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# An almost-periodic solution of Hasegawa–Wakatani equations with vanishing resistivity

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We consider the zero-resistivity limit for Hasegawa–Wakatani equations in a cylindrical domain when the initial data are Stepanov almost-periodic in the axial direction. First, we prove the existence of a solution to Hasegawa–Wakatani equations with zero resistivity; second, we obtain uniform *a priori* estimates with respect to resistivity. Such estimates can be obtained in the same way as for our previous results; therefore, the most important contribution of this paper is the proof of the existence of a local-in-time solution to Hasegawa–Wakatani equations with zero resistivity. We apply the theory of Bohr–Fourier series of Stepanov almost-periodic functions to such a proof.

*Keywords:* Hasegawa–Wakatani equations; drift wave turbulence; almost-periodic function

2010 *Mathematics subject classification:* Primary 35Q60; 35K45; 42A75; 42B05; 82D10

## 1. Introduction

There are many applications of plasma physics, for example, in light sources, surface treatments and nuclear fusion. Specifically, controlled thermonuclear fusion has, thus far, been a challenging problem, because there are many kinds of instability in nuclear fusion plasmas. To confine high-temperature plasma in a vacuum vessel using magnetic forces, we must control both microscopic and macroscopic instabilities [21]. Following studies that began with the Vlasov equation, new instabilities were discovered when the wavelength of the perturbed fields was taken to be as small as the Larmor radius of the charged particles. These instabilities were termed microscopic instabilities [28]. Therefore, to study nuclear fusion plasmas, we must use both magnetohydrodynamic equations, which describe macroscopic instabilities, and other model equations [21].

In this paper, we are concerned with the Hasegawa–Wakatani equations, which are useful in the study of resistive drift wave turbulence and the related anomalous transport, and result in a dramatic reduction in confinement time in a tokamak. A tokamak is the most advanced magnetic confinement device for thermonuclear fusion; in this device, an axisymmetric plasma is confined by a strong magnetic field.

Turbulence that is excited by unstable drift waves is called drift wave turbulence; drift wave instability is classed as a microscopic instability [28]. Drift waves are quasi-electrostatic waves that propagate perpendicularly to both the magnetic field and the density gradient [22]. Hasegawa–Wakatani equations are fluid models; thus, to research the effect of non-Maxwellian distributions in velocity space, we must consider the problem for Vlasov–Poisson equations (kinetic equations) [24, 25].

In 1983, Hasegawa and Wakatani [12, 13] proposed the following equations for the perturbations of plasma density $n$ and the electrostatic potential $\phi$ to describe the resistive drift wave turbulence in the tokamak:

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}-(\nabla\phi\times\boldsymbol{e})\cdot\nabla\right)\Delta\phi&=-\frac{c_1}{n^*}\frac{\partial^2}{\partial x_3^2}(\phi-n)+c_2\Delta^2\phi,\\ \left(\frac{\partial}{\partial t}-(\nabla\phi\times\boldsymbol{e})\cdot\nabla\right)(n+\log n^*)&=-\frac{c_1}{n^*}\frac{\partial^2}{\partial x_3^2}(\phi-n).\end{aligned}\right\}\tag{1.1}$$

These are Hasegawa–Wakatani equations from the two-fluid models in a strong homogeneous magnetic field $\mathbb{B}=B_0\boldsymbol{e}$ and an inhomogeneous plasma equilibrium density $n^*=n^*(|x'|)$ $(x=(x_1,x_2,x_3)=(x',x_3),\ |x'|^2=x_1^2+x_2^2)$. Here, the total density $N$ is divided into equilibrium and fluctuating parts, $N=n^*+n^1$; and the normalizations $e\phi/T_{\mathrm{e}}\equiv\phi$, $n^1/n^*\equiv n$, $\omega_{\mathrm{ci}}t\equiv t$ and $x/\rho_s\equiv x$ are used. Here, $B_0$ is the strength of the magnetic field (assumed to be a constant), $\boldsymbol{e}=(0,0,1)$, $c_1=T_{\mathrm{e}}/(e^2\eta\omega_{\mathrm{ci}})$, $c_2=\mu/(\rho_s^2\omega_{\mathrm{ci}})$, $T_{\mathrm{e}}$ is the electron temperature, $e$ is the elementary charge, $\mu$ is the kinematic ion-viscosity coefficient, $\eta$ is the resistivity, $m_{\mathrm{i}}$ is the ion mass, $\omega_{\mathrm{ci}}=eB_0/m_{\mathrm{i}}$ is the cyclotron frequency and $\rho_s=\sqrt{T_{\mathrm{e}}}/(\omega_{\mathrm{ci}}\sqrt{m_{\mathrm{i}}})$ is the ion Larmor radius. For simplicity, we assume that $c_1$ and $c_2$ are positive constants.

In 1977, before the advent of Hasegawa–Wakatani equations, Hasegawa and Mima [10, 11] proposed the following equation:

$$\left(\frac{\partial}{\partial t}-(\nabla\phi\times\boldsymbol{e})\cdot\nabla\right)(\Delta\phi-\phi-\log n^*)=0.\tag{1.2}$$

This is the Hasegawa–Mima equation from the one-fluid model under the same magnetic field and plasma equilibrium state as used for the Hasegawa–Wakatani equations. Concerning the mathematical results for (1.2) we refer the reader to [16] and the references therein.

Our results address the mathematical issues inherent in (1.1). The existence and uniqueness of a strong solution to the initial–boundary-value problems for (1.1) in a cylindrical domain were proven when the initial data are periodic in the axial direction [16], and when the initial data are Stepanov almost-periodic [19]. In nuclear fusion research, it is important to consider an irrational magnetic surface on which the line of force covers the surface ergodically without closing [27]. However, research into plasma phenomena in an irrational magnetic surface is difficult; therefore, we consider a simple problem as the first step in researching plasma phenomena in a tokamak. In [17, 18] we proved that, as the resistivity tends to zero, the solution of the Hasegawa–Wakatani equations established in [16] converges strongly to that of the model equations of drift wave turbulence with zero resistivity. When the temperature of the plasma is very high, the resistivity of the plasma approaches zero; therefore, it is important for nuclear fusion plasma research to consider the case of zero resistivity. In [14] we obtained two useful lemmas for Stepanov almost-periodic functions for the purpose of obtaining uniform *a priori* estimates for resistivity; additionally, we proved that the Stepanov almost-periodic solution of linearized Hasegawa–Wakatani equations converges strongly to that of linearized Hasegawa–Wakatani equations with zero resistivity as the resistivity tends to zero when the initial data are Stepanov almost-periodic. In [15], we used the lemmas presented in [14] to prove that the Stepanov almost-periodic solutions of the Hasegawa–Wakatani equations established in [19] converge strongly to that of the Hasegawa–Wakatani equations with zero resistivity under the additional condition of $\bar{n} = 0$ as the resistivity tends to zero. Note that Hasegawa–Wakatani equations with zero resistivity under the additional condition $\bar{n} = 0$ are similar to the Hasegawa–Mima equation with a higher-order correction term.

By defining $\varepsilon = 1/c_1$, (1.1) can clearly be written as

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - (\nabla\phi \times \boldsymbol{e}) \cdot \nabla\right)(\Delta\phi - n - \log n^*) &= c_2 \Delta^2 \phi, \\ \varepsilon\left(\frac{\partial}{\partial t} - (\nabla\phi \times \boldsymbol{e}) \cdot \nabla\right)(n + \log n^*) &= -\frac{1}{n^*}\frac{\partial^2}{\partial x_3^2}(\phi - n). \end{aligned}\right\} \tag{1.3}$$

Note that $\partial^2 n/\partial x_1^2$ and $\partial^2 n/\partial x_2^2$ do not appear on the right-hand side of $(1.3)_2$. We shall show that when studying the zero-resistivity limit for (1.3) with almost-periodic initial data this anisotropy causes unexpected difficulties. Generally, when looking for almost-periodic solutions, one looks for almost-periodicity in the time variable. However, in this paper, we consider another problem, as follows.

For a given initial electrostatic potential $\phi_0^\varepsilon$, an initial plasma density $n_0^\varepsilon$, and a background density $n^* = n^*(|x'|)$, let $(\phi^\varepsilon, n^\varepsilon) = (\phi^\varepsilon, n^\varepsilon)(x, t)$ be a solution to the initial–boundary-value problem for (1.3) with $\varepsilon > 0$ in $\omega \times \mathbb{R} \times (0, \infty) \equiv \Omega \times (0, \infty)$ under the following initial and boundary conditions:

$$\left.\begin{aligned} \phi^\varepsilon(x, 0) = \phi_0^\varepsilon(x), \quad n^\varepsilon(x, 0) = n_0^\varepsilon(x) \quad &\text{for } x \in \Omega, \\ \phi^\varepsilon(x, t) = \Delta\phi^\varepsilon(x, t) = n^\varepsilon(x, t) = 0 \quad &\text{for } x \in \Gamma,\ t > 0, \end{aligned}\right\} \tag{1.4}$$

when the initial data are Stepanov almost-periodic in the direction $\boldsymbol{e}$. Here, $\omega = \{x' = (x_1, x_2) \in \mathbb{R}^2 \mid |x'| < R\}$, $\partial\omega = \{x' = (x_1, x_2) \in \mathbb{R}^2 \mid |x'| = R\}$, $\Gamma = \{x \in \mathbb{R}^3 \mid x' \in \partial\omega\}$ and $R$ is a positive real number.

There are several technological applications of almost-periodic functions, for example, in time–frequency analysis of audio signals (e.g. piano, singing voice and violin tones). The fast Fourier transform (FFT) is generally used in time–frequency analysis; however, the FFT is influenced by its window. On the other hand, as suggested by Wiener, generalized harmonic analysis (GHA) using almost-periodic functions can analyse and synthesize signals without introducing window effects. GHA has a very high resolution. Furthermore, a waveform reconstructed by GHA can enable prediction by extrapolation of a reconstructed waveform. However, because this method has a high computational cost, there was no attempt to pursue technological applications until Hirata developed an efficient GHA algorithm in 1994. In [23, 26, 29], GHA is applied to non-stationary signals. Because GHA is a new field, further research is required to clarify its fundamental characteristics.

It is more convenient to change $n^\varepsilon(x,t)$ and $n_0^\varepsilon(x)$ into $n^\varepsilon(x,t)+\log n^*(|x'|)-\log n^*(R)$ and $n_0^\varepsilon(x)+\log n^*(|x'|)-\log n^*(R)$, respectively, while keeping the same notation $n^\varepsilon(x,t)$ and $n_0^\varepsilon(x)$. Then, (1.3) becomes

$$\left.\begin{aligned}\Big(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla\Big)(\Delta\phi^\varepsilon-n^\varepsilon)&=c_2\Delta^2\phi^\varepsilon,\\ \varepsilon\Big(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla\Big)n^\varepsilon&=-\frac{1}{n^*}\frac{\partial^2}{\partial x_3^2}(\phi^\varepsilon-n^\varepsilon)\quad\text{for } x\in\Omega,\ t>0,\end{aligned}\right\}\tag{1.5}$$

whereas (1.4) remains unchanged.

