Proceedings of the Edinburgh Mathematical Society (2018) **61**, 387–400
doi:10.1017/S0013091517000256

RECURRENCE TO SHRINKING TARGETS ON TYPICAL SELF-AFFINE FRACTALS

HENNA KOIVUSALO[1]* AND FELIPE A. RAMÍREZ[2]

[1] *Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 1090 Vienna, Austria* (henna.koivusalo@univie.ac.at)
[2] *Wesleyan University, Middletown, CT, USA* (framirez@wesleyan.edu)

(Received 30 October 2016)

Abstract We explore the problem of finding the Hausdorff dimension of the set of points that recur to shrinking targets on a self-affine fractal. To be exact, we study the dimension of a certain related symbolic recurrence set. In many cases, this set is equivalent to the recurring set on the fractal.

Keywords: self-affine sets; shrinking target; Hausdorff dimension

2010 *Mathematics subject classification:* Primary 37C45
Secondary 28A80

1. Introduction

Given a dynamical system (X, T) and a sequence $\mathcal{B} = (B_k)$ of subsets of X, a *shrinking target problem* is the study of the size of the recurring set

$$R(\mathcal{B}) = \left\{x \in X : T^k(x) \in B_k \text{ for infinitely many } k \in \mathbb{N}\right\}.$$

Viewing the sets B_k as 'targets,' the set $R(\mathcal{B})$ is the set of points whose orbits under the dynamics hit the targets infinitely often.

Interest in recurrence is classical in the study of dynamical systems. There are many ways to interpret the 'size' of $R(\mathcal{B})$. One way is with respect to an invariant measure on X. In this context, for example, for any positive measure set $B \subset X$, the Poincaré recurrence theorem implies that almost all points of X return to B infinitely often, so that $R(\mathcal{B})$ has full measure when $B_k = B$ for all k. It is thus fairly natural to ask what happens when the targets shrink. In many settings $R(\mathcal{B})$ has either full or zero measure depending on the rate of shrinking of B_k. See, for example, Chernov and Kleinbock [5] for a general treatment of such a problem.

For exponentially decreasing balls on Julia sets, Hill and Velani calculated first the Hausdorff dimension [10] and then the Hausdorff measure [12] of the recurring set, in

* Corresponding author.

the latter case proving a zero–one law for $R(\mathcal{B})$. Further, in [**11**] they considered the dimension of $R(\mathcal{B})$ for toral automorphisms. More recently, the problem has been studied for infinitely branching Markov maps [**24**], piecewise expanding maps of the unit interval [**22**] and the dynamical system of continued fraction expansions [**20**]. In all of these cases, the dimension of the recurring set is given as a zero point of a generalized pressure functional.

Recurring points are not only interesting objects in abstract settings; they also appear in number theory in the study of Diophantine approximation. The doubly metric inhomogeneous version of Khintchine's theorem, usually stated as a zero–one law for the set of 'approximable' pairs in $\mathbb{R}^d \times \mathbb{R}^d$, can be interpreted through Fubini's theorem as a zero–one law for the recurring set of shrinking targets on a torus, which holds for almost any toral translation. This connection to toral translations is made pleasantly visible by Fayad [**8**], including a proof of a result of Kurzweil [**19**] from this point of view. Similarly, one can interpret the classical Jarník–Besicovitch theorem as a dimension result for a recurring set of shrinking targets on the torus. For a more recent example, see Bugeaud *et al.* [**4**].

The above results mainly focus on conformal dynamical systems. In this note, we study the shrinking target problem in a simple non-conformal case, namely, under affine dynamics. In particular, we will calculate the dimension of the recurring set on a certain fractal set, the definition of which will be given next.

Hutchinson [**13**] originated the systematic study of iterated function systems (IFSs) $\{f_1, \dots, f_m\}$, which define a fractal set through the identity $E = \cup_{i=1}^m f_i(E)$. In particular, the *self-affine fractal* is the unique, non-empty, compact set E satisfying

$$E_{(a_1,\dots,a_m)} = E = \bigcup_{i=1}^{m} T_i(E) + a_i = \bigcup_{i=1}^{m} f_i(E),$$

generated by iterating the affine maps $\{f_i = T_i + a_i\}_{i=1,\dots,m}$. Here, $\{T_1, \dots, T_m\}$ is a collection of bijective linear contractions on $\mathbb{R}^d$, and $a_1, \dots, a_m \in \mathbb{R}^d$ are translation vectors. If the images $f_i(E)$ are disjoint, it is possible to define a mapping $F : E \to E$ by setting

$$F(x) = f_i^{-1}(x)$$

for points $x \in f_i(E)$, for all $i = 1, \dots, m$. This sets up a dynamical system (E, F), and so it makes sense to formulate a shrinking target problem on E.

The study of dimensional properties of self-affine sets comes in three branches. First, as self-affine carpets owing to, originally, Bedford [**1**] and McMullen [**21**]; second, for generic self-affine sets in the footsteps of Falconer [**6**]; and, third, for *every* self-affine set in some family, as in [**7**]. In this paper, we consider a shrinking target problem associated with the set E in the generic sense, that is, for generic translations $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{R}^{dm}$. A complementary problem of mass escape for carpet-type sets has been studied by Ferguson *et al.* in [**9**].

The main result of this article can be stated informally as follows.

Main Theorem (an informal version of Theorem 5.1). *The dimension of the recurring set to shrinking targets on self-affine fractals is almost surely the unique zero point* $s_0 \in \mathbb{R}^+$ *of a modified pressure function* $P(s)$.

We will impose a quasi-multiplicativity condition (4.3) on the maps T_i, and give the modified pressure formula (5.1). In this article, the target sets B_k are dynamical balls corresponding to cylinder sets in the symbolic space (for definitions, see § 2.1). These target sets match the dynamics on the set. The method of proof is largely classical, and the main difficulty lies in finding the right definition for the pressure functional. Also, in order to obtain the lower bound we need to construct a Cantor-type subset of the recurring set. We do this in the spirit of the proof of the mass transference principle of Beresnevich and Velani [**2**]. Their result cannot be applied to the affine setting directly.

