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HAMEL SPACES AND DISTAL EXPANSIONS

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Abstract. In this note, we construct a distal expansion for the structure $(\mathbb{R};+,<,H)$, where $H \subseteq \mathbb{R}$ is a dense $\mathbb{Q}$-vector space basis of $\mathbb{R}$ (a so-called *Hamel basis*). Our construction is also an expansion of the dense pair $(\mathbb{R};+,<,\mathbb{Q})$ and has full quantifier elimination in a natural language.

§1. Introduction. Distal theories were introduced by Simon in [11] as a way to isolate those NIP theories which are purely unstable. For example, all o-minimal theories and all P-minimal theories are distal, whereas the theory of algebraically closed valued fields is non-distal because of the presence of a stable algebraically closed residue field. Distality has useful combinatorial consequences. For instance, in [4], Chernikov and Starchenko show that distal theories enjoy a version of the *strong Erdös-Hajnal Property*. These combinatorial consequences also apply to theories which have a distal expansion, i.e., to reducts of a distal theory. This article shows that a certain class of non-distal theories have a distal expansion. In this introduction, we describe the most important case of this construction.

In [9], Hieronymi and Nell showed that two particular structures are not distal: $(\mathbb{R};+,<,\mathbb{Q})$ and $(\mathbb{R};+,<,H)$, where $H \subseteq \mathbb{R}$ is a dense $\mathbb{Q}$-vector space basis of $\mathbb{R}$. These findings were initially unexpected, as these structures are closely related to their common o-minimal reduct $(\mathbb{R};+,<)$. Simon then asked whether these structures at least have some distal expansion. In [10], Nell constructed a distal expansion of $(\mathbb{R};+,<,\mathbb{Q})$, essentially by equipping the quotient $\mathbb{R}/\mathbb{Q}$ with a linear order. For the structure $(\mathbb{R};+,<,H)$, a similar trick could not be employed since this structure has elimination of imaginaries [6]. In this article, we construct a distal expansion of $(\mathbb{R};+,<,H)$. We now describe our construction, as it applies to $(\mathbb{R};+,<,H)$, in greater detail.

We expand $(\mathbb{R};+,<,H)$ to a structure

$$\big(\mathbb{R}\cup\{\infty\};+,<,H,v,<_1,\infty\big)$$

with three new primitives: a unary function v, a binary relation $<_1$, and a constant symbol ∞ which names a new element we are adding to the underlying universe.

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First, we define $v : \mathbb{R}\setminus\{0\} \to H \subseteq \mathbb{R}$ as follows. Let $\alpha \in \mathbb{R}\setminus\{0\}$ be nonzero. As H is a basis of $\mathbb{R}$ as a vector space over $\mathbb{Q}$, there are $n \geq 1$, basis vectors $h_1, \ldots, h_n \in H$ and scalars $q_1, \ldots, q_n \in \mathbb{Q}^\times$ such that $\alpha = q_1 h_1 + \cdots + q_n h_n$ and $h_1 < \cdots < h_n$. The data $(n, h_1, \ldots, h_n, q_1, \ldots, q_n)$ is uniquely determined by the requirement that $h_1 < \cdots < h_n$. Thus we define $v(\alpha) := h_1$, and this will be well defined.

We define the binary relation $<_1$ as the unique ordering which makes $(\mathbb{R}; +)$ into an ordered group such that

$$0 <_1 \alpha \;:\Longleftrightarrow\; 0 < q_1, \quad \text{for every } \alpha = q_1 h_1 + \cdots + q_n h_n \neq 0 \text{ as above.}$$

Finally, we introduce a new element $\infty \notin \mathbb{R}$ as a default value:

$$v(0) = v(\infty) = \alpha + \infty = \infty + \alpha = \infty + \infty := \infty, \quad \alpha < \infty, \quad \text{and} \quad \alpha <_1 \infty$$

for every $\alpha \in \mathbb{R}$. We introduce this element ∞ mainly for aesthetic reasons. Indeed, the function $v : \mathbb{R} \to H \cup \{\infty\}$ is in fact a convex valuation on the ordered abelian group $(\mathbb{R}; +, <_1)$ with value set $(H \cup \{\infty\}, <)$.

In certain areas of mathematics where infinite-dimensional vector spaces are commonplace (e.g., Banach space theory) a *basis* in the usual linear algebra sense where only finite sums are allowed is often called a *Hamel basis* in order to distinguish it from a *Schauder basis* which allows for infinite sums (e.g., see [8, Chapter 4]). Accordingly, since H above is a Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, we call the valuation v constructed above a *Hamel valuation*, as its image is H. Likewise, we call the resulting structure a *Hamel space*. In this article, we consider Hamel spaces over an arbitrary ordered field C, not just $\mathbb{Q}$. In a natural language $\mathcal{L}_{\text{Ham}}$, we formulate a certain theory T_{Ham} of these Hamel spaces. The main result of the article is

THEOREM (5.1). *T_{Ham} is distal.*

In Section 2 we recall the definition of distality for a theory T, as well as for a partitioned formula $\varphi(x; y)$. We also provide the statement of Distal Criterion 2.6, a criterion we later use to prove that the theory T_{Ham} is distal.

In Section 3 we introduce *I-ordered C-vector spaces*. These are vector spaces over an ordered field C, equipped with a family of compatible linear orderings $<_i$, indexed by $i \in I$. We construct a complete theory $T_{C,I}$ of these spaces, and show that it is distal. The arguments in this section are routine; however it establishes the foundation language and theory that we will expand when constructing T_{Ham}.

In Section 4 we introduce *Hamel valuations* and *Hamel spaces over C*. We give a language $\mathcal{L}_{\text{Ham}}$ of Hamel spaces over C, and prove that a certain $\mathcal{L}_{Ham}$-theory T_{Ham} admits quantifier elimination and is complete. This section is the main technical part of the article. In Section 5 we prove the main result of the article. This involves various lemmas on how indiscernible sequences behave in Hamel spaces and applying Distal Criterion 2.6. We also point out several consequences of distality which are also of model-theoretic interest.

In Section 6 we show in what sense models of T_{Ham} are expansions of an o-minimal structure with a dense independent set (like $(\mathbb{R}; +, <, H)$). As a bonus, our models of T_{Ham} are also expansions of dense pairs (like $(\mathbb{R}; +, <, \mathbb{Q})$), and we show how this works as well. We also point out some natural follow-up questions. Finally, in Section 7 we show that T_{Ham} is not strongly dependent, in contrast to the theories of dense independent sets or dense pairs.

1.1. Conventions. Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \dots\}$.

All orderings are total. Let S be a set and let $<$ be an ordering on S. We say that a subset $P \subseteq S$ is *downward closed*, or is a *cut*, if for all $a, b \in S$, if $b \in P$ and $a < b$, then $a \in P$. A *well-indexed sequence* is a sequence (a_ρ) whose terms a_ρ are indexed by the elements ρ of an infinite well-ordered set without a greatest element. Given linear orders I_1 and I_2, we denote their concatenation by $I_1 + I_2$.