For convenience, we introduce

$$\bar{f}^\varepsilon(x')=\mathcal{M}\{f^\varepsilon(x)\}\equiv\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}f^\varepsilon(x)\,\mathrm{d}x_3,$$
$$\tilde{f}^\varepsilon(x)=f^\varepsilon(x)-\mathcal{M}\{f^\varepsilon(x)\}\equiv(\mathcal{I}-\mathcal{M})\{f^\varepsilon(x)\}.$$

Then, problem (1.5) is equivalent to the following problem:

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla\Big)(\Delta\phi^\varepsilon-n^\varepsilon)&=c_2\Delta^2\phi^\varepsilon,\\ \varepsilon(\mathcal{I}-\mathcal{M})\Big\{\Big(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla\Big)n^\varepsilon\Big\}&=-\frac{1}{n^*}\frac{\partial^2}{\partial x_3^2}(\tilde{\phi}^\varepsilon-\tilde{n}^\varepsilon),\\ \mathcal{M}\Big\{\Big(\frac{\partial}{\partial t}-(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla\Big)n^\varepsilon\Big\}&=0\quad\text{for } x\in\Omega,\ t>0,\end{aligned}$$

and (1.4) remains the same.

Setting $\varepsilon=0$ in this problem, we have

$$\left.\begin{aligned}\Big(\frac{\partial}{\partial t}-(\nabla\phi^0\times\boldsymbol{e})\cdot\nabla\Big)(\Delta\phi^0-n^0)&=c_2\Delta^2\phi^0,\\ \frac{1}{n^*}\frac{\partial^2}{\partial x_3^2}(\tilde{\phi}^0-\tilde{n}^0)&=0,\\ \mathcal{M}\Big\{\Big(\frac{\partial}{\partial t}-(\nabla\phi^0\times\boldsymbol{e})\cdot\nabla\Big)n^0\Big\}&=0\quad\text{for } x\in\Omega,\ t>0.\end{aligned}\right\}\tag{1.6}$$

The aim of this paper is to establish both the unique existence of a strong Stepanov almost-periodic solution to the initial–boundary-value problem for (1.4) and (1.6) with $\varepsilon=0$ when the initial data are Stepanov almost-periodic in the direction $\boldsymbol{e}$ and the convergence of $(\phi^\varepsilon,n^\varepsilon)$ to $(\phi^0,n^0)$ as $\varepsilon$ tends to zero on some interval, which corresponds to the vanishing resistivity of Hasegawa–Wakatani equations. Note that a similar problem is considered in [15] under the additional condition of $\bar{n}=0$.

In § 2, Stepanov almost-periodic functions and the function spaces appearing in the following theorems are defined.

The following proposition is established in [15].

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Proposition 1.1. *Let $c^\star > 0$, let $\varepsilon \in (0, c^\star]$ and let $n^*(|x'|) \in W_2^2(\omega)$ satisfy $n^*(|x'|) \geqslant n_*$ with a positive constant $n_*$. Assume that $(\phi_0^\varepsilon, n_0^\varepsilon) \in \tilde{S}^4_{\mathrm{ap}}(\mathbb{R}; L^2(\omega)) \times \tilde{S}^2_{\mathrm{ap}}(\mathbb{R}; L^2(\omega))$ satisfies the compatibility conditions*

$$\phi_0^\varepsilon(x) = \Delta\phi_0^\varepsilon(x) = n_0^\varepsilon(x) = 0 \quad \textit{for } x \in \Gamma. \tag{1.7}$$

*Then, there exists a unique solution $(\phi^\varepsilon, n^\varepsilon)$ to problem (1.4), (1.5) on some interval $[0, T]$ such that $(\phi^\varepsilon, n^\varepsilon) \in L^2(0, T; \tilde{S}^4_{\mathrm{ap}}(\mathbb{R}; L^2(\omega))) \times \tilde{S}^{2;1}_{\mathrm{ap}}(\mathbb{R}; L^2(\omega_T))$, $\partial\phi^\varepsilon/\partial t \in L^2(0, T; \tilde{S}^2_{\mathrm{ap}}(\mathbb{R}; L^2(\omega)))$. Here, $T$ is a constant that is independent of $\varepsilon$.*

The second equation of (1.6) implies $\tilde{\phi}^0 - \tilde{n}^0 = 0$ by virtue of the almost-periodic condition in $x_3$ and $\mathcal{M}\tilde{\phi}^0 = \mathcal{M}\tilde{n}^0 = 0$. Inserting this relation into (1.6) and (1.4) with $\varepsilon = 0$, we have

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial t} - (\nabla\phi^0 \times \boldsymbol{e})\cdot\nabla\right)(\Delta\phi^0 - \tilde{\phi}^0) + (\nabla\tilde{\phi}^0 \times \boldsymbol{e})\cdot\nabla\bar{n}^0 &= c_2\Delta^2\phi^0,\\
\left(\frac{\partial}{\partial t} - (\nabla\bar{\phi}^0 \times \boldsymbol{e})\cdot\nabla\right)\bar{n}^0 = 0 &\quad \text{for } x \in \Omega,\ t > 0,\\
\phi^0(x, 0) = \phi_0^0(x) &\quad \text{for } x \in \Omega,\\
\bar{n}^0(x', 0) = \bar{n}_0^0(x') &\quad \text{for } x' \in \omega,\\
\phi^0(x, t) = \Delta\phi^0(x, t) = 0 &\quad \text{for } x \in \Gamma,\ t > 0,\\
\bar{n}^0(x', t) = 0 &\quad \text{for } x' \in \partial\omega,\ t > 0.
\end{aligned}\right\} \tag{1.8}$$

Under an additional condition $\bar{n}_0^0(x') = 0$, the equation in (1.8) is similar to the Hasegawa–Mima equation (1.2) with a higher-order correction term.

In § 3, the following theorem is proven through the local-in-time existence and *a priori* estimates. We can easily obtain *a priori* estimates in the same way as in [15]. Note that a similar problem to that in the present paper is considered in [17] under a periodic boundary condition. The most important contribution of our paper is the proof of the local-in-time solution existence theorem. In [14, 15, 19], a proof of the existence of a local-in-time solution to a linear problem is given; in this paper, we modify the proof and show the existence of a local-in-time solution to a nonlinear problem.

Theorem 1.2. *Assume that $(\phi_0^0, \bar{n}_0^0) \in \tilde{S}^4_{\mathrm{ap}}(\mathbb{R}; L^2(\omega)) \times W_2^3(\omega)$ satisfies the compatibility conditions (1.7) with $\varepsilon = 0$. Then, there exists a unique solution $(\phi^0, \bar{n}^0)$ to the problem (1.8) on some interval $[0, T^*]$ such that*

$$(\phi^0, \bar{n}^0) \in L^2(0, T^*; \tilde{S}^4_{\mathrm{ap}}(\mathbb{R}; L^2(\omega))) \times L^\infty(0, T^*; W_2^3(\omega)),$$
$$\partial\phi^0/\partial t \in L^2(0, T^*; \tilde{S}^2_{\mathrm{ap}}(\mathbb{R}; L^2(\omega))) \quad \textit{and} \quad \partial\bar{n}^0/\partial t \in L^2(0, T^*; W_2^2(\omega)).$$

For this solution $(\phi^0, \bar{n}^0)$, let $\tilde{n}^0(x, t) = \tilde{\phi}^0(x, t)$ and $\tilde{n}_0^0(x) = \tilde{\phi}_0^0(x)$. Then, $(\phi^0, n^0)$ satisfies (1.4) and (1.6).

In § 4 the following theorem is proven by virtue of *a priori* estimates, proposition 1.1 and theorem 1.2.

THEOREM 1.3. *Let* $(\phi^\varepsilon, n^\varepsilon)$ *and* $(\phi^0, n^0)$ *be the solutions established in proposition 1.1 and theorem 1.2, respectively. If the initial data*

$$(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0) \quad \text{as } \varepsilon \to 0$$

*in* $\tilde{S}^3(\mathbb{R}; L^2(\omega)) \times \tilde{S}^2(\mathbb{R}; L^2(\omega))$*, then, as* $\varepsilon \to 0$*,*

$$(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0) \quad \text{in } L^2(0, T^\sharp; \tilde{S}^4(\mathbb{R}; L^2(\omega))) \times \tilde{S}^{2,0}(\mathbb{R}; L^2(\omega_{T^\sharp})),$$
$$\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0 \quad \text{in } \tilde{S}^{0,1}(\mathbb{R}; L^2(\omega_{T^\sharp}))$$

*and* $\bar{n}^\varepsilon \to \bar{n}^0$ *in* $W_2^{0,1}(\omega_{T^\sharp})(\omega_{T^\sharp} \equiv \omega \times (0, T^\sharp))$ *on the some time-interval* $[0, T^\sharp]$*, where* $T^\sharp$ *is determined from proposition 1.1 and theorem 1.2.*

## 2. Function spaces

We introduce the function spaces and the almost periodic functions that we use in the following (see [1]).

Let $\Omega$ be a domain in $\mathbb{R}^m$ $(m = 1, 2, 3, \dots)$. We denote by $W_2^l(\Omega)$ $(l \in \mathbb{R},\ l \geqslant 0)$ the space of functions $u(x)$, $x \in \Omega$, equipped with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{|\alpha|<l} \|\mathrm{D}_x^\alpha u\|^2_{L^2(\Omega)} + \|u\|^2_{\dot{W}_2^l(\Omega)},$$

where

$$\|u\|^2_{\dot{W}_2^l(\Omega)} = \begin{cases} \displaystyle\sum_{|\alpha|=l} \|\mathrm{D}_x^\alpha u\|^2_{L^2(\Omega)} & \text{for } l \in \mathbb{Z}, \\ \displaystyle\sum_{|\alpha|=[l]} \int_\Omega \int_\Omega \frac{|\mathrm{D}_x^\alpha u(x) - \mathrm{D}_y^\alpha u(y)|^2}{|x-y|^{m+2(l-[l])}}\,\mathrm{d}x\,\mathrm{d}y & \text{for } l \notin \mathbb{Z}. \end{cases}$$

Here $[l]$ is the integral part of $l$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a multi-index and $\mathrm{D}_x^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}$ is the generalized derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. For $1 \leqslant p \leqslant \infty$, we denote by $\|\cdot\|_{L^p(\Omega)}$ the norm of the Lebesgue space $L^p(\Omega)$.

The anisotropic Sobolev–Slobodetskiĭ space $W_2^{l,l/2}(Q_T)$ $(Q_T \equiv \Omega \times (0,T))$ is defined as $L^2(0, T; W_2^l(\Omega)) \cap L^2(\Omega; W_2^{l/2}(0,T))$, equipped with the norm

$$\begin{aligned} \|u\|^2_{W_2^{l,l/2}(Q_T)} &= \|u\|^2_{W_2^{l,0}(Q_T)} + \|u\|^2_{W_2^{0,l/2}(Q_T)} \\ &\equiv \int_0^T \|u(t)\|^2_{W_2^l(\Omega)}\,\mathrm{d}t + \int_\Omega \|u(x)\|^2_{W_2^{l/2}(0,T)}\,\mathrm{d}x. \end{aligned}$$

Let $X$ be a Banach space with the norm $\|\cdot\|_X$. We denote by $S^p(\mathbb{R}; X)$ $(1 \leqslant p < \infty)$ the subspace of $L^p_{\mathrm{loc}}(\mathbb{R}; X)$ equipped with the finite norm

$$\|u\|^p_{S^p(\mathbb{R};X)} \equiv \sup_{s\in\mathbb{R}} \int_s^{s+1} \|u(x)\|_X^p\,\mathrm{d}x.$$

The function $f(x) \in L^p_{\mathrm{loc}}(\mathbb{R}; X)$ is called Stepanov almost-periodic ($S^p$-a.p.) if, for any $\varepsilon > 0$, the set

$$E_\varepsilon(f) \equiv \left\{ \sigma \in \mathbb{R} \;\middle|\; \sup_{s\in\mathbb{R}} \left( \int_s^{s+1} \|f(x+\sigma) - f(x)\|_X^p\,\mathrm{d}x \right)^{1/p} \leqslant \varepsilon \right\}$$

is relatively dense in $\mathbb{R}$, that is, there exists $L = L(\varepsilon) > 0$ (inclusion length) such that $E_\varepsilon(f) \cap (a, a+L) \neq \emptyset$ for any $a \in \mathbb{R}$. We denote by $S^p_{\rm ap}(\mathbb{R}; X)$ the space of all $S^p$-a.p. functions from $\mathbb{R}$ to $X$.

