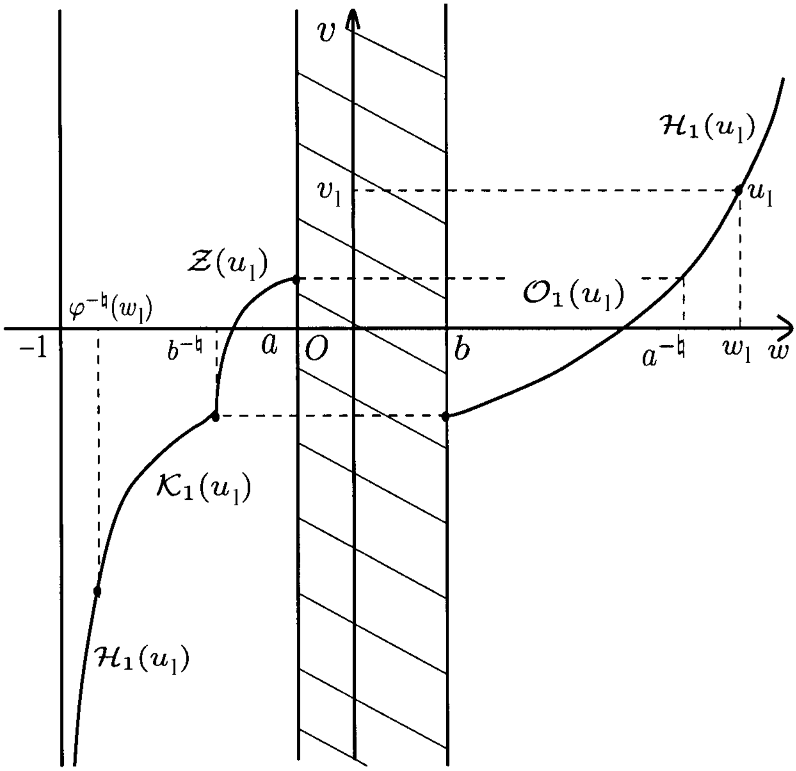

Figure 2. Classical 1-wave curve.

Proposition 2.5 (The 1-wave curve). *Fix a left-hand state* $u_l = (v_l, w_l)$, *with* $w_l > b$. *Based on Liu-admissible shock waves, rarefaction fans, supersonic phase boundaries and stationary phase boundaries, there exists a uniquely defined* 1*-wave curve issuing from* u_l *and given by*

$$\mathcal{W}_1^c(u_l) = \begin{cases} \mathcal{H}_1(u_l), & w \geqslant w_l, \\ \mathcal{O}_1(u_l), & b \leqslant w \leqslant w_l, \\ \mathcal{Z}_1(u_l), & b^{-\natural} \leqslant w \leqslant a, \\ \mathcal{K}_1(u_l), & \varphi^{-\natural}(w_l) \leqslant w \leqslant b^{-\natural}, \\ \mathcal{H}_1(u_l), & -1 < w \leqslant \varphi^{-\natural}(w_l). \end{cases}$$

The wave curve

$$w \in (-1, a] \cup [b, +\infty) \mapsto v = v_1(w) \in \mathbb{R}$$

is continuous, onto, monotone increasing in each of the intervals $(-1, a]$ *and* $[b, +\infty)$, *of class* C^2 *and convex in each interval* $(-1, b^{-\natural})$, $(b^{-\natural}, a)$ *and* $(b, +\infty)$. *Moreover, the wave curve satisfies*

$$v_1(a^{-\natural}) = v_1(a), \quad v_1(b^{-\natural}) = v_1(b),$$
$$\frac{\mathrm{d}v_1}{\mathrm{d}w}(a) = 0, \qquad \frac{\mathrm{d}v_1}{\mathrm{d}w}(b) = 0,$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

and, more precisely,

$$v_1(\varphi_0(w)) = v_1(w) \quad \textit{for all } w \in [b^{-\natural}, a] \cup [b, a^{-\natural}].$$

Similar arguments can be developed to construct the 2-wave curve $\mathcal{W}_2^c(u_\mathrm{r})$, which we omit. Alternatively, we could use the transformation $v \mapsto -v$, $x \mapsto -x$ to construct the (backward) 2-wave curves from the (forward) 1-wave curves. By analogy with (2.12) and (2.14), we define now, for the 2-wave curve,

$$\mathcal{K}_2(u_\mathrm{r}) := \{(v, w) \in \mathcal{H}_2(u^*)/b \leqslant \varphi^\natural(w) \leqslant w_\mathrm{r},\ u^* = (v^*, \varphi^\natural(w)) \in \mathcal{O}_2(u_\mathrm{r})\}$$

and

(1) if $w_\mathrm{r} \geqslant a^{-\natural}$,

$$\mathcal{Z}_2(u_\mathrm{r}) := \{(v, w)/(v, \varphi_0(w)) \in \mathcal{O}_2(u_\mathrm{r}), \text{ with } b \leqslant \varphi_0(w) \leqslant a^{-\natural}\},$$

(2) if $w_\mathrm{r} \in (b, a^{-\natural})$,

$$\begin{aligned}\mathcal{Z}_2(u_\mathrm{r}) := \{(v, w)/(v, \varphi_0(w)) \in \mathcal{O}_2(u_\mathrm{r}), \text{ with } b \leqslant \varphi_0(w) \leqslant w_\mathrm{r}\}\\ \cup \{(v, w)/(v, \varphi_0(w)) \in \mathcal{H}_2(u_\mathrm{r}), \text{ with } w_\mathrm{r} \leqslant \varphi_0(w) \leqslant a^{-\natural}\}.\end{aligned}$$

Proposition 2.6 (The 2-wave curve). *Fix a right-hand state $u_\mathrm{r} = (v_\mathrm{r}, w_\mathrm{r})$ with $w_\mathrm{r} > b$. Based on Liu-admissible shock waves, rarefaction fans, supersonic phase boundaries and stationary phase boundaries, there exists a uniquely defined 2-wave curve issuing from u_r given by*

$$\mathcal{W}_2^c(u_\mathrm{r}) := \begin{cases} \mathcal{H}_2(u_\mathrm{r}), & w \geqslant w_\mathrm{r}, \\ \mathcal{O}_2(u_\mathrm{r}), & b \leqslant w \leqslant w_\mathrm{r}, \\ \mathcal{Z}_2(u_\mathrm{r}), & b^{-\natural} \leqslant w \leqslant a, \\ \mathcal{K}_2(u_\mathrm{r}), & \varphi^{-\natural}(w_\mathrm{r}) \leqslant w \leqslant b^{-\natural}, \\ \mathcal{H}_2(u_\mathrm{r}), & -1 < w \leqslant \varphi^{-\natural}(w_\mathrm{r}). \end{cases}$$

The wave curve

$$w \in (-1, a] \cup [b, +\infty) \mapsto v = v_2(w) \in \mathbb{R}$$

is continuous, onto, monotone decreasing in each of the intervals $(-1, a]$ and $[b, +\infty)$, and of class C^2 and concave in each of the intervals $(-1, b^{-\natural})$, $(b^{-\natural}, a)$ and $(b, +\infty)$. Moreover, the wave curve satisfies

$$v_2(a^{-\natural}) = v_2(a), \quad v_2(b^{-\natural}) = v_2(b),$$
$$\frac{\mathrm{d}v_2}{\mathrm{d}w}(a) = 0, \qquad \frac{\mathrm{d}v_2}{\mathrm{d}w}(b) = 0,$$

and, more precisely,

$$v_2(\varphi_0(w)) = v_2(w) \quad \textit{for all } w \in [b^{-\natural}, a] \cup [b, a^{-\natural}].$$

Similar results as those stated in propositions 2.5 and 2.6 can be obtained when $w_\mathrm{l} < a$ and $w_\mathrm{r} < a$, respectively. Finally, we conclude with the Riemann problem.

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

THEOREM 2.7. *Under the assumption (1.2), the Riemann problem (1.1)–(1.5) admits a unique self-similar solution made of Liu-admissible shock waves, rarefaction fans, supersonic phase boundaries or stationary phase boundaries.*

The following L^1_{loc}-continuous dependence property holds. Denote the solution of (1.1)–(1.5) by $u = u(y)$, with $y = x/t$. Let $u' = u'(y)$ be another Riemann solution corresponding to some other data u'_l and u'_r. Restrict attention to Riemann data remaining in a compact subset and let M be a bound for the maximum wave speed in that region. Then, for some uniform constant $C > 0$, we have the continuous dependence estimate,

$$\int_{-M}^{M} |u(y) - u'(y)| \, \mathrm{d}y \leqslant C(|u_\text{l} - u'_\text{l}| + |u_\text{r} - u'_\text{r}|). \tag{2.15}$$

Proof. Denote by $w \mapsto v_1(w)$ the 1-wave curve of all right-hand states attainable from u_l. Denote by $w \mapsto v_2(w)$ the 2-wave curve of all left-hand states attainable from u_r. In view of propositions 2.5 and 2.6, the gap function

$$\kappa(w) := v_2(w) - v_1(w), \quad w \in (-1, a] \cup [b, +\infty),$$

is monotone decreasing in each of the intervals $(-1, a]$ and $[b, +\infty)$. Solving the Riemann problem is equivalent to finding a root for the function κ. Let us distinguish between three cases.

(i) $\kappa(a) > 0$.

(ii) $\kappa(b) < 0$.

(iii) $\kappa(a) \leqslant 0$ and $\kappa(b) \geqslant 0$.

In case (i), we have $v_1(a) < v_2(a)$. Since the function κ is monotone decreasing in $(-1, a]$ and $\kappa(a) > 0$, it follows that

$$\kappa(w) \geqslant \kappa(a) > 0, \quad w \in (-1, a].$$

Hence the two wave curves cannot meet in the domain $w \in (-1, a]$. On the other hand, for $w \in [b, +\infty)$, propositions 2.5 and 2.6 yield

$$\kappa(a^{-\natural}) = v_2(a^{-\natural}) - v_1(a^{-\natural}) = v_2(a) - v_1(a) > 0.$$

Since κ is monotone decreasing in the interval $[b, +\infty)$ (which contains $a^{-\natural}$), and since $\kappa(w) \to -\infty$ as $w \to +\infty$, we conclude that there exists a unique point $w_\text{m} \in (a^{-\natural}, +\infty)$ such that

$$\kappa(w_\text{m}) = 0.$$

In conclusion, the two wave curves have a unique intersection point $(v_1(w_\text{m}), w_\text{m}) = (v_2(w_\text{m}), w_\text{m})$. Note that, since $w_\text{m} > a^{-\natural}$, the Riemann solution cannot contain a stationary phase boundary.

Dealing with case (ii), for which $v_1(b) > v_2(b)$, is similar. We obtain a unique intersection point $(v_1(w_\text{m}), w_\text{m}) = (v_2(w_\text{m}), w_\text{m})$ with $w_\text{m} \in (-1, b^{-\natural})$.

Finally, consider case (iii), for which we have $v_1(a) \geqslant v_2(a)$ and $v_1(b) \leqslant v_2(b)$. We find

$$\kappa(a) \geqslant 0, \qquad \kappa(b^{-\natural}) = \kappa(b) \leqslant 0$$

and

$$\kappa(b) \leqslant 0, \qquad \kappa(a^{-\natural}) = \kappa(a) \geqslant 0.$$

Therefore, there exist two points $w_{\rm m} \in [b, a^{-\natural}]$ and $w'_{\rm m} \in [b^{-\natural}, a]$ such that

$$\kappa(w_{\rm m}) = \kappa(w'_{\rm m}) = 0.$$

Since the function κ is monotone, these two points are unique and

$$\kappa(\varphi_0(w_{\rm m})) = \kappa(w_{\rm m}) = 0.$$

By a uniqueness argument, we also obtain

$$w'_{\rm m} = \varphi_0(w_{\rm m}).$$

Hence the two wave curves intersect at exactly two points $u_1 := (v_1(w_{\rm m}), w_{\rm m}) = (v_2(w_{\rm m}), w_{\rm m})$ and $u_2 := (v_1(w_{\rm m}), \varphi_0(w_{\rm m})) = (v_2(w_{\rm m}), \varphi_0(w_{\rm m}))$. Note that these two points correspond to solutions having stationary phase boundaries. Recall also that, by construction, a stationary phase boundary connects two points such that (v, w) and $(v, \varphi_0(w))$. The key observation here is that actually both states u_1 and u_2 provide us with the *same Riemann solution*.

First, if the given states $u_{\rm l}$ and $u_{\rm r}$ belong to the same phase, say $w_{\rm l}$, $w_{\rm r} \geqslant b$, then $u_2 \in \mathcal{Z}_1(u_{\rm l}) \cap \mathcal{Z}_2(u_{\rm r})$. The point u_2 cannot be observed in the (x, t)-plane, since, by construction, it would correspond to using a stationary phase boundary from phase 1 to phase 2, followed with another stationary phase boundary from phase 2 to phase 1. These two waves have the same speed and do not separate in the (x, t)-plane. If the given states $u_{\rm l}$ and $u_{\rm r}$ belong to different phases, i.e. $w_{\rm l} \geqslant b$ and $w_{\rm r} \leqslant a$, the solution does contain a stationary phase boundary, precisely the one connecting u_1 to u_2. Therefore, the two points of intersection simply correspond to two ways to describe the solution: the stationary wave being counted together with the non-negative waves or with the non-positive ones.

This completes the description of the Riemann solution.

It can be checked that all the values in the range of the Riemann solutions are continuous functions of the initial data, at least as long as the structure of the Riemann solution remains the same. Importantly, the wave speeds arising in the Riemann solution always change continuously. This is clearly true in each region where the wave curve is smooth. On the other hand, one can check directly on the explicit formulae that the wave speeds are continuous at the critical points where the structure of the Riemann solution changes. This implies the continuous dependence property stated in the theorem. The proof of theorem 2.7 is complete. □

Based on propositions 2.5 and 2.6, we now distinguish between the values of the parameters $u_{\rm l}$ and $u_{\rm r}$ and we list all the possible wave structures of the Riemann solution. The following notation will be used for 1-waves. A classical shock wave connecting a left-hand state u_0 to a right-hand state u_1 is denoted by $C_1(u_0, u_1)$, a supersonic phase boundary by $P_1^{\rm super}(u_0, u_1)$, a rarefaction wave by $R_1(u_0, u_1)$ and a stationary phase boundary by $Z(u_0, u_1)$. A similar notation will be used for 2-waves. Note that stationary phase boundaries can be considered as 1-waves or 2-waves.

