

Rank-one convexity implies quasi-convexity on certain hypersurfaces

Nirmalendu Chaudhuri and Stefan Müller

Max-Planck-Institute for Mathematics in the Sciences,
Inselstrasse 22-26, 04103 Leipzig, Germany
(chaudhur@mis.mpg.de ; sm@mis.mpg.de)

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We show that, if $f : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ is rank-one convex on the hyperboloid $H_D^- := \{X \in S^{2 \times 2} : \det X = -D, X_{11} \geq c > 0\}$, $D \geq 0$, $S^{2 \times 2}$ is the set of 2×2 real symmetric matrices, then f can be approximated by quasi-convex functions on $\mathbb{M}^{2 \times 2}$ uniformly on compact subsets of H_D^- . Equivalently, every gradient Young measure supported on a compact subset of H_D^- is a laminate.

1. Introduction and results

The notion of quasi-convexity was introduced by Morrey in the fundamental paper [7]. He proved that the variational integral

$$I(u) := \int_{\Omega} f(\nabla u(x)) \, dx,$$

defined for sufficiently regular functions $u : \Omega \rightarrow \mathbb{R}^m$, where Ω is a bounded open set in \mathbb{R}^n , $\nabla u(x)$ denotes the gradient of u at x and $f : \mathbb{M}^{m \times n} \rightarrow \mathbb{R}$ is a continuous function, is weakly lower semicontinuous if and only if f satisfies the following so-called quasi-convexity condition: for any open bounded set $U \subset \mathbb{R}^n$,

$$\int_U (f(F + \nabla \phi) - f(F)) \, dx \geq 0 \quad \forall F \in \mathbb{M}^{m \times n} \quad \forall \phi \in C_0^\infty(U).$$

There is no general procedure to verify whether a given function f is quasi-convex or not. A function $f : \mathbb{M}^{m \times n} \rightarrow \mathbb{R}$, on the $m \times n$ real matrices, is called rank-one convex if it is convex on each rank-one line, i.e. all the functions $t \mapsto f(F + ta \otimes b)$ are convex for every $F \in \mathbb{M}^{m \times n}$ and $a \in \mathbb{R}^m, b \in \mathbb{R}^n$. It is easy to prove that quasi-convexity implies rank-one convexity (see, for example, [8]). Whether the converse is true for $m = 2, n \geq 2$, is a major unsolved problem in the calculus of variation. In 1992, Šverák [15] found a striking counterexample showing that rank-one convexity does not imply quasi-convexity for any $n \geq 2, m \geq 3$. Pedregal and Šverák [12] showed that Šverák's idea of the counterexample for $m \geq 3$ could not be used to obtain a counterexample for the 2×2 case. However, in 1999, Müller [9] proved that rank-one convexity implies quasi-convexity on 2×2 diagonal matrices. The aim of this article is to extend this result to the following two-dimensional

nonlinear hypersurface, for any $D \geq 0, c > 0$,

$$H_D^- := \{X = (X_{ij})_{1 \leq i, j \leq 2} \in S^{2 \times 2} : \det X = -D, X_{11} \geq c > 0\},$$

where $S^{2 \times 2}$ is the set of 2×2 real symmetric matrices.

The most concise statement of our result is in terms of gradient Young measures. A Young measure ν is a (weak* measurable) map from a measurable set $\Omega \subset \mathbb{R}^n$ to the space of probability measures on \mathbb{R}^d . The fundamental theorem for Young measures [1, 2, 14, 18, 19] implies that every sequence of maps $u^{(j)} : \Omega \rightarrow \mathbb{R}^d$ that is bounded in L^∞ contains a subsequence (not relabelled) that generates a Young measure ν in the sense that

$$\lim_{j \rightarrow \infty} \int_{\Omega} f(u^{(j)}(x))\phi(x) \, dx = \int_{\Omega} \langle \nu_x, f \rangle \phi(x) \, dx,$$

for all continuous functions f and for all $\phi \in L^1(\Omega)$. Moreover, ν has compact support. Here,

$$\langle \nu_x, f \rangle := \int_{\mathbb{R}^d} f(\lambda) \, d\nu_x(\lambda).$$

We say that ν is a $W^{1,\infty}$ -gradient Young measure if Ω is open and ν is generated by a sequence of gradients $\nabla u^{(j)}$, where $(u^{(j)})$ is bounded in $W^{1,\infty}$. A Young measure is homogeneous if $x \mapsto \nu_x$ is the constant map (a.e.). Kinderlehrer and Pedregal [6] showed that homogeneous Young measures are exactly those probability measures that satisfy Jensen’s inequality for all quasi-convex functions,

$$\langle \nu, f \rangle \geq f(\langle \nu, \text{id} \rangle) \quad \forall f \text{ quasi-convex.}$$

A probability measure μ is called a laminate if Jensen’s inequality holds for all rank-one convex functions (see [11]). It is well known that the question of whether rank-one convexity implies quasi-convexity can be rephrased as: is every homogeneous gradient Young measure a laminate (see, for example, [8])? Our main result is the following.

