Proceedings of the Royal Society of Edinburgh, **137A**, 963–994, *2007*

Symmetry of non-negative solutions of a semilinear elliptic equation with singular nonlinearity

Zongming Guo
Department of Mathematics, Henan Normal University,
Xinxiang 453002, People's Republic of China
and Dong Hua University,
Shanghai 200051, People's Republic of China
(guozm@public.xxptt.ha.cn)

Juncheng Wei
Department of Mathematics, The Chinese University of Hong Kong,
Shatin, Hong Kong (wei@math.cuhk.edu.hk)

(MS received 28 September 2005; accepted 26 July 2006)

We use the method of the moving plane (MMP) to obtain necessary and sufficient conditions for the radial symmetry of positive solutions of the following semi-linear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{1}{u^\nu} = 0 \quad \text{in } \mathbb{R}^n, \quad n \geqslant 2,$$

where $\nu > 0$. In order to apply the MMP, it is crucial to obtain the asymptotic expansion of u at ∞.

1. Introduction

In this paper we investigate the symmetry and local behaviour of non-negative solutions of the equation

$$\Delta u - \frac{1}{u^\nu} = 0, \quad x \in \mathbb{R}^n,\ n \geqslant 2, \quad \nu > 0. \tag{I}$$

We call u a non-negative (positive) solution of (I) if $u \in C^0(\mathbb{R}^n)$, $u \geqslant 0 (u > 0)$, $u \not\equiv 0$ in $\mathbb{R}^n$ and u satisfies (I) a.e. in $\mathbb{R}^n$. (Clearly, $u \equiv 0$ is not a solution of (I).)

Problem (I) arises in the study of steady states of thin films. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u) \tag{1.1}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air–liquid interface. The zero set $\boldsymbol{\Sigma}_u = \{u = 0\}$ is the liquid–solid interface and is sometimes called the set of *ruptures*. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects; a typical choice is $f(u) = u^3$. The coefficient of the second-order term

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m$, $m < 0$. For the background to (1.1), we refer the reader to [2,3,11–13,16] and the references therein.

In general, let us assume that $f(u) = u^p$, $g(u) = u^m$, where $p, m \in \mathbb{R}$. Then if we consider the steady state of (1.1), we see that u satisfying

$$u^p \nabla \Delta u + u^m \nabla u = \mathcal{C}$$

is a steady state of (1.1), where $\mathcal{C} = (C_1, C_2, \ldots, C_n)$ is some constant vector. By assuming that $\mathcal{C} = \mathbf{0}$ (which prevents linear terms on x), we obtain

$$\Delta u + \frac{u^q}{q} - C = 0 \quad \text{in } \Omega, \tag{1.2}$$

where $q = m - p + 1$ and C is some constant. (Here we have assumed that $q \neq 0$. If $q = 0$, we have to replace u^q/q by $\log u$.) Note that solutions to (I) are steady states of (1.1) but the reverse is not true. For thin films under van der Waals forces, we have $f(u) = u^3$, $g(u) = u^m$, $q = m - 2 < -2$. The one-dimensional steady-state problem of (1.1) has been studied thoroughly in [11, 13] and the references therein. It is found that ruptures *never* occur in the one-dimensional case. On the other hand, numerical works on van der Waals driven rupture for (1.1) in two dimensions suggests that the rupture can occur in points [4, 10] or rings [16–18].

In this paper, we consider problem (1.2) in $\mathbb{R}^n$ for $n \geqslant 2$ and assume that the constant $C = 0$. Problem (1.2) becomes (I) with a simple scale of u. It is easy to see that if $u \in C^0_{\mathrm{loc}}(\mathbb{R}^n \setminus \boldsymbol{\Sigma}_u)$, then $u \in C^\infty_{\mathrm{loc}}(\mathbb{R}^n \setminus \boldsymbol{\Sigma}_u)$.

The structure of non-negative solutions of (I) can be complicated since if u is a non-negative solution of (I), then the rupture set $\boldsymbol{\Sigma}_u$ can be non-empty and with a positive Hausdorff dimension [9]. In this paper we are interested in the symmetry property of positive solutions u of (I), i.e. $\boldsymbol{\Sigma}_u = \emptyset$. Note that $\boldsymbol{\Sigma}_u$ can contain at most one element if u is radially symmetric. Indeed, if $\boldsymbol{\Sigma}_u$ contains more than one element, then we claim that u cannot be radially symmetric about some point $x_0 \in \mathbb{R}^n$. In fact, suppose on the contrary that we have a radially symmetric non-negative solution $u \in C^2(\mathbb{R}^n)$ of (I) with $z_0, z_1 \in \boldsymbol{\Sigma}_u$, $z_0 \neq z_1$. We assume that u is radially symmetric about a point x_0. Then there are three cases here: $x_0 = z_0$, $x_0 = z_1$ and $x_0 \neq z_i$, $i = 0, 1$. Now, setting $r = |x - x_0|$, we easily find that $u(x) := u(r)$ satisfies

$$(r^{n-1}u')' = r^{n-1}u^{-\nu}, \quad 0 < r < \infty. \tag{1.3}$$

This implies that $u'(r) \geqslant 0$ for $r \in (0, \infty)$. Now, for the first case, we have $u(r_1) = 0$ with $r_1 := |z_1 - x_0|$, and hence $u \equiv 0$ in $B_{r_1}(x_0)$, which is impossible. We can derive contradictions for other two cases similarly. This implies that our claim holds. On the other hand, the same arguments imply that if $\boldsymbol{\Sigma}_u = \{\text{a single point}\}$, then u must be radially symmetric about this point.

Define

$$\alpha = \frac{2}{\nu + 1}, \quad \lambda = [\alpha(n - 2 + \alpha)]^{-1/(\nu+1)}. \tag{1.4}$$

It is easy to see that

$$u_0(x) = \lambda |x|^\alpha \tag{1.5}$$

is a non-negative radially symmetric solution of (I). For this solution, it is clear that $\boldsymbol{\Sigma}_{u_0} = \{0\}$ and $|x|^{-\alpha}u_0(x) = \lambda$. It may be seen that the limit

$$\lim_{|x|\to+\infty} |x|^{-\alpha}u(x) = \lambda \tag{1.6}$$

plays an important role in the radially symmetric properties of non-negative solutions of (I).

Not all solutions of (I) with a single rupture point are radially symmetric. In fact, let $n = 2$ and $\nu = 3$; then the solution

$$u_\varepsilon(x) = \sqrt{2|x|}(\varepsilon(\cos\tfrac{1}{2}\theta)^2 + \varepsilon^{-1}(\sin\tfrac{1}{2}\theta)^2)^{1/2}, \quad \varepsilon > 0, \tag{1.7}$$

satisfies (I) and has one single point rupture, 0. Note that u_ε is not radially symmetric when $\varepsilon \neq 1$.

Our main goal of this paper is to find *necessary and sufficient* conditions for which solutions of (I) are radially symmetric.

It is very interesting to see that symmetry properties of positive solutions of (I) are related to those of positive solutions of the Lane–Emden equation with positive supercritical exponent

$$\Delta u + u^p = 0, \quad x \in \mathbb{R}^n, \quad p > \frac{n+2}{n-2}. \tag{1.8}$$

The symmetry and local behaviour of positive C^2-solutions of (1.8) were studied by Zhou [19]. He showed that for $n \geqslant 3$ and $(n+2)/(n-2) < p < m$, where

$$m = \begin{cases} \infty, & n = 3, \\ (n+1)/(n-3), & n > 3, \end{cases}$$

a solution u of (1.8) is radially symmetric about some point, provided that u has the following decay:

$$u(x) = O(|x|^{-2/(p-1)}) \quad \text{at } +\infty. \tag{1.9}$$

In a more recent paper [8], Guo extended Zou's result to the cases when $m \leqslant p < \infty$ if $n = 4$ and $p \geqslant n/(n-4)$ if $n \geqslant 5$. More precisely, he showed that, for $n \geqslant 5$ and $p \geqslant n/(n-4)$, a non-negative C^2 solution of (1.8) is radially symmetric about some point in $\mathbb{R}^n$ if and only if $\lim_{|x|\to+\infty} |x|^{2/(p-1)}u(x) = \lambda$ for some $\lambda > 0$. Furthermore, for $n \geqslant 4$ and $(n+1)/(n-3) \leqslant p < n/(n-4)$, u is radially symmetric about some point in $\mathbb{R}^n$ if and only if

$$\lim_{|x|\to+\infty} |x|^{2/(p-1)}u(x) = \lambda \quad \text{and} \quad \lim_{|x|\to+\infty} |x|^{1-(\mu+n)/2}(|x|^{2/(p-1)}u(x) - \lambda) = 0,$$

where $\mu = 4/(p-1) + 4 - 2n$.

Unlike those in [8, 19], positive solutions of (I) do not decay as $|x| \to \infty$. In fact, because of the negative power of u, u grows as $|x| \to +\infty$. We will establish strong asymptotic estimates for the positive solutions of (I) satisfying (1.6), which are *good* enough to obtain their radially symmetric properties.

We remark that the negative exponent $u^{-\nu}$ can be considered as negative *supercritical* in $\mathbb{R}^n$, $n \geqslant 2$. In [1], it is found that, in $\mathbb{R}$, $u^{-\nu}$ is subcritical if $\nu < 3$, critical

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

if $\nu = 3$ and supercritical if $\nu > 3$. Thus, all $u^{-\nu}$ with $\nu > 0$ can be considered supercritical in $\mathbb{R}^n$, $n \geqslant 2$.

In this paper, we will use the devices introduced in [8, 19]. In particular, our arguments in the proofs below are closely related to those of [8]. The key ingredient of our arguments in this paper is the powerful Alexandroff–Serrin method of the moving plane (MMP), which was first developed by Serrin in partial differential equation theory, later extended and generalized by Gidas *et al*. [6, 7] and used by many authors. In contrast to the case of bounded domains or subcritical (critical) nonlinearities, where the Hopf boundary lemma or the Kelvin transform is available to start the MMP, appropriately strong asymptotic estimates of solutions at ∞, replacing boundary lemmas or the Kelvin transform and providing a starting point for the method, are crucial for the moving-plane procedure in the case of the entire space with supercritical nonlinearities.

Define $\mu = 2(\alpha + n - 2)$. We will establish the strong asymptotic estimates for positive solutions $u(x)$ of (I) satisfying (1.6) at ∞ for the following two cases below:

$$\mu - n \geqslant 0, \tag{1.10 a}$$

$$-1 < \mu - n < 0. \tag{1.10 b}$$

It is easily seen that (1.10 a) holds if $n \geqslant 4$ and $\nu > 0$; $n = 3$ and $0 < \nu \leqslant 3$; $n = 2$ and $0 < \nu \leqslant 1$, (1.10 b) holds if $n = 3$ and $\nu > 3$; $n = 2$ and $1 < \nu < 3$. For the case in (1.10 a), the asymptotic estimate (1.6) of u is good enough for us to obtain its symmetry by the moving-plane method. For the case in (1.10 b), the asymptotic estimate (1.6) of u is not good enough to do so. We need to obtain better asymptotic estimates for it.

Our main global results read as follows.

Theorem 1.1. *Let $n \geqslant 2$ be an integer, let $\nu > 0$ and let $u(x)$ be a positive C^0-solution of (I). Suppose that*

$$\mu - n \geqslant 0, \quad i.e. \begin{cases} n \geqslant 4, & \nu > 0, \\ n = 3, & 0 < \nu \leqslant 3, \\ n = 2, & 0 < \nu \leqslant 1. \end{cases} \tag{1.11}$$

Then u is radially symmetric about some point $x_0 \in \mathbb{R}^n$ if and only if

$$\lim_{|x| \to +\infty} |x|^{-\alpha} u(x) = \lambda. \tag{1.12}$$

Theorem 1.2. *Assume that $n = 3$ and $\nu > 3$ or $n = 2$ and $\nu > 1$ (note that $-2 < \mu - n < 0$), $u(x)$ is a positive C^0-solution of (I). Then u is radially symmetric about some point $x_0 \in \mathbb{R}^n$ if and only if*

$$\lim_{|x| \to +\infty} |x|^{-\alpha} u(x) = \lambda \tag{1.13}$$

and

$$\lim_{|x| \to +\infty} |x|^{1+(\mu-n)/2} (|x|^{-\alpha} u(x) - \lambda) = 0. \tag{1.14}$$

We remark that in [19], it is only assumed that

$$u(x) = O(|x|^{-2/(p-1)}) \quad \text{at } +\infty.$$

Here we need the *exact* asymptotics. Example (1.7) shows that it is not enough merely to assume that

$$u(x) = O(|x|^{2/(\nu+1)}) \quad \text{at } +\infty.$$

Example (1.7) also implies that the assumption (1.14) is needed in theorem 1.2 at least for some ν.

However, in the case $n = 2$, $\nu \neq 3$, we can show the following result.

THEOREM 1.3. *If $u(x)$ satisfies (I) and the growth condition*

$$u(x) \geqslant C|x|^{\alpha} \quad \text{at } +\infty, \tag{1.15}$$

then (1.6) holds.

