Macroeconomic Dynamics, **5**, 2001, 272–302. Printed in the United States of America.

LEARNING AND EXCESS VOLATILITY

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We introduce adaptive learning behavior into a general-equilibrium life-cycle economy with capital accumulation. Agents form forecasts of the rate of return to capital assets using least-squares autoregressions on past data. We show that, in contrast to the perfect-foresight dynamics, the dynamical system under learning possesses equilibria that are characterized by persistent excess volatility in returns to capital. We explore a quantitative case for these *learning equilibria*. We use an evolutionary search algorithm to calibrate a version of the system under learning and show that this system can generate data that matches some features of the time-series data for U.S. stock returns and per-capita consumption. We argue that this finding provides support for the hypothesis that the observed excess volatility of asset returns can be explained by changes in investor expectations against a background of relatively small changes in fundamental factors.

Keywords: Learning, Least-Squares Learning, Learning Equilibria, Excess Volatility, Capital Markets

1. INTRODUCTION

1.1. Overview

We present a general-equilibrium economy in which the fact that agents are learning can imply persistent volatility in the economy's state variables. The structure of our model is closely related to many models in use in general-equilibrium macroeconomics today. We argue that our findings provide a plausible explanation for the observed cross-correlations and levels of volatility in economic data coming from markets where expectations seem to play a large role, such as financial markets. Economists have long argued over whether the data in such markets are consistent with fundamental factors, or whether observed prices are instead consistently deviating from the prices implied by underlying fundamentals. Our approach provides a way to frame this debate within the context of standard capital theory.

The environment we examine is a general-equilibrium life-cycle economy with capital accumulation and exogenous growth. Agents live for many periods, and we use standard specifications for preferences and technology. In this environment, agents face a *multistep*-ahead forecast problem, one that has not been studied often

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 1365-1005/01 $9.50

in the learning literature to date. Agents learn by running least-squares regressions using data that are endogenously generated by the economy in which they operate; they form forecasts of future rates of return using their least-squares rule, and then they take optimal actions, given their forecast. The use of a least-squares rule is a common choice in the learning literature, but its effects have not been studied widely in models with capital accumulation.

To make our system as stark as possible, and to maintain relative simplicity, we eliminate all exogenous uncertainty from the economy we study. The volatility that we isolate is thus entirely due to the effects of learning on the system; in the absence of learning, the rate of return to capital in a stationary equilibrium would be a constant that was completely pinned down by unchanging economic fundamentals. Thus, our volatile return to capital is part of the endogenous fluctuation of the economy under learning—the fluctuations arise because expectations are continually being revised in the face of new data. Volatility can persist because agents' expectations affect actual outcomes, and these outcomes in turn feed back into agents' expectations.

We begin by considering a simple benchmark perfect-foresight version of the model and characterize the equilibrium of this system. We then introduce least-squares learning and describe the dynamic system that characterizes the economy under this learning assumption. We show that it is possible to partition the parameter space into two sets—one in which the equilibrium is stable under least-squares learning and one in which it is unstable. In parameter regions where instability arises, we show that a neighborhood of the steady state can contain complicated limiting dynamics for the system under learning. These are the excess volatility or *learning equilibria* of our model. We are able to characterize the situations in which these dynamics arise in terms of a vector of parameters that govern specifications of tastes and technology.

We then consider a more realistic version of our model and explore the possibility of a quantitative case for the learning equilibria that we are able to isolate. We use an evolutionary search algorithm to find interesting calibrations of the economy under least-squares learning. We compare the data generated by these artificial economies to actual data concerning the relationship between per-capita consumption and asset returns in the United States over the past century. Perhaps the most salient feature of the U.S. data is that the percentage standard deviation of returns to capital is large relative to the percentage standard deviation of per-capita consumption growth. We show that under a learning assumption, the model is able to generate data that approach the U.S. data on this dimension, whereas this result is not possible in the perfect-foresight version of the model. We also provide a detailed discussion of other facets of the fit between the model under learning and the data.