THEOREM 1.1. *Every gradient Young measure supported on a compact subset of the hypersurface H_D^- , $D \geq 0$, is a laminate.*

This shows that rank-one convex functions on H_D^- almost admit a quasi-convex extension. More precisely, the following assertion holds.

COROLLARY 1.2. *Let $f : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ be a function that is convex on every rank-one line contained in*

$$H_D^- = \{X = (X_{ij})_{1 \leq i, j \leq 2} \in S^{2 \times 2} : \det X = -D, X_{11} \geq c > 0\}, \quad D \geq 0.$$

Let $K \subset H_D^-$ be compact and let $\epsilon > 0$. Then there exists a quasi-convex function $f_\epsilon : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ such that $\sup_K |f_\epsilon - f| < \epsilon$.

Šverák [17, lemma 3] proved that a probability measure supported on connected subsets of 2×2 matrices without rank-one connections and commuting with the determinant is a Dirac mass. In particular, this argument applies to gradient Young

measures, since the determinant is weakly continuous. Together with [16, proposition 1], it follows that any gradient Young measure supported on the two-sheeted hyperboloid $H_D := \{X \in S^{2 \times 2} : \det X = D\}$ is a Dirac mass for $D > 0$. In contrast, if $A, B \in K \subset \mathbb{M}^{m \times n}$ differs by a matrix of rank-one, then, for any $\lambda \in (0, 1)$, $\lambda\delta_A + (1 - \lambda)\delta_B$ is a non-trivial gradient Young measure supported on the set K . One notices that the one-sheeted hyperboloid

$$\left\{ \begin{pmatrix} z + x & y \\ y & z - x \end{pmatrix} : z^2 - x^2 - y^2 = -1 \right\}$$

is made by two families of straight lines and these lines are exactly the rank-one lines. Presence of these rank-one lines is the main source of difficulty in showing that gradient Young measures are laminates. However, our idea here is to transform the hard Jacobian constraint by means of some coordinate transformations used by Evans and Gariepy [4], inspired by the work of Schoen and Wolfson [13] (see [5] for the corresponding change of variables in the elliptic case), to some linear constraint, and then argue by using [9, theorem 2]. We will make use of the following truncation result, which generalizes an earlier work of Zhang [20].

PROPOSITION 1.3 (cf. theorem 2 of [10]). *Let K be a compact convex set in $\mathbb{M}^{m \times n}$. Suppose that $u^{(j)} \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^m)$ and*

$$\int_{\mathbb{R}^n} \text{dist}(\nabla u^{(j)}(x), K) \, dx \rightarrow 0.$$

Then there exists a sequence $(v^{(j)})$ of Lipschitz functions such that

$$\|\text{dist}(\nabla v^{(j)}, K)\|_\infty \rightarrow 0, \quad \mathcal{L}^n\{u^{(j)} \neq v^{(j)}\} \rightarrow 0.$$

In particular, $(\nabla u^{(j)})$ and $(\nabla v^{(j)})$ generate the same Young measure.

2. Linear constraint

The following lemma quite easily follows from [9, theorem 2], just by rotating and reflecting the coordinate axes. However, we give a proof here, since the idea of the proof will be used later.

LEMMA 2.1. *Let Ω be a bounded domain in \mathbb{R}^2 and $\nu = (\nu_x)_{x \in \Omega}$ be a $W^{1,\infty}$ gradient Young measure supported on*

$$K \subset P := \{X = (X_{ij})_{1 \leq i, j \leq 2} : X_{11} + X_{22} = 0, X_{12} + X_{21} = 0\}.$$

Then ν is a laminate.