The radially symmetric solutions of (I) can be classified according to the following theorem.

THEOREM 1.4. *All radially symmetric solutions of (I) can be classified as follows:*

(a) *the first solution is a solution with a single rupture,*

$$u_0(r) = \left(\frac{\nu+1}{2}\right)^{2/(\nu+1)} r^{2/(\nu+1)};$$

(b) *the other solutions form a one-parameter family $\{u_\eta\}_{\eta>0}$ with $u_\eta(r)$, $r = |x|$, strictly increasing in r, $u_\eta(0) = \eta > 0$, $u_\eta(r) = \eta u_1(\eta^{-(\nu+1)/2} r)$ and, as $r \to +\infty$,*

$$r^{-2/(\nu+1)} u_\eta(r) \to \lambda.$$

As far as we know, our result is the first of its kind in dealing with radial symmetry of non-negative solutions for semi-linear elliptic equations with *negative* power. This paper is organized as follows. In §§ 2 and 3 we provide some key inequalities for the difference $v(y) = r^{-\alpha}u(x) - \lambda$, $y = x/r^2$. In §§ 4 and 5 we study the Lipschitz and the Hölder continuities of v near 0. In § 6 we provide a key auxiliary lemma (lemma 6.2), which is needed for using the MMP. In § 7, we prove the necessary parts of theorems 1.1–1.4. Finally, in § 8, we use the MMP to finish the proofs of sufficient parts of theorems 1.1 and 1.2.

2. Preliminaries

Let $n \geqslant 2$ be a positive integer and let $\mathbb{R}^n$ be the n-Euclidean space. For $\nu > 0$, consider the equation

$$\Delta u = u^{-\nu}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

We are interested in non-negative C^0-solutions u of (2.1) satisfying (1.6).

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

We begin with notation and definitions. Let

$$\alpha = \frac{2}{\nu + 1}, \quad \lambda = [\alpha(\alpha + n - 2)]^{-1/(\nu+1)}. \tag{2.2}$$

Throughout the paper, we shall assume that $\nu > 0$ and

$$\lim_{|x|\to\infty} |x|^{-\alpha} u(x) = \lambda. \tag{2.3}$$

In what follows, we define by $M = M(\cdot)$ positive constants, depending on the arguments inside the parentheses, as well as the structural numbers n and ν, which may vary line from line to line.

For any function $u(x)$ on $\mathbb{R}^n$, we introduce the Kelvin-type transform

$$v(y) = r^{-\alpha} u(x) - \lambda, \quad y = \frac{x}{r^2}, \quad r = |x| > 0. \tag{2.4}$$

Equation (2.1) is converted with singular coefficients at the origin under (2.4). In particular, study the new equation near the origin.

Lemma 2.1. *Let u be a non-negative solution of (2.1), and let v be given by (2.4). Suppose that (2.3) holds. Then v satisfies*

$$\Delta v - \frac{\mu y \cdot \nabla v}{s^2} + \frac{\mu v}{s^2} - \frac{f(v)}{s^2} = 0, \quad y \in \mathbb{R}^n \setminus \{0\}, \tag{2.5}$$

where $s = |y|$ and

$$\mu = 2(\alpha + n - 2), \qquad f(t) = (t + \lambda)^{-\nu} - \lambda^{-\nu} + \nu\lambda^{-(\nu+1)} t.$$

Note that f is real analytic at $t = 0$ and satisfies

$$f(0) = f'(0) = 0, \qquad f''(0) = \nu(\nu + 1)\lambda^{-(\nu+2)} > 0.$$

Moreover, for any integer $\tau \geqslant 0$ there exists a constant $M = M(u) > 0$, $s_0 = s_0(u) > 0$, such that

$$\lim_{s\to 0} v(y) = 0, \quad |\nabla^\tau v(y)| \leqslant \frac{M}{s^\tau} \quad \text{for } s = |y| \leqslant s_0. \tag{2.6}$$

Proof. Using (2.1) and (2.4), equation (2.5) is obtained by direct calculation. The estimates (2.6) can be obtained by arguments identical to those in the proof of [19, lemma 2.1]. □

By lemma 2.1, we are reduced to studying solutions of (2.5) satisfying (2.6). Therefore, in the following, we shall assume that (2.6) is satisfied.

We introduce the function

$$w(s, \theta) = v(s, \theta) - \bar{v}(s), \tag{2.7}$$

where

$$\bar{v}(s) = \frac{1}{\omega_n} \int_{S^{n-1}} v(s, \theta)\, \mathrm{d}\theta, \quad \omega_n = |S^{n-1}|.$$

LEMMA 2.2. *Let v be a solution of (2.5) and let Δ_θ be the Laplace–Beltrami operator on S^{n-1}. Then v, $\bar{v}$ and w respectively satisfy*

$$v'' + \frac{\Delta_\theta v}{s^2} - \frac{\mu - n + 1}{s} v' + \frac{\mu v}{s^2} - \frac{f(v)}{s^2} = 0, \tag{2.8}$$

$$\bar{v}'' - \frac{\mu - n + 1}{s} \bar{v}' + \frac{\mu \bar{v}}{s^2} - \frac{\overline{f(v)}}{s^2} = 0, \tag{2.9}$$

$$w'' + \frac{\Delta_\theta w}{s^2} - \frac{\mu - n + 1}{s} w' + \frac{\mu w}{s^2} - \frac{f(v) - \overline{f(v)}}{s^2} = 0, \tag{2.10}$$

where the prime denotes the derivative with respect to the radius s.

Proof. Equation (2.8) follows directly from (2.5) and

$$\Delta v = v'' + \frac{\Delta_\theta v}{s^2} + \frac{n-1}{s} v', \quad \nabla v \cdot y = v' s.$$

Integrating (2.8) over S^{n-1} yields (2.9) since

$$\overline{\Delta_\theta v} = \frac{1}{\omega_n} \int_{S^{n-1}} \Delta_\theta v(s, \theta) \, \mathrm{d}\theta = 0.$$

Finally, subtracting (2.9) from (2.8) gives (2.10). □

3. A fundamental inequality

The Lipschitz continuity of w at the origin is crucial in proving the expansion of u near ∞, which can be used to obtain the symmetry of u by the MMP. To this end, we first obtain the Hölder-type estimate for v. The function

$$W(s) = \left(\int_{S^{n-1}} w^2(s, \theta) \, \mathrm{d}\theta \right)^{1/2} \tag{3.1}$$

plays an important role in achieving our goal.

THEOREM 3.1. *Let W be given by (3.1). There then exist $s_0 > 0$ and a positive constant $K = K(v, \nu, n, s_0)$ such that, for $0 < s < s_0$,*

$$W(s) \leqslant \begin{cases} K s^{1+\mu-n} & \text{if } -1 < \mu - n < 0, \\ K s & \text{if } \mu - n \geqslant 0. \end{cases} \tag{3.2}$$

The proof of this theorem is related to that of [8, theorem 3.1]. We first obtain the following lemma.

LEMMA 3.2. *For any $0 < \varepsilon < \min\{\frac{1}{2}(1+\mu-n), \frac{1}{2}\}$, there exist $\hat{\delta} = 1+\mu-n-\varepsilon > 0$ for $-1 < \mu - n < 0$, $\hat{\delta} = 1 - \varepsilon$ for $\mu - n \geqslant 0$, $s_0 = s_0(\varepsilon) > 0$ and a positive constant $K = K(v, \hat{\delta}, s_0)$ such that*

$$W(s) \leqslant K s^{\hat{\delta}}, \quad 0 < s < s_0. \tag{3.3}$$

Proof. Let $g(v) = f(v) - \overline{f(v)}$. Then it is known from (2.10) that w satisfies

$$w'' + \frac{\Delta_\theta w}{s^2} - \frac{\mu - n + 1}{s} w' + \frac{\mu w}{s^2} - \frac{g(v)}{s^2} = 0. \tag{3.4}$$

It is known from [14] that the eigenvalues of the problem

$$-\Delta_\theta Q = \sigma Q, \quad \theta \in S^{n-1},$$

are

$$\sigma_k = k(n + k - 2), \quad k \geqslant 0,$$

with multiplicity

$$m_k = \frac{(n - 3 + k)!(n - 2 + 2k)}{k!(n-2)!}.$$

In particular, we have

$$\begin{aligned} &\sigma_0 = 0, && m_0 = 1, && Q_0 \equiv 1, \\ &\sigma_1 = n - 1, && m_1 = n, && Q_i(\theta) = x_i, \quad 1 \leqslant i \leqslant n, \\ &\sigma_2 = 2n. \end{aligned}$$

Here the $Q_i(\theta)$ denote the associated eigenvectors. Therefore, if $u \in L^2(S^{n-1})$ is orthogonal to Q_0, i.e. $\bar{u} = 0$, we have

$$\int_{S^{n-1}} |\nabla_\theta u|^2 \, d\theta \geqslant (n-1) \int_{S^{n-1}} u^2 \, d\theta.$$

Moreover, if u is orthogonal to Q_0, Q_i, $i = 1, 2, \dots, n$, we have

$$\int_{S^{n-1}} |\nabla_\theta u|^2 \, d\theta \geqslant 2n \int_{S^{n-1}} u^2 \, d\theta.$$

Since $w(s, \cdot) \in L^2(S^{n-1})$ and $\bar{w} = 0$, we have $w(s, \theta) = w_1(s, \theta) + w_2(s, \theta)$, where $w_1(s, \theta) = \sum_{i=1}^n w_i(s) Q_i(\theta)$, $\{Q_1(\theta), \dots, Q_n(\theta)\}$ is the basis of the eigenspace H_1 of $-\Delta_{S^{n-1}}$ corresponding to the eigenvalue $n - 1$, $w_2(s, \cdot) \in H_1^\perp$. Now, it follows from (3.4) that $w_i(s)$ satisfies

$$w_i''(s) - \frac{\mu - n + 1}{s} w_i'(s) + \frac{\mu - n + 1}{s^2} w_i(s) - \frac{g_i(s)}{s^2} = 0 \tag{3.5}$$

for $i = 1, 2, \dots, n$, where

$$g_i(s) = \int_{S^{n-1}} f'(\xi(s, \theta)) \sum_{j=1}^n w_j(s) Q_j(\theta) Q_i(\theta) \, d\theta + \int_{S^{n-1}} f'(\xi(s, \theta)) w_2(s, \theta) Q_i(\theta) \, d\theta,$$

and $\xi(s, \theta) = \eta v(s, \theta) + (1 - \eta)\bar{v}$, $\eta \in (0, 1)$ (see [8]).

Let $t = -\ln s$, $z_i(t) = w_i(s)$, $\tilde{\xi}(t, \theta) = \xi(s, \theta)$ and $z_2(t, \theta) = w_2(s, \theta)$. Then $z_i(t)$ satisfies

$$z_i''(t) + (\mu - n + 2) z_i'(t) + (\mu - n + 1) z_i(t) - \tilde{g}_i(t) = 0, \tag{3.6}$$

where

$$\tilde{g}_i(t) = \int_{S^{n-1}} f'(\tilde{\xi}) \sum_{j=1}^n z_j(t) Q_j(\theta) Q_i(\theta) \, d\theta + \int_{S^{n-1}} f'(\tilde{\xi}) z_2(t, \theta) Q_i(\theta) \, d\theta.$$

We first study solutions of

$$y''(t) + (\mu - n + 2)y'(t) + (\mu - n + 1)y(t) = 0. \tag{3.7}$$

A simple calculation implies that (3.7) admits two linearly independent positive solutions

$$y_1(t) = \mathrm{e}^{-(1+\mu-n)t}, \qquad y_2(t) = \mathrm{e}^{-t}.$$

Let $\delta_1 = 1 + \mu - n$, $\delta_2 = 1$. Then if $\mu - n > -1$, we see $\delta_1 > 0$ and $\delta_2 > 0$. By the ordinary differential equation theory, we have

$$z_i(t) = M_1 \mathrm{e}^{-\delta_1 t} + M_2 \mathrm{e}^{-\delta_2 t} + M_3 \int_{t_0}^{t} \frac{\mathrm{e}^{-\delta_1 s}\mathrm{e}^{-\delta_2 t} - \mathrm{e}^{-\delta_1 t}\mathrm{e}^{-\delta_2 s}}{\mathrm{e}^{-(\delta_1+\delta_2)s}} \tilde{g}_i(s)\,\mathrm{d}s, \tag{3.8}$$

where M_j, $j = 1, 2$, are constants depending upon t_0, δ_1 and δ_2, and $|M_3| = |1/(\mu - n)|$ is a constant independent of t_0. Now we consider the only case when $\mu - n \neq 0$; the case when $\mu - n = 0$, i.e. $n = 3$ and $\nu = 3$ or $n = 2$ and $\nu = 1$, will be studied later. It follows from (3.8) that, for t sufficiently large,

$$|z_i(t)| \leqslant C\mathrm{e}^{-\delta t} + C_1 \mathrm{e}^{-\delta t} \int_{t_0}^{t} \mathrm{e}^{\delta s} |\tilde{g}_i(s)|\,\mathrm{d}s, \tag{3.9}$$

where $\delta = \min\{\delta_1, \delta_2\} > 0$ and C_1 is independent of t_0. This implies that

$$|z_i(t)| \leqslant C\mathrm{e}^{-\delta t} + C_1 \mathrm{e}^{-\delta t} \int_{t_0}^{t} \mathrm{e}^{\delta s} \biggl[\sum_{j=1}^{n} |z_j(s)|\,|F_j(s)| + |G_i(s)| \biggr] \mathrm{d}s, \tag{3.10}$$

where

$$|F_j(s)| = \biggl| \int_{S^{n-1}} f'(\tilde{\xi}) Q_i(\theta) Q_j(\theta)\,\mathrm{d}\theta \biggr|,$$

$$|G_i(s)| = \biggl| \int_{S^{n-1}} f'(\tilde{\xi}) z_2(t,\theta) Q_i(\theta)\,\mathrm{d}\theta \biggr|.$$