The classical approach is to study the size of the recurring set in terms of invariant measures, and it turns out that, at least in certain cases, this problem is easily solved based on [**5**]. In particular, in § 6 we notice that as a corollary of [**5**, Theorem 2.1] it is possible to measure the size of the recurring set in terms of the Gibbs measure at the dimension of the self-affine set E. According to [**15**], Gibbs measures are measures of maximal dimension on E, which in part motivates their use in this context. For further reading on general thermodynamic formalism and Gibbs measures we refer to [**3**], and in relation to self-affine sets, to [**15**]).

Corollary to [5] (an informal version of Corollary 6.1). *In terms of the Gibbs measure at the dimension of the self-affine set E, either almost all or almost no points are recurring, depending on whether the sum of the measures of the target sets diverges or converges.*

Even though these formulations of the shrinking target problem give a very natural starting point to the analysis of shrinking targets for non-conformal dynamics, some questions remain. We discuss these at the end of § 6.

2. Shrinking targets, fractals and symbolic dynamics

Let $E_{\boldsymbol{a}}$, f_i, T_i be as in the introduction.

2.1. Self-affine fractals and symbolic dynamics

It is standard to associate with $E_{\boldsymbol{a}}$ a symbolic dynamical system given by the left shift σ on the sequence space $\mathcal{A}^{\mathbb{N}} = \{1, \ldots, m\}^{\mathbb{N}}$. One identifies points $\boldsymbol{i} = (i_1, i_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$ with points on $E_{\boldsymbol{a}}$ through the projection

$$\pi(\boldsymbol{i}) = \lim_{k\to\infty} f_{i_1} \circ \cdots \circ f_{i_k}(0) \in E_{\boldsymbol{a}},$$

and, if this projection is injective, then σ induces a well-defined dynamical system on $E_{\boldsymbol{a}}$, taking the point $\pi(\boldsymbol{i}) = p \in E_{\boldsymbol{a}}$ to $f_{i_1}^{-1}(p)$. In this case, the shrinking targets problem (defined in the next section) on the symbolic dynamical system corresponds to a shrinking targets problem on the fractal with the dynamics induced by σ.

2.2. Shrinking targets on a symbolic dynamical system

A natural way to define a shrinking targets problem on the dynamical system $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is to select a target point $\boldsymbol{j} \in \mathcal{A}^{\mathbb{N}}$ and define the targets to be cylinder sets

$$[\boldsymbol{j}|_k] = \left\{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \boldsymbol{i}|_k = \boldsymbol{j}|_k\right\} \subset \mathcal{A}^{\mathbb{N}}$$

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

of increasing length $k \in \mathbb{N}$ (and hence decreasing diameter) around that point, where $\boldsymbol{j}|_k$ and $\boldsymbol{i}|_k$ denote the truncations $(j_1, \dots, j_k)$ and $(i_1, \dots, i_k)$ of $\boldsymbol{j}$ and $\boldsymbol{i}$, respectively.

In this paper, we treat the shrinking targets problem for $\mathcal{B} = \{B_k\}_{k\in\mathbb{N}}$ where $B_k = [\boldsymbol{j}|_{\ell_k}]$ and $\{\ell_k\} \subseteq \mathbb{N}$ is some fixed (non-decreasing) sequence. We denote the recurring set $R(\mathcal{B})$ by $R(\boldsymbol{j})$. It is exactly

$$\begin{aligned} R(\boldsymbol{j}) &= \left\{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \sigma^k(\boldsymbol{i})|_{\ell_k} = \boldsymbol{j}|_{\ell_k} \text{ for infinitely many } k \in \mathbb{N}\right\} \\ &= \limsup_{k\to\infty} R(\boldsymbol{j}, k) \end{aligned}$$

where $R(\boldsymbol{j}, k) = \left\{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \sigma^k(\boldsymbol{i})|_{\ell_k} = \boldsymbol{j}|_{\ell_k}\right\}$.

2.3. Shrinking targets on self-affine fractals

We project our symbolic shrinking targets problem to the fractal $E_{\boldsymbol{a}}$ through π. That is, we seek the Hausdorff dimension of the set

$$\widetilde{R}(\pi(\boldsymbol{j})) := \pi(R(\boldsymbol{j})) \subset E_{\boldsymbol{a}}.$$

An intuitive interpretation is to think of the shrinking targets around $\pi(\boldsymbol{j})$ as the succession of 'construction sets' of $E_{\boldsymbol{a}}$ to which $\pi(\boldsymbol{j})$ belongs. (One may think of the standard Cantor set, for example, and take the shrinking targets to be subintervals appearing in its standard construction.) Although these are very natural shrinking targets on a fractal, the notion is really only meaningful if the affine maps $\{f_i\}$ satisfy a strong enough separation condition. Still, the set $\widetilde{R}(\pi(\boldsymbol{j}))$ is well defined and can be studied regardless.

3. List of notations

This list of standard notations may be used for reference.

- $\mathcal{A}^{\mathbb{N}} = \{1, \dots, m\}^{\mathbb{N}}$ is the sequence space on m letters.
- $\boldsymbol{j}|_k = j_1 j_2 \dots j_k$ is the string obtained by truncating $\boldsymbol{j} \in \mathcal{A}^{\mathbb{N}}$.
- $|\boldsymbol{i}|$ is the length of a finite word $\boldsymbol{i} = (i_1, \dots, i_k)$, so for example $|\boldsymbol{j}|_k| = k$.
- $\boldsymbol{i}\boldsymbol{j}|_k = i_1 i_2 \dots i_{|\boldsymbol{i}|} j_1 j_2 \dots j_k$ is a concatenation of finite words.
- $\boldsymbol{i} \wedge \boldsymbol{i}'$ is the maximal common beginning of $\boldsymbol{i}, \boldsymbol{i}' \in \mathcal{A}^{\mathbb{N}}$.
- $[\boldsymbol{q}] = \{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \boldsymbol{i}|_{|\boldsymbol{q}|} = \boldsymbol{q}\}$ is the cylinder set defined by the finite word $\boldsymbol{q}$.
- Let $f_i : \mathbb{R}^d \to \mathbb{R}^d$ be the affine maps $f_i(x) = T_i(x) + a_i$.
- For $|\boldsymbol{i}| = k$, denote $f_{\boldsymbol{i}} = f_{i_1} \circ \dots \circ f_{i_k}$, and similarly for $T_{\boldsymbol{i}}$.
- $R(\boldsymbol{j}) = \{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \sigma^k(\boldsymbol{i})|_{\ell_k} = \boldsymbol{j}|_{\ell_k} \text{ infinitely often}\}$ is our symbolic recurring set.
- $\widetilde{R}(\pi(\boldsymbol{j})) = \pi(R(\boldsymbol{j}))$.
- $R(k) = R(\boldsymbol{j}, k) = \{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \sigma^k(\boldsymbol{i})|_{\ell_k} = \boldsymbol{j}|_{\ell_k}\}$, so that we have $R(\boldsymbol{j}) = \limsup R(k)$.