Solving the Riemann problem is equivalent to solving the algebraic equation

$$v_1(w) = v_2(w), \quad w \in (-1, +\infty). \tag{2.16}$$

For definiteness, let us assume $w_l \geqslant b$. To state that a Riemann solution is made of a wave $A(u_1, u_2)$ followed by some wave $B(u_2, u_3)$, we simply write

$$A(u_1, u_2) + B(u_2, u_3).$$

Here is the complete classification of Riemann solutions.

Case I. Assume that $v_1(a) < v_2(a)$, so that (2.16) admits a unique solution w_m satisfying $w_m > a^{-\natural}$.

On one hand, if the states u_l, u_r belong to the same phase (thus $w_l, w_r \geqslant b$), then we have the following.

(I.1) If $w_l \geqslant w_m$ and $w_r \geqslant w_m$, the Riemann solution is $R_1(u_l, u_m) + R_2(u_m, u_r)$.

(I.2) If $w_l \geqslant w_m$ and $b \leqslant w_r < w_m$, then $R_1(u_l, u_m) + C_2(u_m, u_r)$.

(I.3) If $b \leqslant w_l < w_m$ and $w_r \geqslant w_m$, then $C_1(u_l, u_m) + R_2(u_m, u_r)$.

(I.4) If $b \leqslant w_l < w_m$ and $b \leqslant w_r < w_m$, then $C_1(u_l, u_m) + C_2(u_m, u_r)$.

On the other hand, if the states u_l, u_r lie in the different phases (thus $w_l \geqslant b$ and $-1 < w_r \leqslant a$), then we have the following.

(I.5) If $w_l \geqslant w_m$, then $R_1(u_l, u_m) + P_2^{\text{super}}(u_m, u_r)$.

(I.6) If $b \leqslant w_l < w_m$, then $C_1(u_l, u_m) + P_2^{\text{super}}(u_m, u_r)$.

Case II. Suppose next that $v_1(b) > v_2(b)$, so that (2.16) admits a unique solution w_m satisfying $-1 < w_m < a^{-\natural}$.

(II.1) If $-1 < w_r \leqslant w_m$, then the Riemann solution is $P_1^{\text{super}}(u_l, u_m) + R_2(u_m, u_r)$.

(II.2) If $w_m < w_r \leqslant a$, then $P_1^{\text{super}}(u_l, u_m) + C_2(u_m, u_r)$.

Case III. Suppose, finally, that $v_1(a) \geqslant v_2(a)$ and $v_1(b) \leqslant v_2(b)$, so that (2.16) now admits exactly two solutions w_m and $\varphi_0(w_m)$, where $w_m \in [b, a^{-\natural}]$.

If the Riemann data u_l and u_r belong to the same phase, thus $w_l, w_r \geqslant b$, then we have the following.

(III.1) If $w_l \geqslant w_m$ and $w_r \geqslant w_m$, then the Riemann solution is made of

$$R_1(u_l, u_m) + R_2(u_m, u_r).$$

(III.2) If $w_l \geqslant w_m$ and $b \leqslant w_r < w_m$, then $R_1(u_l, u_m) + C_2(u_m, u_r)$.

(III.3) If $b \leqslant w_l < w_m$ and $w_r \geqslant w_m$, then $C_1(u_l, u_m) + R_2(u_m, u_r)$.

(III.4) If $b \leqslant w_l < w_m$ and $b \leqslant w_r < w_m$, then

$$C_1(u_l, u_m) + C_2(u_m, u_r).$$

If the states u_l and u_r belong to different phases (thus $w_l \geqslant b$ and $-1 < w_r \leqslant a$), we have the following.

(III.5) If $w_l \geqslant w_m$ and $-1 < w_r \leqslant \varphi_0(w_m)$, then the Riemann solution is made of

$$R_1(u_l, u_m) + Z(u_m, \varphi_0(u_m)) + R_2(\varphi_0(u_m), u_r),$$

where $\varphi_0(u_m) := (v_1(w_m), \varphi_0(w_m))$.

(III.6) If $w_l \geqslant w_m$ and $a \geqslant w_r > \varphi_0(w_m)$, then

$$R_1(u_l, u_m) + Z(u_m, \varphi_0(u_m)) + C_2(\varphi_0(u_m), u_r).$$

(III.7) If $b \leqslant w_l \leqslant w_m$ and $-1 < w_r \leqslant \varphi_0(w_m)$, then

$$C_1(u_l, u_m) + Z(u_m, \varphi_0(u_m)) + R_2(\varphi_0(u_m), u_r).$$

(III.8) If $b \leqslant w_l \geqslant w_m$ and $a \geqslant w_r > \varphi_0(w_m)$, then

$$C_1(u_l, u_m) + Z(u_m, \varphi_0(u_m)) + C_2(\varphi_0(u_m), u_r).$$

3. Non-classical wave sets based on the entropy inequality

We now return to our initial goal, that is, to describe the family of all Riemann solutions that are solely consistent with the entropy inequality (1.6). We prove in this section that instead of wave curves, we can determine two-dimensional (non-classical) wave sets for each of the two wave families. We focus on the 1-waves and on the construction of the set of all right-hand states attainable via 1-waves only. Relying on the transformation $v \mapsto -v, x \mapsto -x$, it is an easy matter to deduce from the following results similar conclusion for the 2-waves.

First of all, let us investigate the behaviour of the entropy dissipation $E(u_0; u_1)$ given by

$$E(u_0; u_1) := -s(\tfrac{1}{2}(v_1^2 - v_0^2) + \Sigma(w_1) - \Sigma(w_0)) - v_1\sigma(w_1) + v_0\sigma(w_0)$$

for a shock wave with speed s connecting a left-hand state $u_0 = (v_0, w_0)$ to some right-hand state $u_1 = (v_1, w_1)$. The Rankine–Hugoniot relations (2.1), (2.2) lead us to the simpler expression

$$E(v_0, w_0; v_1, w_1) = -s(\Sigma(w_1) - \Sigma(w_0) - \tfrac{1}{2}(\sigma(w_1) + \sigma(w_0))(w_1 - w_0)),$$

which, in particular, tell us that E is independent of v_0 and v_1. In the following, we simply write $E = E(w_0, w_1)$.

In view of the entropy condition (1.6) and the left-hand state u_0 being fixed, our main objective is to determine all the values u_1 for which

$$E(w_0, w_1) = -s(\Sigma(w_1) - \Sigma(w_0) - \tfrac{1}{2}(\sigma(w_1) + \sigma(w_0))(w_1 - w_0)) \leqslant 0. \qquad (3.1)$$

The entropy dissipation function under study is formally analogous to the one associated with the scalar conservation law

$$w_t + \sigma(w)_x = 0,$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

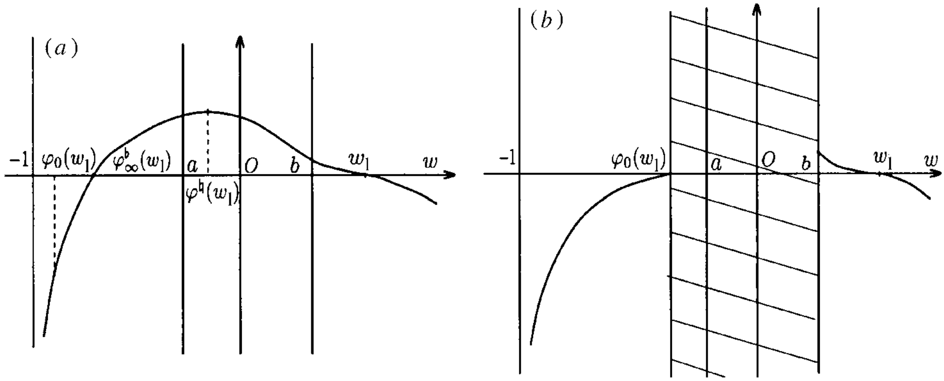

Figure 3. $b \leqslant w_1 < c$. (a) Scalar case. (b) System case.

together with the quadratic entropy $U(w) = \frac{1}{2}w^2$. The corresponding entropy dissipation function is

$$F(w_0, w_1) := \Sigma(w_1) - \Sigma(w_0) - \tfrac{1}{2}(\sigma(w_1) + \sigma(w_0))(w_1 - w_0),$$

which is indeed a factor in (3.1). Namely (for shocks propagating with negative speed), we find

$$E(w_0, w_1) = \sqrt{\frac{\sigma(w_1) - \sigma(w_0)}{w_1 - w_0}}F(w_0, w_1) \leqslant 0. \tag{3.1$'$}$$

It is not difficult to check that F is the area limited by the graph of σ and the line connecting the points with coordinates w_0 and w_1.

From [8] we recall the following.

Lemma 3.1. *The function* $F(w_0, w_1)$ *is monotone increasing in* $(-1, \varphi^\natural(w_0)]$ *and monotone decreasing in* $[\varphi^\natural(w_0), +\infty)$. *More precisely, we have*

$$\begin{aligned} \partial_{w_1} F(w_0, \cdot) > 0 \quad &\textit{in the interval } (-1, \varphi^\natural(w_0)), \\ \partial_{w_1} F(w_0, \cdot) < 0 \quad &\textit{in the interval } (\varphi^\natural(w_0), +\infty), \end{aligned}$$

and

$$F(w_0, w_0) = 0, \qquad F(w_0, \varphi^\natural(w_0)) > 0, \qquad F(w_0, \varphi^{-\natural}(w_0)) < 0.$$

Therefore, there exists a value $\varphi^\flat_\infty(w_0)$ *satisfying*

$$F(w_0, \varphi^\flat_\infty(w_0)) = 0, \quad w_0\varphi^\flat_\infty(w_0) \in (w_0\varphi^{-\natural}(w_0), w_0\varphi^\natural(w_0)). \tag{3.2}$$

Moreover, the function $\varphi^\flat_\infty : (-1, +\infty) \to (-1, +\infty)$ *is monotone decreasing (as are both functions* $\varphi^\natural$ *and* $\varphi^{-\natural}$*),*

$$\frac{\mathrm{d}\varphi^\flat_\infty}{\mathrm{d}w}(w) < 0 \quad \textit{for all } w \in (-1, +\infty).$$

(See part (a) of figures 3, 4, 5 and 6.)

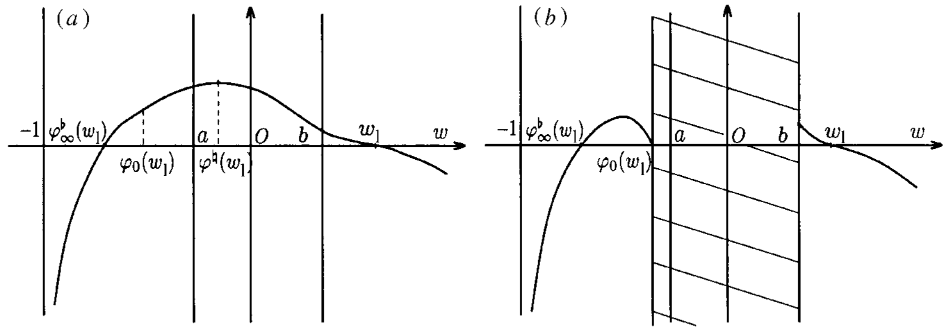

Figure 4. $c \leqslant w_l < a^{-\natural}$. ($a$) Scalar case. ($b$) System case.

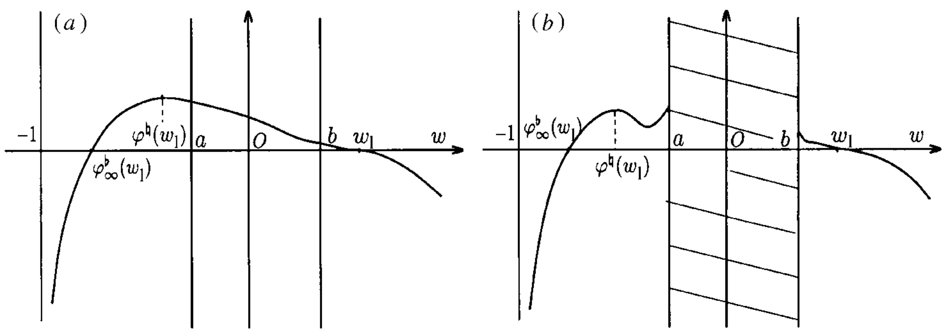

Figure 5. $a^{-\natural} < w_l < d$. (a) Scalar case. (b) System case.

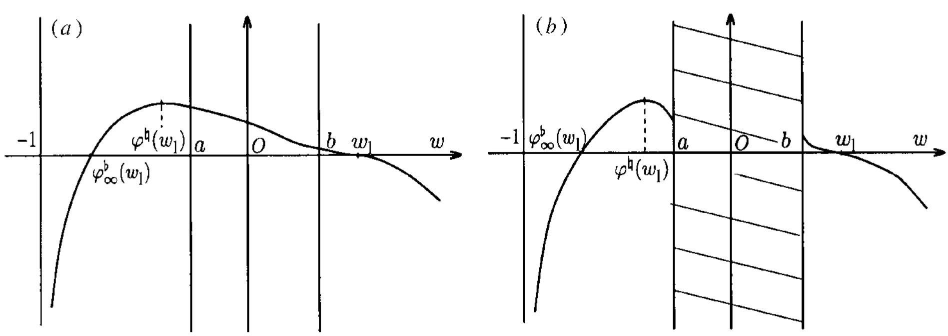

Figure 6. $w_l \geqslant d$. (a) Scalar case. (b) System case.

The function $\varphi^\flat_\infty : (-1, +\infty) \to (-1, +\infty)$ in (3.2) is of class C^1 (by the implicit function theorem), is monotone decreasing and is its own inverse,

$$\varphi^\flat_\infty(\varphi^\flat_\infty(w)) = w, \quad w \in (-1, +\infty).$$

The later property is a direct consequence of the skew-symmetry of F,

$$F(\varphi^\flat_\infty(w), w) = -F(w, \varphi^\flat_\infty(w)) = 0,$$

and of

$$F(\varphi^\flat_\infty(w), \varphi^\flat_\infty(\varphi^\flat_\infty(w))) = 0.$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

The entropy dissipation function $E(w_0, w_1)$ of interest here has a somewhat different behaviour, due to the factor s in the expression (3.1′). Clearly, E vanishes whenever the shock speed s vanishes, so that E may have three roots instead of two. Recalling that the speed s must be a real number, we also see that the constraint

$$s^2 = \frac{\sigma(w_1) - \sigma(w_0)}{w_1 - w_0} \geqslant 0$$

should also be taken into account. With a notation introduced in § 2, this is equivalent to saying $w_1 w_0 \notin (\varphi_0(w_0)w_0, w_0^2)$. On the other hand, by lemma 3.1, the value $w_1 = \varphi_\infty^\flat(w_0)$ is a root of F, and thus of E, provided this value belongs to the interval of interest, that is, $w_0\varphi_\infty^\flat(w_0) \notin (w_0\varphi_0(w_0), w_0^2)$. These observations suggest us to study the equation

$$\varphi_\infty^\flat(w) = \varphi_0(w). \tag{3.3}$$

To this end, we have the following result.