Thus,

$$\sum_{i=1}^{n} |z_i(t)| \leqslant C_2 \mathrm{e}^{-\delta t} + C_3 \mathrm{e}^{-\delta t} \int_{t_0}^{t} \mathrm{e}^{\delta s} \biggl[\sum_{j=1}^{n} |z_j(s)|\,|F_j(s)| + \sum_{i=1}^{n} |G_i(s)| \biggr] \mathrm{d}s, \tag{3.11}$$

where C_3 is independent of t_0. Set $F(t) = \max_{1 \leqslant j \leqslant n} |F_j(t)|$, $G(t) = \sum_{i=1}^{n} |G_i(t)|$ and $Z(t) = \sum_{i=1}^{n} |z_i(t)|$. Then

$$Z(t) \leqslant C_2 \mathrm{e}^{-\delta t} + C_3 \mathrm{e}^{-\delta t} \int_{t_0}^{t} \mathrm{e}^{\delta s} [F(s)Z(s) + G(s)]\,\mathrm{d}s. \tag{3.12}$$

Define $d(t_0) = \max_{t \geqslant t_0} F(t)$. Using the fact that $f'(\tilde{\xi}(t,\theta)) \to 0$ as $t \to \infty$, we find that $d(t_0) \to 0$ as $t_0 \to \infty$. Thus,

$$\mathrm{e}^{\delta t} Z(t) \leqslant C_2 + C_3 d(t_0) \int_{t_0}^{t} \mathrm{e}^{\delta s} Z(s)\,\mathrm{d}s + C_4 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) \mathrm{e}^{\delta s}\,\mathrm{d}s, \tag{3.13}$$

where

$$\tilde{F}(t) = \sum_{i=1}^{n} \bigg[\int_{S^{n-1}} (f'(\tilde{\xi})Q_i(\theta))^2 \, \mathrm{d}\theta \bigg]^{1/2},$$

$$Z_2(t) = \bigg(\int_{S^{n-1}} z_2^2(t,\theta) \, \mathrm{d}\theta \bigg)^{1/2}.$$

Let $\mathrm{e}^{\delta t} Z(t) = h(t)$. Then

$$h(t) \leqslant C_2 + C_3 d(t_0) \int_{t_0}^{t} h(s) \, \mathrm{d}s + C_4 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) \mathrm{e}^{\delta s} \, \mathrm{d}s.$$

Set

$$R(t) = \int_{t_0}^{t} h(s) \, \mathrm{d}s \quad \text{and} \quad l(t) = C_2 + C_4 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) \mathrm{e}^{\delta s} \, \mathrm{d}s.$$

We have

$$R'(t) \leqslant l(t) + C_3 d(t_0) R(t).$$

This implies that

$$R(t) \leqslant \mathrm{e}^{C_3 d(t_0) t} \int_{t_0}^{t} \mathrm{e}^{-C_3 d(t_0) s} l(s) \, \mathrm{d}s.$$

It follows from the integration by parts that

$$\begin{aligned} h(t) &\leqslant l(t) + C_3 d(t_0) \mathrm{e}^{C_3 d(t_0) t} \int_{t_0}^{t} \mathrm{e}^{-C_3 d(t_0) s} l(s) \, \mathrm{d}s \\ &= l(t) - \mathrm{e}^{C_3 d(t_0) t} \int_{t_0}^{t} l(s) \frac{\mathrm{d}}{\mathrm{d}s} (\mathrm{e}^{-C_3 d(t_0) s}) \\ &= \mathrm{e}^{C_3 d(t_0)(t - t_0)} l(t_0) + \int_{t_0}^{t} \mathrm{e}^{C_3 d(t_0)(t-s)} l'(s) \, \mathrm{d}s. \end{aligned}$$

Therefore,

$$Z(t) \leqslant C_5 \mathrm{e}^{-\tilde{\delta} t} + C_6 \int_{t_0}^{t} \tilde{F}(s) Z_2(s) \mathrm{e}^{-\tilde{\delta}(t-s)} \, \mathrm{d}s, \tag{3.14}$$

where $\tilde{\delta} = \delta - C_3 d(t_0)$. Since $d(t_0) \to 0$ as $t_0 \to \infty$, we can choose t_0 sufficiently large such that $\tilde{\delta} > 0$.

On the other hand, we know that $w_2(s,\theta)$ satisfies

$$w_2''(s,\theta) + \frac{\Delta_\theta w_2(s,\theta)}{s^2} - \frac{\mu - n + 1}{s} w_2'(s,\theta) + \frac{\mu}{s^2} w_2(s,\theta) - \frac{g_2(s,\theta)}{s^2} = 0, \tag{3.15}$$

where

$$\begin{aligned} &\int_{S^{n-1}} g_2(s,\theta) w_2(s,\theta) \, \mathrm{d}\theta \\ &\qquad = \int_{S^{n-1}} f'(\xi(s,\theta)) w(s,\theta) w_2(s,\theta) \, \mathrm{d}\theta \\ &\qquad = \int_{S^{n-1}} f'(\xi(s,\theta)) w_1(s,\theta) w_2(s,\theta) \, \mathrm{d}\theta + \int_{S^{n-1}} f'(\xi) w_2^2(s,\theta) \, \mathrm{d}\theta. \end{aligned}$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

Therefore,

$$\left|\int_{S^{n-1}} g_2(s,\theta)w_2(s,\theta)\,\mathrm{d}\theta\right| \leqslant \left[\int_{S^{n-1}}(f'(\xi)w_1(s,\theta))^2\right]^{1/2}\left[\int_{S^{n-1}}(w_2(s,\theta))^2\right]^{1/2} + H(s)\int_{S^{n-1}} w_2^2(s,\theta)\,\mathrm{d}\theta,$$

where $H(s) = \max_{\theta\in S^{n-1}}|f'(\xi(s,\theta))|$ and $H(s)\to 0$ as $s\to 0$.

Let

$$W_2(s) = \left(\int_{S^{n-1}} w_2^2(s,\theta)\,\mathrm{d}\theta\right)^{1/2}.$$

By arguments similar to those in [8, 19], we see that W_2 satisfies

$$W_2''(s) - \frac{\mu-n+1}{s}W_2'(s) + \frac{\mu-2n+H(s)}{s^2}W_2(s) + \frac{[\int_{S^{n-1}}(f'(\xi)w_1(s,\theta))^2]^{1/2}}{s^2} \geqslant 0 \tag{3.16}$$

for $s\in(0,S)$ and some $S>0$. Here we use the inequality

$$\int_{S^{n-1}}|\nabla_\theta w_2(s,\theta)|^2\,\mathrm{d}\theta \geqslant 2n\int_{S^{n-1}} w_2^2(s,\theta)\,\mathrm{d}\theta.$$

Making the transformations $t=-\ln s$, $w_2(s,\theta)=z_2(t,\theta)$ and $Z_2(t)=W_2(s)$, we observe that $Z_2(t)$ satisfies

$$Z_2''(t) + (\mu-n+2)Z_2'(t) + (\mu-2n+H^*(t))Z_2(t) + H_1^*(t)Z(t) \geqslant 0, \tag{3.17}$$

where $H^*(t)=H(s)$ and $H^*(t)\to 0$ as $t\to\infty$. To obtain $H_1^*(t)$, we note that

$$\begin{aligned}\int_{S^{n-1}}(f'(\xi)w_1(s,\theta))^2\,\mathrm{d}\theta &= \int_{S^{n-1}}\left(\sum_{i=1}^n f'(\xi)w_i(s)Q_i(\theta)\right)^2\mathrm{d}\theta\\ &\leqslant C\left(\sum_{i=1}^n|w_i(s)|\right)^2 H_1^2(s),\end{aligned}$$

where

$$H_1^2(s) = \int_{S^{n-1}}(f'(\xi))^2\sum_{i=1}^n Q_i^2(\theta)\,\mathrm{d}\theta$$

and $H_1(s)\to 0$ as $s\to 0$. Under the transformation, we have $H_1^*(t)=H_1(s)$ and $Z(t)=\sum_{i=1}^n|w_i(s)|$. Thus, $H_1^*(t)\to 0$ as $t\to\infty$. Using (3.17) and (3.14), we obtain

$$\begin{aligned}&Z_2''(t) + (\mu-n+2)Z_2'(t) + (\mu-2n+H^*(t))Z_2(t)\\ &\qquad + C_5H_1^*(t)\mathrm{e}^{-\tilde\delta t} + C_6H_1^*(t)\int_{t_0}^t \tilde F(s)Z_2(s)\mathrm{e}^{-\tilde\delta(t-s)}\,\mathrm{d}s \geqslant 0.\end{aligned} \tag{3.18}$$

For any $0<\hat\delta<\tilde\delta$, choose $t^*>t_0$ such that

$$\mu-2n+H^*(t) = 2\left(\frac{2}{\nu+1}-2\right) + H^*(t) < 0 \quad \text{for } t\geqslant t^*,$$

and $K = K(t_0, t^*) > 1$ such that

$$Z_2(t) \leqslant K\mathrm{e}^{-\hat{\delta}t}, \quad t \in [t_0, t^*].$$

Let $\zeta(t) = K\mathrm{e}^{-\hat{\delta}t}$. We claim that

$$\zeta''(t) + (\mu - n + 2)\zeta'(t) + (\mu - 2n + H^*(t))\zeta(t) + C_5 H_1^*(t)\mathrm{e}^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^{t} \tilde{F}(s)\zeta(s)\mathrm{e}^{-\tilde{\delta}(t-s)}\,\mathrm{d}s \leqslant 0, \quad (3.19)$$

for $t \geqslant t^*$. In fact, a simple calculation implies that

$$\begin{aligned}
&\zeta''(t) + (\mu - n + 2)\zeta'(t) + (\mu - 2n + H^*(t))\zeta(t) \\
&\quad + C_5 H_1^*(t)\mathrm{e}^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^{t} \tilde{F}(s)\zeta(s)\mathrm{e}^{-\tilde{\delta}(t-s)}\,\mathrm{d}s \\
&\qquad = K[\hat{\delta}^2 - (\mu - n + 2)\hat{\delta} + (\mu - 2n + H^*(t))]\mathrm{e}^{-\hat{\delta}t} \\
&\qquad\quad + C_5 H_1^*(t)\mathrm{e}^{-\tilde{\delta}t} + C_6 H_1^*(t)\int_{t_0}^{t} \tilde{F}(s)K\mathrm{e}^{-\hat{\delta}s}\mathrm{e}^{-\tilde{\delta}(t-s)}\,\mathrm{d}s \\
&\qquad = \mathrm{e}^{-\hat{\delta}t}\biggl[K(\hat{\delta}^2 - (\mu - n + 2)\hat{\delta} + (\mu - 2n + H_1^*(t))) \\
&\qquad\quad + C_5 H_1^*(t)\mathrm{e}^{-(\tilde{\delta}-\hat{\delta})t} + C_6 H_1^*(t)\int_{t_0}^{t} \tilde{F}(s)K\mathrm{e}^{-(\tilde{\delta}-\hat{\delta})(t-s)}\,\mathrm{d}s\biggr].
\end{aligned}$$

Since $\hat{\delta} < \tilde{\delta}$, we easily see that

$$C_5 H_1^*(t)\mathrm{e}^{-(\tilde{\delta}-\hat{\delta})t} + C_6 H_1^*(t)\int_{t_0}^{t} \tilde{F}(s)K\mathrm{e}^{-(\tilde{\delta}-\hat{\delta})(t-s)}\,\mathrm{d}s \to 0$$

as $t \to \infty$. On the other hand, since, for $\nu > 0$,

$$\mu - 2n = 2\biggl(\frac{2}{\nu + 1} - 2\biggr) < 0, \qquad \hat{\delta} - \mu + n - 2 < -1,$$

we easily see that our claim holds.