This research provides support for the hypothesis that much of the observed volatility in capital asset returns may be due to expectations that are continually being revised, against a backdrop of fundamental factors that are not changing in quantitatively important ways. Our approach provides one method of making this type of argument rigorous and quantifiable, and brings it into contact with a large literature on the nature of asset pricing. Although our model is sparse in some obvious ways that enable us to simplify matters considerably (e.g., all assets pay the same rate of return), in many respects the quantitative fit that we obtain is impressive and suggests that other persistently volatile phenomena (such as business cycles or exchange-rate fluctuations) also might be characterized, in whole or in part, by such learning system dynamics. At a methodological level, we demonstrate how one might go about comparing alternative hypotheses concerning the root causes of asset price volatility to more standard hypotheses, in a manner that allows economists to evaluate the relative merits of each explanation.

1.2. Related Literature

Timmerman (1993, 1996), Arthur et al. (1997), and Brock and Hommes (1998) are among those who have examined whether adaptive learning behavior might help to explain excess volatility in asset market returns. Arthur et al. and Timmerman study learning in the context of the standard, efficient-markets "present-value" model, where the price of a stock is equal to the expected present discounted value of future dividends. In these models, the process governing dividends is exogenously given and therefore is unaffected by the fact that agents are learning. Similarly, Brock and Hommes (1997) study the implications of learning behavior using a version of the cobweb model with exogenously given demand-and-supply schedules. Although the idea we pursue in this paper is similar to this earlier work, our approach differs in that we introduce adaptive learning behavior into a standard *general-equilibrium* macroeconomic model and we explore the implications of our model from a quantitative-theoretic viewpoint.

Sargent (1993), Marimon (1997), and Evans and Honkapohja (1999) survey the literature on learning in macroeconomic models. Grandmont (1998) provides a general discussion of the stability of rational expectations equilibria under adaptive learning behavior and also addresses the possibility of complicated dynamics under learning in general-equilibrium models. Hommes and Sorger (1998a,b) also study the possibility of model consistent endogenous fluctuations due to learning. Bullard (1994) has demonstrated the possibility of endogenous fluctuations under the least-squares learning dynamic that we consider here for a class of general equilibrium endowment economies. One contribution of this paper is to extend this result to production economies.

A number of researchers have recently used adaptive learning models to help explain empirical macroeconomic phenomena. For instance, Arifovic (1996) proposes an adaptive learning explanation of exchange-rate fluctuations, Marcet and Nicolini (1998) suggest that learning may be responsible for recurrent hyperinflationary episodes; Sargent (1999) models the recent history of U.S. inflation and monetary policy as a learning process on the part of monetary policy authorities; and Arifovic et al. (1997) build a learning-based explanation of world growth and development patterns. Here, we go a step beyond these qualitative comparisons and

https://doi.org/10.1017/S1365100501019071 Published online by Cambridge University Press

directly examine how well a calibrated version of our model fits several features of the U.S. data on per-capita consumption and returns to capital.

There is a large literature that addresses the excess volatility of stock prices relative to changes in fundamentals, which is surveyed in Shiller (1989) and Campbell et al. (1997). West (1988) evaluated much of the literature on stock price volatility tests and concluded that excess volatility is a robust empirical phenomenon that is unlikely to be accounted for by standard present-value models. He [along with others, e.g., Shiller (1989)] also dismisses rational bubble explanations as unlikely and argues that the kind of naïve behavior exhibited in "noise trader" models [e.g., DeLong et al. (1990)] does not play a substantial role in the determination of actual stock prices. West concludes that it might be useful to consider some other models of the determination of excess returns, preferably parametric models, "so that the model potentially could be rejected because of implausible parameter estimates or painfully large test statistics." This paper can be viewed as following up on these suggestions.

Many papers, for example, Poterba and Summers (1988), have reported a large forecastable component to real stock returns at long horizons, even though the predictable component at shorter horizons is relatively small. The equilibria we study have this property, as real returns orbit about a constant mean. Campbell (1991) and Campbell and Ammer (1993) used the forecastability result to decompose stock market volatility into the portions attributable to changing forecasts of stock returns, dividend growth, and real (risk-free) interest rates. Of these, the changing forecasts of real stock returns were by far the most important factor in explaining stock market volatility. Our model is consistent with this evidence on expectations-driven volatility, as our fluctuating learning equilibria involve changes in investor sentiment as the sole driving force.