PROPOSITION 3.2. *Equation (3.3) admits a unique value $c \in (b, a^{-\natural})$ satisfying*

$$\varphi_\infty^\flat(c) = \varphi_0(c).$$

The value $\varphi_0(c) \in (b^{-\natural}, a)$ is also a solution of the same equation. By the monotonicity properties of the function $\varphi_\infty^\flat - \varphi_0$ on each of the intervals $[b^{-\natural}, a]$ and $[b, a^{-\natural}]$, we also have

$$\begin{aligned} \varphi_\infty^\flat(w) < \varphi_0(w) \quad & \textit{for } w \in (\varphi_0(c), a] \cup [c, a^{-\natural}), \\ \varphi_\infty^\flat(w) > \varphi_0(w) \quad & \textit{for } w \in [b^{-\natural}, \varphi_0(c)] \cup [a, c). \end{aligned}$$

Proof. First assume $w > 0$. Recall that, by lemma 3.1, the function $F(w_0, w_1)$ vanishes exactly when $w_1 = w_0$ or $w_1 = \varphi_\infty^\flat(w_0)$. Hence it is sufficient for our purpose to check that the equation

$$F(w, \varphi_0(w)) = 0$$

has exactly one solution $w = c \in (b, a^{-\natural})$. For all $w \in [b, a^{-\natural}]$, consider

$$G(w) := F(w, \varphi_0(w)) = \int_{\varphi_0(w)}^{w} \sigma(\tau)\,\mathrm{d}\tau - \sigma(w)(w - \varphi_0(w)).$$

Clearly, the function G is monotone increasing, and we have that $G(b) < 0$ and $G(a^{-\natural}) > 0$. Consequently, there exists a unique value $w = c \in (b, a^{-\natural})$ such that

$$F(c, \varphi_0(c)) = G(c) = 0.$$

By lemma 3.1, we get $\varphi_0(c) = \varphi_\infty^\flat(c)$, which completes the proof of proposition 3.2. □

The behaviour of the function E is determined now, from lemma 3.1 and proposition 3.2 (see part (b) of figures 3, 4, 5 and 6).

THEOREM 3.3. *Consider the entropy dissipation function $E = E(w_0, w_1)$ associated with* 1*-shock waves and restrict attention to values $w_0 \geqslant b$. We distinguish between the following three cases.*

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(i) *If* $w_0 \in [b, c)$, *the entropy dissipation* $E(w_0, w_1)$ *is monotone increasing in* $(-1, \varphi_0(w_0)]$, *monotone decreasing in* $[b, +\infty)$ *and vanishes at exactly two points, i.e.*

$$E(w_0, \varphi_0(w_0)) = E(w_0, w_0) = 0.$$

(ii) *If* $w_0 \in [c, a^{-\natural}]$, *the entropy dissipation* $E(w_0, w_1)$ *is monotone increasing in* $(-1, \theta(w_0)]$, *monotone decreasing in* $[\theta(w_0), \varphi_0(w_0)]$ *and in* $[b, +\infty)$, *and vanishes at exactly three points, with, in particular,*

$$E(w_0, \theta(w_0)) > E(w_0, \varphi_0(w_0)) = E(w_0, \varphi_\infty^\flat(w_0)) = E(w_0, w_0) = 0.$$

(iii) *If* $w_0 > a^{-\natural}$, *the entropy dissipation* $E(w_0, w_1)$ *is monotone increasing in* $(-1, \varphi_\infty^\flat(w_0)]$, *monotone decreasing in* $[b, +\infty)$ *and vanishes at exactly two points, i.e.*

$$E(w_0, \varphi_\infty^\flat(w_0)) = E(w_0, w_0) = 0.$$

For $w_0 \leqslant a$, *the same properties hold, provided we replace* a, $a^{-\natural}$, b *and* c *with* b, $b^{-\natural}$, a *and* $\varphi_0(c)$, *respectively.*

Proof. We already know that $w_1 = w_0$, $\varphi_0(w_0)$ or $\varphi_\infty^\flat(w_0)$ are the roots of E. However, only values remaining in the intervals of interest are relevant. The following arguments focus on deriving the monotonicity of E. First of all, a simple calculation yields

$$\frac{\partial E(w_0, w_1)}{\partial w_1} = \frac{1}{2}\sqrt{\frac{w_1 - w_0}{\sigma(w_1) - \sigma(w_0)}}\frac{1}{(w - w_0)^2} M(w_0, w_1) N(w_0, w_1), \tag{3.4}$$

where

$$M(w_0, w_1) := \sigma(w_1) - \sigma(w_0) - \sigma'(w_1)(w_1 - w_0)$$

and

$$N(w_0, w_1) := 2\int_{w_1}^{w_0} \sigma(w)\,\mathrm{d}w + (3\sigma(w_1) - \sigma(w_0))(w_1 - w_0).$$

Therefore, the sign of E depends on the signs of M and N, which we now investigate.

For the function M, we find

$$\frac{\partial M}{\partial w_1}(w_0, w_1) = -\sigma''(w_1)(w_1 - w_0) > 0 \quad \text{if and only if } 0 < w_1 < w_0.$$

Clearly, from the definition of $\varphi^\natural(w_0)$, we get

$$M(w_0, \varphi^\natural(w_0)) = M(w_0, w_0) = 0.$$

Hence the previous inequality gives us

$$\left.\begin{aligned} &M(w_0, w_1) < 0 \quad \text{for all } w_1 > \varphi^\natural(w_0), \quad w_1 \neq w_0, \\ &M(w_0, w_1) > 0 \quad \text{for all } w_1 < \varphi^\natural(w_0), \quad \text{where } \varphi^\natural(w_0) < 0. \end{aligned}\right\} \tag{3.5}$$

For the function N, we have

$$\frac{\partial N}{\partial w_1}(w_0,w_1) = (w_1-w_0)\bigg(3\sigma'(w_1)+\frac{\sigma(w_1)-\sigma(w_0)}{w_1-w_0}\bigg) > 0 \quad \text{if and only if } w_1 > w_0. \tag{3.6}$$

In the following we study the entropy dissipation E in each interval: $w_1 \in [b,+\infty)$ and $w_1 \in (-1,a]$.

Case A. The interval $w_1 \in [b,+\infty)$.

In view of (3.6), the function N achieves a strict minimum in this interval at the point w_0, i.e.

$$N(w_0,w_1) > N(w_0,w_0) = 0 \quad \text{for all } w_1 \neq w_0.$$

In view of (3.5) and the latter inequality, equation (3.4) yields

$$\frac{\partial E(w_0,w_1)}{\partial w_1} < 0 \quad \text{for all } w_1 \neq w_0. \tag{3.7}$$

Clearly, inequality (3.7) implies that E is strictly monotone in the interval under consideration.

Case B. The interval $w_1 \in (-1,a]$.

Case B.I. Assume also that $w_0 \in [b,c]$.

The function N is well defined only in the interval $w_1 \in (-1,\varphi_0(w_0)]$. It follows from (3.6) that N is decreasing in this interval and, therefore, its (strict) minimum value is $N(w_0,\varphi_0(w_0)) = 0$. In other words, we have

$$N(w_0,w_1) > 0 \quad \text{for all } w_1 \neq \varphi_0(w_0).$$

By (3.5) and the above inequality, equation (3.4) gives

$$\frac{\partial E(w_0,w_1)}{\partial w_1} > 0 \quad \text{for all } w_1 \neq \varphi_0(w_0),$$

which establishes that E is strictly increasing.

Case B.II. Assume also that $w_0 \in [c,a^{-\natural}]$.

The entropy dissipation E is well defined only if $w_1 \in (-1,\varphi_0(w_0)]$. Then we have $\varphi^\natural(w_0) \in [a,0)$. In view of proposition 3.2, we have $\varphi^\flat_\infty(w_0) \leqslant \varphi_0(w_0)$, and from (3.6),

$$\min_{w_1\in(-1,\varphi_0(w_0)]} N(w_0,w_1) = N(w_0,\varphi_0(w_0)) = G(w_0) < 0. \tag{3.8}$$

It is not difficult to see that

$$N(w_0,w_1) \to +\infty \quad \text{as } w_1 \to -1. \tag{3.9}$$

Thus from (3.6), (3.8) and (3.9), we deduce that there exists a unique point, depending on w_0 and denoted by $\theta(w_0) \in (-1,\varphi_0(w_0))$, such that

$$\left.\begin{aligned} &N(w_0,w_1) = 0, && w_1 = \theta(w_0),\\ &N(w_0,w_1) > 0 && \text{for all } w_1 \in (-1,\theta(w_0)),\\ &N(w_0,w_1) < 0 && \text{for all } w_1 \in (\theta(w_0),\varphi_0(w_0)).\end{aligned}\right\} \tag{3.10}$$

Combining (3.5) and (3.10) proves the monotonicity of E.

CASE B.III. Assume also that $w_0 > a^{-\natural}$.

In this case, the function N is monotone decreasing in the interval $(-1, a]$ and thus achieves a strict minimum value at $w_1 = a$, i.e.

$$N(w_0, w_1) > N(w_0, a) \quad \text{for all } w_1 \neq w_0.$$

One needs to investigate the sign of $N(w_0, a)$. It is not difficult to check that the function $w \mapsto N(w, a)$ is monotone increasing in the interval $w \in [a^{-\natural}, +\infty)$, that $N(w, a) \to +\infty$ as $w \to +\infty$ and that

$$N(a^{-\natural}, a) = 2\Big(\int_a^{a^{-\natural}} \sigma(w)\,\mathrm{d}w - \sigma(a)(a^{-\natural} - a)\Big) < 0.$$

Thus there exists a unique root $w = d \in (a^{-\natural}, +\infty)$ of $N(w, a)$,

$$N(d, a) = 0.$$

From this observation, we see that if $w_0 \in [a^{-\natural}, d)$, then $N(w_0, a) < 0$ and, therefore, there exists a unique value, still denoted by $\theta(w_0)$, such that

$$\left.\begin{aligned} &N(w_0, w_1) = 0, && w_1 = \theta(w_0), \\ &N(w_0, w_1) > 0, && w_1 \in (-1, \theta(w_0)), \\ &N(w_0, w_1) < 0, && w_1 \in (\theta(w_0), \varphi_0(w_0)). \end{aligned}\right\} \tag{3.11}$$

From (3.5) and (3.11), one obtains

$$\left.\begin{aligned} &\frac{\partial E(w_0, w_1)}{\partial w_1} > 0 \quad \text{for all } w_1 \in (\min\{\varphi^\natural(w_0), \theta(w_0)\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \max\{\varphi^\natural(w_0), \theta(w_0)\}), \\ &\frac{\partial E(w_0, w_1)}{\partial w_1} \leqslant 0 \quad \text{elsewhere.} \end{aligned}\right\} \tag{3.12}$$

This implies the desired monotonicity property of E. □

Theorem 3.3 provides us with a complete description of the sign of the entropy dissipation. We see that the two functions, $\varphi_\infty^\flat$ and φ_0, play a central role, as they correspond to the (non-trivial) roots of E. However, not both roots are always of interest. The points with component $w \in (\varphi_0(w_0), \varphi_\infty^\flat(w_0))$ cannot be reached, since the associated shock speed would be complex. Based on these considerations, it seems natural to introduce the following (continuous) function, which determines the attainable region:

$$\varphi_{\infty,0}^\flat(w) = \begin{cases} \varphi_\infty^\flat(w), & w \in (-1, \varphi_0(c)] \cup [c, +\infty), \\ \varphi_0(w), & w \in (\varphi_0(c), a] \cup [b, c). \end{cases} \tag{3.13}$$

In view of theorem 3.3, we arrive at the following conclusion.

LEMMA 3.4 (Characterization of propagating discontinuities).

(i) *A left-hand state $u_0 = (v_0, w_0)$ being fixed, the right-hand states $u_1 = (v_1, w_1)$ that can be reached, using propagating discontinuities having non-positive*

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

speeds and satisfying the entropy inequality (1.6), are characterized by

$$w_0 w_1 \leqslant w_0 \varphi^\flat_{\infty,0}(w_0), \quad w_0 w_1 \geqslant w_0^2 \quad \textit{or} \quad w_1 = \varphi_0(w_0). \tag{3.14}$$

(ii) *A right-hand state $u_1 = (v_1, w_1)$ being fixed, the left-hand states $u_0 = (v_0, w_0)$ that can be reached, using propagating discontinuities having non-negative speeds and satisfying the entropy inequality (1.6), are characterized by*

$$w_1 w_0 \leqslant w_1 \varphi^\flat_{\infty,0}(w_1), \quad w_1 w_0 \geqslant w_1^2 \quad \textit{or} \quad w_0 = \varphi_0(w_1). \tag{3.15}$$

We will use the following terminology.

Definition 3.5. A propagating discontinuity connecting a left-hand state u_0 to a left-hand state u_1 is called a subsonic *phase boundary* if it satisfies the entropy inequality (1.6) but does not satisfies the Liu entropy criterion. We denote such waves by $P_1^{\text{sub}}(u_0, u_1)$ or $P_2^{\text{sub}}(u_0, u_1)$ when they have non-zero propagating speeds. Recall that stationary phase boundaries were denoted by $Z(u_0, u_1)$.

Combining lemmas 2.2 and 3.4 yields a characterization of subsonic phase boundaries.

Proposition 3.6 (Characterization of subsonic phase boundaries).

(i) *Given a left-hand state $u_0 = (v_0, w_0)$, the set of all right-hand states $u_1 = (v_1, w_1)$ associated with subsonic* 1*-phase boundaries is determined by*

$$w_0 \varphi^{-\natural}(w_0) < w_0 w_1 \leqslant w_0 \varphi^\flat_{\infty,0}(w_0) \quad \textit{or} \quad w_1 = \varphi_0(w_0). \tag{3.16}$$

(ii) *Given a right-hand state $u_1 = (v_1, w_1)$, the set of all left-hand states $u_0 = (v_0, w_0)$ associated with subsonic* 2*-phase boundaries is determined by*

$$w_1 \varphi^{-\natural}(w_1) < w_1 w_0 \leqslant w_1 \varphi^\flat_{\infty,0}(w_1) \quad \textit{or} \quad w_0 = \varphi_0(w_1). \tag{3.17}$$

It will be convenient to introduce the continuous mapping

$$\varphi^\sharp : (w, w^*) \in (-1, +\infty)^2 \mapsto \varphi^\sharp(w^*, w) \in (-1, +\infty),$$

defined for $w^* \neq w, \varphi^\natural(w), \varphi^{-\natural}(w)$ by

$$\frac{\sigma(w) - \sigma(w^*)}{w - w^*} = \frac{\sigma(\varphi^\sharp(w^*, w)) - \sigma(w^*)}{\varphi^\sharp(w^*, w) - w^*}, \tag{3.18}$$

and extended by continuity. In other words, the points w, w^* and $\varphi^\sharp(w^*, w)$ on the graph of f are aligned. This mapping arises when attempting to combine two shock waves: indeed, it corresponds to the limiting case where the wave connecting w to $\varphi^\sharp(w^*, w)$ travels at the same speed as the wave connecting $\varphi^\sharp(w^*, w)$ to w^*.