Let $X(t) = Z_2(t) - \zeta(t)$. We know that

$$X''(t) + (\mu - n + 2)X'(t) + (\mu - 2n + H^*(t))X(t) + C_6 H_1^*(t)\int_{t_0}^{t} \tilde{F}(s)X(s)\mathrm{e}^{-\tilde{\delta}(t-s)}\,\mathrm{d}s \geqslant 0. \quad (3.20)$$

Since $X(t) \to 0$ as $t \to \infty$ and $X(t^*) \leqslant 0$, the maximum principle implies that

$$X(t) \leqslant 0 \quad \text{for } t \geqslant t^*.$$

This implies that

$$Z_2(t) \leqslant K\mathrm{e}^{-\hat{\delta}t} \quad \text{for } t \geqslant t_0. \quad (3.21)$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

It follows from (3.14) that

$$Z(t) \leqslant K\mathrm{e}^{-\hat{\delta}t} \quad \text{for } t \geqslant t_0. \tag{3.22}$$

This implies that

$$W(s) := \left(\int_{S^{n-1}} w^2(s,\theta)\,\mathrm{d}\theta\right)^{1/2} \leqslant Ks^{\hat{\delta}} \tag{3.23}$$

for $s \in (0, s_0)$, where $s_0 = \mathrm{e}^{-t_0}$.

When $n = 3$ and $\nu = 3$ or $n = 2$ and $\nu = 1$, i.e. $\mu - n = 0$, equation (3.7) has only one characteristic value, -1. By a different variation-of-constants formula and the same steps as above, we also observe that, for any $0 < \hat{\delta} < 1$,

$$W(s) := \left(\int_{S^{n-1}} w^2(s,\theta)\,\mathrm{d}\theta\right)^{1/2} \leqslant Ks^{\hat{\delta}} \tag{3.24}$$

for $s \in (0, s_0)$, where $s_0 = \mathrm{e}^{-t_0}$.

It is clear that, for any $\varepsilon > 0$, we can choose $t_0 = t_0(\varepsilon)$ sufficiently large such that $0 < \hat{\delta} := \delta - \varepsilon < \tilde{\delta}$. Since $\delta = 1 + \mu - n$ for $-1 < \mu - n < 0$ and $\delta = 1$ for $\mu - n \geqslant 0$, we obtain our conclusion. This completes the proof of lemma 3.2. □

We fix the ε in lemma 3.2 for the proofs below, as follows.

Now we study the Hölder estimate for $\bar{v}$. Let $\sigma \in \mathbb{R}$ and

$$\rho(s) = s^{-\sigma}\bar{v}(s).$$

Lemma 3.3. *For any $0 < \sigma < \hat{\delta}$, there exists a positive constant $M = M(v)$ such that*

$$\rho(s) \leqslant M, \quad |\rho'(s)| \leqslant \frac{M}{s}, \quad 0 < s < s_0. \tag{3.25}$$

Proof. The proof is easily obtained from the equation for $\bar{v}$ and a simple calculation in which $\rho(s)$ satisfies

$$\rho''(s) + \frac{2\sigma - \mu + n - 1}{s}\rho' + \frac{\mu_1}{s^2}\rho - \frac{f_1(\bar{v})}{s^2}\rho = g(s), \tag{3.26}$$

where

$$\mu_1 = \sigma(\sigma + n - \mu - 2) + \mu, \qquad f_1(t) = \frac{f(t)}{t}, \quad t \neq 0,$$

and

$$g(s) = \frac{\overline{f(v)} - f(\bar{v})}{s^{2+\sigma}} = o(s^{\hat{\delta}-\sigma-2}).$$

The above identity can be obtained from lemma 3.2 and from the fact that, for s small,

$$f(v) - f(\bar{v}) = o(|v - \bar{v}|),$$

and so

$$|\overline{f(v)} - f(\bar{v})| \leqslant \frac{1}{\omega_n}\int_{S^{n-1}} |f(v) - f(\bar{v})| = o(W) = o(s^{\hat{\delta}}).$$

We first claim that there exist two positive constants $T = T(v)$ and $M = M(v)$ such that

$$\int_t^T \frac{\rho^2}{s} \leqslant M\biggl[1 + (\rho(t))^2 + (\rho'(t))^2 t^2 + \int_t^T s(\rho')^2\biggr] \tag{3.27}$$

for all $0 < t < T$. To see this, fix T and multiply (3.26) by $s\rho(s)$ and integrate from t to T,

$$\mu_1 \int_t^T \frac{\rho^2}{s} = \int_t^T s\rho g(s) + \int_t^T f_1(\bar{v})\frac{\rho^2}{s} - (s\rho'\rho)|_t^T + \int_t^T s(\rho')^2 - \frac{2\sigma - \mu + n - 2}{2}\rho^2\bigg|_t^T. \tag{3.28}$$

Since $\mu_1 = \mu_1(\sigma) = \sigma^2 + (n - \mu - 2)\sigma + \mu$, we have $\mu_1 > \frac{1}{2}$ for $0 < \sigma < \hat{\delta}$. Indeed, we know that $\mu_1(\sigma)$ attains its minimum at $\sigma = 1 + \frac{1}{2}(\mu - n)$. On the other hand, $\hat{\delta} < 1 + \frac{1}{2}(\mu - n)$ for both $-1 < \mu - n < 0$ and $\mu - n \geqslant 0$. (Note that $\hat{\delta} < 1 + \mu - n < 1 + \frac{1}{2}(\mu - n)$ if $-1 < \mu - n < 0$, $\hat{\delta} < 1 \leqslant 1 + \frac{1}{2}(\mu - n)$ if $\mu - n \geqslant 0$.) These imply that $\mu_1(\sigma)$ is decreasing in $(0, 1 + \frac{1}{2}(\mu - n))$ and, for $\sigma \in (0, \hat{\delta})$,

$$\mu_1(\sigma) \geqslant \mu_1(\hat{\delta}) \geqslant \begin{cases} \mu_1(1 + \mu - n) = n - 1 & \text{for } -1 < \mu - n < 0, \\ \mu_1(\hat{\delta}) \geqslant \mu_1(1) = n - 1 & \text{for } \mu - n \geqslant 0. \end{cases}$$

Thus, to obtain (3.27), it suffices to bound the right-hand side of (3.28) in terms of the right-hand side of (3.27). By the condition for f_1, we have

$$\lim_{s \to 0} f_1(\bar{v}) = 0.$$

Hence, by fixing T small enough, we may bound

$$\biggl|\int_t^T f_1(\bar{v})\frac{\rho^2}{s}\biggr| \leqslant \frac{\mu_1}{4}\int_t^T \frac{\rho^2}{s}.$$

By the Schwarz and Young inequalities, we have

$$\biggl|\int_t^T s\rho g(s)\biggr| \leqslant \biggl(\int_t^T \frac{\rho^2}{s}\biggr)^{1/2}\biggl(\int_t^T s^3 g^2(s)\biggr)^{1/2} \leqslant \frac{\mu_1}{4}\int_t^T \frac{\rho^2}{s} + M\int_t^T s^{2\hat{\delta} - 2\sigma - 1},$$

since

$$|g(s)| = o(s^{\hat{\delta} - \sigma - 2}), \quad s \to 0.$$

Therefore,

$$\int_t^T f_1(\bar{v})\frac{\rho^2}{s} + \int_t^T s\rho g(s) \leqslant \frac{\mu_1}{2}\int_t^T \frac{\rho^2}{s} + MT^{2\hat{\delta} - 2\sigma},$$

since $\hat{\delta} > \sigma$. Inserting this into (3.28), we obtain (3.27) immediately, since the last three terms in (3.28) are bounded by the right-hand side of (3.27).

Note that $\sigma < \hat{\delta} < \delta$ implies that $2\sigma < \mu - n + 2$ for both $-1 < \mu - n < 0$ and $\mu - n \geqslant 0$. Indeed,

$$2\sigma < 2\delta = \begin{cases} 2(\mu - n + 1) < \mu - n + 2 & \text{if } -1 < \mu - n < 0, \\ 2 \leqslant \mu - n + 2 & \text{if } \mu - n \geqslant 0. \end{cases}$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

The remainder of the proof of this lemma is a slight variant of the proof of [19, lemma 4.2]. □

As an immediate corollary of lemma 3.3, we obtain the following Hölder-type estimate of $\bar{v}$ and $\bar{v}'$ near $s = 0$.

LEMMA 3.4. *Let $\hat{\delta}$ be given as in lemma 3.2 and let v be a solution of (2.8). There then exists a constant $M = M(v) > 0$ such that*

$$|\bar{v}(s)| \leqslant M s^{\hat{\delta}}, \qquad |\bar{v}'(s)| \leqslant M s^{\hat{\delta}-1}, \tag{3.29}$$

and

$$\int_{S^{n-1}} v^2(s,\theta) \leqslant M s^{2\hat{\delta}}. \tag{3.30}$$

Proof. We show only $(3.29)_1$ and (3.30). The proof of $(3.29)_2$ is left to the reader. We first make the change of variables

$$t = -\ln s, \quad v_1(t) = \bar{v}(s).$$

Then v_1 satisfies

$$v_1''(t) + (\mu - n + 2)v_1'(t) + \mu v_1 = g_1(t), \quad t > 0, \tag{3.31}$$

where

$$g_1(t) = \overline{f(v)} = f(v_1) + (\overline{f(v)} - f(v_1)) = O(|v_1|^2) + o(W) = o(\mathrm{e}^{-\hat{\delta} t})$$

for $2\sigma > \hat{\delta}$ (see lemma 3.3). The two characteristic values of equation (3.31) are

$$k_1 = \frac{n-\mu-2}{2} + \frac{[(\mu-n)^2 - 4(n-1)]^{1/2}}{2},$$
$$k_2 = \frac{n-\mu-2}{2} - \frac{[(\mu-n)^2 - 4(n-1)]^{1/2}}{2}.$$

When $(\mu - n)^2 \leqslant 4(n-1)$ (note that this covers the case $-1 < \mu - n < 0$), we observe that the equation (3.31) has two conjugate characteristic values,

$$k_1 = -\sigma_0 + \sigma_1 \mathrm{i}, \qquad k_2 = -\sigma_0 - \sigma_1 \mathrm{i}$$

with $\sigma_0 = 1 + \frac{1}{2}(\mu - n)$, $\sigma_1 \geqslant 0$. It follows, by the variation-of-constants formula, that there exists a positive constant $M = M(v_1)$ such that

$$|v_1(t)| \leqslant M\mathrm{e}^{-\sigma_0 t}\left(1 + \int_{t_0}^{t} |g_1(s)|\mathrm{e}^{\sigma_0 s}\right) \mathrm{d}s \leqslant M\mathrm{e}^{-\hat{\delta} t}.$$

(Note that $\hat{\delta} < 1 + \frac{1}{2}(\mu - n)$ whether or not $\mu - n \geqslant 0$ or $-1 < \mu - n < 0$.)

When $(\mu - n)^2 > 4(n-1)$, we have $\mu - n > 0$ and $\hat{\delta} < 1$. Moreover, the two characteristic values of (3.31) satisfy

$$k_1 < -1 \quad \text{and} \quad k_2 < -1.$$

Therefore, by arguments similar to those above, we obtain

$$|v_1(t)| \leqslant M\mathrm{e}^{-t} \leqslant M\mathrm{e}^{-\hat{\delta} t}.$$

Since $v(s,\theta) = w(s,\theta) + \bar{v}$, it is easy to see that (3.30) follows from lemma 3.2 and (3.29). This completes the proof. □

By arguments similar to those in the proof of [19, theorem 5.2], we obtain the following proposition from lemmas 3.2 and 3.4.

PROPOSITION 3.5. *Let $\tau \geqslant 0$ be an integer and let v be a solution of (2.8). Then, for the ε given in lemma 3.2, there exists a constant $M = M(v,\varepsilon,\tau) > 0$ (independent of s) such that, for $0 < s < s_0$,*

$$\max_{|y|=s} |D^\tau v(y)| \leqslant \begin{cases} Ms^{1+\mu-n-\varepsilon-\tau} & \text{if } -1 < \mu - n < 0, \\ Ms^{1-\varepsilon-\tau} & \text{if } \mu - n \geqslant 0. \end{cases}$$

Proof of theorem 3.1. By proposition 3.5, we know that

$$|f'(\tilde{\xi})| \leqslant M\mathrm{e}^{-\hat{\delta}t}.$$

Let δ and G be as in the proof of lemma 3.2. We obtain

$$G(t) \leqslant M\mathrm{e}^{-2\hat{\delta}t} = M\mathrm{e}^{-2(\delta-\varepsilon)t}.$$

Choosing ε sufficiently small, it follows from (3.12) that

$$Z(t) \leqslant M\mathrm{e}^{-\delta t} + M\mathrm{e}^{-\delta t}\int_{t_0}^{t} \mathrm{e}^{(\delta-\hat{\delta})s} Z(s)\,\mathrm{d}s. \tag{3.32}$$

Let $R(t) = \mathrm{e}^{\delta t}Z(t)$. Then Gronwall's inequality implies that

$$R(t) \leqslant M.$$

Thus,

$$Z(t) \leqslant M\mathrm{e}^{-\delta t}. \tag{3.33}$$

Arguments similar to those in the proof of lemma 3.2 imply that

$$Z_2(t) \leqslant M\mathrm{e}^{-\delta t}.$$

This completes the proof. □

COROLLARY 3.6. *Let v be a solution of (2.8). There then exists a constant $M = M(v) > 0$ such that, for $0 < s < s_0$,*

$$\bar{v}(s) \leqslant Ms^{1+(\mu-n)}, \quad |\bar{v}'(s)| \leqslant Ms^{\mu-n} \quad \text{if } -1 < \mu - n < 0$$

and

$$|\bar{v}(s)| \leqslant Ms, \qquad |\bar{v}'(s)| \leqslant M \qquad \text{if } \mu - n \geqslant 0.$$

Proof. Since $2\hat{\delta} > \delta$ (by choosing ε small), the proof is similar to that of lemma 3.4. □

Now we can use the estimates obtained in theorem 3.1; corollary 3.6 and arguments similar to those in the proof of [19, theorem 5.2] to obtain the following theorem.