Let $\omega_T \equiv \omega \times (0, T)$ and $l \in \mathbb{Z}$, $l \geqslant 0$. We introduce the following spaces:

$$\tilde{S}^l(\mathbb{R}; X) = \left\{ u \in S^2(\mathbb{R}; X) \;\middle|\; \|u\|^2_{\tilde{S}^l} \equiv \sum_{|\alpha|=0}^{l} \|\mathrm{D}^\alpha_x u\|^2_{S^2(\mathbb{R};X)} < \infty \right\},$$

$$\tilde{S}^l_{\rm ap}(\mathbb{R}; X) = \{u \in \tilde{S}^l(\mathbb{R}; X) \mid \mathrm{D}^\alpha_x u \in S^2_{\rm ap}(\mathbb{R}; X), |\alpha| = 0, 1, \dots, l\},$$

$$\tilde{S}^{l,l/2}(\mathbb{R}; L^2(\omega_T)) = \tilde{S}^l(\mathbb{R}; L^2(\omega_T)) \cap \tilde{S}^0(\mathbb{R}; L^2(\omega; W^{l/2}_2(0, T))),$$

$$\tilde{S}^{l,l/2}_{\rm ap}(\mathbb{R}; L^2(\omega_T)) = \tilde{S}^l_{\rm ap}(\mathbb{R}; L^2(\omega_T)) \cap \tilde{S}^0_{\rm ap}(\mathbb{R}; L^2(\omega; W^{l/2}_2(0, T))).$$

Here $\mathrm{D}^\alpha_x u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$ is the generalized derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ for a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ when $X = L^2(\omega)$, $L^2(\omega_T)$ or $L^2(\omega; W^1_2(0, T))$.

Moreover, we define the norm

$$\|u\|^2_{\tilde{S}^{l,l/2}_T} \equiv \|u\|^2_{\tilde{S}^l(L^2(\omega_T))} + \|u\|^2_{\tilde{S}^0(L^2(\omega; W^{l/2}_2(0,T)))}.$$

We denote by $\|\cdot\|$, $\|\cdot\|_{S^p}$ and $|||\cdot|||_T$ the norms in $S^2(\mathbb{R}; L^2(\omega))$, $S^p(\mathbb{R}; L^p(\omega))$ and $S^2(\mathbb{R}; L^2(\omega_T))$, respectively, and set

$$\|\mathrm{D}^l_x \phi\|^2_{L^p(\Omega)} \equiv \sum_{|\alpha|=l} \|\mathrm{D}^\alpha_x \phi\|^2_{L^p(\Omega)} \quad (1 \leqslant p \leqslant \infty,\ l = 2, 3),$$

$\partial_t = \partial/\partial t$ and $\partial_{x_k} = \partial/\partial x_k$.

## 3. Proof of theorem 1.2

The proof is divided into two parts. In § 3.1 we prove the local-in-time existence in a similar way to that in [14, 15, 19]. In § 3.2 we prove theorem 1.2 with the help of *a priori* estimates.

### 3.1. Local-in-time existence and uniqueness

3.1.1. *Auxiliary lemmas*

Let $X$ be a Hilbert space and let $\psi \in S^2_{\rm ap}(X)$. Note that for any $\xi \in \mathbb{R}$ the mean value

$$\psi_\xi = \mathcal{M}\{\psi(x)\mathrm{e}^{-\mathrm{i}\xi x_3}\} \equiv \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \psi(x)\mathrm{e}^{-\mathrm{i}\xi x_3}\,\mathrm{d}x_3$$

exists in $X$ [5, 30], where $\mathrm{i} = \sqrt{-1}$.

Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\xi_k \neq \xi_{k'}$ for $k \neq k'$. For each $m \in \mathbb{N}$ it is easy to obtain

$$\mathcal{M}\left\{ \left\| \psi(x_3) - \sum_{k=1}^{m} \psi_{\xi_k} \mathrm{e}^{-\mathrm{i}\xi_k x_3} \right\|^2_X \right\} = \mathcal{M}\{\|\psi(x_3)\|^2_X\} - \sum_{k=1}^{m} \|\psi_{\xi_k}\|^2_X,$$

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and hence

$$\sum_{k=1}^{m} \|\psi_{\xi_k}\|_X^2 \leqslant \mathcal{M}\{\|\psi(x_3)\|_X^2\}.$$

This inequality implies that for any $\varepsilon > 0$ there correspond at most a finite number of $\xi_k$ for which $\|\psi_{\xi_k}\|_X > \varepsilon$. From this fact it follows that every $\|\psi_{\xi_k}\|_X (\neq 0)$ belongs to one of the enumerable set of inequalities

$$\|\psi_{\xi_k}\|_X > 1, \quad \frac{1}{m} \geqslant \|\psi_{\xi_k}\|_X > \frac{1}{m+1} \quad (m = 1, 2, 3, \dots),$$

and each of these inequalities is satisfied by at most a finite number of $\xi_k$. Therefore, the quantity $\psi_\xi$ is a non-zero element of $X$ only for at most countable $\xi \in \mathbb{R}$. We call $\sigma(\psi) = \{\xi \in \mathbb{R} \mid \|\psi_\xi\|_X \neq 0\}$ the spectrum of $\psi$, and the formal series $\sum_{\xi\in\sigma(\psi)} \psi_\xi \mathrm{e}^{\mathrm{i}\xi x_3}$ the Bohr–Fourier series of $\psi$, which is written as

$$\psi \sim \sum_{\xi\in\sigma(\psi)} \psi_\xi \mathrm{e}^{\mathrm{i}\xi x_3}.$$

Then the following lemmas hold (see [1, 3, 7, 8]).

LEMMA 3.1. *If $\psi, \psi' \in S^2_{\mathrm{ap}}(X)$ have the same Bohr–Fourier series, then*

$$\|\psi - \psi'\|_{S^2(X)} = 0.$$

LEMMA 3.2. *For any $\psi \in S^2_{\mathrm{ap}}(X)$ Parseval's identity*

$$\mathcal{M}\{\|\psi(x_3)\|_X^2\} = \sum_{\xi\in\sigma(\psi)} \|\psi_\xi\|_X^2$$

*holds.*

Let us consider a generalized trigonometric series

$$\sum_{\xi\in\Lambda} a_\xi \mathrm{e}^{\mathrm{i}\xi x}, \tag{3.1}$$

where $\Lambda$ is a countable subset of $\mathbb{R}$ and $\{a_\xi\}_{\xi\in\Lambda} \subset \mathbb{C}$. Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be a basis of $\Lambda$ [5]. The Bochner–Fejér sum $\mathcal{S}^m(x)$ associated with (3.1) is given by

$$\mathcal{S}^m(x) = \sum_{\nu_1=-(m!)^2}^{(m!)^2} \cdots \sum_{\nu_m=-(m!)^2}^{(m!)^2} \left(1 - \frac{|\nu_1|}{(m!)^2}\right) \cdots \left(1 - \frac{|\nu_m|}{(m!)^2}\right) \times a_\xi^* \exp\left(\mathrm{i}\sum_{j=1}^{m} \nu_j \frac{\gamma_j}{m!} x\right),$$

where, for $\xi \in \Lambda$,

$$a_\xi^* = \begin{cases} a_\xi & \text{if } \displaystyle\sum_{j=1}^{m} \nu_j \frac{\gamma_j}{m!} = \xi, \\ 0 & \text{if } \displaystyle\sum_{j=1}^{m} \nu_j \frac{\gamma_j}{m!} \neq \xi. \end{cases}$$

By introducing an increasing symmetric sequence $\{\Lambda_m\}_{m\in\mathbb{N}}$ of $\Lambda$ converging to $\Lambda$, that is, $-\Lambda_m = \Lambda_m$, $\Lambda_m \subset \Lambda_{m+1}$ and $\Lambda = \bigcup_m \Lambda_m$, $\mathcal{S}^m(x)$ can be written as

$$\mathcal{S}^m(x) = \sum_{\xi\in\Lambda_m} d_\xi^{(m)} a_\xi \mathrm{e}^{\mathrm{i}\xi x}$$

with constants $d_\xi^{(m)}$ satisfying $0 \leqslant d_\xi^{(m)} \leqslant 1$ and $\lim_{m\to\infty} d_\xi^{(m)} = 1$. Note that $d_\xi^{(m)}$ depend on $\xi$ and $m$ but not on $a_\xi$ [8].

We say that $\mathcal{F} \subset S^p_{\mathrm{ap}}(X)$ is $S^p$-equi-almost-periodic if, for any $\varepsilon > 0$, there exists a relatively dense subset $E_\varepsilon$ of $\mathbb{R}$ such that

$$\sup_{s\in\mathbb{R}} \int_s^{s+1} \|f(x+\sigma) - f(x)\|_X^p \,\mathrm{d}x < \varepsilon \quad \text{for } f \in \mathcal{F},\ \sigma \in E_\varepsilon.$$

It is well known that the Riesz–Fischer theorem does not hold for $S^p_{\mathrm{ap}}(X)$ $(1 \leqslant p < \infty)$ [2, 20], while the following lemma holds true (see [6, 9]).

Lemma 3.3. *A necessary and sufficient condition for a generalized trigonometric series (3.1) to be a Bohr–Fourier series of a function $f \in S^p_{\mathrm{ap}}(X)$ $(1 < p < \infty)$ is that a sequence of the Bochner–Fejér sums $\{\mathcal{S}^m(x)\}_{m\in\mathbb{N}}$ associated with the series (3.1) is bounded in $S^p(X)$ and is $S^p$-equi-almost-periodic.*

3.1.2. *Local-in-time existence and uniqueness*

Let us fix the symmetric increasing sequence $\{\Lambda_m\}_{m\in\mathbb{N}}$ of $\Lambda \equiv \sigma(\phi_0^0) = \{\xi \in \mathbb{R} \mid \|\phi^0_{0\xi}\|_{L^2(\omega)} \neq 0\}$ converging to $\Lambda$. For $\xi \in \Lambda_m$ we consider the problem