Keeping w^* fixed, one can check that the function $\varphi^\sharp$ is monotone decreasing in w.

We now start the main construction of this section by focusing mainly on the 1-wave curve (for definiteness). So (v_0, w_0) denotes a fixed left-hand state. We investigate under what conditions a rarefaction wave or a shock wave can be followed with another wave. It is easy to see that the second wave can only be a shock, either a classical one or a (subsonic, supersonic) phase boundary, as described now.

LEMMA 3.7. *A rarefaction wave connecting u_0 to some $u_1 = (v_1, w_1) \in \mathcal{O}_1(u_0)$, with, of course, $0 \leqslant w_1 w_0 \leqslant w_0^2$, can be followed with a shock wave connecting u_1 to some point $u_2 = (v_2, w_2) \in \mathcal{H}_1(u_1)$, provided we have the inequalities*

$$w_1 \varphi^{-\natural}(w_1) \leqslant w_1 w_2 \leqslant w_1 \varphi^{\flat}_{\infty,0}(w_1), \tag{3.19}$$

or else, when $w_1 \in [b, a^{-\natural}]$, the second wave can be a stationary phase boundary.

Proof. We restrict attention to the case $w_0 \geqslant b$. Condition (3.19) rewrites as

$$\varphi^{-\natural}(w_1) \leqslant w_2 \leqslant \varphi^{\flat}_{\infty,0}(w_1). \tag{3.20}$$

On one hand, the assumption $u_2 \in \mathcal{H}_1(u_1)$ with $w_2 \leqslant 0$ yields

$$-1 < w_2 \leqslant \varphi^{\flat}_{\infty,0}(w_1). \tag{3.21}$$

On the other hand, the constraint that the shock from (v_1, w_1) to (v_2, w_2) must follow the rarefaction wave connecting u_0 to u_1 implies that the characteristic speeds in the rarefaction fan be smaller than the shock speed $s_1(w_1, w_2)$. In other words, we must have

$$-\bar{c}(w_1, w_2) \geqslant -c(w_1),$$

or, equivalently,

$$\sigma'(w_1)(w_1 - w_2) - \sigma(w_1) + \sigma(w_2) \geqslant 0.$$

Thanks to (3.10), the last inequality holds true provided

$$w_1 \geqslant \varphi^{\natural}(w_2).$$

Since the function $\varphi^{-\natural}$ is decreasing, we get

$$w_2 \geqslant \varphi^{-\natural}(w_1). \tag{3.22}$$

Combining (3.21) and (3.22) gives (3.20), which completes the proof. □

LEMMA 3.8. *A classical shock connecting the left-hand state u_0 to $u_1 = (v_1, w_1) \in \mathcal{H}_1(u_0)$, with, of course, $w_0 w_1 \geqslant w_0^2$, can be followed with a subsonic 1-phase boundaries connecting to some $u_2 = (v_2, w_2) \in \mathcal{H}_1(u_1)$, provided*

$$w_0 \varphi^{\sharp}(w_1, w_0) \leqslant w_0 w_2 \leqslant w_0 \varphi^{\flat}_{\infty,0}(w_1). \tag{3.23}$$

Proof. We only treat the case $w_0 \geqslant b$, so we have $w_1 \geqslant w_0$. The inequalities (3.23) become

$$\varphi^{\sharp}(w_1, w_0) \leqslant w_2 \leqslant \varphi^{\flat}_{\infty,0}(w_1). \tag{3.24}$$

On one hand, in view of (3.14), the condition $u_2 \in \mathcal{H}_1(u_1)$ imposes

$$w_2 \leqslant \varphi^{\flat}_{\infty,0}(w_1). \tag{3.25}$$

On the other hand, in order for the subsonic phase boundary to follow the classical one, we need

$$s_1(w_0, w_1) \leqslant s_1(w_1, w_2),$$

that is,

$$w_2 \geqslant \varphi^{\sharp}(w_1, w_0). \tag{3.26}$$

□

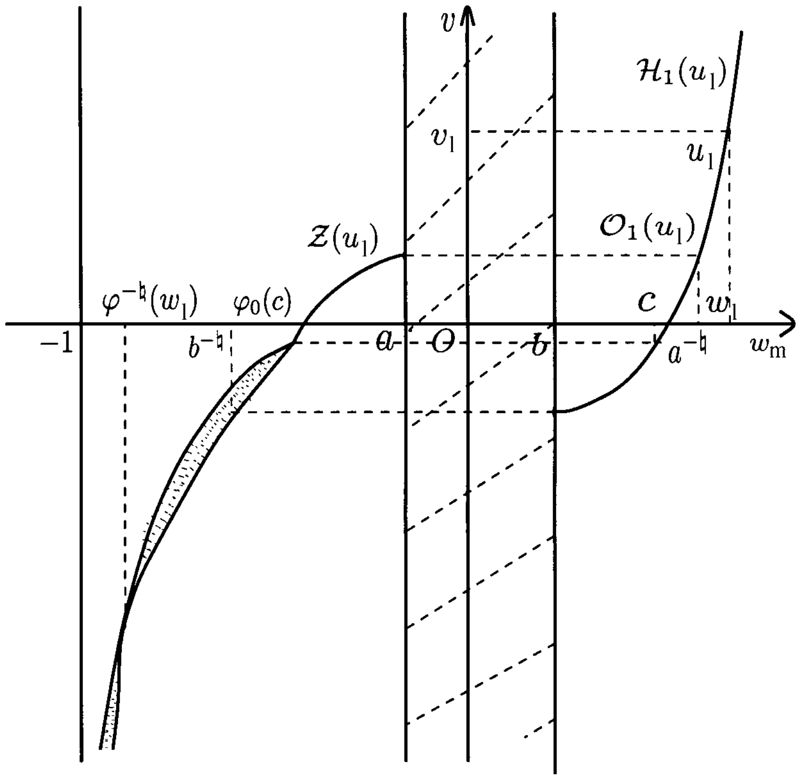

Figure 7. Two-parameter 1-wave set.

Based on lemmas 3.7 and 3.8, we easily arrive to the main results of this section (see figure 7).

THEOREM 3.9 (Two-parameter 1-wave set). *Consider the phase-transition model (1.1) under the assumption (1.2), and restrict attention to weak solutions satisfying the single entropy inequality (1.6). Fix some left-hand state* $u_l = (v_l, w_l)$ *such that* $w_l \geqslant b$ *(for definiteness). Then the set of all right-hand states* $u_m = (v_m, w_m)$ *that can be attained from* u_l *by combining Liu-admissible shock waves, rarefaction fans, subsonic (including stationary) phase boundaries and supersonic phase boundaries of the* 1*-wave family is given as follows.*

(i) *If* $w_m \geqslant w_l$, *the solution consists of a classical shock connecting* u_l *to* $u_m \in \mathcal{H}_1(u_l)$.

(ii) *If* $w_m \in [b, w_l)$, *the solution consists of a rarefaction wave connecting* u_l *to* $u_m \in \mathcal{O}_1(u_l)$. *The* w *component of the solution decreases monotonically from* w_l *to* w_m *as* x *increases in the rarefaction fan.*

(iii) *If* $w_m \in [\varphi_0(c), a]$, *then the solution is a Lax shock* (*respectively, a rarefaction*) *if* $w_l \leqslant \varphi_0(w_m)$ (*respectively,* $w_l \geqslant \varphi_0(w_m)$) *connecting* w_l *to* $\varphi_0(w_m)$, *followed by a stationary phase boundary connecting* $\varphi_0(w_m)$ *to* w_m. *The* w *component of the solution is a non-monotone* (*respectively, monotone*) *function of* x.

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(iv) *If* $w_{\rm m} \in [\varphi^{-\natural}(w_{\rm l}), \varphi_0(c)$,*there exists a one-parameter family of admissible solutions. The solution may consist of a subsonic phase boundary from some intermediate state* u^* *to* $u_{\rm m} \in \mathcal{H}_1(u^*)$ *such that either* $w^* \in [\varphi_0(w_{\rm m}), \varphi^\flat_{\infty,0}(w_{\rm m})]$ *if* $w_{\rm m} \geqslant b^{-\natural}$ *or* $w^* \in [\varphi^\natural(w_{\rm m}), \varphi^\flat_{\infty,0}(w_{\rm m})]$ *if* $w_{\rm m} \geqslant b^{-\natural}$, *preceded with (a) either a rarefaction connecting* $u_{\rm l}$ *to* $u^* \in \mathcal{O}_1(u_{\rm l})$ *if* $w^* < w_{\rm l}$, *the* w *component of the solution being then a monotone function in* x, *or else (b) a classical shock connecting* $u_{\rm l}$ *to* $u^* \in \mathcal{H}_1(u_{\rm l})$ *if* $w^* \geqslant w_{\rm l}$, *the* w *component of the solution being then a non-monotone function in* x.

(v) *If* $w_{\rm m} < \varphi^{-\natural}(w_{\rm l})$, *there exists also a one-parameter family of admissible solutions. The solution can be a composite of a classical shock connecting* $u_{\rm l}$ *to some point* $u^* \in \mathcal{H}_1(u_{\rm l})$ *with* $w^* \in (\varphi^\sharp(w_{\rm m}, w_{\rm l}), \varphi^\flat_{\infty,0}(w_{\rm r})]$ *followed by a subsonic phase boundary connecting* u^* *to* $u_{\rm m} \in \mathcal{H}_1(u^*)$. *This patterns makes sense if* $\varphi^\sharp(w_{\rm m}, w_{\rm l}) < \varphi^\flat_{\infty,0}(w_{\rm r})$, *that is, if the phase boundary propagates faster than the classical shock. It can also be a supersonic phase boundary connecting directly* $u_{\rm l}$ *to* $u_{\rm m}$. *The* w *component of the solution is a non-monotone function of* x.

The discussion of 2-wave patterns is completely analogous. Recall that the wave patterns are always *described from left to right*. The 2-wave set below is made of all attainable left-hand states, while the right-hand state is kept fixed.

THEOREM 3.10 (Two-parameter 2-wave set). *Consider the model of phase transitions (1.1) under the assumption (1.2), and restrict attention to weak solutions satisfying the single entropy inequality (1.6). Fix a right-hand state* $u_{\rm r} = (v_{\rm r}, w_{\rm r})$ *such that* $w_{\rm r} \geqslant b$ *(for definiteness). Then the set of all left-hand states* $u_{\rm m} = (v_{\rm m}, w_{\rm m})$ *that can be attained from* $u_{\rm r}$ *by combining Liu-admissible shock waves, rarefaction fans, subsonic (including stationary) phase boundaries and supersonic phase boundaries of the* 2*-wave family is given as follows.*

(i) *If* $w_{\rm m} > w_{\rm r}$, *the solution consists of a classical shock connecting* $u_{\rm m} \in \mathcal{H}_2(u_{\rm r})$ *to* $u_{\rm r}$.

(ii) *If* $b \leqslant w_{\rm m} < w_{\rm r}$, *the solution consists of a rarefaction wave connecting* $u_{\rm m} \in \mathcal{O}_2(u_{\rm r})$ *to* $u_{\rm r}$. *The* w *component of the solution decreases monotonically from* $w_{\rm m}$ *to* $w_{\rm r}$ *as* x *increases in the rarefaction fan.*

(iii) *If* $w_{\rm m} \in [\varphi_0(c), a]$, *then the solution is a Lax shock* (*respectively, a rarefaction*) *if* $w_{\rm r} \leqslant \varphi_0(w_{\rm m})$ (*respectively,* $w_{\rm r} \geqslant \varphi_0(w_{\rm m})$) *connecting* $\varphi_0(w_{\rm l})$ *to* $w_{\rm r}$, *preceded with a stationary phase boundary from* $w_{\rm m}$ *to* $\varphi_0(w_{\rm m})$. *The* w *component of the solution being a non-monotone* (*respectively, monotone*) *function in* x.

(iv) *If* $w_{\rm m} \in [\varphi^{-\natural}(w_{\rm r}), \varphi_0(c)$, *there exists a one-parameter family of admissible solutions, consisting of a subsonic phase boundary from some intermediate state* $u_{\rm m} \in \mathcal{H}_2(u^*)$ *to* u^* *such that either* $w^* \in [\varphi_0(w_{\rm m}), \varphi^\flat_{\infty,0}(w_{\rm m})]$ *if* $w_{\rm m} \geqslant b^{-\natural}$ *or* $w^* \in [\varphi^\natural(w_{\rm m}), \varphi^\flat_{\infty,0}(w_{\rm m})]$ *otherwise, followed with either a rarefaction connecting* $u_{\rm r}$ *to* $u^* \in \mathcal{O}_2(u_{\rm r})$ *if* $w^* < w_{\rm r}$, *the* w *component of the solution being a monotone function in* x, *or a classical shock connecting* $u^* \in \mathcal{H}_2(u_{\rm r})$ *to* $u_{\rm r}$ *otherwise, the* w*-component of the solution being a non-monotone function in* x.

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(v) *If* $w_{\rm m} < \varphi^{-\natural}(w_{\rm r})$*, there exists a one-parameter family of admissible solutions. The solution can be a composite of a classical shock connecting some point* $u^* \in \mathcal{H}_2(u_{\rm r})$ *to* $u_{\rm r}$*, with* $w^* \in (\varphi^{\sharp}(w_{\rm m}, w_{\rm r}), \varphi^{\flat}_{\infty,0}(w_{\rm m})]$ *preceded with a subsonic phase boundary connecting* $u_{\rm l} \in \mathcal{H}_2(u^*)$ *to* u^**. This patterns makes sense if* $\varphi^{\sharp}(w_{\rm m}, w_{\rm r}) < \varphi^{\flat}_{\infty,0}(w_{\rm m})$*, that is, if the phase boundary propagates slower than the classical shock. It can also be a supersonic phase boundary directly connecting* $u_{\rm r}$ *to* $u_{\rm m}$*. The* w *component of the solution is a non-monotone function of* x*.*

4. Riemann solvers based on a prescribed kinetic relation

At this juncture, we have two-dimensional wave sets of admissible sets, rather than the customary (one-dimensional) wave curves. Our objective now is to impose an additional algebraic relation on the subsonic phase boundaries—the kinetic relation—in order to select uniquely defined wave curves in the wave sets. However, we stress that imposing a kinetic relation is not sufficient, as the classical solution introduced in § 2 and based instead on supersonic phase boundaries is, in principle, also available. Indeed, a precise rule must be given on the use of supersonic phase boundaries as well.