Let I be an index set. An *I-ordering* on S is a family $(<_i)_{i\in I}$ of orderings on S, and S equipped with this family is referred to as an *I-ordered set*. Suppose S is equipped with an I-ordering $(<_i)_{i\in I}$. A subset of S is viewed as an I-ordered set as well by the induced ordering for each i. We put $S_\infty := S \cup \{\infty\}$, $\infty \notin S$, with the I-ordering on S extended to an I-ordering on S_∞ by declaring $S <_i \infty$ for each $i \in I$. Occasionally, we take two distinct elements $-\infty, \infty \notin S$ and extend the I-ordering on S to $S_{\pm\infty} := S \cup \{-\infty, \infty\}$ by declaring $-\infty <_i S <_i \infty$ for each $i \in I$. A *polycut* in S is a family $\big((P_i, Q_i)\big)_{i\in I}$ of pairs of subsets of S such that for each $i \in I$, P_i is a downward closed subset of $(S, <_i)$ and $Q_i = S \setminus P_i$. Given an element b in an I-ordered set extending S, we say that b *realizes* the polycut $\big((P_i, Q_i)\big)_{i\in I}$ if for every $i \in I$, $P_i <_i b <_i Q_i$. Given $i \in I$ and $a, b \in S_{\pm\infty}$ such that $a <_i b$, we define

$$(a, b)_i \ := \ \{s \in S : a <_i s <_i b\}.$$

Suppose $i \in I$ and A is a finite subset of S. Then by $\min_i A$ we mean $\min A$ with respect to the linear order $(S, <_i)$. Note that a cut P in a linear order $(S, <_0)$ is not literally the same thing as a polycut on the set S with respect to the $\{0\}$-ordering $(<_0)$; however, when necessary we will associate to P the pair $(P, S \setminus P)$.

If G is an expansion of an additively written abelian group, then we set $G^{\neq} := G \setminus \{0\}$. For a field C we let $C^\times := C \setminus \{0\} = C^{\neq}$ be its multiplicative group of units.

In general we adopt the model theoretic conventions of Appendix B of [2]. In particular, $\mathcal{L}$ can be a many-sorted language. For a complete $\mathcal{L}$-theory T, we will sometimes consider a model $\mathbb{M} \models T$ and a cardinal $\kappa(\mathbb{M}) > |\mathcal{L}|$ such that $\mathbb{M}$ is $\kappa(\mathbb{M})$-saturated and every reduct of $\mathbb{M}$ is strongly $\kappa(\mathbb{M})$-homogeneous. Such a model is called a *monster model* of T. In particular, every model of T of size $\leq\kappa(\mathbb{M})$ has an elementary embedding into $\mathbb{M}$. All variables are finite multivariables. If x is a variable with initial segment x', then we denote by $x \setminus x'$ the unique variable such that $x = x'(x \setminus x')$. Given a set $A \subseteq \mathbb{M}_s$ for some sort $s \in S$ and a variable x, we denote by A_x the set $\mathbb{M}_x \cap A^{|x|}$, which might be empty even if $A \neq \emptyset$. By convention we will write "indiscernible sequence" when we mean "$\emptyset$-indiscernible sequence."

§2. Preliminaries on distality.

Throughout this section $\mathcal{L}$ is a language and T is a complete $\mathcal{L}$-theory.

2.1. Definition of distality. *In this subsection we fix a monster model $\mathbb{M}$ of T.*

Definition 2.1. We say that T is *distal* if for every $A \subseteq \mathbb{M}$, for every x, for every indiscernible sequence $(a_i)_{i\in I}$ from $\mathbb{M}_x$, and for every $c \in I$ the following holds: if

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(1) $I_1 := \{i \in I : i < c\}$ and $I_2 := \{i \in I : i > c\}$ are infinite, and
(2) $(a_i)_{i \in I_1 + I_2}$ is A-indiscernible,

then $(a_i)_{i \in I}$ is A-indiscernible. Furthermore, we say that an $\mathcal{L}$-structure $\boldsymbol{M}$ is *distal* if $\mathrm{Th}(\boldsymbol{M})$ is distal.

It is also convenient to define what it means for a formula $\varphi(x; y)$ to be distal:

Definition 2.2. We say a formula $\varphi(x; y)$ is *distal* if for every $b \in \mathbb{M}_y$, for every indiscernible sequence $(a_i)_{i \in I}$ from $\mathbb{M}_x$ and for every $c \in I$ the following holds: if

(1) $I_1 := \{i \in I : i < c\}$ and $I_2 := \{i \in I : i > c\}$ are infinite, and
(2) $(a_i)_{i \in I_1 + I_2}$ is b-indiscernible,

then $\models \varphi(a_c; b) \leftrightarrow \varphi(a_i; b)$ for every $i \in I$.

Remark 2.3. The collection of all distal formulas in the variables $(x; y)$ is closed under arbitrary boolean combinations, including negations. In the literature, there is another local notion of distality: that of a formula $\varphi(x; y)$ having a *strong honest definition* (see [3, Theorem 21]). The collection of formulas $\varphi(x; y)$ which have a strong honest definition is in general only closed under *positive* boolean combinations.

Lemma 2.4. *The following are equivalent*:

(1) *T is distal*;
(2) *every $\varphi(x; y) \in \mathcal{L}$ is distal.*

Example 2.5. All o-minimal theories are distal; see [11, Lemma 2.10]. In particular, the theory of ordered vector spaces over an ordered field C is distal.

2.2. A distal criterion. To set the stage for Distal Criterion 2.6 below, we now consider an extension $\mathcal{L}(\mathfrak{f}) := \mathcal{L} \cup \{\mathfrak{f}\}$ of the language $\mathcal{L}$ by a new unary function symbol $\mathfrak{f}$. We assume that the language $\mathcal{L} \cup \{\mathfrak{f}\}$ has the same sorts as $\mathcal{L}$. We also consider $T(\mathfrak{f})$, a complete $\mathcal{L}(\mathfrak{f})$-theory extending T. Given a model $\boldsymbol{M} \models T$ we denote by $(\boldsymbol{M}, \mathfrak{f})$ an expansion of $\boldsymbol{M}$ to a model of $T(\mathfrak{f})$. For a subset X of a model $\boldsymbol{N}$, we let $\langle X \rangle$ denote the $\mathcal{L}$-substructure of $\boldsymbol{N}$ generated by X. If $\boldsymbol{M}$ is a submodel of $\boldsymbol{N}$, we let $\boldsymbol{M}\langle X \rangle$ denote $\langle M \cup X \rangle \subseteq \boldsymbol{N}$. For this subsection we also fix a monster model $\mathbb{M}$ of $T(\mathfrak{f})$.

Distal Criterion 2.6 is a many-sorted generalization of [9, 2.1]. The version we give below is a consequence of [7, 2.6] *In the statement of 2.6, x, x', y, z are variables.*

Distal Criterion 2.6. Suppose T is a distal theory and the following conditions hold:

(1) The theory $T(\mathfrak{f})$ has quantifier elimination.
(2) For every model $(\boldsymbol{N}, \mathfrak{f}) \models T(\mathfrak{f})$, every $\mathcal{L}$-substructure $\boldsymbol{M} \subseteq \boldsymbol{N}$ such that $\mathfrak{f}(M) \subseteq M$, every x, and every $c \in N_x$, there is a y and $d \in \mathfrak{f}\big(\boldsymbol{M}\langle c \rangle\big)_y$ such that
$$\mathfrak{f}\big(\boldsymbol{M}\langle c \rangle\big) \subseteq \big\langle \mathfrak{f}(\boldsymbol{M}), d \big\rangle.$$
(3) Suppose g, h are $\mathcal{L}$-terms of arities xy and $x'z$ respectively, where x' is an initial segment of x. Furthermore, let $b_1 \in \mathbb{M}_y$, and $b_2 \in \mathfrak{f}(\mathbb{M})_z$. If $(a_i)_{i \in I}$ is an indiscernible sequence from $\mathfrak{f}(\mathbb{M})_{x'} \times \mathbb{M}_{x \setminus x'}$ and $c \in I$ are such that

(a) $I_1 := \{i \in I : i < c\}$ and $I_2 := \{i \in I : i > c\}$ are infinite,
(b) $(a_i)_{i \in I_1+I_2}$ is b_1b_2-indiscernible, and
(c) $\mathfrak{f}\big(g(a_i, b_1)\big) = h(a_i, b_2)$ for every $i \in I_1 + I_2$,
then $\mathfrak{f}\big(g(a_c, b_1)\big) = h(a_c, b_2)$.

Then $T(\mathfrak{f})$ is distal.