4. The singular value function

For a linear map $T : \mathbb{R}^d \to \mathbb{R}^d$, define the singular values

$$\sigma_1(T) \geqslant \cdots \geqslant \sigma_d(T)$$

as the lengths of the semiaxes of the ellipsoid $T(B(0,1))$. For any $0 \leqslant t \leqslant d$, let

$$\varphi^t(T) = \sigma_1(T) \dots \sigma_{\lfloor t \rfloor}(T)\sigma_{\lfloor t \rfloor+1}(T)^{\{t\}}$$

where $\lfloor t \rfloor$ is the integer part of t and $\{t\}$ is the fractional part. The function $\varphi^t(T)$ is called the singular value function, and it satisfies

$$\sigma_d(T)^t \leqslant \varphi^t(T) \leqslant \sigma_1(T)^t. \tag{4.1}$$

Furthermore, if $T_1, \dots, T_n$ are linear maps on $\mathbb{R}^d$, then

$$\min_{i=1,\dots,n} \sigma_d(T_i)^\delta \leqslant \frac{\sum_{i=1}^n \varphi^{t+\delta}(T_i)}{\sum_{i=1}^n \varphi^t(T_i)} \leqslant \max_{i=1,\dots,n} \sigma_1(T_i)^\delta \tag{4.2}$$

for all sufficiently small $\delta > 0$. The singular value function is submultiplicative, that is, if $U : \mathbb{R}^d \to \mathbb{R}^d$ is also a linear map, then $\varphi^t(TU) \leqslant \varphi^t(T)\varphi^t(U)$. For proofs see [**6**], for example.

We study collections of linear maps for which there exists some constant $D \in (0,1)$ such that

$$\varphi^t(T_{\boldsymbol{i}\boldsymbol{i}'}) \geqslant D\varphi^t(T_{\boldsymbol{i}})\varphi^t(T_{\boldsymbol{i}'}) \text{ for any finite words } \boldsymbol{i}, \boldsymbol{i}'. \tag{4.3}$$

This condition holds, for example, for similarities and for maps on $\mathbb{R}^2$ satisfying the *cone condition*, meaning that there is a cone that is mapped strictly into itself by all the T_i's (see Käenmäki and Shmerkin [**16**]).

5. The main result and its proof

Given the affine maps $\{f_i\}_{i=1,\dots,m}$, a point $\boldsymbol{j} \in \mathcal{A}^{\mathbb{N}}$ and $s \in \mathbb{R}^+$, we define

$$P(s, \boldsymbol{j}) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\boldsymbol{i}|=k} \varphi^s(T_{\boldsymbol{i}\boldsymbol{j}|_{\ell_k}}). \tag{5.1}$$

This is a modification of the standard pressure function (see [**6**], for example), which is defined to be

$$\mathcal{P}(s) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\boldsymbol{i}|=k} \varphi^s(T_{\boldsymbol{i}}). \tag{5.2}$$

In fact, notice* that in the presence of (4.3) we have

$$P(s, \boldsymbol{j}) = \mathcal{P}(s) + \lim_{k\to\infty} \frac{1}{k} \log \varphi^s(T_{\boldsymbol{j}|_{\ell_k}}). \tag{5.3}$$

Notice also that it is possible for this limit to not exist. However, if it *does* exist for some $\boldsymbol{j}$ and $s \in \mathbb{R}^+$, then in fact $P(s, \boldsymbol{j})$ is a continuous and non-increasing function of s.

* We thank an anonymous referee for making this observation.

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

In this case, we prove that the Hausdorff dimension of the recurring set $\widetilde{R}(\pi(\boldsymbol{j}))$ is almost surely the unique zero point $s_0 \in \mathbb{R}^+$ of $P(\cdot, \boldsymbol{j})$.

Theorem 5.1. *Let $E_{\boldsymbol{a}}$ be the fractal determined by the affine maps $\{T_i + a_i\}_{i=1}^m$, and suppose there is some $D \in (0,1)$ such that (4.3) holds. Furthermore, assume that $\max_i \sigma_1(T_i) < (1/2)$. Consider shrinking targets defined by cylinder sets of length $\{\ell_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ around some fixed $\boldsymbol{j} \in \mathcal{A}^{\mathbb{N}}$.*

If $\lim_{k\to\infty}(1/k)\log\varphi^s(T_{\boldsymbol{j}|_{\ell_k}})$ exists in $[-\infty, 0]$, then with respect to any ergodic σ-invariant probability measure there is almost surely a unique number $s_0 \in \mathbb{R}^+$ satisfying $P(s_0, \boldsymbol{j}) = 0$ and, for almost every $\boldsymbol{a} \in \mathbb{R}^{dm}$, the Hausdorff dimension of the recurring set $\widetilde{R}(\pi(\boldsymbol{j})) \subset E_{\boldsymbol{a}}$ equals s_0.

Remark 5.2. If the maps $\{T_i + a_i\}_{i=1}^m$ satisfy the strong separation condition, then our shrinking targets in the symbolic space correspond naturally to shrinking targets on $E_{\boldsymbol{a}}$, and $\widetilde{R}(\pi(\boldsymbol{j}))$ is their recurring set under the dynamical system on $E_{\boldsymbol{a}}$ induced by the shift in the symbolic space.

Of course, it now becomes a question of *when* $\lim_{k\to\infty}(1/k)\log\varphi^s(T_{\boldsymbol{j}|_{\ell_k}})$ exists. In the following corollary, we show that in particular it exists if $\lim_{k\to\infty}\ell_k/k$ does.

Corollary 5.3. *With the assumptions and notation of Theorem 5.1, if $\lim_{k\to\infty}\ell_k/k$ exists, then Theorem 5.1 applies, and we will have*

$$\dim_{\mathcal{H}} \widetilde{R}(\pi(\boldsymbol{j})) = \begin{cases} \dim_{\mathcal{H}} E_{\boldsymbol{a}} & \text{if } \lim_{k\to\infty}\ell_k/k = 0 \\ s_0 & \text{if } \lim_{k\to\infty}\ell_k/k \in (0,\infty) \\ 0 & \text{if } \lim_{k\to\infty}\ell_k/k = \infty \end{cases} \tag{5.4}$$

almost surely (in the sense of the theorem).