Given condition (C) below, we do obtain uniquely defined wave curves. We also derive some monotonicity properties. It turns out that, when solving the Riemann problem, either the two wave curves intersect at a unique point and the Riemann solution is unique or else, for some range of data, the wave curves intersect at two different points. Contrary to what was observed in § 2, the two intersection points correspond to *distinct* Riemann solutions.

We aim at constructing piecewise monotone wave curves. That is, the wave curves will be such that the v component regarded as a function of the w component will be monotone in each of the two intervals $w \in (-1, a]$ and $w \in [b, +\infty)$. For the selection of the subsonic phase boundaries, we must impose a kinetic relation in each wave family. Let $\varphi^{\flat,i} : (-1, +\infty) \to (-1, +\infty)$ be locally Lipschitz continuous and monotone decreasing *kinetic functions* satisfying (cf. figure 8)

$$w\varphi^{\flat}_{\infty}(w) \leqslant w\varphi^{\flat,i}(w) < w\varphi^{\natural}(w), \quad w \in (-1, \infty), \quad i = 1, 2. \tag{4.1}$$

Their inverse functions, denoted by $\varphi^{-\flat,i} : (-1, \infty) \to (-1, \infty)$, $i = 1, 2$, are monotone decreasing functions as well. It follows from (4.1) that

$$w\varphi^{-\natural}(w) < w\varphi^{-\flat,i}(w) \leqslant w\varphi^{\flat}_{\infty}(w), \quad w \in (-1, \infty). \tag{4.2}$$

Recalling proposition 3.6, we observe that the inverse function of $\varphi^{\flat,1}$ covers admissible subsonic 1-phase boundaries. Indeed, we can connect the left-hand state (v_*, w) to the right-hand state $(v^*, \varphi^{-\flat,1}(w))$. By setting $w_* = \varphi^{-\flat,1}(w)$, we can say equivalently that the kinetic function $\varphi^{\flat,1}$ in (4.1) covers all the possible values of admissible subsonic 1-phase boundaries: the left-hand states $(v_*, \varphi^{\flat,1}(w_*))$ being connected to the right-hand states (v^*, w_*).

We now formulate the following *kinetic relation for subsonic* 1*-phase boundaries*. Any subsonic 1-phase boundary connecting a left-hand state (v_0, w_0) to some right-hand state (v_1, w_1) should satisfy

$$w_0 = \varphi^{\flat,1}(w_1). \tag{4.3}$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

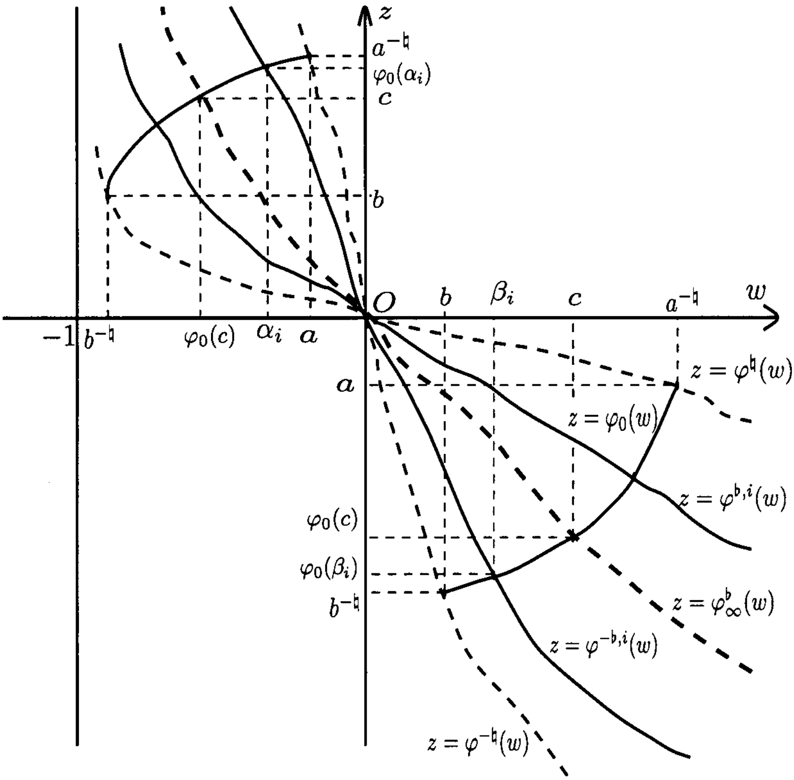

Figure 8. The kinetic function and its inverse.

Similarly, for *subsonic 2-phase boundaries*, we require that any subsonic 2-phase boundary connecting a left-hand state (v_0, w_0) to some right-hand state (v_1, w_1) should satisfy

$$w_1 = \varphi^{\flat,2}(w_0). \tag{4.4}$$

In theorem 3.9 for 1-waves (and similarly in theorem 3.10 for 2-waves), we found in case V two distinct types of Riemann solutions: for values $w \in (\varphi^\flat_{\infty,0}(w_l), \varphi^{-\natural}(w_l))$, the solution can contain a non-stationary subsonic phase boundary or else a stationary phase boundary; for values $w \in (\varphi^{-\natural}(w_l), +\infty)$, the solution can contain a non-stationary subsonic phase boundary or else a supersonic phase boundary. However, such solutions are available only under a condition on the potential wave speed given by the kinetic relation (see the condition $\xi(w) := \varphi^{\flat,1}(w) - \varphi^\sharp(w_l, w) > 0$ in the statement of theorem 3.9, as well as in the proof of theorem 4.3 below).

In the construction proposed here, we postulate that subsonic phase boundaries are preferred. Precisely, the following condition is imposed:

$$\left.\begin{array}{c}\text{the Riemann solution always uses non-stationary}\\ \text{subsonic phase boundaries whenever available.}\end{array}\right\} \tag{C}$$

Alternatively, one could use supersonic phase boundaries whenever they are available. It is not difficult to see that we would then recover the construction in § 2

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

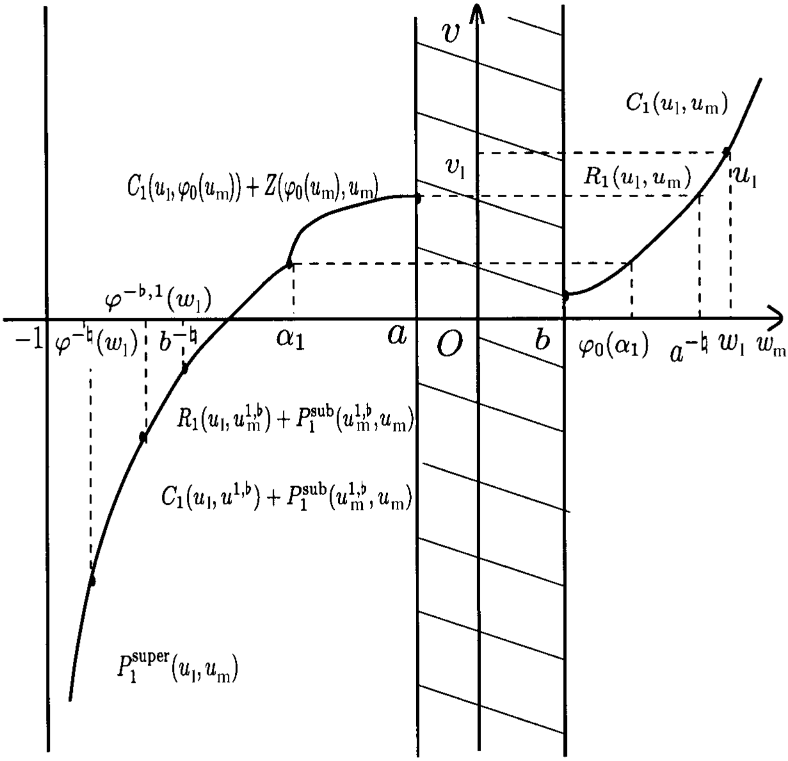

Figure 9. Non-classical 1-wave curve.

based on stationary subsonic phase boundaries only. In a specific application, the actually observed solutions should share properties from both constructions.

Observe that, in view of (4.1) and for $i = 1, 2$, the kinetic function and its inverse intersect the set $\{v = \varphi_0(w)\}$ at some points, denoted, respectively, by $(\beta_i, \varphi_0(\beta_i))$ and $(\alpha_i, \varphi_0(\alpha_i))$, with $\alpha_i \in [b^{-\natural}, \varphi_0(c)], \beta_i \in [c, a^{-\natural}]$. As a consequence of theorem 3.9, we obtain (cf. figure 9) the following result.

THEOREM 4.1 (The 1-wave curve). *Consider all weak solutions of the Riemann problem (1.1)–(1.5). Given some left-hand state $u_l = (v_l, w_l)$, with $w_l \geqslant b$ for definiteness. Under condition (C), the* 1*-wave curve $\mathcal{W}_1(u_l)$ consisting of all right-hand states $u_m = (v_m, w_m)$ attainable from u_l by combining Liu-admissible shock waves, rarefaction fans, subsonic phase boundaries satisfying the kinetic relation (4.3) and supersonic phase boundaries of the* 1*-wave family and stationary phase boundaries is given as follows.*

(i) *If $w_m \geqslant w_l$, then the solution is a classical shock connecting u_l to u_m.*

(ii) *If $w_m \in [b, w_l)$, the solution is a rarefaction wave connecting u_l to u_m.*

(iii) *If $w_m \in [\alpha_1, a]$, the solution consists of either a Lax shock if $\varphi_0(w_m) \geqslant w_l$ or a rarefaction if $\varphi_0(w_m) \in (b, w_l)$, from w_l to $\varphi_0(w_m)$, followed by a stationary phase boundary from $\varphi_0(w_m)$ to w_m.*

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(iv) *If* $w_{\rm m} \in [\varphi^{-\flat,1}(w_{\rm l}), \alpha_1)$, *the solution consists of a rarefaction wave from* $u_{\rm l}$ *to* $u = (v, \varphi^{\flat,1}(w_{\rm m})) \in \mathcal{O}_1(u_{\rm l})$ *followed by a subsonic phase boundary connecting* u *to* $u_{\rm m}$. *The* w *component of the solution is monotone.*

(v) *If* $w_{\rm m} \in (\varphi^{-\natural}(w_{\rm l}), \varphi^{-\flat,1}(w_{\rm l}))$, *then the solution is a Lax shock from* $u_{\rm l}$ *to* $u = (v, \varphi^{\flat,1}(w_{\rm m})) \in \mathcal{H}_1(u_{\rm l})$ *followed by a subsonic phase boundary connecting* u *to* $u_{\rm m}$, *the* w *component of the solution is non-monotone.*

(vi) *If* $w_{\rm m} \leqslant \varphi^{-\natural}(w_{\rm l})$, *then the solution is either a Lax shock from* $u_{\rm l}$ *to* $u = (v, \varphi^{\flat,1}(w_{\rm m})) \in \mathcal{H}_1(u_{\rm l})$, *followed by a subsonic phase boundary connecting* u *to* $u_{\rm m}$, *if and only if* $\varphi^{\flat,1}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm l}, w_{\rm m})$ (*the* w *component of the solution is non-monotone in this case*), *or a supersonic phase boundary connecting directly* $u_{\rm l}$ *with* $u_{\rm m}$ *otherwise.*

Similarly, theorem 3.10 yields the following result.

THEOREM 4.2 (The 2-wave curve). *Consider all weak solutions of the Riemann problem (1.1)–(1.5). Given some right-hand state* $u_{\rm r} = (v_{\rm r}, w_{\rm r})$, *with* $w_{\rm r} \geqslant b$ *for definiteness. Under condition (C), the* 2*-wave curve* $\mathcal{W}_2(u_{\rm r})$ *made of all left-hand states* $u_{\rm m} = (v_{\rm m}, w_{\rm m})$ *attainable from* $u_{\rm r}$ *by combining Liu-admissible shock waves, rarefaction fans, subsonic phase boundaries satisfying the kinetic relation (4.4), and supersonic phase boundaries of the* 2*-wave family and stationary phase boundaries is given as follows.*

(i) *If* $w_{\rm m} \geqslant w_{\rm r}$, *then the solution is a classical shock connecting* $u_{\rm r}$ *to* $u_{\rm m}$.

(ii) *If* $w_{\rm m} \in [b, w_{\rm r})$, *the solution is a rarefaction wave connecting* $u_{\rm r}$ *to* $u_{\rm m}$.

(iii) *If* $w_{\rm m} \in [\alpha_2, a]$, *the solution consists of either a Lax shock if* $w_{\rm r} \leqslant \varphi_0(w_{\rm m})$, *or a rarefaction elsewhere, from* $w_{\rm r}$ *to* $\varphi_0(w_{\rm m})$, *followed by a stationary phase boundary from* $\varphi_0(w_{\rm m})$ *to* $w_{\rm m}$.

(iv) *If* $w_{\rm m} \in [\varphi^{-\flat,2}(w_{\rm r}), \alpha_2)$, *the solution consists of a rarefaction wave from* $u_{\rm r}$ *to* $u = (v, \varphi^{\flat,2}(w_{\rm m})) \in \mathcal{O}_2(u_{\rm r})$, *followed by a subsonic phase boundary connecting* u *to* $u_{\rm m}$. *The* w *component of the solution is monotone.*

(v) *If* $w_{\rm m} \in (\varphi^{-\natural}(w_{\rm r}), \varphi^{-\flat,2}(w_{\rm r}))$, *then the solution is a Lax shock from* $u_{\rm r}$ *to* $u = (v, \varphi^{\flat,2}(w_{\rm m})) \in \mathcal{H}_2(u_{\rm r})$, *followed by a subsonic phase boundary connecting* u *to* $u_{\rm m}$. *The* w *component of the solution is non-monotone.*

(vi) *If* $w_{\rm m} \leqslant \varphi^{-\natural}(w_{\rm r})$, *then the solution is either a Lax shock from* $u_{\rm r}$ *to* $u = (v, \varphi^{\flat,2}(w_{\rm m})) \in \mathcal{H}_2(u_{\rm r})$, *followed by a subsonic phase boundary connecting* u *to* $u_{\rm m}$ *if* $\varphi^{\flat,2}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$ *(the* w *component of the solution is non-monotone in this case), or a supersonic phase boundary connecting directly* $u_{\rm r}$ *with* $u_{\rm m}$ *otherwise.*

The properties of the wave curves are now summarized.