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

THEOREM 3.7. *Let $\tau \geqslant 0$ be an integer and let v be a solution of (2.8). There then exist $s_0 > 0$ and $M = M(v,\tau) > 0$ (independent of s) such that, for $0 < s < s_0$,*

$$\max_{|y|=s} |D^\tau v(y)| \leqslant \begin{cases} M s^{1+\mu-n-\tau} & \text{if } -1 < \mu - n < 0, \\ M s^{1-\tau} & \text{if } \mu - n \geqslant 0. \end{cases} \tag{3.34}$$

4. Local Lipschitz-type estimates and asymptotic expansions for $\mu - n \geqslant 0$

In this section we will obtain the local Lipschitz-type estimate for w. This yields the desired expansion for the application of the moving-plane method. Our main result in this section is the following theorem.

THEOREM 4.1. *Let $\tau \geqslant 0$ be an integer. There then exist $s_0 > 0$ and a constant $M = M(v,\tau) > 0$ (independent of s) such that*

$$\max_{|y|=s} |D^\tau w(y)| \leqslant M s^{1-\tau} \quad \text{for } 0 < s < s_0, \tag{4.1}$$

where w is given by (2.7).

The proof of theorem 4.1 is identical to that of theorem 3.7 and the local maximal principle, as [19, lemma 5.1] plays a role. As before, we first establish a local L^2-estimate for w near the origin and then the rest is routine.

Proof. We show only the case that $\tau = 0$. The rest is left to the reader. Define $\tilde{w}(s,\theta) = w(s,\theta)/s$. Then $\tilde{w}$ satisfies

$$\tilde{w}''(s,\theta) + \frac{\Delta_\theta \tilde{w}}{s^2} - \frac{\mu - n - 1}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} - \frac{f(v) - \overline{f(v)}}{s^3} = 0. \tag{4.2}$$

As in the proof of lemma 3.2, we define

$$\tilde{w}(s,\theta) = \tilde{w}_1(s,\theta) + \tilde{w}_2(s,\theta)$$

and

$$\tilde{w}_1(s,\theta) = \sum_{i=1}^{n} \tilde{w}_i(s) Q_i(\theta).$$

Then $\tilde{w}_i(s)$ satisfies

$$\tilde{w}_i''(s) - \frac{\mu - n - 1}{s}\tilde{w}_i'(s) - \frac{g_i(s)}{s^2} = 0 \tag{4.3}$$

for $i = 1, 2, \dots, n$, where

$$g_i(s) = \int_{S^{n-1}} f'(\xi(s,\theta)) \sum_{j=1}^{n} \tilde{w}_j(s) Q_j Q_i + \int_{S^{n-1}} f'(\xi)\tilde{w}_2(s,\theta) Q_i,$$

and $\xi(s,\theta) = \rho v(s,\theta) + (1-\rho)\bar{v}$, $\rho \in (0,1)$. Let $t = -\ln s$, $z_i(t) = \tilde{w}_i(s)$, $\tilde{\xi}(t,\theta) = \xi(s,\theta)$ and $z_2(t,\theta) = \tilde{w}_2(s,\theta)$. Then $z_i(t)$ satisfies

$$z_i''(t) + (\mu - n) z_i'(t) - \tilde{g}_i(t) = 0, \tag{4.4}$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

where

$$\tilde{g}_i(t) = \int_{S^{n-1}} f'(\tilde{\xi}) \sum_{j=1}^{n} z_j(t) Q_j Q_i + \int_{S^{n-1}} f'(\tilde{\xi}) z_2(t,\theta) Q_i.$$

The two characteristic values of the equation

$$y''(t) + (\mu - n) y'(t) = 0$$

are $\lambda_1 = -(\mu - n)$ and $\lambda_2 = 0$. Note that $\mu - n \geqslant 0$. Arguments similar to those in the proof of lemma 3.2 imply that

$$\sum_{i=1}^{n} |z_i(t)| \leqslant C_7 + C_8 \int_{t_0}^{t} \left[\sum_{j=1}^{n} |z_j(s)| \, |F_j(s)| + \sum_{i=1}^{n} |G_i(s)| \right] \mathrm{d}s, \tag{4.5}$$

where C_8 is independent of t_0. Since $|f'(\tilde{\xi})| = O(\mathrm{e}^{-t})$ and $\sum_{i=1}^{n} |z_i(t)|$ is bounded (see theorem 3.7 and corollary 3.6), we obtain

$$Z(t) \leqslant C_9 + C_{10} \int_{t_0}^{t} \mathrm{e}^{-s} Z_2(s) \, \mathrm{d}s,$$

where $Z(t)$ and $Z_2(t)$ are as in the proof of lemma 3.2. On the other hand, we know that Z_2 satisfies

$$Z_2''(t) + (\mu - n) Z_2'(t) - (n + 1 - \mathrm{e}^{-t}) Z_2(t) + \mathrm{e}^{-t} Z(t) \geqslant 0. \tag{4.6}$$

By the argument in the proof of lemma 3.2 we have

$$Z_2(t) \leqslant M \mathrm{e}^{-t} \quad \text{for } t \geqslant t_0.$$

This implies that $Z(t) \leqslant M$ for $t \geqslant t_0$. Thus, $\tilde{W}(s) \leqslant M$ for $s \in (0, s_0)$, where $\tilde{W}(s) = W(s)/s$, $s_0 = \mathrm{e}^{-t_0}$.

By arguments similar to those in the proof of [19, theorem 6.1], we obtain our conclusion. This completes the proof of theorem 4.1. □

Let

$$\tilde{w}(s,\theta) = \frac{1}{s} w(s,\theta), \tag{4.7}$$

where w is given by (2.7). We view s as a parameter and show that $\tilde{w}$ tends to one of the first eigenfunctions or 0 uniformly in $C^{\tau}(S^{n-1})$ as $s \to 0$ for any $\tau \geqslant 0$. We also obtain an expansion of v in terms of $\bar{v}$ and higher-order terms of w. The following lemma can be obtained from theorem 4.1 and [19, lemma 7.1].

Lemma 4.2. *Let v be a solution of (2.8), let $\tilde{w}$ be given by (4.7) and $\mu - n \geqslant 0$. Then, for any non-negative integers τ and τ_1, there exists a constant $M = M(v, \tau, \tau_1) > 0$ such that*

$$|s^{\tau} D_{\theta}^{\tau_1} D_s^{\tau} \tilde{w}| \leqslant M, \quad y \in B_{s_0}(0), \quad y \neq 0. \tag{4.8}$$

Moreover, $\tilde{w}$ satisfies

$$\tilde{w}''(s,\theta) + \frac{\Delta_{\theta} \tilde{w}}{s^2} - \frac{\mu - n - 1}{s} \tilde{w}' + \frac{n-1}{s^2} \tilde{w} = \frac{f(v) - \overline{f(v)}}{s^3}, \tag{4.9}$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

where

$$|g(s)| = \left|\frac{f(v) - \overline{f(v)}}{s^3}\right| \leqslant Ms^{-1}. \tag{4.10}$$

Now we show the following theorem.

THEOREM 4.3. *Let $\tilde{w}$ be a solution of (4.9). Then, necessarily,*

$$\lim_{s\to 0} \tilde{w}(s,\theta) = V(\theta), \tag{4.11}$$

where V is 0 or one of the first eigenfunctions of $-\Delta$ on S^{n-1}, i.e.

$$\Delta_\theta V + (n-1)V = 0, \quad \bar{V} = 0. \tag{4.12}$$

Proof. Let $\tilde{w}(s,\theta) = \tilde{w}_1(s,\theta) + \tilde{w}_2(s,\theta)$ be as in the proof of theorem 4.1. It is easily seen from the proof of theorem 4.1 that $\tilde{w}_2(s,\theta) \to 0$ as $s \to 0$. (We know that $Z_2(t) \to 0$ as $t \to +\infty$.) On the other hand, we know from the proof of theorem 4.1 that

$$\tilde{w}_1(s,\theta) = \sum_{i=1}^{n} \tilde{w}_i(s)Q_i(\theta)$$

and $\tilde{w}_i(s)$ satisfies equation (4.3). Let $t = -\ln s$, $z_i(t) = \tilde{w}_i(s)$ and $\tilde{g}_i(t) = g_i(s)$. Then z_i satisfies equation (4.4). We easily see that $\tilde{g}_i(t) \leqslant M\mathrm{e}^{-t}$ and $z_i(t)$ is bounded for t sufficiently large. Then z_i''', z_i'' and z_i' remain also bounded when t is sufficiently large.

If $\mu - n = 0$, we easily obtain

$$z_i'(t) \to 0, \quad z_i''(t) \to 0 \quad \text{as } t \to +\infty. \tag{4.13}$$

If $\mu - n > 0$, it follows from (4.4) that

$$(\mu - n)(z_i'(t))^2 = \tilde{g}_i(t)z_i'(t) - (\tfrac{1}{2}(z_i'(t))^2)'. \tag{4.14}$$

This implies that

$$\int_t^\infty (z_i'(s))^2 \,\mathrm{d}s < \infty, \tag{4.15}$$

which implies that (4.13) still holds. Therefore, it follows easily from equation (4.4) that

$$z_i(t) \to z_0^i \quad \text{as } t \to \infty \tag{4.16}$$

for $i = 1, 2, \dots, n$. Where $z_0 = (z_0^1, z_0^2, \dots, z_0^n)$ is a point in $\mathbb{R}^n$, z_0 may be equal to $0 \in \mathbb{R}^n$. This also implies our conclusion. □

Combining theorems 4.1 and 4.3, we establish the following asymptotic expansion at the origin for solutions of (2.8).

THEOREM 4.4 (asymptotic expansion). *Let $\mu - n \geqslant 0$ and let v be a solution of (2.8). Then*

$$v(y) = \bar{v}(s) + s\tilde{w}(s,\theta), \tag{4.17}$$

where

$$\bar{v}(s) = O(s), \qquad \bar{v}'(s) = O(1).$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

Moreover, for any integer $\tau \geqslant 0$*, we have*

$$\tilde{w}(s,\theta) \to V(\theta) \quad \text{as } s \to 0 \tag{4.18}$$

uniformly in $C^\tau(S^{n-1})$*, where* V *is* 0 *or one of the first eigenfunction of* $(-\Delta)$ *on* S^{n-1}*, namely,*

$$\Delta_\theta V + (n-1)V = 0, \quad \bar{V} = 0. \tag{4.19}$$

5. Local Hölder-type estimates and asymptotic expansions for $-1 < \mu - n < 0$

In this section we study the local Hölder-type estimates and asymptotic expansions for the solutions v of (2.8) when $-1 < \mu - n < 0$. We will see that we cannot obtain local Lipschitz-type estimates in this case without extra conditions on v. Our main ideas in all the proofs in this section are similar to those in § 4. We first show the following theorem, which is similar to theorem 4.1.