Proof. If $\lim_{k\to\infty}\ell_k/k = \infty$, then the result follows from Lemma 5.6, and in fact holds for *every* target point $\boldsymbol{j}$ and translation $\boldsymbol{a}$.

If $\lim_{k\to\infty}\ell_k/k \in [0,\infty)$, then Lemma 5.4 implies the existence of $\lim_{k\to\infty}(1/k)\log\varphi^s(T_{\boldsymbol{j}|_{\ell_k}})$, so Theorem 5.1 applies. It is only left to observe that in the case where $\ell_k = o(k)$, we will have $\mathcal{P}(s) = P(s, \boldsymbol{j})$, so the unique zero points coincide, and therefore the recurring set has the same Hausdorff dimension as the fractal. □

5.1. Existence and uniqueness

The next lemma shows that if there is some $L \in [0,\infty)$ for which $\ell_k \sim L \cdot k$, then there exists a unique number $s_0 \in \mathbb{R}^+$ such that $P(s_0, \boldsymbol{j}) = 0$. It is furthermore almost surely independent of $\boldsymbol{j}$ with respect to any σ-invariant measure.

Lemma 5.4. *With the same assumptions and notation as in Theorem 5.1, suppose that $\lim_{k\to\infty}\ell_k/k \in [0,\infty)$. Suppose that μ is a σ-ergodic probability measure on $\mathcal{A}^{\mathbb{N}}$. Then there exists a number $s_0 \in \mathbb{R}^+$ satisfying $P(s_0, \boldsymbol{j}) = 0$ for μ-almost every $\boldsymbol{j} \in \mathcal{A}^{\mathbb{N}}$.*

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

Proof. Suppose $\ell_k \sim L \cdot k$, $L > 0$. Fix s. Let

$$X(m,n) = \log \varphi^s(T_{\sigma^m(\boldsymbol{j})|_{n-m}}).$$

We will apply Kingman's subadditive ergodic theorem to the sequence $X(m,n)$ with respect to the transformation σ, so we must check that X is subadditive and σ-equivariant. Clearly all $X(m,n)$ are integrable, since they only depend on finite words.

Equivariance is verified by the following routine calculation:

$$\begin{aligned} X(m,n) \circ \sigma &= \log \varphi^s(T_{\sigma^m(\sigma(\boldsymbol{j}))|_{n-m}}) \\ &= \log \varphi^s(T_{\sigma^{(m+1)}(\boldsymbol{j})|_{(n+1)-(m+1)}}) = X(m+1, n+1). \end{aligned}$$

For *subadditivity*, we must check that $X(0,n) \leqslant X(0,m) + X(m,n)$. Again, this is routine:

$$\begin{aligned} X(0,m) + X(m,n) &= \log \varphi^s(T_{\boldsymbol{j}|_m}) + \log \varphi^s(T_{\sigma^m(\boldsymbol{j})|_{n-m}}) \\ &= \log \varphi^s(T_{\boldsymbol{j}|_m})\varphi^s(T_{\sigma^m(\boldsymbol{j})|_{n-m}}) \geqslant \log \varphi^s(T_{\boldsymbol{j}|_n}) = X(0,n). \end{aligned}$$

Therefore, by the subadditive ergodic theorem, there almost surely exists a limit

$$\lim_{n\to\infty} \frac{X(0,n)}{n} = \lim_{k\to\infty} \frac{X(0,\ell_k)}{\ell_k}$$

which is σ-invariant and so almost everywhere constant in $\boldsymbol{j}$ with respect to any σ-ergodic measure. Hence,

$$\lim_{k\to\infty} \frac{X(0,\ell_k)}{k} = \lim_{k\to\infty} \frac{X(0,\ell_k)}{\ell_k} \cdot \frac{\ell_k}{k} = L \lim_{k\to\infty} \frac{X(0,\ell_k)}{\ell_k}$$

also exists and is almost surely constant in $\boldsymbol{j}$, and similarly for

$$P(s,\boldsymbol{j}) = \mathcal{P}(s) + \lim_{k\to\infty} \frac{X(0,\ell_k)}{k}.$$

Notice that the limit is continuous and strictly decreasing in s, since by (4.2),

$$\sigma_-^{(k+\ell_k)\delta} \leqslant \frac{\sum_{|\boldsymbol{i}|=k} \varphi^{s+\delta}(T_{\boldsymbol{ij}|_{\ell_k}})}{\sum_{|\boldsymbol{i}|=k} \varphi^{s}(T_{\boldsymbol{ij}|_{\ell_k}})} \leqslant \sigma_+^{(k+\ell_k)\delta},$$

where $0 < \sigma_- = \min_i \sigma_d(T_i)$ and $\sigma_+ = \max_i \sigma_1(T_i) < 1$.

Since $P(0,\boldsymbol{j}) > 0$ and $P(s,\boldsymbol{j}) \to -\infty$ when $s \to \infty$, the existence of a unique zero follows. □

5.2. Upper bound

The following lemma shows that s_0 is an upper bound for the dimension of the recurring set $\widetilde{R}(\pi(\boldsymbol{j}))$.

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

Lemma 5.5. *Let $\{\ell_k\} \subseteq \mathbb{N}$ be a non-decreasing divergent sequence and suppose that $P(\cdot, \boldsymbol{j})$ is decreasing, with unique zero $s_0 \in \mathbb{R}$. Then the Hausdorff dimension of $\widetilde{R}(\pi(\boldsymbol{j}))$ is bounded above by s_0.*

Proof. We will show that for any $t > s_0$, we have $\mathcal{H}^t(\widetilde{R}(\pi(\boldsymbol{j})) = 0$, where $\mathcal{H}^t$ denotes Hausdorff measure.