THEOREM 4.3. *Under condition (C), the* 1*-wave curve* $\mathcal{W}_1(u_{\rm l})$ (*the* 2*-wave curve* $\mathcal{W}_2(u_{\rm r})$) *is continuous, monotone increasing* (*monotone decreasing, respectively*) *in each interval* $(-1, a]$ *and* $[b, +\infty)$, *and extends from* $(v, w) = (-\infty, -1)$ *to*

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

$(v,w)=(+\infty,+\infty)$. *The curve is locally Lipschitz in* w *whenever we have* $\alpha_1 \neq b^{-\natural}$ *and* $\beta_1 \neq a^{-\natural}$ *(respectively,* $\alpha_2 \neq b^{-\natural}$ *and* $\beta_2 \neq a^{-\natural}$*). Moreover, we have the 'symmetry' property*

$$v(w)=v(\varphi_0(w)) \quad \textit{for all } w \in [\alpha_1,a]\cup[b,\beta_1] \quad (w \in [\alpha_2,a]\cup[b,\beta_2], \textit{ respectively}). \tag{4.5}$$

Proof. We only consider the 1-wave curve $\mathcal{W}_1(u_l)$ leaving from $u_l=(v_l,w_l)$ with $w_l \geqslant b$. The case $w_l \leqslant a$ can be treated similarly. The curve $\mathcal{W}_1(u_l)$, $u_l=(v_l,w_l)$, $w_l \geqslant b$, can be parametrized by a function $v=v(w;u_l)$. The regularity of the curve is an immediate consequence of the construction.

If $w \geqslant b$, the non-classical curve coincides with the classical curve given by the Liu construction. It is based on a classical shock if $w > w_l$ and a rarefaction wave otherwise. As noted in the classical case, the function $v=v(w;v_l,w_l)$ is monotone increasing. Furthermore, in view of (1.2), (2.6), we have, by construction,

$$v \to +\infty \quad \text{as } w \to +\infty.$$

If $\alpha_1 \leqslant w \leqslant a$, the solution consists of either a rarefaction wave if $w_l > \varphi_0(w_r)$, or a classical shock otherwise. This wave connects w_l to $\varphi_0(w_r)$ and is followed with a stationary phase boundary from $\varphi_0(w_r)$ to w_r. By the monotonicity property of the function $\varphi^\flat_{\infty,0}(w)=\varphi_0(w)$ in this interval, the wave curve is monotone increasing as well. Observe that the value of v at w coincides with the one of v at $\varphi_0(w)$. This is due to the fact that we use here stationary jumps. Therefore, condition (4.5) holds.

If $w \in (\varphi^{-\flat,1}(w_l),\alpha_1)$, the solution consists of a rarefaction wave from w_l to $\varphi^{\flat,1}(w)$ followed by a phase boundary connecting $\varphi^{\flat,1}(w)$ to w. The value of v is determined by (2.6) and (2.8),

$$\left.\begin{aligned} v-v(\varphi^{\flat,1}(w)) &= \bar{c}(\varphi^{\flat,1}(w),w)(w-\varphi^{\flat,1}(w)),\\ v(\varphi^{\flat,1}(w))-v_l &= \int_{w_l}^{\varphi^{\flat,1}(w)} c(z)\,\mathrm{d}z. \end{aligned}\right\} \tag{4.6}$$

For w in the interval under consideration, we deduce from (4.6) that[1]

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}w} &= \frac{\theta}{2\sqrt{[\sigma(\varphi^{\flat,1}(w))-\sigma(w)]/[\varphi^{\flat,1}(w)-w]}}\\ \theta &:= -\frac{\mathrm{d}\varphi^{\flat,1}}{\mathrm{d}w}(w)\biggl(\sqrt{\sigma'(\varphi^{\flat,1}(w))}-\sqrt{\frac{\sigma(\varphi^{\flat,1}(w))-\sigma(w)}{\varphi^{\flat,1}(w)-w}}\biggr)^2\\ &\qquad\qquad + \sigma'(w)+\frac{\sigma(\varphi^{\flat,1}(w))-\sigma(w)}{\varphi^{\flat,1}(w)-w} > 0. \end{aligned}$$

This again yields the desired monotonicity property of the wave curve.

If $w \in (-1,\varphi^{-\flat,1}(w_l)]$, the solution is *either* a composite of a classical shock connecting w_l to $\varphi^{\flat,1}(w)$ followed by a subsonic phase boundary connecting $\varphi^{\flat,1}(w)$

[1] In fact, the monotonicity of the kinetic function is not necessary. It is not difficult to see that the weaker requirement $(\sigma(\varphi^{\flat,1})-\sigma)' < 0$ suffices here.

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

with w, provided

$$\xi(w) := \varphi^{\flat,1}(w) - \varphi^{\sharp}(w_l, w) > 0,$$

or a classical shock connecting directly w_l with w. In this proof, let us consider together both cases (v) and (vi) listed in the theorem. We claim that the last inequality is always valid in case (v).

Since the function ξ is continuous, the set

$$\mathcal{N} := \{w \in (-1, \varphi^{\flat,1}(w_l))/\varphi^{\flat,1}(w) - \varphi^{\sharp}(w_l, w) > 0\}$$

is an open set and is therefore a union of open intervals. At an endpoint, say w', of an interval $(w', w'') \subset \mathcal{N}$, the speeds of the classical and subsonic phase boundaries tend to the speed of the classical shock connecting w_l to w'. Therefore, the wave curve is (at least) continuous. In each interval $\subset \mathcal{N}^c$ corresponding to classical solutions, the wave curve is clearly monotone increasing. Consider now some $w \in (w', w'') \subset \mathcal{N}$. From the construction in theorem 4.1, we have

$$\left.\begin{aligned} v - v(\varphi^{\flat,1}(w)) &= \bar{c}(\varphi^{\flat,1}(w), w)(w - \varphi^{\flat,1}(w)), \\ v(\varphi^{\flat,1}(w)) - v_l &= \bar{c}(w_l, \varphi^{\flat,1}(w))(\varphi^{\flat,1}(w) - w_l). \end{aligned}\right\} \tag{4.7}$$

Then (4.7) yields

$$\left.\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}w} &= \tfrac{1}{2}\theta_1\theta_2 + \frac{\sigma'(w) + 1}{2\sqrt{[\sigma(\varphi^{\flat,1}(w)) - \sigma(w)]/[\varphi^{\flat,1}(w) - w]}}, \\ \theta_1 &:= \sqrt{\frac{\sigma(\varphi^{\flat,1}(w)) - \sigma(w_l)}{\varphi^{\flat,1}(w) - w_l}} - \sqrt{\frac{\sigma(\varphi^{\flat,1}(w)) - \sigma(w)}{\varphi^{\flat,1}(w) - w}}, \\ \theta_2 &:= \frac{\sigma'(\varphi^{\flat,1}(w))}{2\sqrt{[\sigma(\varphi^{\flat,1}(w)) - \sigma(w_l)]/[\varphi^{\flat,1}(w) - w_l]}\sqrt{[\sigma(\varphi^{\flat,1}(w)) - \sigma(w)]/[\varphi^{\flat,1}(w) - w]}} - 1. \end{aligned}\right\} \tag{4.8}$$

Using the condition that the shock speed is increasing, one obtains

$$-\sqrt{\frac{\sigma(\varphi^{\flat,1}(w)) - \sigma(w_l)}{\varphi^{\flat,1}(w) - w_l}} \leqslant -\sqrt{\frac{\sigma(\varphi^{\flat,1}(w)) - \sigma(w)}{\varphi^{\flat,1}(w) - w}}. \tag{4.9}$$

By definition, $\varphi^{\flat,1}(w) \geqslant w_l > \varphi^{\natural}(w)$. Then it follows from inequality (3.10) that

$$\left.\begin{aligned} \frac{\sigma(\varphi^{\flat,1}(w)) - \sigma(w_l)}{\varphi^{\flat,1}(w) - w_l} &\leqslant \sigma'(\varphi^{\flat,1}(w)), \\ \frac{\sigma(\varphi^{\flat,1}(w)) - \sigma(w)}{\varphi^{\flat,1}(w) - w} &\leqslant \sigma'(\varphi^{\flat,1}(w)). \end{aligned}\right\} \tag{4.10}$$

From (4.8)–(4.10), we deduce

$$\frac{\mathrm{d}v}{\mathrm{d}w} > 0,$$

so that the wave curve is monotone increasing. Besides, in view of (1.2) and (2.6), our construction gives

$$v \to -\infty \text{ as } w \to -1.$$

The proof of theorem 4.3 is completed. □

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

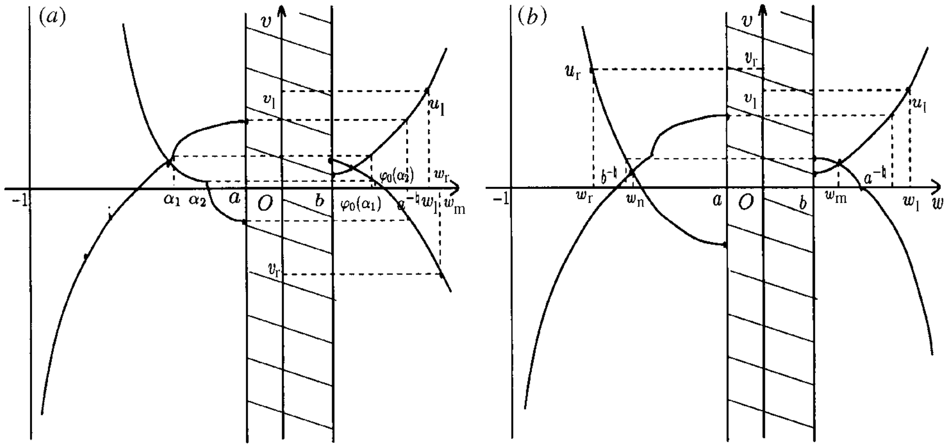

Figure 10. Non-unique Riemann solution.

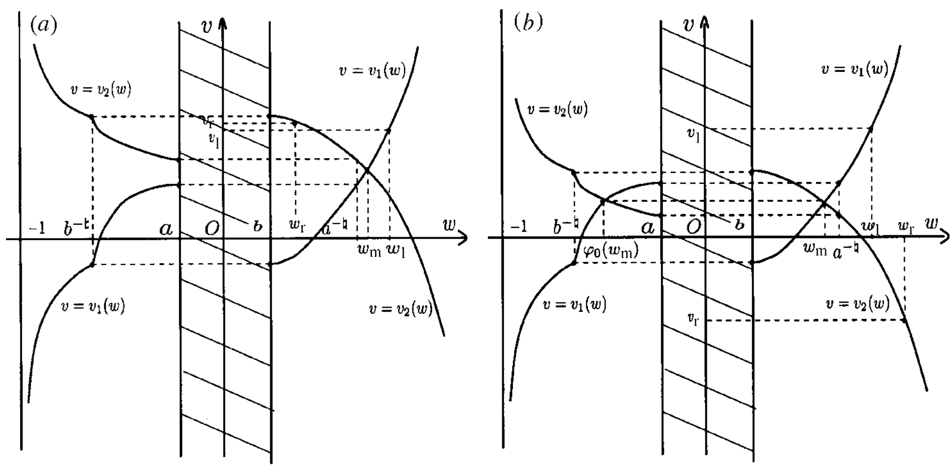

Figure 11. Unique Riemann solution.

It turns out now that the wave curves may intersect at one or two points, depending on the Riemann data (see figures 10 and 11).

THEOREM 4.4. *Consider the two wave curves constructed from condition (C). Given two states in different phases, the wave curves either intersect exactly once and their intersection point leads to a unique solution of the Riemann problem, or else the wave curves intersect at exactly two points located in different phases. In the latter case, if the intersection point corresponds to a solution having a stationary phase boundary, then the two points correspond to the same Riemann solution. In the opposite case, the intersection points are associated with different Riemann solutions.*

This non-uniqueness property of the Riemann solutions was already pointed out by Abeyaratne and Knowles [2] (for the piecewise linear functions) and Shearer and Yang [16] (for the cubic function).

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

The proof of theorem 4.4 follows from the properties of the wave curves established earlier. To select between two solutions when available, one possibility is to follow Abeyaratne and Knowles [1] and impose a further criterion, the *nucleation criterion*. For instance, as long as the initial jump (or some other similar quantity) of the Riemann data remains below a given threshold, one can pick up the single-phase solution. The two-phase solutions is then selected only above the threshold. This approach has the advantage of providing a unique Riemann solution, which, however, is not L^1 continuous with respect to the initial data. This seems to be an actual feature of the dynamics of phase boundaries, and constructing a Riemann solver to be both physically relevant and mathematically sound remains currently open.

A mathematically attracting approach is to restrict attention to a narrow class of kinetic functions, based on the following observation: non-uniqueness does not arise as long as we are concerned with stationary phase boundaries, as they cannot separate in the (x,t)-plane. Indeed, a strong assumption on the kinetic functions can ensure that the two intersection points, when they exist, always correspond to *stationary phase boundaries*. In this way, we guarantee the uniqueness of the Riemann solution, as well as its continuous dependence with respect to the data. This construction proposed now is probably not realistic from the physical standpoint. The more general approach based on a nucleation criterion above should be more relevant in most applications.

THEOREM 4.5. *Suppose that the kinetic functions (4.1) also satisfy the restriction*

$$\varphi^{\flat,i}(a^{-\natural}) = a \quad \textit{and} \quad \varphi^{\flat,i}(b^{-\natural}) = b, \quad i = 1, 2.$$

Under condition (C), the Riemann problem (1.1)–(1.5) admits a unique solution made of Liu-admissible shock waves, rarefaction fans, supersonic phase boundaries and stationary phase boundaries, as well as subsonic phase boundaries satisfying the kinetic relations (4.3), (4.4).

Moreover, this solution depends continuously upon its initial data in the L^1-norm (see (2.15)).