§3. I-ordered C-vector spaces. *In this section C is an ordered field and I is a nonempty index set.* Later we will consider the case where $I = 2 = \{0, 1\}$, but the presentation would not simplify much if we were to restrict ourselves to that special case.

Definition 3.1. An *I-ordered C-vector space* is a C-vector space G equipped with an I-ordering $(<_i)_{i\in I}$ such that G is an ordered C-vector space with respect to each $<_i$. In other words, G is a vector space over C, and for every $\lambda \in C$, $x, y, z \in G$ and $i \in I$:

(1) if $x \leq_i y$, then $x + z \leq_i y + z$, and
(2) if $\lambda > 0$ and $x >_i 0$, then $\lambda x >_i 0$.

Let G, G' be two I-ordered C-vector spaces. We call G an *I-ordered C-subspace* of G', or G' an *extension* of G, if, as C-vector spaces, G is a subspace of G' and the I-ordering on G agrees with the induced I-ordering from G' (notation: $G \subseteq G'$). An *embedding* of I-ordered C-vector spaces is an embedding $j : G \to G'$ of C-vector spaces such that for all $x \in G$ and $i \in I$, if $x >_i 0$ in G, then $j(x) >_i 0$ in G'.

Lemma 3.2. *Let G be an I-ordered C-vector space and $\big((P_i, Q_i)\big)_{i\in I}$ a polycut in G. Then there is an I-ordered C-vector space G' extending G and an element $b \in G'$ such that*

(1) *for any I-ordered C-vector space extension G^* of G and an element $b^* \in G^*$ which realizes the polycut $\big((P_i, Q_i)\big)_{i\in I}$, there is an embedding $G' \to G^*$ over G of I-ordered C-vector spaces which sends b to b^*.*

Furthermore, given any I-ordered C-vector space G' extending G and $b \in G'$ satisfying (1) above, we have

(2) *b realizes the polycut $\big((P_i, Q_i)\big)_{i\in I}$, and*
(3) *$G' = G \oplus Cb$ (internal direct sum of C-vector spaces).*

Proof. As a C-vector space, let $G' := G \oplus Cb$. For each $i \in I$, extend $<_i$ to the unique ordered C-vector space ordering on G' such that $P_i <_i b <_i Q_i$. The universal property then follows by the universal property in [2, 2.4.16]. (2) and (3) are also clear. ⊣

Definition 3.3. We say that the I-ordering $(<_i)_{i\in I}$ on an I-ordered C-vector space G is *independent* if for every n, distinct $i_1, \dots, i_n \in I$, and for all pairs $a_1, b_1, \dots, a_n, b_n \in G_{\pm\infty}$, if $a_k <_{i_k} b_k$ for $k = 1, \dots, n$, then

$$(a_1, b_1)_{i_1} \cap \cdots \cap (a_n, b_n)_{i_n} \quad \text{is nonempty.}$$

Let $\mathcal{L}_{C,I}$ be the natural language of I-ordered C-vector spaces:

$$\mathcal{L}_{C,I} = \{0, +, (\lambda_c)_{c\in C}\} \cup \{<_i : i \in I\}.$$

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Let $T_{C,I}$ be the $\mathcal{L}_{C,I}$-theory whose models are precisely the I-ordered C-vector spaces G such that the orderings $(<_i)_{i\in I}$ on G are independent. By applying Lemma 3.2 iteratively starting with the trivial I-ordered C-vector space with underlying set $\{0\}$, we can construct a model of $T_{C,I}$ and thus $T_{C,I}$ is consistent. Note that since I is nonempty, models of $T_{C,I}$ are necessarily infinite.

PROPOSITION 3.4. *The $\mathcal{L}_{C,I}$-theory $T_{C,I}$ admits quantifier elimination and is complete.*

PROOF. Let G and G^* be models of $T_{C,I}$ and suppose $H \subseteq G$ is a proper $\mathcal{L}_{C,I}$-substructure of G. Furthermore, suppose G^* is $|H|^+$-saturated and $i : H \to G^*$ is an embedding of $\mathcal{L}_{C,I}$-structures. For quantifier elimination, it suffices to find $b \in G \setminus H$ such that i extends to an embedding $H + Cb \to G^*$ (e.g., see [2, B.11.10]).

Take $b \in G \setminus H$ and let $\big((P_i, Q_i)\big)_{i\in I}$ be the unique polycut in H realized by b. Then the image $\big(i(P_i), i(Q_i)\big)_{i\in I}$ determines a partial type in G^* over the parameter set $i(H)$:

$$i(P_i) < x < i(Q_i) \quad \text{for every } i \in I.$$

As the orderings on G^* are independent and G^* is $|H|^+$-saturated, we may take $b^* \in G^*$ realizing this partial type. Then by Lemma 3.2, there is an embedding $H + Cb \to G^*$ which extends i and sends b to b^*.

Completeness follows from quantifier elimination and the fact that the trivial I-ordered C-vector space with a single element embeds into every model of $T_{C,I}$ ⊣

COROLLARY 3.5. *$T_{C,I}$ is distal.*

PROOF. By Proposition 3.4 and Lemma 2.4, it suffices to show each quantifier-free $\mathcal{L}_{C,I}$-formula $\varphi(x; y)$ is distal. Every atomic formula and negated atomic formula is in a reduct to $\{0, +, (\lambda_c)_{c\in C}, <_i\}$ for some $i \in I$. Every such reduct is an ordered C-vector space, and hence by Example 2.5 is distal. Therefore, each formula $\varphi(x; y)$ is equivalent to a boolean combination of distal formulas, hence is distal. ⊣

§4. Hamel spaces. *In this section C is an ordered field.*

4.1. Hamel valuations. Consider a 2-ordered C-vector space G. Recall from Definition 3.1 that G is equipped with a 2-ordering on it, where the index set $2 = \{0, 1\}$ has size two. The two orderings on G are denoted by $<_0$ and $<_1$, and we think of $<_0$ as the "original" ordering, and $<_1$ as the "auxiliary" ordering. Furthermore, recall that by convention the 2-ordering on G extends to a 2-ordering on $G_\infty = G \cup \{\infty\}$ by setting $G <_0 \infty$ and $G <_1 \infty$. We now arrive at our main definition:

A *Hamel valuation* on G is a map $v : G \to G_\infty$ which satisfies the following:

(1) $v : G \to G_\infty$ is a (non-surjective) convex valuation which makes G a valued vector space over C with respect to the ordering $<_1$ on the vector space and the ordering $<_0$ on the value set, i.e., for all $x, y \in G$ and $\lambda \in C^\times$:
  - (a) $vx = \infty$ iff $x = 0$;
  - (b) $v(x + y) \geq_0 \min_0(vx, vy)$;
  - (c) $v(\lambda x) = vx$;
  - (d) if $0 <_1 x <_1 y$, then $vx \geq_0 vy$.

(2) (Idempotence) $vx = v(vx)$ for every $x \in G$.
(3) (Positivity) $vx >_1 0$ for every $x \in G$.

A *Hamel space* (*over* C) is a pair (G, v) where G is a 2-ordered C-vector space, and v is a Hamel valuation on G. *In the rest of this subsection* (G, v) *is a Hamel space.*

DEFINITION 4.1. We say that (G, v) is *independent* if the 2-ordering $(<_0, <_1)$ on G is independent in the sense of Definition 3.3. We say that (G, v) is *dense* if for every $a, b \in G$, if $a <_0 b$, then there is $c \in G$ such that $a <_0 vc <_0 b$.

Note that the set $v(G^{\neq})$ is never dense in $(G, <_1)$; indeed, given $x \in G^{\neq}$, we have $v(G) \cap (vx, 2vx)_1 = \emptyset$. However, if (G, v) is independent, then the set $G \setminus v(G)$ is dense in both orderings:

LEMMA 4.2. *Suppose* (G, v) *is independent. Then for every* $x_0, y_0, x_1, y_1 \in G$ *such that* $x_0 <_0 y_0$ *and* $x_1 <_1 y_1$, *there is* $z \in G$ *such that* $z \neq vz$ *and* $x_0 <_0 z <_0 y_0$ *and* $x_1 <_1 z <_1 y_1$.