Proof. Let $(u^{(j)})$ be a bounded sequence in $W^{1,\infty}(\Omega, \mathbb{R}^2)$ and let $(\nabla u^{(j)})$ generate the Young measure ν . Therefore, $\text{dist}(\nabla u^{(j)}, K) \rightarrow 0$ in $L^p(\Omega)$ for all $p < \infty$ and hence

$$u_{1,1}^{(j)} + u_{2,2}^{(j)} \rightarrow 0 \quad \text{and} \quad u_{1,2}^{(j)} + u_{2,1}^{(j)} \rightarrow 0 \quad \text{in } L^p(\Omega) \quad \text{for all } p < \infty.$$

Let

$$\nabla u^{(j)} = \begin{pmatrix} u_{1,1}^{(j)} & u_{1,2}^{(j)} \\ u_{2,1}^{(j)} & u_{2,2}^{(j)} \end{pmatrix}, \quad u_{\alpha,\beta}^{(j)}(x) := \frac{\partial}{\partial x_\beta} u_\alpha^{(j)}(x), \quad 1 \leq \alpha, \beta \leq 2,$$

and $u^{(j)} \xrightarrow{*} u$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$. Then the centre of mass satisfies

$$\bar{\nu}_x := \langle \nu_x, \text{id} \rangle = \nabla u(x) \quad \text{for a.e. } x \in \Omega.$$

Now consider

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SO(2)$$

and

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in SO(2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $v^{(j)} : T(\Omega) \rightarrow \mathbb{R}^2$ by $v^{(j)}(Tx) := Su^{(j)}(x)$. Then $\nabla v^{(j)}(Tx) = S\nabla u^{(j)}(x)T^{-1}$ and it is easy to see that the non-diagonal terms in the gradient matrix $\nabla v^{(j)}$ converge to zero strongly in $L^p(T(\Omega))$ for all $p < \infty$. Assume that $v^{(j)} \xrightarrow{*} v$ in $W^{1,\infty}(T(\Omega), \mathbb{R}^2)$. Let $\mu = (\mu_y)_{y \in T(\Omega)}$ be the Young measure generated by the sequence $(\nabla v^{(j)})$. The centre of mass satisfies $\bar{\mu}_y = \nabla v(y)$ and μ is supported on the 2×2 diagonal matrices. Hence, by [9, theorem 2], μ is a laminate. Now we need to show that ν is also a laminate. Let $f : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ be a rank-one convex function. Then the function $g : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$, defined by $g(X) := f(SXT)$, is also rank-one convex. By the fundamental theorem of Young measures [1], and by passage to a subsequence, for any $U \subset\subset \Omega$, we obtain

$$\begin{aligned} \int_{T(U)} g(\langle \mu_y, \text{id} \rangle) \, dy &\leq \int_{T(U)} \langle \mu_y, g \rangle \, dy \\ &= \lim_{j \rightarrow \infty} \int_{T(U)} g(\nabla v^{(j)}(y)) \, dy \\ &= \lim_{j \rightarrow \infty} \int_U g(\nabla v^{(j)}(Tx)) \, dx \\ &= \lim_{j \rightarrow \infty} \int_U g(S\nabla u^{(j)}(x)T^{-1}) \, dx \\ &= \lim_{j \rightarrow \infty} \int_U f(\nabla u^{(j)}(x)) \, dx \\ &= \int_U \langle \nu_x, f \rangle \, dx. \end{aligned}$$

By a change of variables and by the definition of g , we have

$$\int_{T(U)} g(\nabla v(y)) \, dy = \int_U f(\nabla u(x)) \, dx,$$

and the proof is finished. □

LEMMA 2.2. *Any gradient Young measure supported on*

$$P_c := \{X = (X_{ij})_{1 \leq i, j \leq 2} : X_{11} + X_{22} = c, X_{12} + X_{21} = 0\}, \quad c \neq 0,$$

is a laminate.

Proof. This follows from the change of variables $u(x) \mapsto u(x) + (0, -cx_2)$. □

3. Proof of theorem 1.1

CASE I: $D > 0$. Without loss of generality, we can assume that $D = 1$, that the Young measure $\nu = (\nu_x)_{x \in \Omega}$ is homogeneous and that $\Omega = (0, 1)^2$. Let $(\nabla u^{(j)}) \subset W^{1,\infty}(\Omega, \mathbb{R}^{2 \times 2})$, generate the Young measure ν , $u^{(j)} \xrightarrow{*} u$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$ and $\text{supp } \nu = K \subset H_D^-$. Since K is compact,

$$K \subset \tilde{K} := B_R \cap \{X \in S^{2 \times 2} : X_{11} \geq c > 0\} \quad \text{for some } R > 0,$$