Notice that when $t > s_0$, then for all large k,

$$\sum_{|\boldsymbol{i}|=k} \varphi^t(T_{\boldsymbol{ij}|_{\ell_k}}) \leqslant \exp(-k\varepsilon)$$

for some $\varepsilon > 0$. Therefore

$$\sum_{k=0}^{\infty} \sum_{|\boldsymbol{i}|=k} \varphi^t(T_{\boldsymbol{ij}|_{\ell_k}}) < \infty. \tag{5.5}$$

For all N, we have that

$$R(\boldsymbol{j}) = \limsup_{k\to\infty} \{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} \mid \sigma^k(\boldsymbol{i})|_{\ell_k} = \boldsymbol{j}|_{\ell_k}\} \subset \bigcup_{k=N}^{\infty} \bigcup_{|\boldsymbol{i}|=k} [\boldsymbol{ij}|_{\ell_k}],$$

hence,

$$\widetilde{R}(\pi(\boldsymbol{j})) \subset \bigcup_{k=N}^{\infty} \bigcup_{|\boldsymbol{i}|=k} \pi[\boldsymbol{ij}|_{\ell_k}].$$

Since the T_is are contractions, there is some $M > 0$ sufficiently large that $f_i(B(0,M)) \subset B(0,M)$ for all $i = 1, \ldots, m$. Then we have the inclusion $\pi[\boldsymbol{ij}|_{\ell_k}] \subset f_{\boldsymbol{ij}|_{\ell_k}}(B(0,M))$, and therefore

$$\widetilde{R}(\pi(\boldsymbol{j})) \subset \bigcup_{k=N}^{\infty} \bigcup_{|\boldsymbol{i}|=k} f_{\boldsymbol{ij}|_{\ell_k}}(B(0,M)).$$

Recall that $\sigma_1(T_{\boldsymbol{ij}|_{\ell_k}}), \ldots, \sigma_d(T_{\boldsymbol{ij}|_{\ell_k}})$ are the lengths of the semiaxes of the ellipsoid $T_{\boldsymbol{ij}|_{\ell_k}}(B(0,1))$, in decreasing order. Therefore, the number of cubes of side-length $\sigma_{\lfloor t\rfloor+1}q(T_{\boldsymbol{ij}|_{\ell_k}})$ required to cover the sets $f_{\boldsymbol{ij}|_{\ell_k}}(B(0,M))$ is bounded by

$$A\sigma_1(T_{\boldsymbol{ij}|_{\ell_k}}) \ldots \sigma_{\lfloor t\rfloor+1}(T_{\boldsymbol{ij}|_{\ell_k}})^{-\lfloor t\rfloor},$$

where $A > 0$ is some constant. We can bound the Hausdorff measure

$$\begin{aligned}\mathcal{H}^t(\widetilde{R}(\pi(\boldsymbol{j}))) &\leqslant A \sum_{k=N}^{\infty} \sum_{|\boldsymbol{i}|=k} \sigma_1(T_{\boldsymbol{ij}|_{\ell_k}}) \ldots \sigma_{\lfloor t\rfloor+1}(T_{\boldsymbol{ij}|_{\ell_k}})^{-\lfloor t\rfloor} \cdot \sigma_{\lfloor t\rfloor+1}(T_{\boldsymbol{ij}|_{\ell_k}})^t \\ &= A \sum_{k=N}^{\infty} \sum_{|\boldsymbol{i}|=k} \varphi^t(T_{\boldsymbol{ij}|_{\ell_k}}).\end{aligned}$$

But $N \in \mathbb{N}$ was arbitrary, so, recalling (5.5), we can take $N \to \infty$ to see that $\mathcal{H}^t(\widetilde{R}(\pi(\boldsymbol{j}))) = 0$. This proves the lemma. □

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

The next lemma shows that if the shrinking targets are projected cylinder sets whose lengths grow superlinearly, then the corresponding recurring set is zero-dimensional.

Lemma 5.6. *Suppose that* $\lim_{k\to\infty} \ell_k/k = \infty$. *Then* $\dim_{\mathcal{H}} \widetilde{R}(\pi(\boldsymbol{j}), \ell_k) = 0$.

Proof. We show that $\mathcal{H}^\delta(\widetilde{R}(\pi(\boldsymbol{j}), \ell_k)) = 0$ for any $\delta > 0$.
Let $g(k) = \ell_k/k$. Then $g(k) \to \infty$ as $k \to \infty$, and

$$\sum_{k\in\mathbb{N}} \sum_{|\boldsymbol{i}|=k} \varphi^\delta(T_{\boldsymbol{ij}|_{\ell_k}}) \leqslant \sum_{k\in\mathbb{N}} m^k (\sigma_+^\delta)^{k+\ell_k} \leqslant \sum_{k\in\mathbb{N}} (m \cdot \sigma_+^{\delta(1+g(k))})^k < \infty$$

because for all k large enough, the parenthetical quantity is $< 1/2$ (say). The argument from the proof of Lemma 5.5 shows that $\mathcal{H}^\delta(\widetilde{R}(\pi(\boldsymbol{j}))) = 0$, and proves the lemma. □

5.3. Lower bound

Fix numbers $s < t < s_0$. Let (n_k) be a sequence of natural numbers. Let $C = \bigcap_{k=1}^\infty R(n_k)$, where

$$\begin{aligned} R(n_k) &= \{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \sigma^{n_k}(\boldsymbol{i})|_{\ell_{n_k}} = \boldsymbol{j}\,|_{\ell_{n_k}}\} \\ &= \left\{\boldsymbol{i} \in \mathcal{A}^{\mathbb{N}} : \sigma^{n_k}(\boldsymbol{i}) \in B_{n_k}\right\}. \end{aligned}$$

Notice that $C(k) = R(n_k)$ are finite unions of compact cylinder sets and thus compact, so that C is compact as well. Furthermore, $C \subset R = \limsup R(k)$, and we hope to prove that $\dim_{\mathcal{H}} C \geqslant s$.

Lemma 5.7. *Let* $E_{\boldsymbol{a}}$ *and* $\{\ell_k\}$ *be as in Theorem 5.1. Fix* $t < s_0$. *Then there is a probability measure* μ *supported on* C, *and a constant* H, *satisfying*

$$\mu[\boldsymbol{q}] \leqslant H\varphi^t(T_{\boldsymbol{q}}) \tag{5.6}$$

for all finite words $\boldsymbol{q}$.

Remark 5.8. A measure μ satisfying (5.6) could possibly be found through a construction similar to one used in [**23**], or perhaps an adaptation of [**14**], and this may relieve us from having to assume quasi-multiplicativity (4.3). However, quasi-multiplicativity is also needed in the observation (5.3). For this reason, we have opted to present a self-contained proof of the existence of μ, even if our proof *does* rely on (4.3).