2. THE ENVIRONMENT

2.1. Preferences and Endowments

Time t is discrete and takes on integer values on the real line. At every date t, a new generation of agents is born. These agents live for n periods, where $n \geq 2$ is a positive integer. The size of each new generation of agents grows at a constant gross rate $\psi \geq 1$, where the size of the time $t = 0$ generation is normalized to unity. There is a single, perishable good that is both consumed and used as input to production, which we call capital, and an unbacked outside asset issued by the government. The representative agent of a generation born at time t seeks to maximize discounted lifetime utility given by

$$U = \sum_{i=0}^{n-1} \delta^i \frac{c_t(t+i)^{1-\rho}}{1-\rho}, \tag{1}$$

where $\delta > 0$ is the per-period discount factor, $\rho > 1$ is a preference parameter describing the inverse of the intertemporal elasticity of substitution, and $c_t(t+i)$

denotes the time $t+i$ consumption of the agent born at time t. Agents are endowed with one unit of leisure in every period along with an effective labor productivity coefficient, e_i, where i indexes the period of life. Agents inelastically supply their unit of leisure in every period in exchange for the competitive market wage rate. Their income is the wage rate multiplied by their labor productivity coefficient. The lifetime sequence of effective labor coefficients, $\{e_i\}_{i=1}^{n}$, $e_i \geq 0$, is the same for all agents.

2.2. Production Technology

The economy contains a number of perfectly competitive firms that all have access to the same constant-returns-to-scale production technology. Aggregate output, $Y(t)$, is determined according to

$$Y(t) = \lambda^{(t-1)(1-\alpha)} K(t)^{\alpha} L(t)^{1-\alpha}, \tag{2}$$

where $\lambda \geq 1$ denotes the exogenous gross rate of labor productivity growth, $\alpha \in (0,1)$ denotes capital's share of output, $K(t)$ denotes the aggregate capital stock at time t, and $L(t)$ denotes the aggregate effective labor supply at date t. This latter quantity is given by

$$L(t) = \sum_{j=0}^{n-1} \psi^{t-j-1} e_{j+1}. \tag{3}$$

Denoting the ratio of capital to effective labor as $k(t) = K(t)/L(t)$, the marginal products of capital and labor determine the rental rate, $r(t)$, and the wage rate, $w(t)$, as

$$\begin{aligned} r(t) &= \lambda^{(t-1)(1-\alpha)} \alpha k(t)^{\alpha-1}, \\ w(t) &= \lambda^{(t-1)(1-\alpha)} (1-\alpha) k(t)^{\alpha}. \end{aligned} \tag{4}$$

2.3. Government

A government that endures forever plays only one role, which is to supply unbacked liabilities to the economy at a constant growth rate $\theta \geq \lambda\psi$. These liabilities pay no interest. The aggregate, time t stock of this outside asset, denoted $H(t)$, evolves according to

$$H(t) = \theta H(t-1). \tag{5}$$

The government's real revenue from seigniorage is endogenously determined by $g(t) = [H(t) - H(t-1)]/P(t)$, where $P(t)$ denotes the price of the consumption good in terms of the outside asset at time t. Substituting the rule for government liabilities into the expression for $g(t)$, we can rewrite real government revenue as

$$g(t) = \left(\frac{\theta - 1}{\theta}\right) \frac{H(t)}{P(t)}. \tag{6}$$

It is assumed that government revenue leaves the economy.

https://doi.org/10.1017/S1365100501019071 Published online by Cambridge University Press

The unbacked government liabilities can be interpreted as unbacked government debt paying zero nominal return, combined with the monetary base. Since the supply of these liabilities is assumed to grow at a gross rate θ that exceeds the growth rate of the economy, the price level will rise over time, so that the unbacked liabilities will pay a low real rate of return equal to $P(t)/P(t+1)$. Arbitrage will force capital to pay the same real return as these unbacked liabilities, and thus by construction we are going to end up with a counterfactually low mean return to capital. This could be remedied, at the cost of some complication, by the inclusion of a simple form of costly financial intermediation for capital in the model, so that the real returns to capital would be higher than the real returns to unbacked government liabilities by a constant sufficient to make the mean return to capital match the data.[1]

2.4. Rates of Return

Agents can rent capital to firms, borrow or lend in the consumption loan market, and hold government liabilities. The gross real return from renting capital or making consumption loans at time t is $R(t) = 1 + r(t+1) - \mu$, where μ is the constant depreciation rate. Arbitrage ensures that the gross rate of return from holding non-interest-bearing government liabilities at time t, $P(t)/P(t+1)$ (the inverse of the gross inflation rate), equals $R(t)$ for all t.