where $B_R := \{X \in \mathbb{M}^{2 \times 2} : |X| \leq R\}$. Since \tilde{K} is a compact convex set and $\text{dist}(\nabla u^{(j)}, \tilde{K}) \rightarrow 0$ in $L^p(\Omega)$ for all $p < \infty$, by proposition 1.3, there exists a sequence $(v^{(j)})$, with uniformly bounded Lipschitz constant, such that $(\nabla v^{(j)})$ generates the same measure ν and $\|\text{dist}(\nabla v^{(j)}, \tilde{K})\|_\infty \rightarrow 0$ as $j \rightarrow \infty$. Hence we can assume that our original generating sequence $(u^{(j)})$ satisfies $u_{1,1}^{(j)} \geq \frac{1}{2}c$ and $|\nabla u^{(j)}| \leq 2R$. By the Ascoli-Arzelà theorem, $u^{(j)} \rightarrow u$ uniformly on Ω . Since ν is supported on H_D^- , it is easy to see that $\det(\nabla u^{(j)}(x)) + 1$ and $u_{1,2}^{(j)} - u_{2,1}^{(j)}$ converge to zero strongly in $L^p(\Omega)$ for all $p < \infty$. Now our idea is to obtain a new sequence of uniformly bounded Lipschitz functions on some suitable domain which generates a new Young measure μ , supported on the set P defined in lemma 2.1. Then, by lemma 2.1, such a measure μ will be a laminate and finally we will argue in a similar way as in the proof of lemma 2.1 to show that the original measure ν is a laminate. This will be obtained through the following steps.

STEP 1 (change of variables). As in [4], consider the maps $T^{(j)}, T : \Omega \rightarrow \mathbb{R}^2$, defined by

$$T^{(j)}(x_1, x_2) := (u_1^{(j)}(x), x_2) \quad \text{and} \quad T(x_1, x_2) := (u_1(x), x_2),$$

respectively. Since $u_1^{(j)}(\cdot, t)$ and $u_1(\cdot, t)$ are strictly monotonically increasing on $(0, 1)$ for each $0 < t < 1$, the maps $T^{(j)} : \Omega \rightarrow T^{(j)}(\Omega)$ and $T : \Omega \rightarrow T(\Omega)$ are bi-Lipschitz, where

$$T^{(j)}(\Omega) = \{(y_1, y_2) : u_1^{(j)}(0, y_2) < y_1 < u_1^{(j)}(1, y_2), 0 < y_2 < 1\}$$

and

$$T(\Omega) = \{(y_1, y_2) : u_1(0, y_2) < y_1 < u_1(1, y_2), 0 < y_2 < 1\}.$$

Hence there exist Lipschitz maps $g^{(j)} : T^{(j)}(\Omega) \rightarrow \mathbb{R}^2$ and $g : T(\Omega) \rightarrow \mathbb{R}^2$ such that

$$x_1 = g_1^{(j)}(u_1^{(j)}(x), x_2), \quad u_2^{(j)}(x) = g_2^{(j)}(u_1^{(j)}(x), x_2) \tag{3.1}$$

and

$$x_1 = g_1(u_1(x), x_2), \quad u_2(x) = g_2(u_1(x), x_2). \tag{3.2}$$

From the definition of $T^{(j)}$, T and differentiating (3.1) with respect to x_1, x_2 , we obtain, for a.e. x ,

$$\nabla T^{(j)}(x) = \begin{pmatrix} u_{1,1}^{(j)}(x) & u_{1,2}^{(j)}(x) \\ 0 & 1 \end{pmatrix}, \quad \nabla T(x) = \begin{pmatrix} u_{1,1}(x) & u_{1,2}(x) \\ 0 & 1 \end{pmatrix}$$

and

$$\left. \begin{aligned} 1 &= g_{1,1}^{(j)}(T^{(j)}(x))u_{1,1}^{(j)}(x), \\ 0 &= g_{1,1}^{(j)}(T^{(j)}(x))u_{1,2}^{(j)}(x) + g_{1,2}^{(j)}(T^{(j)}(x)), \\ u_{2,1}^{(j)}(x) &= g_{2,1}^{(j)}(T^{(j)}(x))u_{1,1}^{(j)}(x), \\ u_{2,2}^{(j)}(x) &= g_{2,1}^{(j)}(T^{(j)}(x))u_{1,2}^{(j)}(x) + g_{2,2}^{(j)}(T^{(j)}(x)). \end{aligned} \right\} \tag{3.3}$$

From (3.3), we have

$$\nabla g^{(j)}(T^{(j)}(x)) = \frac{1}{u_{1,1}^{(j)}(x)} \begin{pmatrix} 1 & -u_{1,2}^{(j)}(x) \\ u_{2,1}^{(j)}(x) & \det \nabla u^{(j)}(x) \end{pmatrix}, \tag{3.4}$$

and similarly, from (3.2), we obtain

$$\nabla g(T(x)) = \frac{1}{u_{1,1}(x)} \begin{pmatrix} 1 & -u_{1,2}(x) \\ u_{2,1}(x) & \det \nabla u(x) \end{pmatrix} \text{ for a.e. } x \in \Omega. \tag{3.5}$$