THEOREM 5.1. *Let* $\tau \geqslant 0$ *be an integer, let* $-1 < \mu - n < 0$ *and and let* v *be a solution of (2.8). There then exist* $s_0 > 0$ *and a constant* $M = M(v,\tau) > 0$ *(independent of* s*) such that*

$$\max_{|y|=s} |D^\tau w(y)| \leqslant M s^{1+\mu-n-\tau} \quad \text{for } 0 < s < s_0, \tag{5.1}$$

where w *is given by (2.7).*

Proof. The proof is similar to that of theorem 4.1. Define

$$\tilde{w} = \frac{w}{s^{1+\mu-n}} \quad \text{and} \quad \tilde{w}(s,\theta) = \sum_{i=1}^{n} \tilde{w}_i(s) Q_i(\theta) + \tilde{w}_2(s,\theta), \tag{5.2}$$

as in the proof of theorem 4.1. We observe that $\tilde{w}(s)$ satisfies

$$\tilde{w}'' + \frac{\Delta_\theta \tilde{w}}{s^2} + \frac{1+\mu-n}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} = g(y), \tag{5.3}$$

where

$$|g(y)| = \left| \frac{\overline{f(v)} - f(v)}{s^{2+1+\mu-n}} \right|$$

and $\tilde{w}_i(s)$ satisfies

$$\tilde{w}_i''(s) + \frac{1+\mu-n}{s}\tilde{w}_i'(s) - \frac{g_i(s)}{s^2} = 0, \tag{5.4}$$

where

$$g_i(s) = \int_{S^{n-1}} f'(\xi) \left[\sum_{j=1}^{n} \tilde{w}_j(s) Q_j Q_i + \tilde{w}_2(s,\theta) Q_i \right] \mathrm{d}\theta. \tag{5.5}$$

Let $t = -\ln s$, $z_i(t) = \tilde{w}_i(s)$ and $\tilde{g}_i(t) = g_i(s)$. Then

$$z_i'' - (\mu - n)z_i' - \tilde{g}_i(t) = 0. \tag{5.6}$$

The two characteristic values of the equation

$$y''(t) - (\mu - n)y'(t) = 0 \tag{5.7}$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

are $\lambda_1 = \mu - n$ and $\lambda_2 = 0$. We know that $\lambda_1 < 0$ if $-1 < \mu - n < 0$. Therefore, exactly the same arguments as those in the proof of theorem 4.1 imply that

$$Z_2(t) \leqslant M\mathrm{e}^{-(1+\mu-n)t} \quad \text{for } t \geqslant t_0 \tag{5.8}$$

and

$$Z(t) \leqslant M \quad \text{for } t \geqslant t_0. \tag{5.9}$$

These also imply that

$$\tilde{W}(s) \leqslant M \quad \text{for } s \in (0, s_0), \tag{5.10}$$

where $\tilde{W}(s) = s^{-(1+\mu-n)}W(s)$. The rest of the proof identical to that of theorem 4.1. □

The following lemma, which is similar to lemma 4.2, can be obtained by theorem 5.1 and [19, lemma 7.1].

LEMMA 5.2. *Let v be a solution of (2.8) and let $\tilde{w}$ be given by (5.2). Then, for any non-negative integers τ and τ_1, there exists a constant $M = M(v, \tau, \tau_1) > 0$ such that*

$$|s^\tau D_\theta^{\tau_1} D_s^\tau \tilde{w}| \leqslant M, \quad y \in B_{s_0}(0), \quad y \neq 0. \tag{5.11}$$

Moreover, $\tilde{w}$ satisfies

$$\tilde{w}'' + \frac{\Delta_\theta \tilde{w}}{s^2} + \frac{1+\mu-n}{s}\tilde{w}' + \frac{n-1}{s^2}\tilde{w} = g(y), \tag{5.12}$$

where

$$|g(y)| = \left| \frac{\overline{f(v)} - f(v)}{s^{2+1+\mu-n}} \right| \leqslant M s^{\mu-n-1}. \tag{5.13}$$

Now we claim the following theorem.

THEOREM 5.3. *Let $\tilde{w}$ be a solution of (5.12). Then, necessarily,*

$$\lim_{s\to 0} \tilde{w}(s, \theta) = V(\theta), \tag{5.14}$$

where V is 0 or one of the first eigenfunctions of $-\Delta$ on S^{n-1} (with eigenvalue $(n-1)$), i.e.

$$\Delta_\theta V + (n-1)V = 0, \quad \bar{V} = 0. \tag{5.15}$$

Proof. This theorem can be obtained by the same arguments as those in the proof of theorem 4.3 or [19, theorem 7.1]. □

Combining theorems 5.1 and 5.3, we establish the following asymptotic expansion at the origin for solutions of (2.8).

THEOREM 5.4 (asymptotic expansion). *Let $-1 < \mu - n < 0$ and let v be a solution of (2.8). Then*

$$v(y) = \bar{v}(s) + s^{1+\mu-n}\tilde{w}(s, \theta), \tag{5.16}$$

where

$$\bar{v}(s) = O(s^{\hat{\sigma}}), \qquad \bar{v}'(s) = O(s^{\hat{\sigma}-1}).$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

Here $\hat{\sigma} = \min\{1 + \frac{1}{2}(\mu - n), 2(1 + \mu - n)\}$. *Moreover, for any integer* $\tau \geqslant 0$, *we have*

$$\tilde{w}(s, \theta) \to V(\theta) \quad \text{as } s \to 0 \tag{5.17}$$

uniformly in $C^\tau(S^{n-1})$, *where* V *is* 0 *or one of the first eigenfunctions of* $-\Delta$ *on* S^{n-1}, *namely,*

$$\Delta_\theta V + (n-1)V = 0, \quad \bar{V} = 0. \tag{5.18}$$

Remark 5.5. When $\mu - n = -1$, i.e. $n = 2$ and $\nu = 3$, we easily obtain the expansion of v of (2.8) by

$$v(y) = \bar{v}(s) + w(s, \theta), \tag{5.19}$$

where $w(s, \theta)$ is defined in (2.7) and $w(s, \theta) \to 0$ as $s \to 0$. Moreover, since $w(s, \theta)$ satisfies

$$w'' + \frac{\Delta_\theta w}{s^2} + \frac{n-1}{s^2} w - \frac{f(v) - \overline{f(v)}}{s^2} = 0, \tag{5.20}$$

it follows by the same arguments as those in the proof of [19, lemma 7.3] that

$$\lim_{s \to 0} s w'(s, \theta) = 0, \qquad \lim_{s \to 0} s^2 w''(s, \theta) = 0 \tag{5.21}$$

in $C^\tau(S^{n-1})$ uniformly for any integer $\tau \geqslant 0$.

6. An auxiliary lemma for $\mu - n \geqslant 0$

In this section we obtain an auxiliary lemma for the moving-plane procedure. The main idea is similar to that of [19, § 8].

Using the transform (2.4), we immediately obtain an asymptotic expansion for non-negative solutions of (I) at ∞ by combining theorem 4.4 and [19, lemma 8.1] under assumptions $\mu - n \geqslant 0$ and (2.3).

Theorem 6.1. *Let* $\mu - n \geqslant 0$ *and let* u *be a non-negative solution of (I). Suppose that the assumption (2.3) holds. Then we have the expansion*

$$u(x) = r^\alpha \left(\lambda + \xi(r) + \frac{\eta(r, \theta)}{r} \right), \tag{6.1}$$

where (r, θ) *is the spherical coordinates with* $r = |x|$. *Furthermore, the following properties are satisfied.*

(i) $\xi(r) = r^{-\alpha}\bar{u}(r) - \lambda$, *and there exist* $R_0 (= s_0^{-1}) > 0$ *and a constant* $M = M(u) > 0$ *such that*

$$|\xi(r)| \leqslant M r^{-1}, \quad |\xi'(r)| \leqslant M r^{-2} \quad \text{for } r > R_0. \tag{6.2}$$

(ii) *Let* τ *and* τ_1 *be two non-negative integers. There then exists a positive constant* $M = M(u, \tau, \tau_1)$ *such that*

$$|r^\tau D_\theta^{\tau_1} D_r^\tau \eta| \leqslant M, \quad r > R_0. \tag{6.3}$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

(iii) *Let τ be a non-negative integer. Then $\eta(r,\theta)$ tends to $V(\theta)$ uniformly in $C^\tau(S^{n-1})$ as $r\to\infty$, where*

$$V(\theta)=\theta\cdot x_0 \tag{6.4}$$

for some $x_0\in\mathbb{R}^n$ fixed and $\theta=x/r\in S^{n-1}$.

The theorem enables us to establish the precise limit property below (lemma 6.2) for non-negative solutions of (I), which we need in order to begin the moving-plane procedure.

We first introduce some notation.

For $\gamma\in\mathbb{R}$, let Σ_γ be the following hyperplane:

$$\Sigma_\gamma=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_1=\gamma\}.$$

For $x\in\mathbb{R}^n$, denote by x^γ the reflection point of x about Σ_γ, that is,

$$x^\gamma=(2\gamma-x_1,x_2,\dots,x_n).$$

As a corollary of the expansions (6.1)–(6.4), we have the following result.

LEMMA 6.2. *Let $\mu-n\geqslant 0$ and u be a non-negative solution of (I). Suppose that (2.3) holds. Then*

(i) *If $\gamma^j\in\mathbb{R}\to\gamma$ and $\{x^j\}\to\infty$, with $x_1^j<\gamma^j$, then*

$$\lim_{j\to\infty}\frac{u(x^j)-u(x^{j^\gamma})}{(\gamma^j-x_1^j)|x^j|^{\alpha-2}}=-2\alpha\lambda\gamma-2(x_0)_1, \tag{6.5}$$

where $(x_0)_1$ is the first component of x_0 in (6.4).

(ii) *Define*

$$\gamma_0=-\frac{(x_0)_1}{\alpha\lambda}. \tag{6.6}$$

There then exists a constant $M=M(u)>0$ such that

$$u_1(x)\geqslant 0 \quad \textit{if}\ \ x_1\geqslant\gamma_0+1\ \textit{and}\ |x|\geqslant M. \tag{6.7}$$

Proof. To prove (6.5), without loss of generality, we assume that

$$\lim_{j\to\infty}\frac{x^j}{|x^j|}=\bar\theta\in S^{n-1}.$$

For simplicity, we also assume that

$$\gamma^j\equiv\gamma,\quad j=1,2,\dots,$$

since the following arguments work equally well for the sequence $\{\gamma^j\}$.

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

Using the expansion (6.1), we have

$$\frac{u(x^j)-u(x^{j^\gamma})}{(\gamma-x_1^j)|x^j|^{\alpha-2}} = \frac{\lambda}{(\gamma-x_1^j)|x^j|^{\alpha-2}}(|x^j|^\alpha - |x^{j^\gamma}|^\alpha) + \frac{1}{(\gamma-x_1^j)|x^j|^{\alpha-2}}(\xi(|x^j|)|x^j|^\alpha - \xi(|\xi^{j^\gamma}|)|x^{j^\gamma}|^\alpha) + \frac{1}{(\gamma-x_1^j)|x^j|^{\alpha-2}}(|x^j|^{\alpha-1}\eta(|x^j|,\theta^j) - |x^{j^\gamma}|^{\alpha-1}\eta(|x^{j^\gamma}|,\theta^{j^\gamma})).$$

By the mean-value theorem, we have

$$|x^j|^\alpha - |x^{j^\gamma}|^\alpha = -\frac{4\alpha\gamma\beta_j^{\alpha-1}(\gamma-x_1^j)}{|x^j|+|x^{j^\gamma}|},$$

where β_j is a number between $|x^j|$ and $|x^{j^\gamma}|$. Therefore,

$$\frac{\lambda}{(\gamma-x_1^j)|x^j|^{\alpha-2}}(|x^j|^\alpha - |x^{j^\gamma}|^\alpha) = -\frac{4\lambda\alpha\gamma\beta_j^{\alpha-1}}{|x^j|^{\alpha-2}(|x^j|+|x^{j^\gamma}|)} = -4\alpha\lambda\gamma(\tfrac{1}{2}+o(1)) \to -2\alpha\lambda\gamma \quad \text{as } j\to\infty,$$

since $|x^j|/|x^{j^\gamma}| \to 1$. Similarly, for some β_j between $|x^j|$ and $|x^{j^\gamma}|$ we have

$$\xi(|x^j|)|x^j|^\alpha - \xi(|\xi^{j^\gamma}|)|x^{j^\gamma}|^\alpha = [\alpha\beta_j^{\alpha-1}\xi(|x^j|) + |x^{j^\gamma}|^\alpha\xi'(\beta_j)]\frac{-4\gamma(\gamma-x_1^j)}{|x^j|+|x^{j^\gamma}|},$$

and, in turn,

$$\frac{1}{|x^j|^{\alpha-2}(\gamma-x_1^j)}(\xi(|x^j|)|x^j|^\alpha - \xi(|x^{j^\gamma}|)|x^{j^\gamma}|^\alpha) = O(|x^j|^{\alpha-2})\frac{4\gamma}{|x^j|^{\alpha-2}(|x^j|+|x^{j^\gamma}|)} = O(|x^j|^{-1}) \to 0 \quad \text{as } j\to\infty.$$

Here we have used the estimate (6.2). We write

$$\frac{1}{|x^j|^{\alpha-2}(\gamma-x_1^j)}(|x^j|^{\alpha-1}\eta(|x^j|,\theta^j) - |x^{j^\gamma}|^{\alpha-1}\eta(|x^{j^\gamma}|,\theta^{j^\gamma})) = \frac{\eta(|x^{j^\gamma}|,\theta^{j^\gamma})}{|x^j|^{\alpha-2}(\gamma-x_1^j)}(|x^j|^{\alpha-1} - |x^{j^\gamma}|^{\alpha-1}) + \frac{|x^j|}{\gamma-x_1^j}(\eta(|x^j|,\theta^{j^\gamma}) - \eta(|x^{j^\gamma}|,\theta^{j^\gamma})) + \frac{|x^j|}{\gamma-x_1^j}(\eta(|x^j|,\theta^j) - \eta(|x^j|,\theta^{j^\gamma})).$$

As before, by (6.3) we bound

$$\frac{\eta(|x^{j^\gamma}|,\theta^{j^\gamma})}{|x^j|^{\alpha-2}(\gamma-x_1^j)}(|x^j|^{\alpha-1} - |x^{j^\gamma}|^{\alpha-1}) = O(|x^j|^{-1}) \to 0 \quad \text{as } j\to\infty.$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

We can obtain the estimates

$$\frac{|x^j|}{\gamma - x_1^j}[\eta(|x^j|,\theta^{j^\gamma}) - \eta(|x^{j^\gamma}|,\theta^{j^\gamma})] = O(|x^j|^{-1}) \to 0 \quad \text{as } j \to \infty,$$
$$\frac{|x^j|}{\gamma - x_1^j}[\eta(|x^j|,\theta^{j}) - \eta(|x^{j}|,\theta^{j^\gamma})] \to -2(x_0)_1$$

by the same idea as that in [19]. Thus,

$$\frac{1}{|x^j|^{\alpha-2}(\gamma - x_1^j)}(|x^j|^{\alpha-1}\eta(|x^j|,\theta^j) - |x^{j^\gamma}|^{\alpha-1}\eta(|x^{j^\gamma}|,\theta^{j^\gamma})) \to -2(x_0)_1 \quad \text{as } j \to \infty. \tag{6.8}$$

These imply that (6.5) holds.