PROOF. By independence, there is $z' \in G$ such that $x_0 <_0 z' <_0 y_0$ and $x_1 <_1 z' <_1 y_1$. If $z' \neq vz'$, then $z := z'$ will work. Otherwise, necessarily $z' = vz' >_1 0$. Applying independence again, we get $z \in G$ such that $x <_0 z <_0 y_0$ and $z' <_1 z <_1 \min_1(2z', y_1)$. By convexity, we have $vz = vz' \neq z$, so this z works. ⊣

In general, the set $v(G^{\neq})$ will not span G as a C-vector space. However, $v(G^{\neq})$ will always be C-linearly independent:

LEMMA 4.3. *The set* $v(G^{\neq})$ *is* C*-linearly independent.*

PROOF. Suppose $g_1, \dots, g_n \in v(G^{\neq})$ are such that $g_1 <_0 \cdots <_0 g_n$. Take $\lambda_1, \dots, \lambda_n \in C^{\times}$. Then $v(\lambda_i g_i) = vg_i = g_i$ for all $i = 1, \dots, n$, and so $v(\lambda_i g_i) \neq v(\lambda_j g_j)$ for $i \neq j$. Thus

$$v\big(\textstyle\sum_{i=1}^n \lambda_i g_i\big) = \min_0\big\{v(\lambda_i g_i) : i = 1, \dots, n\big\} = \min_0\{g_i : i = 1, \dots, n\} = g_1 \neq \infty.$$

In particular, $\sum_{i=1}^n \lambda_i g_i \neq 0$. ⊣

The following will be used in our proof of Theorem 5.1, namely, to verify condition (2) in Distal Criterion 2.6, with (G, v) playing the role of "$(\boldsymbol{N}, \mathfrak{f})$." We actually prove something more general:

PROPOSITION 4.4. *Suppose* (G, v) *is a Hamel space and* $G_0 \subseteq G$ *is a* C*-linear subspace of* G. *Given* $c_1, \dots, c_m \in G \setminus G_0$, *we have*

$$\#\big(v(G_0 + \textstyle\sum_{i=1}^m Cc_i) \setminus v(G_0)\big) \leq m.$$

In particular, there is $n \leq m$ *and distinct*

$$d_1, \dots, d_n \in v(G_0 + \textstyle\sum_{i=1}^m Cc_i) \setminus v(G_0)$$

such that

$$v(G_0 + \textstyle\sum_{i=1}^m Cc_i) \subseteq v(G_0) \cup \{d_1, \dots, d_n\}.$$

PROOF. Assume towards a contradiction that there are $m+1$ distinct $d_1, \dots, d_{m+1} \in v(G_0 + \sum_{i=1}^m Cc_i) \setminus v(G_0)$. For each $j = 1, \dots, m+1$, let $e_j \in G_0 + \sum_{j=1}^m Cc_i$ be such that $ve_j = d_j$. We claim that $e_1, \dots, e_{m+1}$ are C-linearly independent over G_0. This follows from the fact that for $g \in G_0$ and $\lambda_1, \dots, \lambda_{m+1} \in C$, we have

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$$v\big(g+\textstyle\sum_{j=1}^{m+1}\lambda_j e_j\big) \;=\; \min_0\big(\{d_j : \lambda_j \neq 0\} \cup \{vg\}\big).$$

Thus $\dim_C(G_0 + \sum_{i=1}^{m} Cc_i)/G_0 \geq m+1$, a contradiction. ⊣

4.2. Extensions of Hamel spaces. *In this subsection (G, v) and (G', v') are Hamel spaces.* We call (G, v) a *Hamel subspace* of (G', v'), or (G', v') an *extension* of (G, v), if $G \subseteq G'$ as 2-ordered C-vector spaces, and for all $x \in G$, $v(x) = v'(x)$; notation: $(G, v) \subseteq (G', v')$. An *embedding* $i : (G, v) \to (G', v')$ of Hamel spaces is an embedding $i : G \to G'$ of the underlying 2-ordered C-vector spaces such that for all $x \in G$, $i(vx) = v'i(x)$.

As is typical in valuation theory, we consider three different types of extensions. The first type of extension deals with the situation where we want to adjoin a new element to the value set $v(G)$:

Lemma 4.5 (Growing the value set). *Suppose $P \subseteq G$ is a cut in $(G, <_0)$. Then there is an extension (G', v') of (G, v) and an element $h \in G'$ such that*

(1) *$h = v'h$,*
(2) *$P <_0 h <_0 G \setminus P$, and*
(3) *given any embedding $i : (G, v) \to (G^*, v^*)$ and an element $h^* \in G^*$ such that*

$$i(P) <_0 h^* = v^*h^* <_0 i(G \setminus P),$$

there is an extension of i to an embedding $(G', v') \to (G^*, v^*)$ which sends h to h^*.

Proof. First, we will define the polycut over G that such an element h must realize. Set $P_0 := P$, $Q_0 := G \setminus P$,

$$P_1 \;:=\; \{g \in G : g \leq_1 0\} \cup \{g \in G : vg \in Q_0\},$$

and $Q_1 := G \setminus P_1$. Let $G' := G + Ch$ be the extension of G of 2-ordered C-vector spaces given in Lemma 3.2 for the polycut $\big((P_i, Q_i)\big)_{i=0,1}$. Next, define the map $v' : G' \to G'_\infty$ by

$$v'(g+ch) \;:=\; \begin{cases} vg & \text{if } vg \in P_0 \text{ or } c = 0,\\ h & \text{if } vg \notin P_0 \text{ and } c \neq 0,\end{cases}$$

for $g \in G$ and $c \in C$. It is easily checked that (G', v') is an extension of (G, v) with the desired universal property. ⊣

Before we proceed with Lemma 4.7, we first recall the notion of *pseudocauchy sequence* and *pseudolimit* from valuation theory (in the specific context of Hamel valuations):

Definition 4.6. Suppose (a_ρ) is a well-indexed sequence from G.

(1) We say (a_ρ) is a *pseudocauchy sequence in G* (or *pc-sequence in G*) if for some index ρ_0:

$$\rho_0 < \rho < \sigma < \tau \implies v(a_\rho - a_\sigma) <_0 v(a_\sigma - a_\tau).$$

(2) Given $a \in G$, we say (a_ρ) *pseudoconverges to a*, written $a_\rho \rightsquigarrow a$, if for some index ρ_0:

$$\rho_0 < \rho < \sigma \implies v(a - a_\rho) <_0 v(a - a_\sigma).$$

In this case we also say that *a is a pseudolimit of* (a_ρ).

(3) A *divergent* pc-sequence in G is a pc-sequence in G without a pseudolimit in G.

See [2, Section 2.2] for more about pc-sequences in the broader context of valued abelian groups.

Our next type of extension deals with the case where we want to adjoin a pseudolimit to a divergent pc-sequence. This will result in a so-called *immediate extension*. We state in the proof explicitly what an immediate extension means in this context and refer the reader to [2, Chapters 2 and 3] for a discussion of immediate extensions in the broader context of valuation theory.

Lemma 4.7 (Immediate extension). *Suppose $P \subseteq G$ is a cut in $(G, <_0)$ and (b_ρ) is a divergent pc-sequence in G. Then there is an extension (G', v') of (G, v) and an element $h \in G'$ such that*

(1) $b_\rho \rightsquigarrow h$,
(2) $P <_0 h <_0 G \setminus P$, *and*
(3) *given any embedding $i : (G, v) \to (G^*, v^*)$ and an element $h^* \in G^*$ such that*
$$i(b_\rho) \rightsquigarrow h^* \quad \text{and} \quad i(P) <_0 h^* <_0 i(G \setminus P),$$
there is an extension of i to an embedding $(G', v') \to (G^*, v^*)$ which sends h to h^*.