Proof. Let (n_k) be a sequence of natural numbers to be fixed later, with the property that $n_k \geqslant n_{k-1} + \ell_{n_{k-1}}$ for all $k \in \mathbb{N}$. Let

$$\mu_k = \frac{\sum_{\boldsymbol{i}\in A(k)} \varphi^t(T_{\boldsymbol{i}})\delta_{\boldsymbol{i}}}{\sum_{\boldsymbol{i}\in A(k)} \varphi^t(T_{\boldsymbol{i}})}$$

where

$$A(k) = \left\{\boldsymbol{i} : |\boldsymbol{i}| = k \text{ and } \exists \boldsymbol{j} \in \mathcal{A}^{\mathbb{N}} \text{ with } \boldsymbol{ij} \in C\right\}$$

is the set of $\boldsymbol{i}$ of length k that can be extended to be elements of C (as defined above). In this way, μ_{n_k} is supported on the set of points that recur to the shrinking targets at times

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

$n_1, n_2, \ldots, n_k$. Define the probability measure μ as the limit of a convergent subsequence of $\{\mu_k\}$.

Let $|\boldsymbol{q}| = q$. With condition (4.3) we have that

$$\begin{aligned}\mu_k[\boldsymbol{q}] &= \frac{\sum_{\substack{\boldsymbol{i}\in A(k)\\ \boldsymbol{i}|_q=\boldsymbol{q}}} \varphi^t(T_{\boldsymbol{i}})}{\sum_{\boldsymbol{i}\in A(k)} \varphi^t(T_{\boldsymbol{i}})} \leqslant \frac{\varphi^t(T_{\boldsymbol{q}}) \sum_{\substack{\boldsymbol{i}\in A(k)\\ \boldsymbol{i}|_q=\boldsymbol{q}}} \varphi^t(T_{\sigma^q(\boldsymbol{i})})}{D\sum_{\boldsymbol{i}\in A(k)} \varphi^t(T_{\boldsymbol{i}|_q})\varphi^t(T_{\sigma^q(\boldsymbol{i})})} \\ &= \frac{\varphi^t(T_{\boldsymbol{q}}) \sum_{\substack{\boldsymbol{i}\in A(k)\\ \boldsymbol{i}|_q=\boldsymbol{q}}} \varphi^t(T_{\sigma^q(\boldsymbol{i})})}{D\sum_{\boldsymbol{i}\in A(q)} \varphi^t(T_{\boldsymbol{i}}) \sum_{\substack{\boldsymbol{i}\in A(k)\\ \boldsymbol{i}|_q=\boldsymbol{q}}} \varphi^t(T_{\sigma^q(\boldsymbol{i})})} = \frac{\varphi^t(T_{\boldsymbol{q}})}{D\sum_{\boldsymbol{i}\in A(q)} \varphi^t(T_{\boldsymbol{i}})} \end{aligned} \tag{5.7}$$

for all $k > q$. We will now prove that the sequence $\{n_k\}$ can be chosen so that there is some $\kappa > 0$ such that

$$\Sigma_q := \sum_{\boldsymbol{i}\in A(q)} \varphi^t(T_{\boldsymbol{i}}) \geqslant \kappa$$

for all sufficiently large q. This will prove the lemma with $H = (D\kappa)^{-1}$.

Clearly, Σ_q decreases in intervals of the form $n_k \leqslant q \leqslant n_k + \ell_{n_k}$, so we first show that given a non-decreasing $\iota : \mathbb{N} \to \mathbb{R}$, we can choose $\{n_k\}$ so that $\Sigma_{n_k+\ell_{n_k}} \geqslant \iota(k)$ for all k.

Recall that $P(s, \boldsymbol{j})$ is decreasing in s. Therefore, since $t < s_0$, there is some $\varepsilon > 0$ such that

$$\sum_{|\boldsymbol{i}|=k} \varphi^t(T_{\boldsymbol{i}\boldsymbol{j}|_{\ell_k}}) \geqslant \exp(k\varepsilon) \tag{5.8}$$

for all k sufficiently large. Hence there is a number n_1 such that

$$\Sigma_{n_1+\ell_{n_1}} = \sum_{|\boldsymbol{i}|=n_1} \varphi^t(T_{\boldsymbol{i}\boldsymbol{j}|_{\ell_{n_1}}}) \geqslant \exp(n_1\varepsilon) \geqslant \iota(1).$$

Proceeding inductively, assume we have already chosen $n_1, n_2, \ldots, n_{k-1}$ such that $\Sigma_{n_{k-1}+\ell_{n_{k-1}}} \geqslant \iota(k-1)$. Then

$$\begin{aligned}\Sigma_{n_k+\ell_{n_k}} &= \sum_{|\boldsymbol{i}_1|=n_1} \sum_{|\boldsymbol{i}_2|=n_2-n_1-\ell_{n_1}} \cdots \sum_{|\boldsymbol{i}_k|=n_k-n_{k-1}-\ell_{n_{k-1}}} \varphi^t(T_{\boldsymbol{i}_1\boldsymbol{j}|_{\ell_{n_1}}\cdots \boldsymbol{i}_k\boldsymbol{j}|_{\ell_{n_k}}}) \\ &\overset{(4.3)}{\geqslant} \sum_{|\boldsymbol{i}_1|=n_1} \cdots \sum_{|\boldsymbol{i}_k|=n_k-n_{k-1}-\ell_{n_{k-1}}} D\varphi^t(T_{\boldsymbol{i}_1\boldsymbol{j}|_{\ell_{n_1}}\cdots \boldsymbol{i}_{k-1}\boldsymbol{j}|_{\ell_{n_{k-1}}}})\varphi^t(T_{\boldsymbol{i}_k\boldsymbol{j}|_{\ell_{n_k}}}) \\ &= \Sigma_{n_{k-1}+\ell_{n_{k-1}}} D \sum_{|\boldsymbol{i}|=n_k-n_{k-1}-\ell_{n_{k-1}}} \varphi^t(T_{\boldsymbol{i}\boldsymbol{j}|_{\ell_{n_k}}}).\end{aligned}$$

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

Now, by the induction hypothesis,

$$\geqslant \iota(k-1)D\bigg(\sum_{|\boldsymbol{i}|=n_{k-1}+\ell_{n_{k-1}}}\varphi^t(T_{\boldsymbol{i}})\bigg)^{-1}\sum_{|\boldsymbol{i}|=n_k}\varphi^t(T_{\boldsymbol{i}\boldsymbol{j}|_{\ell_{n_k}}})$$
$$\overset{(5.8)}{\geqslant}\Bigg[\iota(k-1)D\bigg(\sum_{|\boldsymbol{i}|=n_{k-1}+\ell_{n_{k-1}}}\varphi^t(T_{\boldsymbol{i}})\bigg)^{-1}\Bigg]\exp(n_k\varepsilon), \tag{5.9}$$

as long as we choose n_k sufficiently large. Notice that the quantity in square brackets is a positive number only depending on our choices of $n_1,\dots,n_{k-1}$, so we can certainly choose n_k large enough that (5.9) exceeds $\iota(k)$, as claimed.