To prove (6.7), we may use similar arguments to those in the proof of (6.5). Indeed, suppose that (6.7) is false. There then exists a sequence $\{x^j\} \to \infty$ such that

$$u_1(x^j) < 0, \quad x_1^j \geqslant \gamma_0 + 1, \quad j = 1, 2, \dots.$$

It follows that there exists a sequence of *bounded positive* numbers $\{d_j\}$ such that

$$u(x^j) > u(x_{d_j}), \quad x_{d_j} = x^j + (2d_j, 0, \dots, 0), \quad j = 1, 2, \dots.$$

Define

$$\gamma^j = x_1^j + d_j > x_1^j.$$

By assumption, we have

$$\frac{1}{(\gamma^j - x_1^j)|x^j|^{\alpha-2}}[u(x^j) - u(x^{j^\gamma})] > 0, \quad j = 1, 2, \dots. \tag{6.9}$$

There are two possibilities, that is,

$$\lim_{j\to\infty}\inf \gamma^j < \infty \quad \text{or} \quad \lim_{j\to\infty} \gamma^j = \infty.$$

If the first case occurs, we choose a convergent subsequence of $\{\gamma^j\}$ (with limit $\gamma \geqslant \gamma_0 + 1$, still denoted by $\{\gamma^j\}$) and, applying (6.5), (6.6), we obtain

$$\frac{1}{(\gamma^j - x_1^j)|x^j|^{\alpha-2}}[u(x^j) - u(x^{j^\gamma})] \to -2\alpha\lambda\gamma - 2(x_0)_1 \leqslant -2\alpha\lambda < 0.$$

This contradicts (6.9). We can derive a contradiction for the second case similarly. The proof is a slight variant of the proof of [19, lemma 8.2]. Thus, neither the first or the second case can occur and (6.7) is shown. □

7. Necessary conditions

In this section we will prove that, if u is a non-negative radially symmetric solution of (I), then the limits (1.12) and (1.13), (1.14) hold for $\mu - n \geqslant 0$ and $-2 < \mu - n < 0$, respectively. Furthermore, we classify all radially symmetric solutions and prove theorems 1.3 and 1.4.

THEOREM 7.1. *Let $n \geqslant 2$, let $\nu > 0$ and let u be a non-negative solution of (I). If u is radially symmetric about some point $x_0 \in \mathbb{R}^n$, then*

$$\lim_{|x|\to\infty} |x|^{-\alpha} u(x) = \lambda, \tag{7.1}$$

where α and λ are as in (1.4). If $-2 < \mu - n < 0$, then

$$\lim_{|x|\to\infty} |x|^{1+(\mu-n)/2}(|x|^{-\alpha}u(x) - \lambda) = 0. \tag{7.2}$$

Proof. Without loss of generality, we assume that $x_0 = 0$. First define a new independent variable $t = -\ln|x|$, $r = |x|$, and set

$$v(-\ln(|x|)) \equiv |x|^{-2/(\nu+1)}u(x). \tag{7.3}$$

Then the new function $v(t)$ satisfies

$$v''(t) - (n + 2\alpha - 2)v'(t) + \alpha(n + \alpha - 2)v(t) = v^{-\nu}. \tag{7.4}$$

Now look at the phase-plane portrait for this equation in the (v, v_t)-plane. The only equilibrium point is $(v^*, 0)$ with $(v^*)^{-(\nu+1)} = \alpha(n + \alpha - 2)$, which is an unstable equilibrium. This implies that $v(t) \to v^*$ as $t \to -\infty$ and thus

$$\lim_{|x|\to\infty} |x|^{-\alpha}u(x) = \lambda.$$

To prove (7.2), we define

$$v(s) = |x|^{-\alpha}u(x) - \lambda, \quad s = \frac{1}{|x|}, \quad \tilde{v}(s) = s^{-\sigma_0}v(s),$$

where $\sigma_0 = 1 + \frac{1}{2}(\mu - n)$ with $-2 < \mu - n < 0$. Then, by (3.26), $\tilde{v}(s)$ satisfies

$$\tilde{v}'' + \frac{1}{s}\tilde{v}' - \frac{\frac{1}{4}((\mu-n)^2 - 4n + 4)}{s^2}\tilde{v} - \frac{f(v)}{s^{2+\sigma_0}} = 0. \tag{7.5}$$

Since $v(s) \to 0$ as $s \to 0$ (see (7.1)), by arguments similar to those in the proofs of lemmas 3.3 and 3.4, we have $\tilde{v}(s) \leqslant M$ for s sufficiently small. Indeed, if we use the notation in the proof of lemmas 3.3 and 3.4, we may claim that, for any $0 < \sigma < \sigma_0$, $v(s) = O(s^\sigma)$. In fact, noting that $g(s)$ in the proof of lemma 3.3 is 0 here and $2\sigma < \mu - n + 2$ if $0 < \sigma < \sigma_0$, this claim can be obtained from a variant of the proof of lemma 3.3 (since $\mu_1(\sigma_0) > 0$ for $-2 < \mu - n < 0$). This implies that the $g_1(t)$ in the proof of lemma 3.4 satisfies

$$g_1(t) = O(|v_1|^2) = O(\mathrm{e}^{-2\sigma t})$$

here. Choose $0 < \sigma < \sigma_0$ and $2\sigma > \sigma_0$. The proof of lemma 3.4 shows that

$$|v_1(t)| \leqslant M\mathrm{e}^{-\sigma_0 t}\left(1 + \int_{t_0}^{t} |g_1(s)|\mathrm{e}^{\sigma_0 s}\right) \mathrm{d}s \leqslant M\mathrm{e}^{-\sigma_0 t}.$$

This implies that $\tilde{v}(s) \leqslant M$. Let $t = -\ln s$, $\hat{v}(t) = \tilde{v}(s)$. Then $\hat{v}(t)$ satisfies

$$\hat{v}'' - \tfrac{1}{4}((\mu-n)^2 - 4n + 4)\hat{v} + O(\mathrm{e}^{-\sigma_0 t})\hat{v} = 0 \tag{7.6}$$

and $\hat{v}$ is bounded for sufficiently large t. Arguments identical to those in the proof of theorem 4.3 imply that

$$\lim_{t\to\infty} \hat{v}'(t) = 0 = \lim_{t\to\infty} \hat{v}''(t).$$

This implies that

$$\lim_{t\to\infty} \hat{v}(t) = 0$$

(note that $(\nu - n)^2 - 4n + 4 < 0$ for $-2 < \mu - n < 0$, $\nu > 0$ and $n \geqslant 2$). This completes the proof. □

Proof of theorem 1.3. This follows from results in [5].

Suppose that $u(x) \geqslant C|x|^{2/(\nu+1)}$ for $|x|$ large. We now consider the function v defined in (7.3) and satisfying (7.4). As $t \to -\infty$, $v(t,\theta) \geqslant C$ and $v^{-\nu} \leqslant C$. Hence, by the Harnack inequality, $v(t,\theta) \leqslant C$ as $t \to -\infty$. By the results in [15], $v(t,\theta) \to v(\theta)$, where $v(\theta)$ satisfies

$$v_{\theta\theta} + \frac{4}{(\nu+1)^2} v - \frac{1}{v^\nu} = 0, \quad v \text{ is } 2\pi\text{-periodic}.$$

By [5, theorem 2.1], $v(\theta) \equiv \text{const}$. This proves theorem 1.3. □

Proof of theorem 1.4. Let $u = u(r)$ be a radially symmetric solution of (I). If $u(0) = 0$, then we have

$$(r^{n-1} u_r)_r = \frac{r^{n-1}}{u^\nu},$$

which implies that $u_r \geqslant 0$ and $r^{n-1}u_r \to 0$ as $r \to 0$. Hence,

$$r^{n-1} u_r = \int_0^r \frac{s^{n-1}}{u^\nu(s)}\, \mathrm{d}s \geqslant \frac{1}{n u^\nu(r)} r^n,$$

which implies that

$$u(r) \geqslant C r^\alpha \quad \text{for all } r \geqslant 0. \tag{7.7}$$

We now consider the function v defined in (7.3) and satisfying (7.4). As we know, $v(t) \to v^*$ as $t \to -\infty$. Next we consider the case when $t \to +\infty$. From (7.7), we see that $v(t) \geqslant C$ for all t. Since $\mathrm{e}^{-2/(\nu+1)t} v(t) \to 0$ as $t \to +\infty$ and $v^{-\nu} \leqslant C$, a simple ordinary differential equation theory shows that $v(t)$ is bounded as $t \to +\infty$ and $v(t) \to v^*$ as $t \to +\infty$ (since v^* is the only positive equilibrium point).

Now multiplying the equation for $v(t)$ by $v'(t)$ and integrating over $(-\infty, +\infty)$, we see that

$$-(n + 2\alpha - 2) \int_{-\infty}^{+\infty} (v'(t))^2\, \mathrm{d}t = 0,$$

which implies that $v(t) \equiv v^*$. Thus, $u = u_0(r) = (\frac{1}{2}(\nu+1))^{2/(\nu+1)} r^{2/(\nu+1)}$.

If $u(0) = \eta > 0$, by theorems 1.1 and 1.2, we have

$$\lim_{|x|\to+\infty} |x|^{-\alpha} u(x) = \lambda.$$

Then, by scaling invariance, all solutions of (I) form a one-parameter family of solutions. □

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

8. The moving-plane method: proof of the main results

In this section we use the moving-plane method to give the proofs of theorems 1.1 and 1.2.

The following special form of maximum principles is useful.

LEMMA 8.1. *Let $\gamma \in \mathbb{R}^1$ and let u be a positive solution of (I). Suppose that*

$$u(x) \leqslant u(x^\gamma), \quad u(x) \not\equiv u(x^\gamma) \quad \text{if } x_1 < \gamma.$$

Then

$$u(x) < u(x^\gamma) \quad \text{if } x_1 < \gamma \tag{8.1}$$

and

$$u_1 > 0 \quad \text{on } x_1 = \gamma, \tag{8.2}$$

where x^γ is the reflection point of x with respect to Σ_γ.

Proof. Consider the function

$$v(x) = u(x) - u(x^\gamma) \leqslant 0, \quad x_1 < \gamma.$$

Then v satisfies

$$\Delta v = -\nu h(x) v(x), \quad x_1 < \gamma,$$

where

$$h(x) = \int_0^1 \xi_\rho^{-(\nu+1)} \, \mathrm{d}\rho$$

and $\xi_\rho = \rho u(x) + (1-\rho) u(x^\gamma)$. Since $u(x^\gamma) > 0$ and $u(x) > 0$ for $x_1 < \gamma$, we have $h(x) > 0$ for $x_1 < \gamma$. Hence, by the strong maximum principle, v assumes non-negative maximal values only on the boundary $x_1 = \gamma$, which implies (8.1), while (8.2) is a direct consequence of the Hopf boundary lemma, since $v = 0$ on $x_1 = \gamma$. □

Proof of theorem 1.1. We need only to prove sufficiency. We first claim that there exists $\gamma' > 0$ such that

$$u(x) < u(x^\gamma) \quad \text{if } x_1 < \gamma \text{ and } \gamma \geqslant \gamma'. \tag{8.3}$$

Suppose for contradiction that (8.3) is not true. There then exist two sequences, $\{\gamma^i\} \to \infty$ and $\{x^i\}$, with $x_1^i < \gamma^i$ such that

$$u(x^i) \geqslant u(y^i), \quad y^i = x^{i^{\gamma^i}}, \quad i = 1, 2, \dots. \tag{8.4}$$

Obviously, $y^i \to \infty$, so $u(y^i) \to \infty$. In turn $|x^i| \to \infty$. By lemma 6.2, we must have

$$x_1^i \leqslant \gamma_0 + 1 \quad \text{for } i \text{ large}.$$

It follows that, for any $\gamma_1 > \gamma_0 + 1$,

$$u(x^i) \geqslant u(y^i) \geqslant u(x^{i^{\gamma_1}}) \quad \text{for } i \text{ large},$$

since $x_1^{i^{\gamma^i}} \gg x_1^{i^{\gamma_1}}$ for i large and $u(x) \to \infty$ as $|x| \to \infty$. On the other hand, by lemma 6.2 again, we conclude that

$$0 \leqslant \frac{1}{(\gamma_1 - x_1^i)|x^i|^{\alpha-2}}[u(x^i) - u(x^{i^{\gamma_1}})] \to -2\alpha\gamma_1\lambda - 2(x_0)_1 < 0.$$

This is a contradiction and (8.3) follows.