Now observe that

$$\nabla(g^{(j)} \circ T^{(j)})(x) = \nabla g^{(j)}(T^{(j)}(x))\nabla T^{(j)}(x) = \begin{pmatrix} 1 & 0 \\ u_{2,1}^{(j)}(x) & u_{2,2}^{(j)}(x) \end{pmatrix},$$

and hence, from (3.5), we conclude that $\nabla(g^{(j)} \circ T^{(j)}) \xrightarrow{*} \nabla(g \circ T)$ in $L^\infty(\Omega, \mathbb{M}^{2 \times 2})$. From (3.1) and (3.2), it follows that $g^{(j)} \circ T^{(j)} \xrightarrow{*} g \circ T$ in $W^{1,\infty}(\Omega, \mathbb{R}^2)$.

STEP 2 (domain selection). Define

$$v_\alpha^{(j)}(t) := u_1^{(j)}(\alpha, t) \quad \text{and} \quad v_\alpha(t) := u_1(\alpha, t) \quad \text{for } \alpha = 0, 1 \text{ on } (0, 1).$$

Since $u_{1,1}^{(j)}(x) \geq \frac{1}{2}c$ on Ω , it follows that $v_1^{(j)}(t) - v_0^{(j)}(t) \geq \frac{1}{2}c > 0$ on $(0, 1)$ and, from the uniform convergence of $(u^{(j)})$, we have

$$\inf_{t \in (0,1)} (v_1(t) - v_0(t)) \geq \frac{1}{2}c.$$

Choose

$$0 < \epsilon < \frac{1}{4} \inf_{t \in (0,1)} (v_1(t) - v_0(t)).$$

Then, for sufficiently large j_0 ,

$$V_\epsilon := \{(y_1, y_2) : v_0(y_2) + \epsilon < y_1 < v_1(y_2) - \epsilon, 0 < y_2 < 1\} \subset \cap T^{(j)}(\Omega), \tag{3.6}$$

and, trivially, $V_\epsilon \subset T(\Omega)$. Define $f^{(j)} := g^{(j)}|_{V_\epsilon}$. We need to prove that the sequence $(f^{(j)})$ is uniformly Lipschitz on V_ϵ . Observe that, for $y \in V_\epsilon$, there exists $x^{(j)} \in \Omega$ such that $y = T^{(j)}(x^{(j)})$, so

$$\nabla f^{(j)}(y) = \nabla g^{(j)}(T^{(j)}(x^{(j)})) = \nabla(g^{(j)} \circ T^{(j)})(x^{(j)})(\nabla T^{(j)}(x^{(j)}))^{-1}.$$

Hence, from step 1 and from the fact that $u_{1,1}^{(j)} \geq \frac{1}{2}c$, it follows that

$$\|f^{(j)}\|_{W^{1,\infty}(V_\epsilon, \mathbb{R}^2)} \leq M \quad \text{for some } M > 0.$$

Suppose that $f^{(j)} \xrightarrow{*} f$ in $W^{1,\infty}(V_\epsilon, \mathbb{R}^2)$. We prove that $f = g$ on the smaller domain

$$\tilde{V}_\epsilon := \{(y_1, y_2) : v_0(y_2) + \frac{3}{2}\epsilon < y_1 < v_1(y_2) - \frac{3}{2}\epsilon, 0 < y_2 < 1\} \subset V_\epsilon.$$

Let $y = Tx \in \tilde{V}_\epsilon \subset T(\Omega)$ for some $x \in \Omega$. Then, by the definition of \tilde{V}_ϵ , $T^{(j)}(x) \in V_\epsilon$. Since $f^{(j)}$ is uniformly Lipschitz on V_ϵ and $T^{(j)} \rightarrow T$ on Ω , we get

$$\lim_{j \rightarrow \infty} (f^{(j)} \circ T^{(j)}(x) - f^{(j)} \circ T(x)) = 0. \tag{3.7}$$

From step 1 and (3.7), we obtain

$$\begin{aligned} f(T(x)) &= \lim_{j \rightarrow \infty} f^{(j)}(Tx) \\ &= \lim_{j \rightarrow \infty} g^{(j)}(Tx) \\ &= \lim_{j \rightarrow \infty} [g^{(j)}(T^{(j)}(x)) + (g^{(j)}(Tx) - g^{(j)}(T^{(j)}(x)))] \\ &= g(T(x)), \end{aligned}$$

and hence $f = g$ on \tilde{V}_ϵ .