Now, suppose that we are in an interval of the form $n_k+\ell_{n_k}\leqslant q\leqslant n_{k+1}$, and $q-n_k-\ell_{n_k}=\ell$. Then (4.3) implies that

$$\Sigma_q\geqslant D\Sigma_{n_k+\ell_{n_k}}\sum_{\boldsymbol{i}=\ell}\varphi^t(T_{\boldsymbol{i}}).$$

Since $t<s_0<\dim E_{\boldsymbol{a}}$, the expression $\sum_{\boldsymbol{i}=\ell}\varphi^t(T_{\boldsymbol{i}})$ is bounded below by a constant that is uniform over $\ell\in\mathbb{N}$, which proves that there is some $\kappa>0$ such that $\Sigma_q\geqslant\kappa$ for q sufficiently large, as wanted.

Returning to (5.7), the lemma is proved with $H=(D\kappa)^{-1}$. □

Lemma 5.9. *The image measure $\pi_*\mu$ given by Lemma 5.7 has finite s-energy for almost all $\boldsymbol{a}$.*

Proof. Using [6, Lemma 3.1], [25, Proposition 3.1] and Fubini's theorem, the problem inside any ball of radius ρ reduces to the study of finiteness of

$$I=\iint_{R\times R}\frac{\mathrm{d}\mu(\boldsymbol{i})\,\mathrm{d}\mu(\boldsymbol{k})}{\varphi^s(T_{\boldsymbol{i}\wedge\boldsymbol{k}})}.$$

Now, using (5.6), the fact that μ is a probability measure and $t>s$,

$$\begin{aligned}I&\leqslant\sum_{q=1}^{\infty}\sum_{|\boldsymbol{q}|=q}\sum_{i\neq k}\varphi^s(T_{\boldsymbol{q}})^{-1}\mu[\boldsymbol{q}i]\mu[\boldsymbol{q}k]\\&\leqslant Hm\sum_{q=1}^{\infty}\sum_{|\boldsymbol{q}|=q}\sum_{k=1}^{m}\frac{\varphi^t(T_{\boldsymbol{q}})}{\varphi^s(T_{\boldsymbol{q}})}\mu[\boldsymbol{q}k]\\&\leqslant Hm\sum_{q=1}^{\infty}\sigma_+^{q(t-s)}<\infty.\end{aligned}$$

Since ρ is arbitrary, the claim follows for almost all $\boldsymbol{a}$. □

Proof of Theorem 5.1. Fix a centre $\boldsymbol{j}$ and a sequence (ℓ_k) of target sizes such that the limit $\lim_{k\to\infty}\frac{1}{k}\log\varphi^s(T_{\boldsymbol{j}|_{\ell_k}})$ exists. The almost-sure existence and uniqueness of the

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

zero point s_0 of the pressure follow from Lemma 5.4, and the estimate $\dim_{\mathcal{H}} \widetilde{R}(\pi(\boldsymbol{j})) \leqslant s_0$ from Lemma 5.5.

For the lower bound, let $s < s_0$. By Lemma 5.9 and the potential theoretic characterization of the Hausdorff dimension, since μ is supported on C, we have $\dim_{\mathcal{H}} \pi(C) \geqslant s$ almost surely. Approaching s_0 along a sequence gives the lower bound for the dimension of $\pi(C) \subset \widetilde{R}(\pi(\boldsymbol{j}))$. □

Remark 5.10. Notice that the dimension of the recurring set is not affected by the set of points where π is not an injection, as long as this set has dimension smaller than s_0. This is the case for large classes of self-affine sets, for a Lebesgue-positive proportion of translation vectors $\boldsymbol{a}$, and in these cases there is the more geometric interpretation of § 2.3 for the result of Theorem 5.1.

6. Some further observations

It is classical to study the shrinking target problem for an invariant measure. Unfortunately, there is no canonical choice for a measure on a self-affine set. There is, however, a natural family of measures, called Gibbs measures, which we now define. In general, a Gibbs measure for the potential φ^t, or a *Gibbs measure at t*, is a σ-invariant probability measure μ on the symbolic space $\mathcal{A}^{\mathbb{N}}$ satisfying

$$C^{-1} \leqslant \frac{\mu[\boldsymbol{q}]}{\varphi^t(T_{\boldsymbol{q}}) \exp(-|\boldsymbol{q}|\mathcal{P}(t))} \leqslant C. \tag{6.1}$$

Unfortunately, as demonstrated by [**18**, Example 6.4], Gibbs measures do not always exist. However, under the assumption (4.3) they do, and, further, a Gibbs measure at the zero point of the pressure $\mathcal{P}$ is a measure of maximal dimension, that is,

$$\dim_{\mathcal{H}}(\pi_*\mu) = \dim_{\mathcal{H}} E_{\boldsymbol{a}}$$

for almost all $\boldsymbol{a} \in \mathbb{R}^{dm}$. These facts were proved in [**18**, Theorem 5.2], [**18**, p. 8–9] and [**17**, Theorem 2.2]. Also see [**15**, Theorem 4.1]. This means that the Gibbs measure is a natural candidate as a measure of the size of the recurring set. Now we are in the position to formulate and prove the following corollary.