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}((\Delta' - \xi^2) d_\xi^{(m)} \phi_\xi^m - (\mathcal{I} - \mathcal{M})\{d_\xi^{(m)} \phi_\xi^m\}) - c_2 (\Delta' - \xi^2)^2 d_\xi^{(m)} \phi_\xi^m \\
&\quad = \sum_{\eta+\theta=\xi} (\nabla d_\eta^{(m)} \phi_\eta^m \times \boldsymbol{e}) \cdot \nabla((\Delta' - \theta^2) d_\theta^{(m)} \phi_\theta^m - (\mathcal{I} - \mathcal{M})\{d_\theta^{(m)} \phi_\theta^m\}) \\
&\qquad\qquad - (\nabla(\mathcal{I} - \mathcal{M})\{d_\xi^{(m)} \phi_\xi^m\} \times \boldsymbol{e}) \cdot \nabla n^m, \\
&\left(\frac{\partial}{\partial t} - (\nabla\phi_0^m \times \boldsymbol{e}) \cdot \nabla\right) n^m = 0 \qquad \text{for } x' \in \omega,\ t > 0, \\
&(\phi_\xi^m, n^m)|_{t=0} = (\phi^0_{0\xi}, \bar{n}_0^0) \quad \text{for } x' \in \omega, \\
&(\phi_\xi^m, n^m) = (0, 0) \qquad \text{for } x' \in \partial\omega,\ t > 0,
\end{aligned}\right\} \tag{3.2}$$

where $\phi^0_{0\xi} = \mathcal{M}\{\phi_0^0 \mathrm{e}^{-\mathrm{i}\xi x_3}\}$. The existence of a unique solution $(\phi_\xi^m, n^m)$ to (3.2) can be proven by the method of characteristics and successive approximations. We define $\boldsymbol{v}(x', t) \equiv -\nabla\phi_0^m(x', t) \times \boldsymbol{e}$ and introduce the characteristic transformation $\Pi_{\xi'}^{x'}: x' \mapsto \xi' = (\xi_1, \xi_2) \equiv X(0; x', t)$, where $X(\tau; x', t)$ is the solution curve of the ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} X(\tau; x', t) = \boldsymbol{v}(X(\tau; x', t), \tau), \quad X(t; x', t) = x' \quad (0 \leqslant \tau \leqslant t). \tag{3.3}$$

The unique existence of such a solution curve $X(\tau; x', t)$ $(x' \in \Omega,\ 0 \leqslant \tau \leqslant t)$ of (3.3) is due to the fundamental existence theorem of ordinary differential equations provided that $v$ is suitably smooth. Let $x' = X^{-1}(t;\xi',0)$ be the inverse of $X(0;x',t) = \xi'$. Then (3.3) implies that $X^{-1}$ is a solution curve of

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X^{-1}(\tau;\xi',0) = \boldsymbol{u}(\xi',\tau) \tag{3.4}$$

with $\boldsymbol{u}(\xi',t) \equiv \boldsymbol{v}(X^{-1}(t;\xi',0),t) = \boldsymbol{v}(x',t)$ whose solution is expressed by

$$x' = X^{-1}(t;\xi',0) = \xi' + \int_0^t \boldsymbol{u}(\xi',\tau)\,\mathrm{d}\tau \equiv X_u(\xi',t). \tag{3.5}$$

According to the condition $\boldsymbol{v} = \mathbf{0}$ on $\partial\omega$, $\Pi_{\xi'}^{x'}$ is a one-to-one mapping from $\bar{\omega}$ and $\partial\omega$ onto $\bar{\omega}$ and $\partial\omega$, respectively for each $t > 0$.

Then $(3.2)_2$ yields $\partial\bar{\rho}/\partial t = 0$ for $\bar{\rho}(\xi',t) \equiv n^m(X_u(\xi',t),t)$, which is easily solved as

$$\bar{\rho}(\xi',t) = \bar{\rho}(\xi',0) = \bar{n}_0^0(X_u(\xi',0)) = \bar{n}_0^0(\xi'). \tag{3.6}$$

Using the results above we can transform (3.2) by using the characteristic transformation $\Pi_{\xi'}^{x'}$ and then we can prove the local-in-time existence theorem for the transformed problem by successive approximations. Finally, we can prove the local-in-time existence theorem for the problem (3.2) by using the inverse transformations of $\Pi_{\xi'}^{x'}$. This proof is similar to that in [17], and hence we omit it.

Then it is obvious that $(\mathcal{S}_{\phi^0}^m, n^m) = (\sum_{\xi\in\Lambda_m} d_\xi^{(m)}\phi_\xi^m \mathrm{e}^{\mathrm{i}\xi x_3}, n^m)$ is a solution of the problem

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial t} - (\nabla\mathcal{S}_{\phi^0}^m \times \boldsymbol{e})\cdot\nabla\right)&(\Delta\mathcal{S}_{\phi^0}^m - (\mathcal{I}-\mathcal{M})\{\mathcal{S}_{\phi^0}^m\}) \\
&+ (\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}_{\phi^0}^m\}\times\boldsymbol{e})\cdot\nabla n^m = c_2\Delta^2\mathcal{S}_{\phi^0}^m, \\
\left(\frac{\partial}{\partial t} - (\nabla\phi_0^m\times\boldsymbol{e})\cdot\nabla\right)n^m = 0 &\qquad \text{for } x\in\Omega,\ t>0, \\
\mathcal{S}_{\phi^0}^m(x,0) = \mathcal{S}_{\phi_0^0}^m(x) &\qquad \text{for } x\in\Omega, \\
n^m(x',0) = \bar{n}_0^0(x') &\qquad \text{for } x'\in\omega, \\
\mathcal{S}_{\phi^0}^m(x,t) = \Delta\mathcal{S}_{\phi^0}^m(x,t) = 0 &\qquad \text{for } x\in\Gamma,\ t>0, \\
n^m(x',t) = 0 &\qquad \text{for } x'\in\partial\omega,\ t>0,
\end{aligned}\right\} \tag{3.7}$$

where

$$\mathcal{S}_{\phi_0^0}^m = \sum_{\xi\in\Lambda_m} d_\xi^{(m)}\phi_{0\xi}^0 \mathrm{e}^{\mathrm{i}\xi x_3}.$$

In order to obtain *a priori* estimates of $(\mathcal{S}_{\phi^0}^m, n^m)$, let us introduce the following cut-off function $\eta_s$:

> let $s,\delta\in\mathbb{R}$, let $\delta > 1$ and let $\eta_s(x_3)\in C^1(\mathbb{R})$ be a cut-off function such that $\eta_s \equiv 1$ on $[s, s+\delta]$, $\eta_s \equiv 0$ on $(-\infty, s-\delta]\cup[s+2\delta,+\infty)$, $0 \leqslant \eta_s(x_3) \leqslant 1$, and $|\eta_s'(x_3)| \leqslant c/\delta$, $|\eta_s''(x_3)| \leqslant c/\delta^2$ with a constant $c$ independent of $\delta$ and $\eta_s'(x_3+2\delta) = -\eta_s'(x_3)$ for $x_3\in[s-\delta, s]$.

The following lemma was obtained in [14].

LEMMA 3.4. *Let*

$$f \in S^1_{\mathrm{ap}}(\mathbb{R}; L^1(\omega \times (0,t))), \quad s \in \mathbb{R}.$$

*Then for any $\eta > 0$ there exists $\delta \in \mathbb{R}$ such that the following inequality holds:*

$$\left| \int_{s-\delta}^{s+2\delta} \int_0^t \int_\omega f(x,\tau)\eta_s'(x_3)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}x_3 \right| \leqslant \eta.$$

We shall now prove the following. Since the regularity of the solution is not sufficient, the arguments of the proof are formal. However, one can justify them by using the method of difference quotients or mollifiers. Throughout this subsection, we denote by $c$ a constant, which may differ at each occurrence.

LEMMA 3.5. *For any $t \in [0,T]$, $\eta > 0$, there exists a positive constant $\delta$ such that the following estimates hold:*

$$\|n^m(t)\|^2_{L^\infty(\omega)} = \|\bar{n}^0_0\|^2_{L^\infty(\omega)}, \tag{3.8}$$

$$\begin{aligned} \|\nabla\mathcal{S}^m_{\phi^0}(t)\|^2 + \|(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\|^2 + c_2|\!|\!|\Delta\mathcal{S}^m_{\phi^0}|\!|\!|^2_t \\ \leqslant c(3\delta+1)(\|\nabla\mathcal{S}^m_{\phi^0_0}\|^2 + \|(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0_0}\}\|^2) + \eta \\ \equiv c^*, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \|\Delta\mathcal{S}^m_{\phi^0}(t)\|^2 + \|\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\|^2 + |\!|\!|\nabla\Delta\mathcal{S}^m_{\phi^0}|\!|\!|^2_t \\ \leqslant c(3\delta+1)(\|\Delta\mathcal{S}^m_{\phi^0_0}\|^2 + \|\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0_0}\}\|^2 \\ + c^*(1 + (\|\bar{n}^0_0\|^2_{L^\infty(\omega)} + c^{*4})t)) + \eta \\ \equiv C^{**}(t). \end{aligned} \tag{3.10}$$

*Proof.* The solution of $(3.7)_2$ is given by $\Pi^{x'}_{\xi'}\bar{\rho}(\xi',t)$ via (3.6), which yields (3.8).

Multiplying $(3.7)_1$ by $\mathcal{S}^m_{\phi^0}\eta_s$ and integrating over

$$\Omega^s \equiv \omega \times (s-\delta, s+2\delta),$$

we have, by virtue of integration by parts,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\mathcal{S}^m_{\phi^0}(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + \|(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)}) \\ + c_2\|\Delta\mathcal{S}^m_{\phi^0}(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} = \int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x. \end{aligned}$$

Integrating this over $[0,t]$ and taking the supremum over $s \in \mathbb{R}$, we obtain (3.9) with the help of lemma 3.4 and the inequalities

$$\left.\begin{aligned} \sup_{s\in\mathbb{R}}\|f\|^p_{L^p(\Omega^s)} \leqslant (3\delta+1)\|f\|^p_{S^p}, \\ \|f\|^2 \leqslant \sup_{s\in\mathbb{R}}\|f\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} \quad \text{for } f \in S^p(L^p(\omega)) \quad (1 \leqslant p < \infty). \end{aligned}\right\} \tag{3.11}$$

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Multiplying $(3.7)_1$ by $\Delta\mathcal{S}^m_{\phi^0}\eta_s$ and integrating over $\Omega^s$, we have, by virtue of integration by parts and the Gagliardo–Nirenberg and Young inequalities,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta\mathcal{S}^m_{\phi^0}(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + \|\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)}) \\
&\qquad + c_2\|\nabla\Delta\mathcal{S}^m_{\phi^0}(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} \\
&\quad = -\int_{\Omega^s}(\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}\times \boldsymbol{e})\cdot\nabla n^m\Delta\mathcal{S}^m_{\phi^0}\eta_s\,\mathrm{d}x \\
&\qquad + \int_{\Omega^s}(\nabla\mathcal{S}^m_{\phi^0}\times\boldsymbol{e})\cdot\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}\Delta\mathcal{S}^m_{\phi^0}\eta_s\,\mathrm{d}x + \int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x \\
&\quad \leqslant c\|\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\|_{L^2(\Omega^s)}\|n^m(t)\|_{L^\infty(\omega)}\|\nabla\Delta\mathcal{S}^m_{\phi^0}(t)\|_{L^2(\Omega^s)} \\
&\qquad + c\|\nabla\Delta\mathcal{S}^m_{\phi^0}(t)\|_{L^2(\Omega^s)}\|\nabla\mathcal{S}^m_{\phi^0}(t)\|_{L^4(\Omega^s)}\|(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\|_{L^4(\Omega^s)} \\
&\qquad + \int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x \\
&\quad \leqslant (\varepsilon_1+\varepsilon_2)\|\nabla\Delta\mathcal{S}^m_{\phi^0}(t)\|^2 + c(3\delta+1)^2\left(\frac{c^*}{\varepsilon_1}\|\bar{n}^0_0\|^2_{L^\infty(\omega)} + \frac{1}{\varepsilon_2}(c^{*5}+\|\Delta\mathcal{S}^m_{\phi^0}(t)\|^2)\right) \\
&\qquad + \int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x.
\end{aligned}$$

Here we used the estimates (3.8), (3.9) and (3.11) and the inequality

$$\|\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}\| \leqslant \|\nabla\mathcal{S}^m_{\phi^0}\| + \|\nabla\mathcal{M}\{\mathcal{S}^m_{\phi^0}\}\| \leqslant 2\|\nabla\mathcal{S}^m_{\phi^0}\|.$$

Integrating this over $[0,t]$ and taking the supremum over $s\in\mathbb{R}$ and taking $\varepsilon_1,\varepsilon_2$ sufficiently small, we obtain (3.9) with the help of lemma 3.4, (3.9) and (3.11). □

Since lemma 3.4 enables us to easily obtain the following in the same way as in [17], we omit the proof.