Now let Γ be a subset of $\mathbb{R}$ defined by

$$\Gamma = \{\gamma \in (\gamma_0, \infty) : (8.3) \text{ holds}\}.$$

We shall prove that

$$\Gamma = (\gamma_0, \infty). \tag{8.5}$$

We first show that Γ is open. Suppose for contradiction that, for some $\gamma \in \Gamma$, there exist two sequences $\{\gamma^i\} \to \gamma$ and $\{x^i\}$ with $x_1^i < \gamma^i$ such that (8.4) holds. Obviously, there is a subsequence of $\{x^i\}$ tending to either ∞ or $\hat{x} \in \mathbb{R}^n$ as $i \to \infty$. If the first case occurs, we simply use lemma 6.2 and derive a contradiction, since $\gamma > \gamma_0$. If the second case occurs, we can infer, from the definition of γ, that

$$\hat{x}_1 = \gamma.$$

It follows that

$$u_1(\hat{x}) \leqslant 0, \quad \hat{x}_1 = \gamma.$$

This simply cannot happen because of (8.2), that is, Γ is open.

Set

$$\tilde{\gamma} = \inf\{\gamma \in (\gamma_0, \infty) : (\gamma, \infty) \subset \Gamma\}.$$

We want to show that

$$\tilde{\gamma} = \gamma_0. \tag{8.6}$$

Suppose for contradiction that this is not true, i.e. $\tilde{\gamma} > \gamma_0$. By continuity, we have

$$u(x) \leqslant u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}.$$

By lemma 8.1, we see that either

$$u(x) \equiv u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}$$

or

$$u(x) < u(x^{\tilde{\gamma}}) \quad \text{for } x_1 < \tilde{\gamma}, \text{ i.e. } \tilde{\gamma} \in \Gamma.$$

The latter cannot occur because $(\tilde{\gamma}, \infty)$ is maximal and Γ is open. The former cannot occur either because it contradicts lemma 6.2 since $\tilde{\gamma} > \gamma_0$. Thus, $\tilde{\gamma} = \gamma_0$ and (8.5) is proved.

By continuity again, we have

$$u(x) \leqslant u(x^{\gamma_0}) \quad \text{for } x_1 < \gamma_0.$$

Reversing the x_1-axis, we conclude that

$$u(x) \leqslant u(x^{\gamma_0}) \quad \text{for } x_1 > \gamma_0.$$

That is, u is symmetric about the plane $x_1 = \gamma_0$. Since this argument applies for any direction, we finally obtain the radial symmetry of u about some point $x_0 \in \mathbb{R}^n$. The proof of theorem 1.1 is thus complete. □

Proof of theorem 1.2. It is enough to prove the sufficiency. First, we note that the asymptotic expansion obtained in theorem 5.4 is not good enough for us to use the moving-plane method. This implies that the assumption (1.6) is not sufficient to guarantee the symmetry of u; we need stronger assumptions.

The following lemma implies our conclusion.

LEMMA 8.2. *Let $-2 < \mu - n < 0$, $v(s,\theta)$, $w(s,\theta)$ be defined as in (2.4) and (2.7). Assume that $v(s,\theta)$ satisfies*

$$s^{-\sigma_0} v(s,\theta) \to 0 \quad \text{as } s \to 0, \tag{8.7}$$

where $\sigma_0 = 1 + \frac{1}{2}(\mu - n)$. Then v has a local Lipschitz-type estimate and an asymptotic expansion similar to (4.17).

Proof. Let $\tilde{w}(s,\theta) = s^{-\sigma_0} w(s,\theta)$. We see that $\tilde{w}$ satisfies

$$\tilde{w}'' + \frac{1}{s}\tilde{w}' + \frac{\Delta_\theta \tilde{w}}{s^2} - \frac{\frac{1}{4}((\mu-n)^2 - 4n + 4)}{s^2}\tilde{w} - \frac{f(v) - \overline{f(v)}}{s^{2+\sigma_0}} = 0. \tag{8.8}$$

Define

$$\tilde{W}(s) = \left(\int_{S^{n-1}} \tilde{w}^2(s,\theta)\,\mathrm{d}\theta\right)^{1/2}.$$

Arguments similar to those in the proof of [19, theorem 3.1] imply that $\tilde{W}(s)$ satisfies the inequality

$$\tilde{W}'' + \frac{1}{s}\tilde{W} - \frac{(\mu-n)^2/4 - F(s)}{s^2}\tilde{W} \geqslant 0, \tag{8.9}$$

and $\tilde{W}(s) \to 0$ as $s \to 0$, where

$$F(s) = \max_{\theta \in S^{n-1}} |f'(v(s,\theta))|.$$

Using the comparison principle as in [19], we obtain the following fundamental inequality similar to that in theorem 3.1 or [19, theorem 3.2]:

$$\tilde{W}(s) \leqslant M s^{\tilde{\delta}}, \quad 0 < s < 1, \tag{8.10}$$

for any $0 < \tilde{\delta} < |\mu - n|/2 - F(s)$. (We know that $F(s) \to 0$ as $s \to 0$.) This implies that, for any $0 < \tilde{\delta} < \frac{1}{2}|\mu - n|$, there exist $s_0 = s_0(\tilde{\delta}) > 0$ sufficiently small and a positive constant $M = M(\tilde{\delta}, v) > 0$ such that

$$\tilde{W}(s) \leqslant M s^{\tilde{\delta}}, \quad 0 < s < s_0. \tag{8.11}$$

Let $W(s)$ be the same as that in theorem 3.1. Note that $0 < -\frac{1}{2}(\mu - n) < 1$ for $n = 3$, $\nu > 3$ and $n = 2$, $\nu > 1$. We can then obtain the same conclusion as in [19, theorem 3.2] (note that $W(s) = s^{\sigma_0}\tilde{W}(s)$). That is, for any

$$0 < \max\{-\tfrac{1}{2}(\mu - n), (1 + \mu - n)\} < \delta < 1,$$

https://doi.org/10.1017/S0308210505001083 Published online by Cambridge University Press

there exist $\hat{s}_0 = \hat{s}_0(\delta) > 0$ and a positive constant $M = M(\delta, v)$ such that

$$W(s) \leqslant M s^{\delta}, \quad 0 < s < \hat{s}_0.$$

We can also obtain the same conclusion as in [19, lemma 6.1]. Indeed, define $\hat{W}(s) = W(s)/s$. We infer by a similar argument to that in the proof of [19, lemma 6.1] that there exists a constant $M = M(v) > 0$ such that $\hat{W}$ satisfies

$$\hat{W}'' + \frac{1-(\mu-n)}{s}\hat{W}' + M s^{\sigma_0 - 2}\hat{W} \geqslant 0, \quad 0 < s < \hat{s}_0. \tag{8.12}$$

We also have

$$\hat{W} \leqslant M s^{\delta-1}, \qquad |\hat{W}'| \leqslant M s^{\delta-2}. \tag{8.13}$$

For any $T > 0$, $T \leqslant \hat{s}_0$, multiply (8.12) by $s^{1-(\mu-n)}$ and integrate from $T > t > 0$ to T to obtain

$$s^{1-(\mu-n)}\hat{W}'|_t^T + M\int_t^T \hat{W} s^{\sigma_0-(\mu-n)-1}\,\mathrm{d}s \geqslant 0. \tag{8.14}$$

By (8.13), we see that

$$\lim_{t\to 0}|t^{1-(\mu-n)}\hat{W}'(t)| = 0, \qquad \int_t^T \hat{W} s^{\sigma_0-(\mu-n)-1}\,\mathrm{d}s \leqslant M T^{\delta+\sigma_0-(\mu-n)-1}, \tag{8.15}$$

since $\delta > 1 + \mu - n$ and $\sigma_0 - (\mu - n) - 1 = -\frac{1}{2}(\mu - n) > 0$. Thus, letting $t \to 0$ in (8.14) yields

$$\hat{W}'(T) + M T^{\delta+\sigma_0-2} \geqslant 0.$$

For any $s \leqslant T$, we integrate from s to T to obtain

$$\hat{W}(T) - \hat{W}(s) + M\int_s^T t^{\delta+\sigma_0-2} \geqslant 0,$$

that is,

$$\hat{W}(s) \leqslant \hat{W}(T) + M T^{\delta+\sigma_0-1},$$

since $\delta + \sigma_0 - 1 > 0$ (note that $\delta > -\frac{1}{2}(\mu - n)$). We now obtain

$$W(s) \leqslant M s, \quad 0 < s < \hat{s}_0.$$

This implies that a conclusion similar to that of theorem 4.1 holds for this case.

By the same procedure as in the proofs of lemma 4.2 and theorems 4.3 and 4.4, we obtain the local Lipschitz estimate for v and the asymptotic expansion of v similar to that in (4.17). □

From lemma 8.2 we obtain conclusions similar to those in theorem 6.1 for u. The proof of sufficiency is then obtained by the moving-plane method as in §§ 6 and 8. This completes the proof of theorem 1.2. □

Acknowledgments

This paper was written while Z.M.G. was visiting the Department of Mathematics, Chinese University of Hong Kong. He thanks the department for its hospitality. His research is supported by NSFC Grant no. 10571022. The research of J.W. is partly supported by Earmarked Grants from RGC of Hong Kong. We thank the referee for many helpful suggestions.

References

1 J. Ai, K. S. Chou and J. Wei. Self-similar solutions for the anisotropic affine curvature shortening problem. *Calc. Var. PDEs* **13** (2001), 311–337.

2 A. L. Bertozzi and M. C. Pugh. Long-wave instabilites and saturation in thin film equations. *Commun. Pure Appl. Math.* **51** (1998), 625–661.

3 A. L. Bertozzi and M. C. Pugh. Finite-time blow-up of solutions of some long-wave unstable thin film equations. *Indiana Univ. Math. J.* **49** (2000), 1323–1366.

4 J. P. Burelbach, S. G. Bankoff and S. H. Davis. Nonlinear stability of evaporating/condensing liquid films. *J. Fluid Mech.* **195** (1988), 463–494.

5 K.-S. Chou and J. Wei. Classification of isolated point ruptures in $\mathbb{R}^2$. Preprint.

6 B. Gidas, W. Ni and L. Nirenberg. Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **68** (1979), 209–243.

7 B. Gidas, W. Ni and L. Nirenberg. Symmetry properties of positive solutions of nonlinear equations in $\mathbb{R}^n$. In *Mathematical Analysis*, Advances in Mathematics Supplementary Studies Series A, vol. 7, pp. 369–402 (Academic, 1981).

8 Z. M. Guo. On the symmetry of positive solutions of the Lane–Emden equation with supercritical exponent. *Adv. Diff. Eqns* **7** (2002), 641–666.

9 Z. M. Guo and J. Wei. Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity. *Manuscr. Math.* **120** (2006), 193–209.

10 C.-C. Hwang, C.-K. Lin and W.-Y. Uen. A nonlinear three-dimensional rupture theory of thin liquid films. *J. Colloid Interface Sci.* **190** (1997), 250–252.

11 R. S. Laugesen and M. C. Pugh. Properties of steady states for thin film equations. *Eur. J. Appl. Math.* **11** (2000), 293–351.

12 R. S. Laugesen and M. C. Pugh. Finite-time blow-up of solutions of some long-wave unstable thin film equations. *Indiana Univ. Math. J.* **49** (2000), 1323–1366.

13 R. S. Laugesen and M. C. Pugh. Linear stability of steady states for thin film and Cahn–Hilliard type equations. *Arch. Ration. Mech. Analysis* **154** (2000), 3–51.

14 O. Rey. The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent. *J. Funct. Analysis* **89** (1990), 1–52.

15 L. Simon. Asymptotics for a class of nonlinear evolution equations with applications to geometric problems. *Ann. Math.* **118** (1983), 525–571.

16 T. P. Witelski and A. J. Bernoff. Dynamics of three-dimensional thin film rupture. *Physica* D **147** (2000), 155–176.

17 S. G. Yiantsios and R. H. Davis. Close approach and deformation of two viscous drops due to gravity and van der Waals forces. *J. Colloid Interface Sci.* **144** (1991), 412–433.

18 S. G. Yiantsios and B. G. Higgins. Rupture of thin films: nonlinear stability analysis. *J. Colloid Interface Sci.* **147** (1991), 341–350.

19 H. Zou. Symmetry of positive solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^n$. *J. Diff. Eqns* **120** (1995), 46–88.

(*Issued 12 October 2007*)