Proof. Since the arguments are quite similar to the ones in the proof of theorem 2.7, we only sketch the proof. We parametrize the two wave curves $\mathcal{W}_1(u_\mathrm{l})$ and $\mathcal{W}_2(u_\mathrm{r})$ by the functions

$$\begin{aligned} \mathcal{W}_1(u_\mathrm{l})&: \quad v = v_1(w) \quad \text{for } w \in (-1, a] \cup [b, +\infty), \\ \mathcal{W}_2(u_\mathrm{r})&: \quad v = v_2(w) \quad \text{for } w \in (-1, a] \cup [b, +\infty). \end{aligned}$$

From theorem 4.3, we see that the function

$$\kappa(w) := v_2(w) - v_1(w), \quad w \in (-1, a] \cup [b, +\infty), \tag{4.11}$$

is monotone decreasing in each interval $(-1, a]$ and $[b, +\infty)$.

We distinguish three case, as in the proof of theorem 2.7.

CASE 1. On one hand, the function κ in (4.11) is monotone decreasing in $(-1, a]$, with $\kappa(a) > 0$. This implies that $\kappa(w) \geqslant \kappa(a) > 0$ for all $w \in (-1, a]$ and, therefore, the two wave curves can not intersect in the region $w \in (-1, a]$. On the other hand,

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

equation (4.5) and the restriction on the kinetic functions imposed in the theorem yield

$$\kappa(a^{-\natural}) = v_2(a^{-\natural}) - v_1(a^{-\natural}) = v_2(a) - v_1(a) > 0.$$

In view of this inequality and since the function κ is monotone decreasing in the interval $[b, +\infty) \ni a^{-\natural}$, and that $\kappa(w) \to -\infty$ as $w \to +\infty$, we see that there exists a unique value $w_{\rm m} \in (a^{-\natural}, +\infty)$ such that

$$\kappa(w_{\rm m}) = 0.$$

The two wave curves thus intersect at a unique point $(v_1(w_{\rm m}), w_{\rm m})$. Since $w_{\rm m} > a^{-\natural}$, the solution does not contain stationary phase boundaries.

CASE 2. The proof is completely similar. Now the intersection point $(v_1(w_{\rm m}), w_{\rm m})$ satisfies $w_{\rm m} \in (-1, b^{-\natural})$.

CASE 3. In this case, the two wave curves actually meet at exactly two points, $u_1 := (v_1(w_{\rm m}), w_{\rm m})$ and $u_2 := (v_1(w_{\rm m}), \varphi_0(w_{\rm m}))$. Here we have $w_{\rm m} \in [b, a^{-\natural}]$. This value corresponds to a stationary phase boundary, as follows from the additional assumption made on the kinetics. Clearly, stationary phase boundary appear if and only if $u_i \in \mathcal{Z}_i$, $i = 1, 2$. As in the construction of § 2, the points u_1 and u_2 correspond to a unique stationary phase boundary. The Riemann solution is thus unique. The continuous dependence is checked similarly as in the proof of theorem 2.7. The proof of theorem 4.5 is completed. □

REMARK 4.6. Finally, based on the general description of the wave curves in theorems 4.1 and 4.2, we now list all the possibilities of Riemann solutions. Consider the roots $w_{\rm m}$ of the equation

$$v_1(w) = v_2(w), \quad w \in (-1, \infty), \tag{4.12}$$

and set

$$u_{\rm m}^{i,\flat} := (v_i(\varphi^{\flat,i}(w_{\rm m})), w_{\rm m}), \quad i = 1, 2.$$

For definiteness, we will assume that $w_{\rm l} \geqslant b$ and consider $u_{\rm r}$ as a parameter.

CASE I. Suppose that $v_2(a) > v_1(a)$. Then (4.12) admits a unique solution $w_{\rm m}$ satisfying $w_{\rm m} > a^{-\natural}$.

If the states $u_{\rm l}$ and $u_{\rm r}$ belong to the same phase, i.e. $w_{\rm l}$, $w_{\rm r} \geqslant b$, then we have the following.

(I.2) If $w_{\rm l} \geqslant w_{\rm m}$ and $w_{\rm r} \geqslant w_{\rm m}$, then the Riemann solution contains the following waves: $R_1(u_{\rm l}, u_{\rm m}) + R_2(u_{\rm m}, u_{\rm r})$.

(I.2) If $w_{\rm l} \geqslant w_{\rm m}$ and $b \leqslant w_{\rm r} < w_{\rm m}$, then $R_1(u_{\rm l}, u_{\rm m}) + C_2(u_{\rm m}, u_{\rm r})$.

(I.3) If $b \leqslant w_{\rm l} < w_{\rm m}$ and $w_{\rm r} \geqslant w_{\rm m}$, then $C_1(u_{\rm l}, u_{\rm m}) + R_2(u_{\rm m}, u_{\rm r})$.

(I.4) If $b \leqslant w_{\rm l} < w_{\rm m}$ and $b \leqslant w_{\rm r} < w_{\rm m}$, then $C_1(u_{\rm l}, u_{\rm m}) + C_2(u_{\rm m}, u_{\rm r})$.

If the states $u_{\rm l}$ and $u_{\rm r}$ belong to different phases, i.e. $w_{\rm l} \geqslant b$ and $-1 < w_{\rm r} \leqslant a$, then we find the following.

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(I.5) If $w_{\rm l} \geqslant w_{\rm m}$ and $-1 < w_{\rm r} < \varphi^{\flat,2}(w_{\rm m})$, then

$$R_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + R_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(I.6) If $w_{\rm l} \geqslant w_{\rm m}$ and $\varphi^{\flat,2}(w_{\rm m}) \leqslant w_{\rm r} \leqslant \varphi^{\natural}(w_{\rm m})$, then

$$R_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(I.7) If $w_{\rm l} \geqslant w_{\rm m}$ and $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm r} \leqslant a$, then the solution is either

$$R_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,2}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or $R_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm super}(u_{\rm m}, u_{\rm r})$ otherwise.

(I.8) If $b \leqslant w_{\rm l} < w_{\rm m}$ and $-1 < w_{\rm r} < \varphi^{\flat,2}(w_{\rm m})$, then

$$C_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + R_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(I.9) If $b \leqslant w_{\rm l} < w_{\rm m}$ and $\varphi^{\flat,2}(w_{\rm m}) \leqslant w_{\rm r} \leqslant \varphi^{\natural}(w_{\rm m})$, then

$$C_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(I.10) If $b \leqslant w_{\rm l} < w_{\rm m}$ and $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm r} \leqslant a$, then the solution is either

$$C_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,2}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or $R_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm super}(u_{\rm m}, u_{\rm r})$ otherwise.

Case II. Suppose that $v_2(b) < v_1(b)$. Then (4.12) admits a unique solution $w_{\rm m}$ satisfying $-1 < w_{\rm m} < a^{-\natural}$.

If the states $u_{\rm l}$ and $u_{\rm r}$ belong to the same phase, i.e. $w_{\rm l}, w_{\rm r} \geqslant b$, then we have the following.

(II.1) If $w_{\rm l} > \varphi^{\flat,1}(w_{\rm m})$ and $w_{\rm r} > \varphi^{\flat,2}(w_{\rm m})$, then the Riemann solution contains the following waves:

$$R_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + R_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(II.2) If $w_{\rm l} > \varphi^{\flat,1}(w_{\rm m})$ and $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm r} \leqslant \varphi^{\flat,2}(w_{\rm m})$, then

$$R_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(II.3) If $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm l} \leqslant \varphi^{\flat,1}(w_{\rm m})$ and $w_{\rm r} > \varphi^{\flat,2}(w_{\rm m})$, then

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + R_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(II.4) If $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm l} \leqslant \varphi^{\flat,1}(w_{\rm m})$ and $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm l} \leqslant \varphi^{\flat,2}(w_{\rm m})$, then

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(II.5) If $w_{\rm l} > \varphi^{\flat,1}(w_{\rm m})$ and $a \leqslant w_{\rm r} < \varphi^{\natural}(w_{\rm m})$, then the solution is either

$$R_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,2}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or $R_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm super}(u_{\rm m}, u_{\rm r})$ otherwise.

(II.6) If $a \leqslant w_{\rm l} < \varphi^{\natural}(w_{\rm m})$ and $w_{\rm r} > \varphi^{\flat,2}(w_{\rm m})$, then the solution is either

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + R_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,1}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or $P_1^{\rm super}(u_{\rm l}, u_{\rm m}) + R_2(u_{\rm m}, u_{\rm r})$ otherwise.

(II.7) If $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm l} \leqslant \varphi^{\flat,1}(w_{\rm m})$ and $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm l} \leqslant \varphi^{\flat,2}(w_{\rm m})$, then

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}).$$

(II.8) If $a \leqslant w_{\rm l} < \varphi^{\natural}(w_{\rm m})$ and $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm r} \leqslant \varphi^{\flat,2}(w_{\rm m})$, then the solution is either

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,1}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or $P_1^{\rm super}(u_{\rm l}, u_{\rm m}) + C_2(u_{\rm m}, u_{\rm r})$ otherwise.

(II.9) If $\varphi^{\natural}(w_{\rm m}) \leqslant w_{\rm l} < \varphi^{\flat,1}(w_{\rm m})$ and $a \leqslant w_{\rm r} \leqslant \varphi^{\natural}(w_{\rm m})$, then the solution is either

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,2}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or $C_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm super}(u_{\rm m}, u_{\rm r})$ otherwise.

(II.10) If $a \leqslant w_{\rm l} < \varphi^{\natural}(w_{\rm m})$ and $a \leqslant w_{\rm r} \leqslant \varphi^{\natural}(w_{\rm m})$, then the solution is either

$$C_1(u_{\rm l}, u_{\rm m}^{1,\flat}) + P_1^{\rm sub}(u_{\rm m}^{1,\flat}, u_{\rm m}) + P_2^{\rm sub}(u_{\rm m}, u_{\rm m}^{2,\flat}) + C_2(u_{\rm m}^{2,\flat}, u_{\rm r}) \quad \text{if } \varphi^{\flat,i}(w_{\rm m}) > \varphi^{\sharp}(w_{\rm r}, w_{\rm m})$$

or

$$P_1^{\rm super}(u_{\rm l}, u_{\rm m}) + C_2(u_{\rm m}, u_{\rm r}) \quad \text{if } \varphi^{\flat,1}(w_{\rm m}) \leqslant \varphi^{\sharp}(w_{\rm r}, w_{\rm m}) < \varphi^{\flat,2}(w_{\rm m})$$

or

$$C_1(u_{\rm l}, u_{\rm m}) + P_2^{\rm super}(u_{\rm m}, u_{\rm r}) \quad \text{if } \varphi^{\flat,2}(w_{\rm m}) \leqslant \varphi^{\sharp}(w_{\rm r}, w_{\rm m}) < \varphi^{\flat,1}(w_{\rm m})$$

or, finally, $P_1^{\rm super}(u_{\rm l}, u_{\rm m}) + P_2^{\rm super}(u_{\rm m}, u_{\rm r})$ otherwise.

If the states belong to different phases, i.e. $w_{\rm l} \geqslant b$ and $-1 < w_{\rm r} \leqslant a$, we have the following.

(II.11) If $w_l > \varphi^{\flat,1}(w_m)$ and $-1 < w_r \leqslant w_m$, then the solution is

$$R_1(u_l, u_m^{1,\flat}) + P_1^{sub}(u_m^{1,\flat}, u_m) + R_2(u_m, u_r).$$

(II.12) If $w_l > \varphi^{\flat,1}(w_m)$ and $w_m < w_r \leqslant a$, then the solution is

$$R_1(u_l, u_m^{1,\flat}) + P_1^{sub}(u_m^{1,\flat}, u_m) + C_2(u_m, u_r).$$

(II.13) If $\varphi^{\natural}(w_m) \leqslant w_l \leqslant \varphi^{\flat,1}(w_m)$ and $-1 < w_r \leqslant w_m$, then

$$C_1(u_l, u_m^{1,\flat}) + P_1^{sub}(u_m^{1,\flat}, u_m) + R_2(u_m, u_r).$$

(II.14) If $\varphi^{\natural}(w_m) \leqslant w_l \leqslant \varphi^{\flat,1}(w_m)$ and $w_m < w_r \leqslant a$, then

$$C_1(u_l, u_m^{1,\flat}) + P_1^{sub}(u_m^{1,\flat}, u_m) + C_2(u_m, u_r).$$

(II.15) If $b \leqslant w_l < \varphi^{\natural}(w_m)$ and $-1 < w_r \leqslant w_m$, then the solution is either

$$C_1(u_l, u_m^{1,\flat}) + P_1^{sub}(u_m^{1,\flat}, u_m) + P_2^{sub}(u_m, u_m^{2,\flat}) + R_2(u_m^{2,\flat}, u_r) \quad \text{if } \varphi^{\flat,1}(w_m) > \varphi^{\sharp}(w_r, w_m)$$

or $P_1^{super}(u_l, u_m) + R_2(u_m, u_r)$ otherwise.

(II.16) If $b \leqslant w_l < \varphi^{\natural}(w_m)$ and $w_m < w_r \leqslant a$, then the solution is either

$$C_1(u_l, u_m^{1,\flat}) + P_1^{sub}(u_m^{1,\flat}, u_m) + P_2^{sub}(u_m, u_m^{2,\flat}) + C_2(u_m^{2,\flat}, u_r) \quad \text{if } \varphi^{\flat,1}(w_m) > \varphi^{\sharp}(w_r, w_m)$$

or $P_1^{super}(u_l, u_m) + C_2(u_m, u_r)$ otherwise.

Case III. Suppose, finally, that $v_2(a) \leqslant v_1(a)$ and $v_2(b) \geqslant v_1(b)$. Then (4.12) admits exactly two solutions $w_m \in [b, +\infty)$ and $w_n \in (-1, a]$. When $w_n \neq \varphi_0(w_m)$, the Riemann solutions are not unique as one can see in the following.

If the states u_l and u_r belong to the same phase, i.e. w_l and $w_r \geqslant b$, we have the following.

(III.1) If $w_l \geqslant w_m$ and $w_r \geqslant w_m$, then the solution is $R_1(u_l, u_m) + R_2(u_m, u_r)$.

(III.2) If $w_l \geqslant w_m$ and $b \leqslant w_r < w_m$, then $R_1(u_l, u_m) + C_2(u_m, u_r)$.

(III.3) If $b \leqslant w_l < w_m$ and $w_r \geqslant w_m$, then $C_1(u_l, u_m) + R_2(u_m, u_r)$.

(III.4) If $b \leqslant w_l < w_m$ and $b \leqslant w_r < w_m$, then $C_1(u_l, u_m) + C_2(u_m, u_r)$.

However, the solution is not unique and can also contain subsonic phase boundaries, as follows.