Lemma 3.6. *For any $\eta>0$ there exist positive constants $\delta$ and $T^*$ such that the estimates*

$$\begin{aligned}
\|\nabla\Delta\mathcal{S}^m_{\phi^0}(t)\|^2 + \|\Delta(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}(t)\|^2 + \|\nabla n^m(t)\|^2_{L^2(\omega)} + c_2|\!|\!|\Delta^2\mathcal{S}^m_{\phi^0}|\!|\!|^2_t \\
\leqslant \frac{C'(t)}{1-cC'(t)t},
\end{aligned}$$

$$\|\mathrm{D}^\alpha_{x'}n^m(t)\|^2_{L^2(\omega)} \leqslant \left(\sum_{|\alpha'|\leqslant|\alpha|}\|\mathrm{D}^{\alpha'}_{x'}\bar{n}^0_0\|^2_{L^2(\omega)}\right)C_\alpha(t) \equiv C'_\alpha(t), \quad |\alpha|=1,2,3,$$

$$\|\partial_t\mathrm{D}^\alpha_{x'}n^m(t)\|^2_{L^2(\omega)} \leqslant \frac{C'(t)C'_{\alpha+1}(t)}{1-cC'(t)t}, \quad |\alpha|=0,1,2,$$

*hold for any $t\in[0,T^*)$. Here*

$$\begin{aligned}
C'(t) = (3\delta+1)(\|\nabla\Delta\mathcal{S}^m_{\phi^0_0}\|^2 + \|\Delta(\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0_0}\}\|^2 \\
+ \|\nabla\bar{n}^0_0\|^2_{L^2(\omega)} + c(C^{**}(t) + c^*C^{**}(t)^4 + c^{*2}+1)) + \eta
\end{aligned}$$

*and $C_\alpha(t)$ is a monotonically increasing function of $t$ ($|\alpha|=1,2,3$).*

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Now we prove that $\{(\mathcal{S}^m_{\phi^0}, n^m, \partial_t \mathcal{S}^m_{\phi^0}, \partial_t n^m)\}_{m=1}^\infty$ forms a sequence bounded in $L^2(0,T;\tilde{S}^4(\mathbb{R};L^2(\omega)))\times L^\infty(0,T;W_2^3(\omega))\times L^2(0,T;\tilde{S}^2(\mathbb{R};L^2(\omega)))\times L^2(0,T;W_2^2(\omega))$ with the help of lemmas 3.5 and 3.6 and the well-known fact (see [2, 4, 6, 8])

$$\|\mathcal{S}^m_\psi\|_{S^p(X)} \leqslant \|\psi\|_{S^p(X)}, \tag{3.12}$$

$$\|\mathcal{S}^m_\psi - \psi\|_{S^p(X)} \to 0 \quad \text{as } m \to \infty \tag{3.13}$$

for any $\psi \in S^p_{\text{ap}}(X)$ defined on a Banach space $X$ $(1 \leqslant p \leqslant \infty)$.

Indeed, the boundedness of

$$\{(\mathcal{S}^m_{\phi^0}, n^m, \partial_t \mathcal{S}^m_{\phi^0}, \partial_t n^m)\}_{m=1}^\infty$$

in $L^2(0,T;\tilde{S}^4(\mathbb{R};L^2(\omega))) \times L^\infty(0,T;W_2^3(\omega)) \times L^2(0,T;\tilde{S}^2(\mathbb{R};L^2(\omega))) \times L^2(0,T;W_2^2(\omega))$ followed from $\|\mathcal{S}^m_{\phi^0_0}\|^2_{\tilde{S}^2} \leqslant \|\phi^0_0\|^2_{\tilde{S}^2}$ directly derived from (3.12) and lemmas 3.5 and 3.6. Hence, there exists a subsequence of $\{(\mathcal{S}^m_{\phi^0}, n^m, \partial_t \mathcal{S}^m_{\phi^0}, \partial_t n^m)\}_{m=1}^\infty$, and some function $(\phi^0, \bar{n}^0, \partial_t \phi^0, \partial_t \bar{n}^0)$ satisfies

$$\begin{aligned}(\mathcal{S}^m_{\phi^0}, \partial_t \mathcal{S}^m_{\phi^0}, \partial_t n^m) \rightharpoonup (\phi^0, \partial_t \phi^0, \partial_t \bar{n}^0)&\\ \text{in } L^2(0,T;\tilde{S}^4(\mathbb{R};L^2(\omega)))&\\ \times L^2(0,T;\tilde{S}^2(\mathbb{R};L^2(\omega))) \times L^2(0,T;W_2^2(\omega))&\\ \text{weakly as } m \to \infty,&\\ n^m \rightharpoonup \bar{n}^0 \quad \text{in } L^\infty(0,T;W_2^3(\omega)) \quad \text{weakly}^* \text{ as } m \to \infty.&\end{aligned}$$

According to the Rellich theorem, we have

$$\begin{aligned}(\mathcal{S}^m_{\phi^0}, \partial_t \mathcal{S}^m_{\phi^0}, \partial_t n^m) \to (\phi^0, \partial_t \phi^0, \partial_t \bar{n}^0)&\\ \text{in } L^2(0,T;\tilde{S}^3(\mathbb{R};L^2(\omega)))&\\ \times L^2(0,T;\tilde{S}^1(\mathbb{R};L^2(\omega))) \times L^2(0,T;W_2^1(\omega))&\\ \text{strongly as } m \to \infty.&\end{aligned}$$

From these we have

$$\begin{aligned}(\nabla \mathcal{S}^m_{\phi^0} \times \boldsymbol{e}) \cdot \nabla(\Delta \mathcal{S}^m_{\phi^0} - (\mathcal{I} - \mathcal{M})\{\mathcal{S}^m_{\phi^0}\}) \rightharpoonup (\nabla \phi^0 \times \boldsymbol{e}) \cdot \nabla(\Delta \phi^0 - \tilde{\phi}^0),&\\ (\nabla(\mathcal{I} - \mathcal{M})\{\mathcal{S}^m_{\phi^0}\} \times \boldsymbol{e}) \cdot \nabla n^m \rightharpoonup (\nabla \tilde{\phi}^0 \times \boldsymbol{e}) \cdot \nabla \bar{n}^0,&\\ (\nabla \phi^m_0 \times \boldsymbol{e}) \cdot \nabla n^m \rightharpoonup (\nabla \bar{\phi}^0 \times \boldsymbol{e}) \cdot \nabla \bar{n}^0&\\ \text{in } L^2(0,T;\tilde{S}^0(\mathbb{R};L^2(\omega)))&\\ \text{weakly as } m \to \infty.&\end{aligned}$$

Therefore, we obtain

$$\left.\begin{aligned}\int_s^{s+1} \int_{\omega_t} \bigg\{ \bigg( \frac{\partial}{\partial t} - (\nabla \phi^0 \times \boldsymbol{e}) \cdot \nabla \bigg) (\Delta \phi^0 - \tilde{\phi}^0)&\\ + (\nabla \tilde{\phi}^0 \times \boldsymbol{e}) \cdot \nabla \bar{n}^0 - c_2 \Delta^2 \phi^0 \bigg\} \varphi \, \mathrm{d}x' \, \mathrm{d}t \, \mathrm{d}x_3 = 0,&\\ \int_{\omega_t} \bigg( \frac{\partial}{\partial t} - (\nabla \bar{\phi}^0 \times \boldsymbol{e}) \cdot \nabla \bigg) \bar{n}^0 \varphi \, \mathrm{d}x' \, \mathrm{d}t = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega).&\end{aligned}\right\} \tag{3.14}$$

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Here $\omega_t \equiv \omega \times (0,t)$, $s \in \mathbb{R}$ and $C_0^\infty$ is constituted by all infinitely differentiable functions with compact support in $\Omega$.

Now we shall prove that $(\phi^0, \partial_t \phi^0)$ belongs to $L^2(0,T;\tilde{S}^4_{\mathrm{ap}}(\mathbb{R};L^2(\omega))) \times L^2(0,T;\tilde{S}^2_{\mathrm{ap}}(\mathbb{R};L^2(\omega)))$ with the help of lemma 3.3. Let

$$\mathcal{S}^{m[\sigma]}_{\phi^0}(x,t) = \mathcal{S}^m_{\phi^0}(x', x_3+\sigma, t) \quad \text{and} \quad \mathcal{V}^m_{\phi^0\sigma}(x,t) = \mathcal{S}^m_{\phi^0}(x', x_3+\sigma, t) - \mathcal{S}^m_{\phi^0}(x', x_3, t)$$

for any $\sigma \neq 0$. Then $\mathcal{V}^m_{\phi^0\sigma}$ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}(\Delta \mathcal{V}^m_{\phi^0\sigma} - (\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0\sigma}\}) - (\nabla \mathcal{S}^{m[\sigma]}_{\phi^0} \times \boldsymbol{e}) \cdot \nabla(\Delta \mathcal{V}^m_{\phi^0\sigma} - (\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0\sigma}\}) \\
&\quad - (\nabla \mathcal{V}^m_{\phi^0\sigma} \times \boldsymbol{e}) \cdot \nabla(\Delta \mathcal{S}^m_{\phi^0} - (\mathcal{I}-\mathcal{M})\{\mathcal{S}^m_{\phi^0}\}) \\
&\qquad + (\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0\sigma}\} \times \boldsymbol{e}) \cdot \nabla n^m = c_2 \Delta^2 \mathcal{V}^m_{\phi^0\sigma} \quad \text{for } x \in \Omega,\ t > 0, \\
&\mathcal{V}^m_{\phi^0\sigma}(x,0) = \mathcal{V}^m_{\phi^0_0\sigma}(x) \quad \text{for } x \in \Omega, \\
&\mathcal{V}^m_{\phi^0\sigma}(x,t) = \Delta \mathcal{V}^m_{\phi^0\sigma}(x,t) = 0 \quad \text{for } x \in \Gamma,\ t > 0.
\end{aligned}$$

Since this is a linear problem, we can easily obtain the following lemmas with the help of lemmas 3.5 and 3.6.