Proof. Let $G' := G \oplus Ch$ be a C-vector space extension of the underlying C-vector space of G. We now have to extend v to a map v' on G', as well as extend the orderings $<_0$ and $<_1$ to orderings on G'.

First, by [2, 2.3.1] we extend $v : G \to G_\infty$ uniquely to a map $v : G' \to G'_\infty$ which makes (G', v) a *valued vector space over C* such that $b_\rho \rightsquigarrow h$; then (G', v') will be an *immediate extension* of (G, v). In other words, there is a unique way to extend v to a map $v' : G' \to G'_\infty$ and a unique way to extend the $<_0$-ordering on $v(G)$ to $v'(G')$ such that:

(1) (G', v') is a *valued vector space over C*, i.e., for every $x, y \in G'$ and $\lambda \in C^\times$:
  (a) $v'x = \infty$ iff $x = 0$,
  (b) $v'(x + y) \geq_0 \min_0(v'x, v'y)$,
  (c) $v'(\lambda x) = vx$,
(2) $b_\rho \rightsquigarrow h$ in the sense of (2) in Definition 4.6.

Note that *a priori* the value set $v'(G')$ might now be bigger than $v(G)$ and it could include elements of $G \setminus v(G)$, and thus the ordering $<_0$ on $v'(G')$ could include elements of $G' \setminus G$ or be inconsistent with the existing $<_0$-ordering on G. However, as a byproduct of [2, 2.3.1] this is not the case and we actually have arranged in addition:

(3) $v(G') = v(G)$, and so $<_0$ has not yet been extended to include any elements of $G' \setminus G$,
(4) the map $v' : G' \to G'_\infty$ satisfies Idempotence and Positivity, and
(5) (G', v') is an *immediate extension* of (G, v') as valued vector spaces over C, i.e., for every $x \in (G')^{\neq}$ there is $y \in G$ such that $v'(x - y) >_0 v'(x)$.

Next, by [2, 2.4.20], there is just one ordering $<_1$ on G' which extends $<_1$ on G which makes $(G', <_1)$ an ordered C-vector space and v' a convex valuation with

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respect to $<_1$, i.e., for all $x, y \in G'$, if $0 <_1 x <_1 y$, then $vx \geq_0 vy$. We equip G' with this ordering.

Finally, by [2, 2.4.16] there is a unique ordering $<_0$ on G' which extends $<_0$ on G such that $P <_0 h <_0 G \setminus P$; we also equip G' with this ordering. It is easily checked that, equipped with these orderings, (G', v') is an extension of (G, v) with the desired universal property. ⊣

Given $\alpha \in v(G^{\neq})$, we define the sets

$$\overline{B}(\alpha) := \{g \in G : vg \geq_0 \alpha\}, \quad \text{and} \quad B(\alpha) := \{g \in G : vg >_0 \alpha\}.$$

The sets $\overline{B}(\alpha)$ and $B(\alpha)$ are convex with respect to the $<_1$-ordering. Furthermore, we will construe them as ordered C-vector subspaces of G with respect to the $<_1$-ordering. As $\overline{B}(\alpha) \supseteq B(\alpha)$, the $<_1$-ordering induces an ordering on the quotient $G(\alpha) := \overline{B}(\alpha)/B(\alpha)$, giving it a natural structure as an ordered C-vector space. Moreover, given an embedding $i : (G, v) \to (G', v')$ and $\alpha \in v(G^{\neq})$, i induces a natural ordered C-vector space embedding $i : G(\alpha) \to G'(i\alpha)$.

We define an α-*cut* to be a subset $P \subseteq \overline{B}(\alpha)$ such that

(1) P is downward closed in $(\overline{B}(\alpha), <_1)$, and
(2) for all $x, y \in \overline{B}(\alpha)$, if $x - y \in B(\alpha)$, then $x \in P$ iff $y \in P$.

In other words, an α-cut is essentially a lift of a cut in the ordered C-vector space $G(\alpha)$.

Lemma 4.8 (Growing a quotient space). *Suppose $\alpha \in v(G^{\neq})$, P is an α-cut, and $P' \subseteq G$ is a cut in $(G, <_0)$. Then there is an extension (G', v') of (G, v) and an element $h \in G'$ such that*

(1) $P' <_0 h <_0 G \setminus P'$,
(2) $vh = \alpha$,
(3) $P <_1 h <_1 \overline{B}(\alpha) \setminus P$, and
(4) given any embedding $i : (G, v) \to (G^*, v^*)$ and an element $h^* \in G^*$ such that
  (a) $i(P') <_0 h^* < i(G \setminus P')$,
  (b) for all $g \in G$ and $c \in C$,

$$v^*\big(i(g) + ch^*\big) := \begin{cases} \min_0\big(v^* i(g), i(\alpha)\big) & \text{if } c \neq 0, \\ v^* i(g) & \text{otherwise, and} \end{cases}$$

  (c) $i(P) <_1 h^* <_1 i\big(\overline{B}(\alpha) \setminus P\big)$,
  there is an extension of i to an embedding $(G', v') \to (G^*, v^*)$ which sends h to h^*.

Proof. First, we will define the polycut over G that such an element h must realize. Set $P_0 := P'$, $Q_0 := G \setminus P_0$,

$$P_1 := \{g \in G : g <_1 \overline{B}(\alpha)\} \cup P,$$

and $Q_1 := G \setminus P_1$. Let $G' := G + Ch$ be the extension of G of 2-ordered C-vector spaces given in Lemma 3.2 for the polycut $\big((P_i, Q_i)\big)_{i=0,1}$. Next, define the map $v' : G' \to G'_\infty$ by

$$v'(g + ch) := \begin{cases} \min_0(vg, \alpha) & \text{if } c \neq 0, \\ vg & \text{otherwise,} \end{cases}$$

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for $g \in G$ and $c \in C$. It is easily checked that (G', v') is an extension of (G, v) with the desired property. $\dashv$

4.3. Model theory of Hamel spaces. Now let $\mathcal{L}_{\text{Ham}}$ be the natural language of Hamel spaces over C, i.e.,

$$\mathcal{L}_{\text{Ham}} := \mathcal{L}_{C,2} \cup \{v, \infty\} = \{0, +, (\lambda_c)_{c \in C}, <_0, <_1, v, \infty\}.$$

We consider a Hamel space (G, v) as an $\mathcal{L}_{\text{Ham}}$-structure with underlying set G_∞ and the obvious interpretation of the symbols in $\mathcal{L}_{\text{Ham}}$, with ∞ as a default value:

$$g + \infty = \infty + g = \lambda_c(\infty) = v(\infty) = \infty + \infty = \infty$$

for all $g \in G$ and $c \in C$. We let T_{Ham} be the $\mathcal{L}_{\text{Ham}}$-theory whose models are the independent and dense Hamel spaces over C.