(III.5) If $w_l \geqslant \varphi^{\flat,1}(w_m)$ and $w_r \geqslant \varphi^{\flat,2}(w_n)$, then the Riemann solution can also be

$$R_1(u_l, u_n^{1,\flat}) + P_1^{sub}(u_n^{1,\flat}, u_n) + P_2^{sub}(u_n, u_n^{2,\flat}) + R_2(u_n^{2,\flat}, u_r).$$

(III.6) If $w_{\mathrm{l}} \geqslant \varphi^{\flat,1}(w_{\mathrm{m}})$ and $b \leqslant w_{\mathrm{r}} < \varphi^{\flat,2}(w_{\mathrm{n}})$, then

$$R_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + P_2^{\mathrm{sub}}(u_{\mathrm{n}}, u_{\mathrm{n}}^{2,\flat}) + C_2(u_{\mathrm{n}}^{2,\flat}, u_{\mathrm{r}}).$$

(III.7) If $b \leqslant w_{\mathrm{l}} < \varphi^{\flat,1}(w_{\mathrm{m}})$ and $w_{\mathrm{r}} \geqslant \varphi^{\flat,2}(w_{\mathrm{n}})$, then the Riemann solution can be

$$C_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + P_2^{\mathrm{sub}}(u_{\mathrm{n}}, u_{\mathrm{n}}^{2,\flat}) + R_2(u_{\mathrm{n}}^{2,\flat}, u_{\mathrm{r}}).$$

(III.8) If $b \leqslant w_{\mathrm{l}} < \varphi^{\flat,1}(w_{\mathrm{m}})$ and $b \leqslant w_{\mathrm{r}} < \varphi^{\flat,2}(w_{\mathrm{n}})$, then

$$C_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + P_2^{\mathrm{sub}}(u_{\mathrm{n}}, u_{\mathrm{n}}^{2,\flat}) + C_2(u_{\mathrm{n}}^{2,\flat}, u_{\mathrm{r}}).$$

Assume now that the states u_{l} and u_{r} belong to different phases, i.e. $w_{\mathrm{l}} \geqslant b$ and $-1 < w_{\mathrm{r}} \leqslant a$. If $w_{\mathrm{n}} = \varphi_0(w_{\mathrm{m}})$, then the Riemann solution is unique and it contains stationary phase boundaries (but no other subsonic waves).

(III.9) If $w_{\mathrm{l}} \geqslant w_{\mathrm{m}}$ and $-1 < w_{\mathrm{r}} \leqslant \varphi_0(w_{\mathrm{m}})$, then the solution is

$$R_1(u_{\mathrm{l}}, u_{\mathrm{m}}) + Z(u_{\mathrm{m}}, \tilde{u}_{\mathrm{m}}) + R_2(\tilde{u}_{\mathrm{m}}, u_{\mathrm{r}}),$$

where $\tilde{u}_{\mathrm{m}} := (v_1(w_{\mathrm{m}}), \varphi_0(w_{\mathrm{m}}))$.

(III.10) If $w_{\mathrm{l}} \geqslant w_{\mathrm{m}}$ and $a \geqslant w_{\mathrm{r}} > \varphi_0(w_{\mathrm{m}})$, then

$$R_1(u_{\mathrm{l}}, u_{\mathrm{m}}) + Z(u_{\mathrm{m}}, \tilde{u}_{\mathrm{m}}) + C_2(\tilde{u}_{\mathrm{m}}, u_{\mathrm{r}}).$$

(III.11) If $b \leqslant w_{\mathrm{l}} \leqslant w_{\mathrm{m}}$ and $-1 < w_{\mathrm{r}} \leqslant \varphi_0(w_{\mathrm{m}})$, then

$$C_1(u_{\mathrm{l}}, u_{\mathrm{m}}) + Z(u_{\mathrm{m}}, \tilde{u}_{\mathrm{m}}) + R_2(\tilde{u}_{\mathrm{m}}, u_{\mathrm{r}}).$$

(III.12) If $b \leqslant w_{\mathrm{l}} \geqslant w_{\mathrm{m}}$ and $a \geqslant w_{\mathrm{r}} > \varphi_0(w_{\mathrm{m}})$, then

$$C_1(u_{\mathrm{l}}, u_{\mathrm{m}}) + Z(u_{\mathrm{m}}, \tilde{u}_{\mathrm{m}}) + C_2(\tilde{u}_{\mathrm{m}}, u_{\mathrm{r}}).$$

If $w_{\mathrm{n}} \neq \varphi(w_{\mathrm{m}})$, then the Riemann problem may admit two solutions.

(III.13) If $w_{\mathrm{l}} \geqslant w_{\mathrm{m}}$ and $-1 < w_{\mathrm{r}} \leqslant w_{\mathrm{n}}$, then the Riemann solution is either

$$R_1(u_{\mathrm{l}}, u_{\mathrm{m}}) + P_2^{\mathrm{sub}}(u_{\mathrm{m}}, u_{\mathrm{m}}^{2,\flat}) + \begin{cases} R_2(u_{\mathrm{m}}^{2,\flat}, u_{\mathrm{r}}) & \text{if } w_{\mathrm{r}} \leqslant \varphi^{\flat,2}(w_{\mathrm{m}}), \\ C_2(u_{\mathrm{m}}^{2,\flat}, u_{\mathrm{r}}) & \text{if } w_{\mathrm{r}} > \varphi^{\flat,2}(w_{\mathrm{m}}), \end{cases}$$

or

$$\begin{aligned} &R_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + R_2(u_{\mathrm{n}}, u_{\mathrm{r}}) \quad \text{if } w_{\mathrm{l}} \geqslant \varphi^{\flat,1}(w_{\mathrm{n}}), \\ &C_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + R_2(u_{\mathrm{n}}, u_{\mathrm{r}}) \quad \text{if } w_{\mathrm{l}} < \varphi^{\flat,1}(w_{\mathrm{n}}). \end{aligned}$$

(III.14) If $w_{\mathrm{l}} \geqslant w_{\mathrm{m}}$ and $a \geqslant w_{\mathrm{r}} > w_{\mathrm{n}}$, then the Riemann solution is either

$$R_1(u_{\mathrm{l}}, u_{\mathrm{m}}) + P_2^{\mathrm{sub}}(u_{\mathrm{m}}, u_{\mathrm{m}}^{2,\flat}) + \begin{cases} R_2(u_{\mathrm{m}}^{2,\flat}, u_{\mathrm{r}}) & \text{if } w_{\mathrm{r}} \leqslant \varphi^{\flat,2}(w_{\mathrm{m}}), \\ C_2(u_{\mathrm{m}}^{2,\flat}, u_{\mathrm{r}}) & \text{if } w_{\mathrm{r}} > \varphi^{\flat,2}(w_{\mathrm{m}}), \end{cases}$$

or

$$\begin{aligned} &R_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + C_2(u_{\mathrm{n}}, u_{\mathrm{r}}) \quad \text{if } w_{\mathrm{l}} \geqslant \varphi^{\flat,1}(w_{\mathrm{n}}), \\ &C_1(u_{\mathrm{l}}, u_{\mathrm{n}}^{1,\flat}) + P_1^{\mathrm{sub}}(u_{\mathrm{n}}^{1,\flat}, u_{\mathrm{n}}) + C_2(u_{\mathrm{n}}, u_{\mathrm{r}}) \quad \text{if } w_{\mathrm{l}} < \varphi^{\flat,1}(w_{\mathrm{n}}). \end{aligned}$$

https://doi.org/10.1017/S030821050000158X Published online by Cambridge University Press

(III.15) If $b \leqslant w_\mathrm{l} \leqslant w_\mathrm{m}$ and $-1 < w_\mathrm{r} \leqslant w_\mathrm{n}$, then the Riemann solution is either

$$C_1(u_\mathrm{l}, u_\mathrm{m}) + P_2^{\mathrm{sub}}(u_\mathrm{m}, u_\mathrm{m}^{2,\flat}) + \begin{cases} R_2(u_\mathrm{m}^{2,\flat}, u_\mathrm{r}) & \text{if } w_\mathrm{r} \leqslant \varphi^{\flat,2}(w_\mathrm{m}), \\ C_2(u_\mathrm{m}^{2,\flat}, u_\mathrm{r}) & \text{if } w_\mathrm{r} > \varphi^{\flat,2}(w_\mathrm{m}), \end{cases}$$

or

$$\begin{aligned} &R_1(u_\mathrm{l}, u_\mathrm{n}^{1,\flat}) + P_1^{\mathrm{sub}}(u_\mathrm{n}^{1,\flat}, u_\mathrm{n}) + R_2(u_\mathrm{n}, u_\mathrm{r}) \quad \text{if } w_\mathrm{l} \geqslant \varphi^{\flat,1}(w_\mathrm{n}), \\ &C_1(u_\mathrm{l}, u_\mathrm{n}^{1,\flat}) + P_1^{\mathrm{sub}}(u_\mathrm{n}^{1,\flat}, u_\mathrm{n}) + R_2(u_\mathrm{n}, u_\mathrm{r}) \quad \text{if } w_\mathrm{l} < \varphi^{\flat,1}(w_\mathrm{n}). \end{aligned}$$

(III.16) If $b \leqslant w_\mathrm{l} \leqslant w_\mathrm{m}$ and $a \geqslant w_\mathrm{r} > w_\mathrm{n}$, then the Riemann solution is either

$$C_1(u_\mathrm{l}, u_\mathrm{m}) + P_2^{\mathrm{sub}}(u_\mathrm{m}, u_\mathrm{m}^{2,\flat}) + \begin{cases} R_2(u_\mathrm{m}^{2,\flat}, u_\mathrm{r}) & \text{if } w_\mathrm{r} \leqslant \varphi^{\flat,2}(w_\mathrm{m}), \\ C_2(u_\mathrm{m}^{2,\flat}, u_\mathrm{r}) & \text{if } w_\mathrm{r} > \varphi^{\flat,2}(w_\mathrm{m}), \end{cases}$$

or

$$\begin{aligned} &R_1(u_\mathrm{l}, u_\mathrm{n}^{1,\flat}) + P_1^{\mathrm{sub}}(u_\mathrm{n}^{1,\flat}, u_\mathrm{n}) + C_2(u_\mathrm{n}, u_\mathrm{r}) \quad \text{if } w_\mathrm{l} \geqslant \varphi^{\flat,1}(w_\mathrm{n}), \\ &C_1(u_\mathrm{l}, u_\mathrm{n}^{1,\flat}) + P_1^{\mathrm{sub}}(u_\mathrm{n}^{1,\flat}, u_\mathrm{n}) + C_2(u_\mathrm{n}, u_\mathrm{r}) \quad \text{if } w_\mathrm{l} < \varphi^{\flat,1}(w_\mathrm{n}). \end{aligned}$$

Acknowledgments

P.L.F. was supported by the Centre National de la Recherche Scientifique. This work was done while M.D.T. was visiting the Ecole Polytechnique during the academic year 1999–2000 while being financially supported by the French–Vietnamese Institute 'ForMathVietnam'.

References

1 R. Abeyaratne and J. K. Knowles. On the dissipative response due to discontinuous strains in bars of unstable elastic materials. *Int. J. Solids Struct.* **24** (1988), 1021–1044.

2 R. Abeyaratne and J. K. Knowles. Kinetic relations and the propagation of phase boundaries in solids. *Arch. Ration. Mech. Analysis* **114** (1991), 119–154.

3 B. T. Hayes and P. G. LeFloch. Nonclassical shocks and kinetic relations: scalar conservation laws. *Arch. Ration. Mech. Analysis* **139** (1997), 1–56.

4 B. T. Hayes and P. G. LeFloch. Nonclassical shocks and kinetic relations: strictly hyperbolic systems. *SIAM J. Math. Analysis* **31** (2000), 941–991.

5 R. D. James. The propagation of phase boundaries in elastic bars. *Arch. Ration. Mech. Analysis* **73** (1980), 125–158.

6 P. D. Lax. Shock wave and entropy. In *Contributions to functional analysis* (ed. E. A. Zarantonello), pp. 603–634 (Academic, 1971).

7 P. G. LeFloch. Propagating phase boundaries: formulation of the problem and existence via the Glimm method. *Arch. Ration. Mech. Analysis* **123** (1993), 153–197.

8 P. G. LeFloch. An introduction to nonclassical shocks of systems of conservation laws. In *Proc. of the International School on Theory and Numerics for Conservation Laws, Freiburg/Littenweiler, Germany, 20–24 October 1997* (ed. D. Kröner, M. Ohlberger and C. Rohde), Lecture Notes in Computational Science and Engineering, pp. 28–72 (Springer, 1998).

9 P. G. LeFloch. *Hyperbolic systems of conservation laws: the theory of classical and nonclassical shock waves*. ETH Lecture Notes Series (Basel: Birkhäuser, 2001).

10 P. G. LeFloch and M. D. Thanh. Nonclassical Riemann solvers and kinetic relations. III. A nonconvex hyperbolic model for van der Waals fluids. *Electron. J. Diff. Eqns* **72** (2000), 1–19.

11 P. G. LeFloch and M. D. Thanh. Nonclassical Riemann solvers and kinetic relations. I. An hyperbolic model of elastodynamics. *Z. Angew. Math. Phys.* **52** (2001), 597–619.

12 T.-P. Liu. The Riemann problem for general 2×2 conservation laws. *Trans. Am. Math. Soc.* **199** (1974), 89–112.

13 T. J. Pence. On the mechanical dissipation of solutions to the Riemann problem for impact involving a two-phase elastic material. *Arch. Ration. Mech. Analysis* **117** (1992), 1–52.

14 P. Rosakis and J. K. Knowles. Unstable kinetic relations and the dynamics of solid–solid phase transitions. *J. Mech. Phys. Solids* **45** (1997), 2055–2081.

15 M. Shearer. The Riemann problem for a class of conservation laws of mixed type. *J. Diff. Eqns* **46** (1982), 426–443.

16 M. Shearer and Y. Yang. The Riemann problem for a system of mixed type with a cubic nonlinearity. *Proc. R. Soc. Edinb.* A **125** (1995), 675–699.

17 M. Slemrod. Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Ration. Mech. Analysis* **81** (1983), 301–315.

18 M. Slemrod. A limiting 'viscosity' approach to the Riemann problem for materials exhibiting a change of phase I. *Arch. Ration. Mech. Analysis* **105** (1989), 327–365.

19 L. Truskinovsky. Dynamics of non-equilibrium phase boundaries in a heat conducting non-linear elastic medium. *J. Appl. Math. Mech.* **51** (1987), 777–784.

20 L. Truskinovsky. Kinks versus shocks. In *Shock induced transitions and phase structures in general media* (ed. R. Fosdick, E. Dunn and H. Slemrod). IMA Volumes on Mathematics and its Applications, vol. 52, pp. 185–229 (Springer, 1993).

(*Issued 15 February 2002*)