LEMMA 3.7. *For any $t \in [0,T]$, $\eta > 0$ there exists a positive constant $\delta$ such that the following estimates hold:*

$$\begin{aligned}
&\|\nabla \mathcal{V}^m_{\phi^0\sigma}(t)\|^2 + \|(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0\sigma}\}(t)\|^2 + c_2 |||\Delta \mathcal{V}^m_{\phi^0\sigma}|||_t^2 \\
&\qquad \leqslant C(t,\delta)(\|\nabla \mathcal{V}^m_{\phi^0_0\sigma}\|^2 + \|(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0_0\sigma}\}\|^2) + \eta, \\
&\|\Delta \mathcal{V}^m_{\phi^0\sigma}(t)\|^2 + \|\nabla(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0\sigma}\}(t)\|^2 + |||\nabla \Delta \mathcal{V}^m_{\phi^0\sigma}|||_t^2 \\
&\qquad \leqslant C(t,\delta)(\|\nabla \mathcal{V}^m_{\phi^0_0\sigma}\|^2_{\tilde{S}^1} + \|(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0_0\sigma}\}\|^2_{\tilde{S}^1}) + \eta.
\end{aligned}$$

LEMMA 3.8. *For any $\eta > 0$ there exist positive constants $\delta$ and $T^*$ such that the estimates*

$$\begin{aligned}
&\|\nabla \Delta \mathcal{V}^m_{\phi^0\sigma}(t)\|^2 + \|\Delta(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0\sigma}\}(t)\|^2 + c_2 |||\Delta^2 \mathcal{V}^m_{\phi^0\sigma}|||_t^2 \\
&\qquad \leqslant C(t,\delta)(\|\nabla \mathcal{V}^m_{\phi^0_0\sigma}\|^2_{\tilde{S}^2} + \|(\mathcal{I}-\mathcal{M})\{\mathcal{V}^m_{\phi^0_0\sigma}\}\|^2_{\tilde{S}^2}) + \eta
\end{aligned}$$

*hold for any $t \in [0, T^*)$.*

Now we prove that $\{(\mathcal{S}^m_{\phi^0}, \partial_t \mathcal{S}^m_{\phi^0})\}_{m=1}^\infty$ is $L^2(0,T;\tilde{S}^4(\mathbb{R};L^2(\omega))) \times L^2(0,T;\tilde{S}^2(\mathbb{R};L^2(\omega)))$-equi-almost-periodic with the help of lemmas 3.7, 3.8 and (3.12).

Let

$$\Phi^0_{0\sigma}(x) = \phi^0_0(x', x_3+\sigma) - \phi^0_0(x', x_3) \quad \text{for any } \sigma \neq 0.$$

It is easy to see that

$$\Phi^0_{0\sigma\xi} = (\mathrm{e}^{\mathrm{i}\xi\sigma} - 1)\phi^0_{0\xi} \quad \text{and} \quad \mathcal{V}^m_{\phi^0_0\sigma}(x) = \mathcal{S}^m_{\Phi^0_{0\sigma}}(x).$$

Then (3.12) yields $\|\mathcal{V}^m_{\phi^0_0\sigma}\|^2_{\tilde{S}^2} \leqslant \|\Phi^0_{0\sigma}\|^2_{\tilde{S}^2}$. From this we find that $\{(\mathcal{S}^m_{\phi^0}, \partial_t \mathcal{S}^m_{\phi^0})\}_{m=1}^\infty$ is $L^2(0,T;\tilde{S}^4(\mathbb{R};L^2(\omega))) \times L^2(0,T;\tilde{S}^2(\mathbb{R};L^2(\omega)))$-equi-almost-periodic by virtue of lemmas 3.7 and 3.8.

Lemma 3.3 implies that $(\phi^0, \partial_t\phi^0)$ belongs to $L^2(0,T;\tilde{S}^4_{\mathrm{ap}}(\mathbb{R};L^2(\omega))) \times L^2(0,T;\tilde{S}^2_{\mathrm{ap}}(\mathbb{R};L^2(\omega)))$. Moreover, $(\phi^0, \partial_t\phi^0)$ is unique in the same class according to lemma 3.1. From these results and (3.14), we find that $(\phi^0, \bar{n}^0)$ is a unique solution of problem (1.8). Thus, the proof of local-in-time existence and uniqueness is complete.

### 3.2. *A priori* estimates

Let $T$ be an arbitrary positive number and let $(\phi^0, \bar{n}^0)$ be a solution of problem (1.8) belonging to $(L^2(0,T;\tilde{S}^4_{\mathrm{ap}}(\mathbb{R};L^2(\omega))) \cap W^1_2(0,T;\tilde{S}^2_{\mathrm{ap}}(\mathbb{R};L^2(\omega)))) \times (L^\infty(0,T;W^3_2(\omega)) \cap W^1_2(0,T;W^2_2(\omega)))$. Throughout this subsection, we denote by $c$ a constant, which may differ at each occurrence. We can obtain the following in the same way as in lemmas 3.5 and 3.6.

Lemma 3.9. *For any $t \in [0,T]$, $\eta > 0$ there exists a positive constant $\delta$ such that the following estimates hold*

$$
\begin{aligned}
\|\bar{n}^0(t)\|^2_{L^\infty(\omega)} &= \|\bar{n}^0_0\|^2_{L^\infty(\omega)},\\
\|\nabla\phi^0(t)\|^2 + \|\tilde{\phi}^0(t)\|^2 + c_2|\!|\!|\Delta\phi^0|\!|\!|^2_t &\leqslant c(3\delta+1)(\|\nabla\phi^0_0\|^2 + \|\tilde{\phi}^0_0\|^2) + \eta\\
&\equiv c^*,\\
\|\Delta\phi^0(t)\|^2 + \|\nabla\tilde{\phi}^0(t)\|^2 + |\!|\!|\nabla\Delta\phi^0|\!|\!|^2_t &\leqslant c(3\delta+1)(\|\Delta\phi^0_0\|^2 + \|\nabla\tilde{\phi}^0_0\|^2\\
&\quad + c^*(1 + (\|\bar{n}^0_0\|^2_{L^\infty(\omega)} + c^{*4})t)) + \eta\\
&\equiv C^{**}(t).
\end{aligned}
$$

Lemma 3.10. *For any $\eta > 0$ there exist positive constants $\delta$ and $T^*$ such that the estimates*

$$
\|\nabla\Delta\phi^0(t)\|^2 + \|\Delta\tilde{\phi}^0(t)\|^2 + \|\nabla\bar{n}^0(t)\|^2_{L^2(\omega)} + c_2|\!|\!|\Delta^2\phi^0|\!|\!|^2_t \leqslant \frac{C'(t)}{1 - cC'(t)t},
$$

$$
\|\mathrm{D}^\alpha_{x'}\bar{n}^0(t)\|^2_{L^2(\omega)} \leqslant \Bigg(\sum_{|\alpha'|\leqslant|\alpha|} \|\mathrm{D}^{\alpha'}_{x'}\bar{n}^0_0\|^2_{L^2(\omega)}\Bigg) C_\alpha(t) \equiv C'_\alpha(t), \quad |\alpha| = 1,2,3,
$$

$$
\|\partial_t \mathrm{D}^\alpha_{x'}\bar{n}^0(t)\|^2_{L^2(\omega)} \leqslant \frac{C'(t)C'_{\alpha+1}(t)}{1 - cC'(t)t}, \quad |\alpha| = 0,1,2,
$$

*hold for any $t \in [0,T^*)$. Here*

$$
C'(t) = (3\delta+1)(\|\nabla\Delta\phi^0_0\|^2 + \|\Delta\tilde{\phi}^0_0\|^2 + \|\nabla\bar{n}^0_0\|^2_{L^2(\omega)} + c(C^{**}(t) + c^*C^{**}(t)^4 + c^{*2} + 1)) + \eta
$$

*and $C_\alpha(t)$ is a monotonically increasing function of $t$ $(|\alpha| = 1,2,3)$.*

By standard arguments based upon the *a priori* estimates in lemmas 3.9 and 3.10, the solution $(\phi^0, \bar{n}^0)$ established above can be extended up to $T^*$ indicated in the proof of lemma 3.10. In the same way as in [19], we can easily prove the Stepanov almost-periodicity of the solution. Thus, the proof of theorem 1.2 is complete.

## 4. Proof of theorem 1.3

Let $T^\sharp = \min(T, T^*)$, where $T$ and $T^*$ are found in proposition 1.1 and theorem 1.2, respectively. Subtracting (1.6) from (1.5), and setting $\phi \equiv \phi^\varepsilon - \phi^0$ and $n \equiv n^\varepsilon - n^0$,

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we have

$$\left.\begin{aligned}&\left(\frac{\partial}{\partial t}-(\nabla\phi^{\varepsilon}\times\boldsymbol{e})\cdot\nabla\right)(\Delta\phi-n)-(\nabla\phi\times\boldsymbol{e})\cdot\nabla(\Delta\phi^{0}-n^{0})=c_{2}\Delta^{2}\phi,\\&\varepsilon\left(\frac{\partial}{\partial t}-(\nabla\phi^{\varepsilon}\times\boldsymbol{e})\cdot\nabla\right)n=-\varepsilon(\mathcal{I}-\mathcal{M})\left\{\left(\frac{\partial}{\partial t}-(\nabla\phi^{0}\times\boldsymbol{e})\cdot\nabla\right)n^{0}\right\}\\&\qquad\qquad-\frac{1}{n^{*}}\frac{\partial^{2}(\phi-n)}{\partial x_{3}^{2}}+\varepsilon(\nabla\phi\times\boldsymbol{e})\cdot\nabla n^{0}\\&\qquad\qquad\qquad\text{for } x\in\Omega,\ 0<t<T^{\sharp},\\&\phi(x,0)=\phi_{0}^{\varepsilon}-\phi_{0}^{0},\quad n(x,0)=n_{0}^{\varepsilon}-n_{0}^{0}\quad\text{for } x\in\Omega,\\&\phi(x,t)=\Delta\phi(x,t)=n(x,t)=0\quad\text{for } x\in\Gamma,\ 0<t<T^{\sharp}.\end{aligned}\right\}\tag{4.1}$$

For the solution $(\phi^{\varepsilon}, n^{\varepsilon})$ of problem (1.4), (1.5), the following *a priori* estimates are established in [15].

Lemma 4.1. *Let $c^{\star}>0$ and $\varepsilon\in(0,c^{\star}]$. For any $t\in[0,T]$, $\eta>0$ there exists a positive constant $\delta$ independent of $\varepsilon$ such that the following estimates hold:*

$$\begin{aligned}\|\nabla\phi^{\varepsilon}(t)\|^{2}+\|n^{\varepsilon}(t)\|^{2}+c_{2}|\!|\!|\Delta\phi^{\varepsilon}|\!|\!|_{t}^{2}&\leqslant c(3\delta+1)(\|\nabla\phi_{0}^{\varepsilon}\|^{2}+\|n_{0}^{\varepsilon}\|^{2})+\eta,\\|\!|\!|\partial_{x_{3}}(\phi^{\varepsilon}-n^{\varepsilon})|\!|\!|_{t}^{2}&\leqslant\varepsilon c((3\delta+1)(\|\nabla\phi_{0}^{\varepsilon}\|^{2}+\|n_{0}^{\varepsilon}\|^{2})+\eta).\end{aligned}$$

*Here $c$ is a positive constant independent of $\varepsilon$ and $\delta$.*

Lemma 4.2. *Let $c^{\star}>0$ and $\varepsilon\in(0,c^{\star}]$. For any $\eta>0$ there exist positive constants $\delta$ and $T$ independent of $\varepsilon$ such that the estimate*

$$\begin{aligned}&\varepsilon(\|\nabla n^{\varepsilon}(t)\|^{2}+\|\Delta\phi^{\varepsilon}(t)\|^{2}+\|\Delta n^{\varepsilon}(t)\|^{2}+\|\nabla\Delta\phi^{\varepsilon}(t)\|^{2})\\&\quad+\varepsilon c_{2}(|\!|\!|\nabla\Delta\phi^{\varepsilon}|\!|\!|_{t}^{2}+|\!|\!|\Delta^{2}\phi^{\varepsilon}|\!|\!|_{t}^{2})+|\!|\!|\partial_{x_{3}}\nabla(\phi^{\varepsilon}-n^{\varepsilon})|\!|\!|_{t}^{2}+|\!|\!|\partial_{x_{3}}\Delta(\phi^{\varepsilon}-n^{\varepsilon})|\!|\!|_{t}^{2}\\&\qquad\leqslant\varepsilon\left(\left(\frac{1}{S_{0}^{*}+\|\nabla\phi_{0}^{\varepsilon}\|^{2}+\|n_{0}^{\varepsilon}\|^{2}+\eta}-C(\delta)t\right)^{-1}-(\|\nabla\phi_{0}^{\varepsilon}\|^{2}+\|n_{0}^{\varepsilon}\|^{2}+\eta)\right)\end{aligned}$$

*holds for any $t\in[0,T)$. Here $c$ is a positive constant independent of $\varepsilon$ and $\delta$, and $C(\delta)$ is a positive constant depending increasingly on $\delta$ and $S_{0}^{*}\equiv(3\delta+1)(\|\nabla n_{0}^{\varepsilon}\|^{2}+\|\Delta\phi_{0}^{\varepsilon}\|^{2}+\|\Delta n_{0}^{\varepsilon}\|^{2}+\|\nabla\Delta\phi_{0}^{\varepsilon}\|^{2}+c(\|\nabla\phi_{0}^{\varepsilon}\|^{2}+\|n_{0}^{\varepsilon}\|^{2}+1))+\eta$.*

From lemmas 3.9, 3.10, 4.1 and 4.2 we can obtain uniform *a priori* estimates for $(\phi,n)$ with respect to resistivity that are necessary to prove theorem 1.3. In order to get *a priori* estimates, we use lemma 3.4 and the following lemmas, which are proved in [14, 15].