Lemma 4.9. *T_{Ham} is consistent.*

Proof. To show T_{Ham} is consistent, we need to show that it has a model. The following two claims will help us with this:

Claim 4.10. *Every Hamel space (G, v) has an extension $(G_{\text{d}}, v_{\text{d}})$ which is dense.*

Proof of claim. Let $\big((a_\lambda, b_\lambda)\big)_{\lambda<\kappa}$ be an enumeration of all pairs $(a, b) \in G^2$ such that $a <_0 b$. We will recursively construct an increasing tower of extensions $\big((G_\lambda, v_\lambda)\big)_{\lambda<\kappa}$ of $(G, v) = (G_0, v_0)$ with the property that for each $\lambda < \kappa$, there is $h \in G_{\lambda+1}$ such that $a_\lambda <_0 vh < b_\lambda$. Suppose for some ordinal $0 < \nu < \kappa$ we have already constructed $\big((G_\lambda, v_\lambda)\big)_{\lambda<\nu}$. We have to construct (G_ν, v_ν) and we have two cases:

(1) if ν is a limit ordinal, then we set $(G_\nu, v_\nu) := \bigcup_{\lambda<\nu}(G_\lambda, v_\lambda)$,
(2) if $\nu = \mu + 1$, then we define $P := \{g \in G_\mu : g \leq_0 a_\mu\}$, a cut in $(G_\mu, <_0)$. Then we define (G_ν, v_ν) to be the extension of (G_μ, v_μ) given by Lemma 4.5 for this cut P. By construction, we have $P <_0 v_\nu h < G_\mu \setminus P$, with $a_\mu \in P$ and $b_\mu \in G_\mu \setminus P$ and so $a_\mu <_0 v_\nu h <_0 b_\mu$.

This process terminates eventually with the desired tower $\big((G_\lambda, v_\lambda)\big)_{\lambda<\kappa}$. Now we set $(G_+, v_+) := \bigcup_{\lambda<\kappa}(G_\lambda, v_\lambda)$. We have just shown that every Hamel space (G, v) has an extension (G_+, v_+) with the property that for every $a, b \in G$ such that $a <_0 b$, there is $c \in G_+$ such that $a <_0 v_+ c <_0 b$.

Now we define another tower increasing tower of extensions $\big((G^n, v^n)\big)_{n<\omega}$ of $(G, v) = (G^0, v^0)$ such that $(G^{n+1}, v^{n+1}) = \big((G^n)_+, (v^n)_+\big)$ for each n. Finally, we set $(G_{\text{d}}, v_{\text{d}}) := \bigcup_n (G^n, v^n)$. It is clear that $(G_{\text{d}}, v_{\text{d}})$ is an extension of (G, v) which is dense. $\dashv$

Claim 4.11. *Every Hamel space (G, v) has an extension $(G_{\text{i}}, v_{\text{i}})$ which is independent.*

Proof of claim. By mimicking the construction done in the proof of Claim 4.10 above, it suffices to show that given $a, b, c, d \in G_{\pm\infty}$ such that $a <_0 b$ and $c <_1 d$, there is an extension (G', v') of (G, v) with an element $h \in G'$ such that $a <_0 h <_0 b$ and $c <_1 h <_1 d$.

Fix $a, b, c, d \in G_{\pm\infty}$ such that $a <_0 b$ and $c <_1 d$. First, by passing to an extension, we may assume that $G \neq \{0\}$ (if $G = \{0\}$, then apply Lemma 4.5 to the cut $P = \{0\}$ to get a proper extension such that $G \neq \{0\}$). Since $G \neq \{0\}$, we

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also arrange that $a, b, c, d \in G$. Next, by subtracting c from each of a, b, c, d, we arrange that $c = 0$, so $0 <_1 d$. Next, set $\alpha := vd$ and define the α-cut

$$P := \{g \in G : vg \geq_0 \alpha \ \& \ v <_1 0\} \cup \{g \in G : vg >_0 \alpha\},$$

i.e., the downward closure of $B(\alpha)$ inside $\overline{B}(\alpha)$ (with respect to the $<_1$-ordering). In particular, $d \in \overline{B}(\alpha) \setminus P$. Furthermore, define the cut

$$P' := \{g \in G : g \leq_0 a\}$$

in $(G, <_0)$. Finally, we let (G', v') be the extension of (G, v) given by Lemma 4.8 for this P and P'. In particular, there is an element $h \in G'$ such that $P' <_0 h <_0 G \setminus P'$ and $P <_1 h <_1 \overline{B}(\alpha) \setminus P$. Thus $a <_0 h <_0 b$ and $0 <_1 h <_1 d$, as desired. ⊣

Finally, let (G, v) be any Hamel space. By alternating between Claims 4.10 and 4.11 above and taking a union, we can construct an extension of (G, v) which is a model of T_{Ham}. ⊣

THEOREM 4.12. *The $\mathcal{L}_{Ham}$-theory T_{Ham} admits quantifier elimination and is complete.*

PROOF. Let (G, v) and (G^*, v^*) be models of T_{Ham} and suppose $(H, v) \subseteq (G, v)$ is a proper $\mathcal{L}_{\text{Ham}}$-substructure of (G, v). Furthermore, suppose (G^*, v^*) is $|H|^+$-saturated and $i : (H, v) \to (G^*, v^*)$ is an embedding of $\mathcal{L}_{\text{Ham}}$-structures. For quantifier elimination, it suffices to find $h \in G \setminus H$ such that i extends to an embedding $(H + Ch, v) \to (G^*, v^*)$ (e.g., see [2, B.11.10]). We consider three cases:

CASE 1. *There is $h \in v(G) \setminus v(H)$.* Choose such an $h \in G$. Set $P := \{g \in H : g <_0 h\}$ and $Q := H \setminus P$. By saturation and denseness of (G^*, v^*), there is $h^* \in v^*(G^*) \subseteq G^*$ such that $i(P) <_0 h^* <_0 i(Q)$. By Lemma 4.5, i extends to an embedding $(H + Ch, v) \to (G^*, v^*)$ which sends h to h^*.

CASE 2. *There is $h \in G \setminus H$ such that $v(h - H)$ does not have a largest element.* Choose such an $h \in G$. Take a well-indexed sequence (b_ρ) in H such that $\big(v(h - b_\rho)\big)$ is strictly increasing and cofinal in $v(h - H)$. Then (b_ρ) is a divergent pc-sequence in H such that $b_\rho \rightsquigarrow h$. Set $P := \{g \in H : g <_0 h\}$ and $Q := H \setminus P$. Then by saturation and independence of (G^*, v^*) there is $h^* \in G^*$ such that $i(b_\rho) \rightsquigarrow h^*$ and $i(P) <_0 h^* <_0 i(Q)$. By Lemma 4.7, i extends to an embedding $(H + Ch, v) \to (G^*, v^*)$ which sends h to h^*.

CASE 3. *There is $h \in G$ such that $vh \in H$ and there is no $g \in H$ such that $v(h - g) >_0 vh$.* Choose such an $h \in G$. Set $P' := \{g \in H : g <_0 h\}$, a cut in $(H, <_0)$. Furthermore, define

$$P := \{g \in H : vg \geq_0 vh \text{ and } g \leq_1 h\},$$

which is a vh-cut in H by the assumption on h. Next, in the quotient space $G^*(ivh)$, pick an element $\bar{h}$ such that $i(P) + B(ivh) <_1 \bar{h} <_1 i\big(\overline{B}(vh) \setminus P\big) + B(ivh)$, which can be done by saturation of the ordered C-vector space $G^*(ivh)$, an interpretable structure in (G^*, v^*). By independence and saturation of (G^*, v^*), there is an element $h^* \in \overline{B^*}(ivh) \subseteq G^*$ such that $h^* + B^*(ivh) = \bar{h}$ and $i(P') <_0 h^* <_0 i(H \setminus P)$ (note that this also uses the fact that $B^*(ivh)$ has at least two elements, a consequence of denseness and independence). Then by Lemma 4.8, i extends to an embedding $(H + Ch, v) \to (G^*, v^*)$ which sends h to h^*.

Completeness follows from quantifier elimination and the fact that the trivial Hamel space with underlying set $\{0, \infty\}$ embeds into every model of T_{Ham}. ⊣

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§5. Distality for Hamel spaces. In this section we prove the main result of this article:

THEOREM 5.1. *T_{Ham} is distal.*

This has the following consequences, also of interest:

COROLLARY 5.2. *T_{Ham} has the nonindependence property (NIP).*

PROOF. It is well known that distality implies NIP; e.g., see [7, Proposition 2.8] for a proof. ⊣

COROLLARY 5.3. *No model of T_{Ham} interprets an infinite field of positive characteristic.*

PROOF. See [4, Corollary 6.3]. ⊣

In the rest of this section $\mathbb{M}$ is a monster model of T_{Ham} with underlying set G_∞.