STEP 3 (transformed Young measure). Let $\mu = (\mu_y)_{y \in V_\epsilon}$ be the Young measure generated by the sequence $(\nabla f^{(j)})$ obtained in step 2. Suppose that E is the support of the measure μ . Now observe that, for any $p < \infty$,

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{V_\epsilon} |f_{1,1}^{(j)} + f_{2,2}^{(j)}|^p dy &= \lim_{j \rightarrow \infty} \int_{V_\epsilon} |g_{1,1}^{(j)} + g_{2,2}^{(j)}|^p dy \\ &\leq \lim_{j \rightarrow \infty} \int_{T^{(j)}(\Omega)} |g_{1,1}^{(j)} + g_{2,2}^{(j)}|^p dy \\ &= \lim_{j \rightarrow \infty} \int_{\Omega} |g_{1,1}^{(j)}(T^{(j)}(x)) + g_{2,2}^{(j)}(T^{(j)}(x))|^p u_{1,1}^{(j)}(x) dx \\ &\leq M \lim_{j \rightarrow \infty} \int_{\Omega} |\det \nabla u^{(j)}(x) + 1|^p dx \\ &= 0, \end{aligned}$$

and similarly, we can show that

$$\lim_{j \rightarrow \infty} \int_{V_\epsilon} |f_{1,2}^{(j)} + f_{2,1}^{(j)}|^p dy = 0.$$

Thus the support E of μ is contained in

$$P := \{X = (X_{ij})_{1 \leq i, j \leq 2} : X_{11} + X_{22} = 0, X_{12} + X_{21} = 0\}$$

and hence, by lemma 2.1, μ is a laminate.

STEP 4 (conclusion of the proof). Define

$$\mathbb{M}_+^{2 \times 2} := \{X = (X_{ij})_{1 \leq i, j \leq 2} \in \mathbb{M}^{2 \times 2} : X_{11} > 0\}$$

and consider the map $\Phi : \mathbb{M}_+^{2 \times 2} \rightarrow \mathbb{M}_+^{2 \times 2}$,

$$\Phi(X) := \frac{1}{X_{11}} \begin{pmatrix} 1 & -X_{12} \\ X_{21} & \det X \end{pmatrix}. \tag{3.8}$$

From the definition of the map Φ , it follows that $\Phi = \Phi^{-1}$, and, by using the formula $\det(A - B) = \det(A) - \text{Cof}(A) : B + \det(B)$ for 2×2 matrices A, B , one obtains

$$\det(\Phi(X) - \Phi(Y)) = -\frac{1}{X_{11}Y_{11}} \det(X - Y) \quad \text{for any matrices } X, Y \in \mathbb{M}_+^{2 \times 2}.$$

Hence $\text{rank}(X - Y) = 1$ if and only if $\text{rank}(\Phi(X) - \Phi(Y)) = 1$. Since $\det : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ is linear along any rank-one direction, by direct computation it follows that

$$\Phi(\lambda X + (1 - \lambda)Y) = \frac{\lambda X_{11}}{\lambda X_{11} + (1 - \lambda)Y_{11}} \Phi(X) + \frac{(1 - \lambda)Y_{11}}{\lambda X_{11} + (1 - \lambda)Y_{11}} \Phi(Y)$$

for any $X, Y \in \mathbb{M}_+^{2 \times 2}$, $\text{rank}(X - Y) = 1$ and $0 \leq \lambda \leq 1$. Let $h : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ be a rank-one convex function and define $\tilde{h} : \mathbb{M}_+^{2 \times 2} \rightarrow \mathbb{R}$ by

$$\tilde{h}(X) := X_{11}h(\Phi(X)) \quad \text{for } X \in \mathbb{M}_+^{2 \times 2}.$$

Now we show that $\tilde{h}(X)$ is rank-one convex on $\mathbb{M}_+^{2 \times 2}$. Let

$$X, Y \in \mathbb{M}_+^{2 \times 2}, \quad \det(X - Y) = 0 \quad \text{and} \quad \tilde{\lambda} := \frac{\lambda X_{11}}{\lambda X_{11} + (1 - \lambda)Y_{11}}.$$

Then (3.8) and the rank-one convexity of h imply that

$$\begin{aligned} \tilde{h}(\lambda X + (1 - \lambda)Y) &= (\lambda X_{11} + (1 - \lambda)Y_{11})h(\tilde{\lambda}\Phi(X) + (1 - \tilde{\lambda})\Phi(Y)) \\ &\leq \lambda X_{11}h(\Phi(X)) + (1 - \lambda)Y_{11}h(\Phi(Y)) \\ &= \lambda \tilde{h}(X) + (1 - \lambda)\tilde{h}(Y). \end{aligned}$$