Lemma 4.3. *Let $\psi\in S_{\mathrm{ap}}^{2}(\mathbb{R};L^{2}(\omega))$, $\partial_{x_{3}}\psi\in S_{\mathrm{ap}}^{2}(\mathbb{R};L^{2}(\omega))$, $\mathcal{M}\{\psi(x)\}=0$ in $L^{2}(\omega)$ and $s,\delta\in\mathbb{R}$, $\delta>1$. Then*

$$\int_{s-\delta}^{s+2\delta}\|\psi(x_{3})\|_{L^{2}(\omega)}^{2}\,\mathrm{d}x_{3}\leqslant c\int_{s-\delta}^{s+2\delta}\|\partial_{x_{3}}\psi(x_{3})\|_{L^{2}(\omega)}^{2}\,\mathrm{d}x_{3}.$$

*Here $c$ is a positive constant independent of $s$ and $\delta$.*

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The following are the generalization of lemma 3.4 to the case of Stepanov almost-periodic functions depending on a parameter.

Lemma 4.4. *Let $c^\star > 0$ and $\varepsilon \in (0, c^\star]$; $f^\varepsilon$ depends on $\varepsilon$, $f^\varepsilon \in S^1_{\mathrm{ap}}(\mathbb{R}; L^1(\omega\times(0,t)))$; $s \in \mathbb{R}$. Assuming that there exists a positive constant $M$ independent of $\varepsilon$ such that the inequality*

$$-M < R^\varepsilon(s,t) \equiv \int_{s-\delta}^{s+2\delta}\int_0^t\int_\omega f^\varepsilon(x,\tau)\eta_s'(x_3)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}x_3$$

*holds, for any $\eta > 0$ there then exists a positive constant $\delta$ independent of $\varepsilon$ such that the inequality $|R^\varepsilon(s,t)| \leqslant \eta$ holds.*

Corollary 4.5. *Let $c^\star, c^* > 0$, $\varepsilon \in (0, c^\star]$, $f^\varepsilon$ and $g^\varepsilon$ depend on $\varepsilon$, $f^\varepsilon \in S^1_{\mathrm{ap}}(\mathbb{R}; L^1(\omega \times (0,t)))$, $g^\varepsilon \in L^1(0,t)$, $|g^\varepsilon(t)| \leqslant c^*$ and $s \in \mathbb{R}$. Assuming that there exists a positive constant $M$ independent of $\varepsilon$ such that the inequality*

$$-M < R^\varepsilon(s,t) \equiv \int_0^t \left(g^\varepsilon(\tau)\int_{s-\delta}^{s+2\delta}\int_\omega f^\varepsilon(x,\tau)\eta_s'(x_3)\,\mathrm{d}x'\,\mathrm{d}x_3\right)\mathrm{d}\tau$$

*holds, for any $\eta > 0$ there then exists a positive constant $\delta$ independent of $\varepsilon$ such that the inequality $|R^\varepsilon(s,t)| \leqslant \eta$ holds.*

Again, as in § 3.1, the following arguments are formal. However, they are justified by the method of difference quotients or mollifiers. We denote by $c$ a constant independent of $t$ and by $C(t)$ a constant dependent on both $t$ and the bounds of $\phi^\varepsilon, n^\varepsilon, \phi^0, n^0$, which may differ at each occurrence. We now prove the following.

Lemma 4.6. *Let $c^\star > 0$ and $\varepsilon \in (0, c^\star]$. For any $t \in [0, T^\sharp]$, $\eta > 0$ there exists a positive constant $\delta$ independent of $\varepsilon$ such that the following estimate holds:*

$$\begin{aligned}\varepsilon(\|\nabla\phi(t)\|^2 + \|n(t)\|^2 + |||\Delta\phi|||_t^2) + |||\partial_{x_3}(\phi - n)|||_t^2 \\ \leqslant \varepsilon(C(t,\delta)(\|\nabla\phi(0)\|^2 + \|n(0)\|^2 + \varepsilon) + \eta). \quad (4.2)\end{aligned}$$

*Proof.* Multiplying $(4.1)_1$ by $\phi\eta_s$ and integrating over $\Omega^s$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + c_2\|\Delta\phi(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + \int_{\Omega^s}(\partial_t - (\nabla\phi^\varepsilon \times \boldsymbol{e})\cdot\nabla)n\phi\eta_s\,\mathrm{d}x \\ = \int_{\Omega^s}(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla\phi\Delta\phi\eta_s\,\mathrm{d}x + \int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x \\ \leqslant \varepsilon_1\|\Delta\phi(t)\|^2_{L^2(\Omega^s)} + \frac{c}{\varepsilon_1}\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega^s)}\|\nabla\phi(t)\|^2_{L^2(\Omega^s)} + \int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x. \\ (4.3)\end{aligned}$$

Here we used integration by parts and Schwarz's inequality.

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Similarly, multiplying $(4.1)_2$ by $(\tilde{\phi}-\tilde{n})\eta_s$ and integrating over $\Omega^s$, we have

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + \left\|\partial_{x_3}(\tilde{\phi}-\tilde{n})(t)\sqrt{\frac{\eta_s}{n^*}}\right\|^2_{L^2(\Omega^s)} \\
&\quad - \varepsilon\int_{\Omega^s}(\partial_t - (\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla)n\phi\eta_s\,\mathrm{d}x \\
&\quad\; = \varepsilon\int_{\Omega^s}(\partial_t - (\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla)n(\bar{\phi}-\bar{n})\eta_s\,\mathrm{d}x - \int_{\Omega^s}\frac{1}{n^*}\partial_{x_3}(\tilde{\phi}-\tilde{n})(\tilde{\phi}-\tilde{n})\eta_s'\,\mathrm{d}x \\
&\qquad + \varepsilon\int_{\Omega^s}(\mathcal{I}-\mathcal{M})\{(\partial_t-(\nabla\phi^0\times\boldsymbol{e})\cdot\nabla)n^0\}(\tilde{\phi}-\tilde{n})\eta_s\,\mathrm{d}x \\
&\qquad - \varepsilon\int_{\Omega^s}(\nabla\phi\times\boldsymbol{e})\cdot\nabla n^0(\tilde{\phi}-\tilde{n})\eta_s\,\mathrm{d}x \\
&\quad\; \leqslant \varepsilon^2 c\left\|\int_c^{x_3}(\mathcal{I}-\mathcal{M})\{(\partial_t-(\nabla\phi^0\times\boldsymbol{e})\cdot\nabla)n^0\}\,\mathrm{d}x_3\right\|^2_{L^2(\Omega^s)} \\
&\qquad + \frac{c}{\delta}\left\|\partial_{x_3}(\tilde{\phi}-\tilde{n})(t)\sqrt{\frac{1}{n^*}}\right\|^2_{L^2(\Omega^s)} + \varepsilon\int_{\Omega^s}\{\cdot\}\eta_s'\,\mathrm{d}x \\
&\qquad + \varepsilon\|\nabla\phi(t)\|_{L^2(\Omega^s)}\|\nabla n^0(t)\|_{L^\infty(\Omega^s)}(\|\phi(t)\|_{L^2(\Omega^s)} + \|n(t)\|_{L^2(\Omega^s)}). \qquad (4.4)
\end{aligned}$$

Here we use lemma 4.3 and $\mathcal{M}\{(\partial_t-(\nabla\phi^\varepsilon\times\boldsymbol{e})\cdot\nabla)n\} = \mathcal{M}\{(\nabla\phi\times\boldsymbol{e})\cdot\nabla n^0\}$.

Adding (4.4) and (4.3) multiplied by $\varepsilon$ yields

$$\begin{aligned}
&\varepsilon\left(\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\phi(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} + \|n(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)}) + c_2\|\Delta\phi(t)\sqrt{\eta_s}\|^2_{L^2(\Omega^s)}\right) \\
&\quad + \left\|\partial_{x_3}(\tilde{\phi}-\tilde{n})(t)\sqrt{\frac{\eta_s}{n^*}}\right\|^2_{L^2(\Omega^s)} \\
&\qquad \leqslant \varepsilon c\left(\frac{1}{\varepsilon_1}\|\nabla\phi^\varepsilon(t)\|^2_{L^\infty(\Omega^s)} + \|\nabla n^0(t)\|_{L^\infty(\Omega^s)}\right) \\
&\qquad\quad \times(\|\nabla\phi(t)\|^2_{L^2(\Omega^s)} + \|n(t)\|^2_{L^2(\Omega^s)}) \\
&\qquad\quad + \varepsilon^2 c\|(\mathcal{I}-\mathcal{M})\{(\partial_t-(\nabla\phi^0\times\boldsymbol{e})\cdot\nabla)n^0\}\|_{L^2(\Omega^s)} + \varepsilon\varepsilon_1\|\Delta\phi(t)\|^2_{L^2(\Omega^s)} \\
&\qquad\quad + \frac{c}{\delta}\left\|\partial_{x_3}(\tilde{\phi}-\tilde{n})(t)\sqrt{\frac{1}{n^*}}\right\|^2_{L^2(\Omega^s)} + \varepsilon\int_{\Omega^s}\{\cdot\}\eta_s'(x_3)\,\mathrm{d}x. \qquad (4.5)
\end{aligned}$$

Setting

$$S(t) \equiv \|\nabla\phi(t)\|^2 + \|n(t)\|^2,$$
$$I^\varepsilon(x,t) \equiv \varepsilon\int_0^t\int_{\Omega^s}\{\cdot\}\eta_s'(x_3)\,\mathrm{d}x\,\mathrm{d}\tau,$$

integrating (4.5) over $[0,t]$, taking the supremum over $s\in\mathbb{R}$ and taking $\delta$ sufficiently large and $\varepsilon_1$ sufficiently small, we have

$$S(t) - \sup_{s\in\mathbb{R}} I^\varepsilon(x,t) \leqslant c(3\delta+1)\left\{\int_0^t S(\tau)\,\mathrm{d}\tau + S(0) + \varepsilon\right\}. \qquad (4.6)$$

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Here we used (3.11). Similarly, we obtain

$$-I^{\varepsilon}(x,t) \leqslant c(3\delta+1)\biggl\{\int_0^t S(\tau)\,\mathrm{d}\tau + S(0) + \varepsilon\biggr\}. \tag{4.7}$$

We shall prove that there exists a positive constant $M$ independent of $\varepsilon$ such that $-M < I^{\varepsilon}(x,t)$ holds. Now we consider the following cases:

(i) $\sup_{s\in\mathbb{R}} I^{\varepsilon}(x,t) \leqslant 0$, and

(ii) $\sup_{s\in\mathbb{R}} I^{\varepsilon}(x,t) > 0$, $I^{\varepsilon}(x,t) \leqslant -\frac{1}{2}\sup_{s\in\mathbb{R}} I^{\varepsilon}(x,t)$.