Corollary 6.1 (Corollary to Theorem 2.1 of [5]). *Assume E is a self-affine set as in Theorem 5.1. Let $(\ell_k) \subset \mathbb{N}$ be an increasing sequence tending to ∞, and $\boldsymbol{j} \in \mathcal{A}^{\mathbb{N}}$. Denote by μ the Gibbs measure at the point t where $\mathcal{P}(t) = 0$. Then the measure of the recurrence set $\pi_*\mu(\widetilde{R}(\pi(\boldsymbol{j})))$ is either 0 or 1 according to whether the sum*

$$\sum_{k=1}^{\infty} \mu([\boldsymbol{j}_{\ell_k}]) \tag{6.2}$$

converges or diverges.

Proof. First, the fact that the convergence of the sum (6.2) implies $\pi_*\mu(\tilde{R}(\pi(\boldsymbol{j}))) = 0$ follows from the classical Borel–Cantelli lemma. This is the case since $\pi_*\mu(\widetilde{R}(\pi(\boldsymbol{j}))) = \mu(R(\boldsymbol{j}))$ and μ is σ-invariant.

https://doi.org/10.1017/S0013091517000256 Published online by Cambridge University Press

Now assume that the sum (6.2) diverges. Notice that while the definition of Gibbs measures in the symbolic space corresponding to self-affine sets does not necessarily coincide with the classical one, under the assumption (4.3) the measure μ satisfies Facts 1–3 in § 3 of [**5**]. As these facts are all the knowledge on μ needed to prove Theorem 2.1 of [**5**], it follows that $\mu(R(\boldsymbol{j})) = \pi_*\mu(\widetilde{R}(\pi(\boldsymbol{j})))$ has measure 1. □

We finish with two further questions related to Theorem 5.1.

- Will the claim of Theorem 5.1 remain true without condition (4.3)?
- What about recurrence under F to geometric sets B_k, for example, to balls in $\mathbb{R}^d$? What could be the counterpart of s_0? One has to be a little careful when stating a result of this type since, as pointed out above, in the overlapping cases F is not even well defined, and in most cases the corresponding symbolic problem becomes very hard to track. In our treatment, it is possible that points fairly close to the target point $\pi(\boldsymbol{j})$ on the fractal set are not considered to be recurring, if they are far away from $\boldsymbol{j}$ in the symbolic space. Taking this kind of short cut becomes impossible in the geometric case.

Acknowledgements. We are grateful to Thomas Jordan for suggesting the problem to the first author, and for helpful discussions. We also thank Antti Käenmäki for taking the time to read our manuscript, and for giving valuable feedback. Finally, we thank the referees for their careful reading. H.K. was supported by EPSRC Programme Grant EP/L001462 and F.A.R. by EP/J018260/1.

References

1. T. Bedford, Crinkly curves, Markov partitions and box dimensions in self-similar sets. PhD Thesis, The University of Warwick, 1984.
2. V. Beresnevich and S. Velani, A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures, *Ann. of Math. (2)* **164**(3) (2006), 971–992.
3. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Volume 470 (Springer, Berlin, New York, 1975).
4. Y. Bugeaud, S. Harrap, S. Kristensen and S. Velani, On shrinking targets for $\mathbb{Z}^m$ actions on tori, *Mathematika* **56**(2) (2010), 193–202.
5. N. Chernov and D. Kleinbock, Dynamical Borel-Cantelli lemmas for Gibbs measures, *Israel J. Math.* **122** (2001), 1–27.
6. K. Falconer, The Hausdorff dimension of self-affine fractals, *Math. Proc. Cambridge Philos. Soc.* **103**(2) (1988), 339–350.
7. K. J. Falconer, The dimension of self-affine fractals. II, *Math. Proc. Cambridge Philos. Soc.* **111**(1) (1992), 169–179.
8. B. Fayad, Mixing in the absence of the shrinking target property, *Bull. Lond. Math. Soc.* **38**(5) (2006), 829–838.
9. A. Ferguson, T. Jordan and M. Rams, Dimension of self-affine sets with holes, *Ann. Acad. Sci. Fenn. Math.* **40**(1) (2015), 63–88.
10. R. Hill and S. L. Velani, The ergodic theory of shrinking targets, *Invent. Math.* **119**(1) (1995), 175–198.
11. R. Hill and S. L. Velani, The shrinking target problem for matrix transformations of tori, *J. Lond. Math. Soc. (2)* **60**(2) (1999), 381–398.

12. R. Hill and S. L. Velani, A zero-infinity law for well-approximable points in Julia sets, *Ergodic Theory Dynam. Systems* **22**(6) (2002), 1773–1782.
13. J. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30**(5) (1981), 713–747.
14. A. Käenmäki, On natural invariant measures on generalised iterated function systems, *Ann. Acad. Sci. Fenn. Math.* **29**(2) (2004), 419–458.
15. A. Käenmäki and H. W. J. Reeve, Multifractal analysis of Birkhoff averages for typical infinitely generated self-affine sets, *J. Fractal Geom.* **1**(1) (2014), 83–152.
16. A. Käenmäki and P. Shmerkin, Overlapping self-affine sets of Kakeya type, *Ergodic Theory Dynam. Systems* **29**(3) (2009), 941–965.
17. A. Käenmäki and M. Vilppolainen, Separation conditions on controlled Moran constructions, *Fund. Math.* **200**(1) (2008), 69–100.
18. A. Käenmäki and M. Vilppolainen, Dimension and measures on sub-self-affine sets, *Monatsh. Math.* **161**(3) (2010), 271–293.
19. J. Kurzweil, On the metric theory of inhomogeneous diophantine approximations, *Studia Math.* **15** (1955), 84–112.
20. B. Li, B.-W. Wang, J. Wu and J. Xu, The shrinking target problem in the dynamical system of continued fractions, *Proc. Lond. Math. Soc. (3)* **108**(1) (2014), 159–186.
21. C. McMullen, The Hausdorff dimension of general Sierpiński carpets, *Nagoya Math. J.* **96** (1984), 1–9.
22. T. Persson and M. Rams, On shrinking targets for piecewise expanding rational maps, available at http://arxiv.org/abs/1406.6785.
23. T. Rajala and M. Vilppolainen, Weakly controlled moran constructions and iterated function systems in metric spaces, *Illinois J. Math.* **55**(3) (2011), 1015–1051.
24. H. Reeve, Shrinking targets for countable Markov maps, available at http://arxiv.org/abs/1107.4736.
25. B. Solomyak, Measure and dimension for some fractal families, *Math. Proc. Cambridge Philos. Soc.* **124**(3) (1998), 531–546.