It is well known that rank-one convex functions are locally Lipschitz (see [3, p. 157]). Since $\|\nabla f^{(j)}\|_\infty \leq R$, then $\|\tilde{h}\|_{L^\infty(B_R)} \leq M$, where $B_R = \{X \in \mathbb{M}^{2 \times 2} : |X| \leq R\}$. Recall the definitions

$$\tilde{V}_\epsilon = \{(y_1, y_2) : v_0(y_2) + \frac{3}{2}\epsilon < y_1 < v_1(y_2) - \frac{3}{2}\epsilon, 0 < y_2 < 1\}$$

and

$$T(\Omega) = \{(y_1, y_2) : u_1(0, y_2) < y_1 < u_1(1, y_2), 0 < y_2 < 1\}.$$

It follows that $\mathcal{L}^2(T(\Omega) \setminus \tilde{V}_\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Since μ is a laminate and the generating sequence satisfies $\nabla f^{(j)}(y) \in \mathbb{M}_+^{2 \times 2}$ a.e. $y \in V_\epsilon$, we have, for a.e. $y \in V_\epsilon$,

$$\tilde{h}(\nabla f(y)) = \tilde{h}(\langle \mu_y, \text{id} \rangle) \leq \langle \mu_y, \tilde{h} \rangle. \tag{3.9}$$

Hence, for any $0 < \epsilon < \frac{1}{4} \inf_{t \in (0,1)} (v_1(t) - v_0(t))$, we have

$$\begin{aligned}
 \int_{\tilde{V}_\epsilon} \langle \mu_y, \tilde{h} \rangle dy &= \lim_{j \rightarrow \infty} \int_{\tilde{V}_\epsilon} \tilde{h}(\nabla f^{(j)}(y)) dy \\
 &= \lim_{j \rightarrow \infty} \int_{\tilde{V}_\epsilon} \tilde{h}(\nabla g^{(j)}(y)) dy \\
 &= \lim_{j \rightarrow \infty} \left[\int_{T^{(j)}(\Omega)} \tilde{h}(\nabla g^{(j)}(y)) dy - \int_{T^{(j)}(\Omega) \setminus \tilde{V}_\epsilon} \tilde{h}(\nabla g^{(j)}(y)) dy \right] \\
 &\leq \lim_{j \rightarrow \infty} \left[\int_{T^{(j)}(\Omega)} \tilde{h}(\nabla g^{(j)}(y)) dy + M \mathcal{L}^2(T^{(j)}(\Omega) \setminus \tilde{V}_\epsilon) \right] \\
 &= \lim_{j \rightarrow \infty} \left[\int_{\Omega} \tilde{h}(\nabla g^{(j)}(T^{(j)}(x))) u_{1,1}^{(j)} dx + M \mathcal{L}^2(T^{(j)}(\Omega) \setminus \tilde{V}_\epsilon) \right] \\
 &= \lim_{j \rightarrow \infty} \left[\int_{\Omega} \tilde{h}(\Phi(\nabla u^{(j)}(x))) u_{1,1}^{(j)} dx + M \mathcal{L}^2(T^{(j)}(\Omega) \setminus \tilde{V}_\epsilon) \right] \\
 &= \lim_{j \rightarrow \infty} \left[\int_{\Omega} h(\nabla u^{(j)}(x)) dx + M \mathcal{L}^2(T^{(j)}(\Omega) \setminus \tilde{V}_\epsilon) \right] \\
 &= \int_{\Omega} \langle \nu, h \rangle dx + M \mathcal{L}^2(T(\Omega) \setminus \tilde{V}_\epsilon) \\
 &= \langle \nu, h \rangle + M \mathcal{L}^2(T(\Omega) \setminus \tilde{V}_\epsilon)
 \end{aligned} \tag{3.10}$$

Therefore, from (3.9) and (3.10), for sufficiently small ϵ ,

$$\int_{\tilde{V}_\epsilon} \tilde{h}(\nabla g(y)) dy = \int_{\tilde{V}_\epsilon} \tilde{h}(\nabla f(y)) dy \leq \langle \nu, h \rangle + M \mathcal{L}^2(T(\Omega) \setminus \tilde{V}_\epsilon),$$

and hence, by passing to the limit $\epsilon \rightarrow 0$, we obtain

$$\int_{T(\Omega)} \tilde{h}(\nabla g(y)) dy \leq \langle \nu, h \rangle. \tag{3.11}$$

On the other hand, by a change of variables, the definition of \tilde{h} and Φ , and by using $\nabla g(T(x)) = \Phi(\nabla u(x))$, we obtain

$$\int_{T(\Omega)} \tilde{h}(\nabla g(y)) dy = \int_{\Omega} h(\nabla f(x)) dx = h(\langle \nu, \text{id} \rangle). \tag{3.12}$$

Hence theorem 1.1 follows from (3.11) and (3.12).