Let $\alpha > 2$ and add (4.6) to (4.7) multiplied by $\alpha$. Then

$$S(t) - \alpha^* I^{\varepsilon}(x,t) \leqslant c(1+\alpha)(3\delta+1)\biggl\{\int_0^t S(\tau)\,\mathrm{d}\tau + S(0) + \varepsilon\biggr\}. \tag{4.8}$$

Here $\alpha^* \equiv \alpha$ in case (i) and $\alpha^* \equiv \alpha - 2$ in case (ii). Let $C(\delta) \equiv c(3\delta+1)$. Then Gronwall's lemma yields

$$\begin{aligned}\int_0^t S(\tau)\,\mathrm{d}\tau \mathrm{e}^{-(1+\alpha)C(\delta)t} &\leqslant \int_0^t \{(1+\alpha)C(\delta)(S(0)+\varepsilon) + \alpha^* I^{\varepsilon}(x,t)\}\mathrm{e}^{-(1+\alpha)C(\delta)\tau}\,\mathrm{d}\tau \\ &\equiv \int_0^t (1+\alpha)C(\delta)(S(0)+\varepsilon)\mathrm{e}^{-(1+\alpha)C(\delta)\tau}\,\mathrm{d}\tau + R^{\varepsilon}(s,t).\end{aligned} \tag{4.9}$$

Since $\varepsilon \in (0, c^\star] \geqslant 0$ and $\int_0^t S(\tau)\,\mathrm{d}\tau \mathrm{e}^{-(1+\alpha)C(\delta)t} \geqslant 0$, we can apply corollary 4.5 to $R^{\varepsilon}(s,t)$. Then we find that for any $\eta > 0$ there exists a positive constant $\delta$ independent of $\varepsilon$ such that $|R^{\varepsilon}(s,t)| < \eta$ holds. Hence, (4.9) yields

$$\int_0^t S(\tau)\,\mathrm{d}\tau + S(0) + \varepsilon \leqslant (\eta + S(0) + \varepsilon)\mathrm{e}^{(1+\alpha)C(\delta)t}$$

From this and (4.7) we have $-I^{\varepsilon}(x,t) \leqslant C(\delta)(\eta + S(0) + \varepsilon)\mathrm{e}^{(1+\alpha)C(\delta)t}$. If

$$\sup_{s\in\mathbb{R}} I^{\varepsilon}(x,t) > 0, \quad I^{\varepsilon}(x,t) > -\frac{1}{2}\sup_{s\in\mathbb{R}} I^{\varepsilon}(x,t),$$

we can obtain the same estimate by using the above result. Above all, we can apply lemmas 4.4 to $I^{\varepsilon}(x,t)$. For any $\eta > 0$ there exists a positive constant $\delta$ independent of $\varepsilon$ such that $|I^{\varepsilon}(x,t)| < \eta$ holds. Hence, (4.6) yields

$$S(t) \leqslant c(3\delta+1)\biggl\{\int_0^t S(\tau)\,\mathrm{d}\tau + S(0) + \varepsilon\biggr\} + \eta.$$

From this inequality, (4.5) and Gronwall's lemma, we have (4.2). □

Since lemmas 4.3 and 4.4 and corollary 4.5 enable us to easily obtain the following lemma in the same way as in [17], we omit its proof.

Lemma 4.7. *Let $c^\star > 0$ and $\varepsilon \in (0, c^\star]$. For any $t \in [0, T^\sharp]$, $\eta > 0$ there exists a positive constant $\delta$ independent of $\varepsilon$ such that the following estimates hold:*

$$\varepsilon(\|\Delta\phi(t)\|^2 + \|\nabla n(t)\|^2 + |||\nabla\Delta\phi|||_t^2) + |||\partial_{x_3}\nabla(\phi - n)|||_t^2 \leqslant \varepsilon(C(t,\delta)(\|\nabla\phi(0)\|_{\tilde S^1}^2 + \|n(0)\|_{\tilde S^1}^2) + \varepsilon + \eta),$$

$$\varepsilon(\|\nabla\Delta\phi(t)\|^2 + \|\Delta n(t)\|^2 + |||\Delta^2\phi|||_t^2) + |||\partial_{x_3}\Delta(\phi - n)|||_t^2 \leqslant \varepsilon(C(t,\delta)(\|\nabla\phi(0)\|_{\tilde S^2}^2 + \|n(0)\|_{\tilde S^2}^2) + \varepsilon + \eta),$$

$$|||\partial_\tau(\Delta\phi - n)|||_t^2 \leqslant C(t,\delta)(\|\nabla\phi(0)\|_{\tilde S^2}^2 + \|n(0)\|_{\tilde S^2}^2 + \varepsilon) + \eta,$$

$$\int_0^t \|\partial_\tau\mathcal{M}\{n\}(\tau)\|_{L^2(\omega)}^2 \mathrm{d}\tau \leqslant C(t,\delta)(\|\nabla\phi(0)\|_{\tilde S^1}^2 + \|n(0)\|_{\tilde S^1}^2 + \varepsilon) + \eta.$$

From lemmas 4.6 and 4.7, it is easy to see that if the initial data $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ as $\varepsilon \to 0$ in $\tilde S^3(\mathbb{R}; L^2(\omega)) \times \tilde S^2(\mathbb{R}; L^2(\omega))$, then, as $\varepsilon \to 0$, $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ in $L^2(0, T^\sharp; \tilde S^4(\mathbb{R}; L^2(\omega))) \times \tilde S^{2,0}(\mathbb{R}; L^2(\omega_{T^\sharp}))$, $\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0$ in $\tilde S^{0,1}(\mathbb{R}; L^2(\omega_{T^\sharp}))$ and $\bar n^\varepsilon \to \bar n^0$ in $W_2^{0,1}(\omega_{T^\sharp})$. Thus, the proof of theorem 1.3 is complete.

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**References**


1 L. Amerio and G. Prouse. *Almost-periodic functions and functional equations* (New York: Van Nostrand Reinhold, 1971).

2 A. S. Besicovitch. On generalized almost periodic functions. *Proc. Lond. Math. Soc.* **2** (1926), 495–512.

3 A. S. Besicovitch. *Almost periodic functions* (New York: Dover, 1955).

4 A. S. Besicovitch and H. Bohr. Some remarks on generalisations of almost periodic functions. *Det Kgl. Danske Videnskabernes selskab. Math.-flys. Meddelelser* **8** (1927), 1–33.

5 H. Bohr. *Almost periodic functions* (New York: Chelsea, 1947).

6 G. Bruno, R. Grande and R. Iannacci. Almost-periodic multipliers. *Acta Appl. Math.* **65** (2001), 137–151.

7 R. Cooke. Almost-periodic functions. *Am. Math. Mon.* **88** (1981), 515–526.

8 C. Corduneanu. *Almost periodic functions*, 2nd Engl. edn (New York: Chelsea, 1989).

9 R. Doss. Contribution to the theory of almost-periodic functions. *Annals Math.* **46** (1945), 196–219.

10 A. Hasegawa and K. Mima. Stationary spectrum of strong turbulence in magnetized plasma. *Phys. Rev. Lett.* **39** (1977), 205–208.

11 A. Hasegawa and K. Mima. Pseudo-three-dimensional turbulence in magnetized nonuniform plasma. *Phys. Fluids* **21** (1978), 87–92.

12 A. Hasegawa and M. Wakatani. Plasma edge turbulence. *Phys. Rev. Lett.* **50** (1983), 682–686.

13 A. Hasegawa and M. Wakatani. A collisional drift wave description of plasma edge turbulence. *Phys. Fluids* **27** (1984), 611–618.

14 S. Kondo. Almost-periodic solution of linearized Hasegawa–Wakatani equations with vanishing resistivity. *Rend. Sem. Mat. Univ. Padova* **133** (2015), 215–239.

15 S. Kondo. On the almost-periodic solution of Hasegawa–Wakatani equations. *J. Evol. Eqns* **16** (2016), 155–172.

16 S. Kondo and A. Tani. Initial boundary value problem for model equations of resistive drift wave turbulence. *SIAM J. Math. Analysis* **43** (2011), 925–943.
17 S. Kondo and A. Tani. Initial boundary value problem of Hasegawa–Wakatani equations with vanishing resistivity. *Adv. Math. Sci. Appl.* **21** (2011), 223–253.
18 S. Kondo and A. Tani. On the Hasegawa–Wakatani equations with vanishing resistivity. *Proc. Jpn Acad.* A **87** (2011), 156–161.
19 S. Kondo and A. Tani. Almost-periodic solutions to initial boundary value problem for model equations of resistive drift wave turbulence. *Annali Scuola Norm. Sup. Pisa* **16** (2016), 291–333.
20 G. T. LaVarnway and R. Cooke. A characterization of Fourier series of Stepanov almost-periodic functions. *J. Fourier Analysis Applic.* **7** (2001), 127–142.
21 K. Miyamoto. *Fundamentals of plasma physics and controlled fusion* (Tokyo: Iwanami, 1997).
22 K. Nishikawa and M. Wakatani. *Plasma physics: basic theory with fusion applications* (Springer, 1999).
23 H. Noda and A. Nishihara. Fast and accurate generalized harmonic analysis and its parallel computation by GPU. *IEICE Trans. Fundamentals* A **E92** (2009), 745–752.
24 A. L. Skubachevskii. Initial–boundary-value problems for the Vlasov–Poisson equations in a half-space. *Proc. Steklov Inst. Math.* **283** (2013), 197–225.
25 A. L. Skubachevskii. Vlasov–Poisson equations for a two-component plasma in a homogeneous magnetic field. *Russ. Math. Surv.* **69** (2014), 291–330.
26 T. Terada, H. Nakajima, M. Tohyama and Y. Hirata. Nonstationary waveform analysis and synthesis using generalized harmonic analysis. In *Proc. IEEE–SP Int. Symp. on Time-Frequency and Time-Scale Analysis*, pp. 429–432 (New York: IEEE Press, 1994).
27 M. Wakatani. *Stellarator and heliotron devices* (Oxford University Press, 1998).
28 J. Wesson. *Tokamaks* (Oxford University Press, 1987).
29 Y. Yamasaki, S. Kudo, Y. Oikawa, K. Endo, M. Nishikawa and H. Ohta. High-efficiency, high-speed, 1-bit coding using generalized harmonic analysis. *J. Acoust. Soc. Am.* **100** (1996), 2601.
30 S. Zaidman. *Almost-periodic functions in abstract spaces*, Research Notes in Mathematics, vol. 126 (Boston, MA: Pitman Advanced Publishing Program, 1985).