5.1. Indiscernible lemmas. In this subsection we will prove the main lemmas involving indiscernible sequences in G_∞ that we need for verifying condition (3) in Distal Criterion 2.6. *We assume in this subsection that $I = I_1 + (c) + I_2$ is an ordered index set with infinite I_1 and I_2, and i, j, k range over I.*

LEMMA 5.4. *Let $(a_i)_{i\in I}$ be a nonconstant indiscernible sequence from G and suppose $(b, b') \in G \times v(G)$ is such that $(a_i)_{i\in I_1+I_2}$ is bb'-indiscernible. If $v(a_i - b) = b'$ for all $i \neq c$, then $v(a_c - b) = b'$.*

PROOF. First assume that $0 <_1 a_i - b <_1 a_j - b$ for all $i < j$. Now for $i \in I_1$ and $j \in I_2$ we have $v(a_i - b) = v(a_j - b)$ as well as $0 <_1 a_i - b \leq_1 a_c - b \leq_1 a_j - b$, so as v is convex with respect to the $<_1$-ordering, $v(a_c - b) = b'$ as well. The other cases are similar. ⊣

LEMMA 5.5. *Let $(a_i a'_i)_{i\in I}$ be an indiscernible sequence from $G \times v(G)$ such that (a_i) and (a'_i) are each nonconstant, and suppose $b \in G$ is such that $(a_i a'_i)_{i\in I_1+I_2}$ is b-indiscernible. If $v(a_i - b) = a'_i$ for all $i \neq c$, then $v(a_c - b) = a'_c$.*

PROOF. Without loss of generality, assume that $(a'_i)_{i\in I}$ is strictly increasing in the $<_0$-ordering. Suppose $i, j \in I_1 + I_2$ with $i < j$. Then

$$v(a_i - a_j) \;=\; v\big((a_i - b) + (b - a_j)\big) \;=\; \min_0(a'_i, a'_j) \;=\; a'_i.$$

By indiscernibility, for $j \in I_2$, $v(a_c - a_j) = a'_c$ and so

$$v(a_c - b) \;=\; v\big((a_c - a_j) + (a_j - b)\big) \;=\; \min_0(a'_c, a'_j) \;=\; a'_c. \qquad ⊣$$

In the next two lemmas $\mathcal{L} := \{0, +, (\lambda_c)_{c\in C}, <_0, <_1, \infty\} \subseteq \mathcal{L}_{\text{Ham}}$.

LEMMA 5.6. *Let g, h be $\mathcal{L}$-terms of arities $n + k$ and $m + l$ respectively with $m \leq n$, $b_1 \in \mathbb{M}^k$, $b_2 \in v(\mathbb{M})^l$, $(a_i)_{i\in I}$ be an indiscernible sequence from $v(\mathbb{M})^m \times \mathbb{M}^{n-m}$ such that*

(1) *$\big(g(a_i, b_1)\big)_{i\in I_1+I_2}$ is a constant sequence, and*
(2) *$v\big(g(a_i, b_1)\big) = h(a_i, b_2)$ for every $i \neq c$.*

Then $v\big(g(a_c, b_1)\big) = h(a_c, b_2)$.

PROOF. This is routine and left to the reader. See the proof of [7, Lemma 4.3]. ⊣

Lemma 5.7. *Let $h(x, y)$ be an $\mathcal{L}$-term of arity $m + n$, $b \in \mathbb{M}^n$, and $(a_i)_{i\in I}$ an indiscernible sequence from $v(\mathbb{M})^m$, with $a_i = (a_{i,1}, \dots, a_{i,m})$. Assume that $h(a_i, b) \in v(\mathbb{M})$ for infinitely many i. Then one of the following is true*:

(1) $h(a_i, b) = \infty$ *for every* i;
(2) *there is* $\beta \in v(G^{\neq})$ *such that* $h(a_i, b) = \beta$ *for every* i;
(3) *there is* $l \in \{1, \dots, m\}$ *such that* $h(a_i, b) = a_{i,l}$ *for every* i.

Proof. This is an exercise in simplification and bookkeeping which mimics the proof of [7, Lemma 4.4], except that it uses the fact that the value set $v(G^{\neq})$ is a C-linearly independent subset of G (Lemma 4.3). ⊣

5.2. Proof of Theorem 5.1. In this subsection we prove Theorem 5.1 by verifying the hypotheses of Distal Criterion 2.6. In the language of 2.6, the role of T will be played by the reduct $T := T_{\text{Ham}} \restriction \mathcal{L}$, where $\mathcal{L} := \mathcal{L}_{\text{Ham}} \setminus \{v\} = \{0, +, (\lambda_c)_{c\in C}, <_0, <_1, \infty\}$. The $\mathcal{L}$-theory T is bi-interpretable with the $\mathcal{L}_{C,2}$-theory $T_{C,2}$. Indeed, T is essentially the same thing as $T_{C,2}$, except that T has an extra point ∞ at infinity with respect to both orders which serves as a default value with respect to the C-vector space structure. As distality is preserved under bi-interpretability, by Corollary 3.5 we have that T is distal.

In the language of 2.6, we also construe T_{Ham} as $T_{\text{Ham}} = T(v)$, and in particular, $\mathcal{L}_{\text{Ham}} = \mathcal{L}(v)$. Since T_{Ham} has quantifier elimination (Theorem 4.12), this verifies Condition (1) in 2.6. Condition (2) in 2.6 follows from Proposition 4.4.

Finally we will verify condition (3) in 2.6. Let g, h be $\mathcal{L}$-terms of arities $n+k$ and $m + l$ respectively, with $m \le n$, $b_1 \in \mathbb{M}^k$, $b_2 \in v(\mathbb{M})^l$, $(a_i)_{i\in I}$ be an indiscernible sequence from $v(\mathbb{M})^m \times \mathbb{M}^{n-m}$ such that

(a) $I = I_1 + (c) + I_2$ where I_1 and I_2 are infinite, and $(a_i)_{i\in I_1+I_2}$ is b_1b_2-indiscernible, and
(b) $v\big(g(a_i, b_1)\big) = h(a_i, b_2)$ for every $i \in I_1 + I_2$.

Our job is to show that $v\big(g(a_c, b_1)\big) = h(a_c, b_2)$. We have several cases to consider:

Case 1. $\big(g(a_i, b_1)\big)_{i\in I_1+I_2}$ *is a constant sequence.* In this case, $v\big(g(a_c, b_1)\big) = h(a_c, b_2)$ follows from Lemma 5.6.

For the remainder of the proof, we assume that $\big(g(a_i, b_1)\big)_{i\in I_1+I_2}$ *is not a constant sequence*. In particular, the symbol ∞ does not play a nondummy role in $g(a_i, b_1)$, so the $\mathcal{L}$-term $g(x, y)$ is essentially a C-linear combination of its arguments. Specifically, we may assume there are $c_1, \dots, c_n, d_1, \dots, d_k \in C$ such that

$$g(x, y) \;=\; \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{k} d_j y_j,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_k)$. Then for each $i \in I$ we define

$$a_i' \;:=\; \sum_{j=1}^{n} c_j a_{i,j} \quad \text{and} \quad b \;:=\; \sum_{j=1}^{k} d_j b_{1,j}$$

where $a_i = (a_{i,1}, \dots, a_{i,n})$ and $b_1 = (b_{1,1}, \dots, b_{1,k})$. Ultimately, this gives us $b \in G$, and a nonconstant indiscernible sequence $(a_i')_{i\in I}$ from $\mathbb{M}$ such that

(c) $g(a_i, b_1) = a_i' - b$ for every $i \in I$,

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(d) $(a_i a'_i)_{i \in I}$ is an indiscernible sequence from $v(\mathbb{M})^m \times \mathbb{M}^{n-m+1}$,
(e) $(a_i a'_i)_{i \in I+I_1}$ is $b_1 b_2 b$-indiscernible, and
(f) $v(a'_i - b) = h(a_i, b_2)$ for every $i \in I_1 + I_2$.