CASE II: $D = 0$. In this case, we follow the same steps as for $D > 0$. In step 1, equation (3.5) becomes

$$\nabla g(T(x)) = \frac{1}{u_{1,1}(x)} \begin{pmatrix} 1 & -u_{1,2}(x) \\ u_{2,1}(x) & 0 \end{pmatrix},$$

and step 2 remains unchanged. The only difference to be noticed in step 3 is that $\int_{\tilde{V}_\epsilon} |f_{1,1}^{(j)} + f_{2,2}^{(j)} - 1|^p \rightarrow 0$, instead of $\int_{\tilde{V}_\epsilon} |f_{1,1}^{(j)} + f_{2,2}^{(j)}|^p \rightarrow 0$. This shows that the Young measure μ , generated by the sequence $(\nabla f^{(j)})$, is supported on

$$P_1 = \{X = (X_{ij})_{1 \leq i, j \leq 2} : X_{11} + X_{22} = 1, X_{12} + X_{21} = 0\},$$

and hence, by lemma 2.2, μ is a laminate. By step 4, it again follows that the original measure is laminate.

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References

- 1 J. M. Ball. A version of the fundamental theorem for Young measures. In *PDEs and continuum models of phase transitions*, Lecture Notes in Physics, vol. 344, pp. 207–215 (Springer, 1988).
- 2 H. Berliocchi and J. M. Lasry. Intégrales normales et mesures paramétrées en calcul des variations. *Bull. Soc. Math. France* **101** (1973), 129–184.
- 3 B. Dacorogna. *Direct methods in the calculus of variations* (Springer, 1989).
- 4 L. C. Evans and R. F. Gariepy. On the partial regularity of energy-minimizing, area-preserving maps. *Calc. Var. PDEs* **9** (1999), 357–372.
- 5 E. Heinz. Über die Lösungen der Minimalflächengleichung. *Nachr. Akad. Wiss. Gött. Math. Phys.* **51** (1952), 51–56.
- 6 D. Kinderlehrer and P. Pedregal. Characterization of Young measures generated by gradients. *Arch. Ration. Mech. Analysis* **115** (1991), 329–365.
- 7 C. B. Morrey. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pac. J. Math.* **2** (1952), 25–53.
- 8 S. Müller. Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems*, Lecture Notes in Mathematics, vol. 1713, pp. 85–210 (Springer, 1999).
- 9 S. Müller. Rank-one convexity implies quasi-convexity on diagonal matrices. *Int. Math. Res. Not.* **20** (1999), 1087–1095.
- 10 S. Müller. A sharp version of Zhang’s theorem on truncating sequences of gradients. *Trans. Am. Math. Soc.* **351** (1999), 4585–4597.
- 11 P. Pedregal. Laminates and microstructure. *Eur. J. Appl. Math.* **4** (1993), 121–149.
- 12 P. Pedregal and V. Šverák. A note on quasi-convexity and rank-one convexity for 2×2 matrices. *J. Convex Analysis* **5** (1998), 107–117.
- 13 R. Schoen and J. Wolfson. Minimizing volume among Lagrangian submanifolds. In *Proc. Sympos. Pure Math. Differential equations: La Pietra, Florence, 1996*, vol. 65, pp. 181–199 (Providence, RI: American Mathematical Society, 1999).
- 14 L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear Analysis and Mechanics, Heriot-Watt Symp.*, Research Notes in Mathematics, vol. 39, pp. 136–212 (Boston, MA: Pitman, 1979).
- 15 V. Šverák. Rank-one convexity does not imply quasi-convexity. *Proc. R. Soc. Edinb. A* **120** (1992), 185–189.
- 16 V. Šverák. New examples of quasi-convex functions. *Arch. Ration. Mech. Analysis* **119** (1992), 293–300.
- 17 V. Šverák. On Tartar’s conjecture. *Annls Inst. H. Poincaré Analyse Non Linéaire* **10** (1993), 405–412.
- 18 L. C. Young. Generalized curves and the existence of an attained absolute minimum in the calculus of variations. *C. R. Soc. Sci. Lett. Varsovie* **30** (1937), 191–234.
- 19 L. C. Young. *Lectures on the calculus of variations and optimal control theory* (Philadelphia, PA: Saunders, 1969).
- 20 K. Zhang. A construction of quasi-convex functions with linear growth at infinity. *Annli Scuola Norm. Sup. Pisa* **19** (1992), 313–326.

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