We now must show that $v(a'_c - b) = h(a_c, b_2)$. Since $h(a_i, b_2) \in v(\mathbb{M})$ for all $i \in I_1 + I_2$, by Lemma 5.7 we have three more cases to consider:

Case 2. *$h(a_i, b_2) = \infty$ for every $i \in I$.* In this case, we have $v(a'_i - b) = \infty$ for every $i \in I_1 + I_2$, so $a'_i = b$ for every $i \in I_1 + I_2$. Thus $(a'_i)_{i \in I}$ is a constant sequence and so $v(a'_c - b) = \infty$ as well.

Case 3. *There is $\beta \in v(G^{\neq})$ such that $h(a_i, b_2) = \beta$ for every $i \in I$.* This case follows from Lemma 5.4.

Case 4. *There is $l \in \{1, \dots, m\}$ such that $h(a_i, b_2) = a_{i,l}$ for every $i \in I$.* This case follows from Lemma 5.5.

§6. Connection to dense pairs and independent sets.

In [9], Hieronymi and Nell considered whether certain commonly studied pair structures were distal. These include expansions of o-minimal structures by dense independent sets [6] and dense pairs of o-minimal structures [14]. That is, expansions by a dense independent set and expansions by proper, dense elementary substructures. We now show that a model of T_{Ham} interprets both the expansion of an ordered C-vector space by a dense C-independent set and the expansion by a proper, dense elementary substructure.

Corollary 6.1. *Let $\boldsymbol{M}$ be a model of T_{Ham} with underlying set G_∞. Then there are*

(1) *a definable, dense, C-linearly independent $H \subseteq G$ and*
(2) *a definable $S \subsetneq G$ that is the underlying set of an elementary substructure of G as an ordered C-vector space.*

Hence T_{Ham} interprets a distal expansion for both independent pairs of ordered C-vector spaces and dense pairs of ordered C-vector spaces.

Proof. Note that $H := v(G^{\neq})$ is C-linearly independent by Lemma 4.3 and $<_0$-dense in G, since $\boldsymbol{M}$ is dense as a Hamel space. Thus the structure $(G; +, <_0, (\lambda_c)_{c \in C}, H)$ is an independent pair of ordered C-vector spaces.

With H as above, consider the upward-closed subset of the value set:

$$H_0 := \{h \in H : h >_0 0\} \cup \{\infty\}.$$

This yields a certain "generalized ball":

$$S := \{g \in G : vg \in H_0\}.$$

S is closed under C-linear combinations. Furthermore, as $\boldsymbol{M}$ is independent, S is dense in G with respect to the $<_0$-ordering. Thus the pair $(G; +, <_0, (\lambda_c)_{c \in C}, S)$ is a dense pair of ordered C-vector spaces. ⊣

In particular, both independent pairs and dense pairs of ordered C-vector spaces admit a distal expansion. While this was known previously for dense pairs [10], this was unknown for independent pairs. The strategy used for dense pairs relied on manipulating certain imaginary sorts. However, independent pairs eliminate imaginaries in this setting [6]. Thus a new approach was necessary for this case.

Of course, in this article we restricted our attention to the case where the base o-minimal theory is an ordered vector space. The case of expanding a field is also of interest:

QUESTION 6.2. *Suppose* $\mathfrak{R}$ *is an o-minimal expansion of* $(\mathbb{R};+,\cdot)$ *and* $H\subseteq\mathbb{R}$ *is a dense and* $\mathrm{dcl}_{\mathfrak{R}}$*-independent subset of* $\mathbb{R}$. *Does* $(\mathfrak{R},H)$ *have a distal expansion*?

We believe the answer to be yes, and perhaps such a distal expansion can be constructed in a manner similar to our construction here (with a new valuation v and auxiliary ordering $<_1$). However, such an expansion will undoubtedly require stronger results from o-minimality than we used here.

§7. DP-rank. For the sake of completeness, in this final section we characterize the complexity of the theory T_{Ham} with regard to the notion of dp-*rank*. Among NIP theories, dp-rank provides a finer form of classification of the complexity of a theory. In this scale, dp-*minimal* is the simplest, followed by *having finite* dp-*rank*, and then *being strongly dependent*. We show that T_{Ham} is *not strongly dependent*, which is on the complicated end of the scale. For definitions of these concepts, see [13] or [12, Chapter 4].

THEOREM 7.1. T_{Ham} *is not strongly dependent. Therefore it is not* dp*-minimal and does not have finite* dp*-rank.*

A theory is strongly dependent iff it is NIP and *strong* (see [1, Corollary 11]), where *strong* is another property that a theory may or may not have. Thus, in order to show that T_{Ham} is not strongly dependent, it suffices to show that T_{Ham} is not strong. For this we will use the following:

PROPOSITION 7.2. [5, 2.14] *Suppose that* $\boldsymbol{M}=(M;+,<,\dots)$ *is an expansion of a densely-ordered abelian group. Let* $\boldsymbol{N}$ *be a saturated model of* $\mathrm{Th}(\boldsymbol{M})$, *and suppose that for every* $\varepsilon>0$ *in* $\boldsymbol{N}$ *there is an infinite definable discrete set* $X\subseteq\boldsymbol{N}$ *such that* $X\subseteq(0,\varepsilon)$. *Then* $\mathrm{Th}(\boldsymbol{M})$ *is not strong.*

PROOF OF THEOREM 7.1. Let $\boldsymbol{N}$ be a saturated model of T_{Ham} with underlying set G_∞. We cannot apply Proposition 7.2 directly to $\boldsymbol{N}$ since in a very strict sense $\boldsymbol{N}$ is not an expansion of an abelian group, due to the presence of the element ∞. Instead, we will apply it to the structure G_{ind}, the full induced structure on the definable set G. We recall the definition of G_{ind}:

We introduce the one-sorted language $\mathcal{L}_{\mathrm{ind}}$ which contains, for each $\mathcal{L}_{\mathrm{Ham}}$-formula $\varphi(y_1,\dots,y_n)$ an n-ary relation symbol R_φ; then G_{ind} is the $\mathcal{L}_{\mathrm{ind}}$-structure with underlying set G where each relation symbol R_φ is interpreted by $\varphi^{\boldsymbol{N}}\cap G^n$.

The structure G_{ind} is bi-interpretable with $\boldsymbol{N}$ and it actually is an expansion of a densely-ordered abelian group. Essentially, G_{ind} is the same thing as $\boldsymbol{N}$ except with the element ∞ removed. We will apply Proposition 7.2 to this structure. Let $\varepsilon>_1 0$. Pick $g\in G$ such that $g=vg>_0 v\varepsilon$, which is possible by denseness. It is easily checked that the definable subset

$$X\ :=\ \{x\in G: x=vx\ \&\ x>_0 g\}$$

of G is an infinite discrete set (with respect to the order topology induced by $<_1$) such that $X\subseteq(0,\varepsilon)_1$. Thus T_{Ham} is not strong. ⊣

In general, among NIP theories there is no clear correlation between distality and dp-rank. We started with a non-distal structure $(\mathbb{R};+,<,H)$ which is strongly

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dependent [6, 2.28], and constructed a distal expansion $(\mathbb{R}; +, <, <_1, v, \infty)$ which is not strongly dependent. Perhaps there is a "milder" distal expansion out there:

QUESTION 7.3. *Does* $(\mathbb{R}; +, <, H)$ *admit a strongly dependent distal expansion*?

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