Since we consider the behavior of our model under an adaptive learning assumption in which agents lack perfect foresight, we want to draw a sharp distinction between past rates of return that are known at each date t and expected rates of return that must be forecast. We use the notation $R(t-i-1) = P(t-i-1)/P(t-i)$, $i = 0, 1, \ldots,$ to define past *realized* gross rates of return. For future *expected* gross rates of return, we need to keep track of the date at which these expectations were formed because when agents are learning, forecasts can change from one date to the next. We imagine that agents forecast the future gross *inflation rate*, and we use the notation $\beta_{t-1}(t-i+j)$ to denote the commonly held expectation, formed at date $t-i$, of the gross inflation rate at time $t-i+j$, where $i, j = 0, 1, \ldots, n-2$. The expected gross inflation rate is the inverse of the expected gross rate of return in this economy.

2.5. The Household's Problem

The representative agent born at time t faces the following sequence of budget constraints, one for each of the n periods of life:

$$\begin{aligned} c_t(t) &\leq w(t)e_1 - a_t(t), \\ c_t(t+i) &\leq w(t+i)e_{i+1} - a_t(t+i) + a_t(t+i-1)\beta_t(t+i-1)^{-1}, \\ c_t(t+n-1) &\leq w(t+n-1)e_n + a_t(t+n-2)\beta_t(t+n-2)^{-1}, \end{aligned} \tag{7}$$

https://doi.org/10.1017/S1365100501019071 Published online by Cambridge University Press

for $i = 1, 2, \ldots, n-2$. Here, $a_t(t+i)$ denotes asset holdings accumulated by the agent born at time t as of period $t+i$. These constraints can be combined into the single lifetime constraint,

$$c_t(t) + \sum_{i=1}^{n-1} c_t(t+i) \prod_{j=0}^{i-1} \beta_t(t+j) \leq w(t)e_1 + \sum_{i=1}^{n-1} w(t+i)e_{i+1} \prod_{j=0}^{i-1} \beta_t(t+j). \quad \textbf{(8)}$$

The household's problem is to maximize (1) subject to (8).

2.6. Perfect-Foresight Equilibria

We first consider the case in which households have perfect-foresight knowledge of rates of return. The perfect-foresight case serves as a useful benchmark for later comparison with the results we obtain for the same model under a learning assumption and allows us to characterize the conditions under which a rational expectations equilibrium exists. Under perfect foresight, we do not need to draw a distinction between past realized gross rates of return and future expected rates of return. For this reason, in this section, we replace the notation of expected gross inflation with the perfect-foresight realization; that is, we set $\beta_{t-i}(t-i+j) = R(t-i+j)^{-1}$ for all i, j. Thus, under perfect foresight, the representative agent's budget constraint can be rewritten as

$$c_t(t) + \sum_{i=1}^{n-1} c_t(t+i) \prod_{j=0}^{i-1} R(t+j)^{-1} \leq w(t)e_1 + \sum_{i=1}^{n-1} w(t+i)e_{i+1} \prod_{j=0}^{i-1} R(t+j)^{-1}. \quad \textbf{(9)}$$

Solving the representative agent's problem, we can determine each generation's consumption demand at time t in terms of interest rates and wage rates. Rewriting wage rates as a function of interest rates, we can determine each generation's asset holdings at date t as a function of interest rates alone. Aggregate asset holdings, $A(t)$, is then given by $A(t) = \sum_{t=0}^{n-2} \psi^{t-i-1} a_{t-i}(t)$. This expression is a complicated function of interest rates dating from $t-n+2$ to $t+n-2$.

The asset-market-clearing condition is that aggregate asset holdings equal the real stock of unbacked liabilities plus the capital stock,

$$A(t) = \frac{H(t)}{P(t)} + K(t+1). \quad \textbf{(10)}$$

Combining (5) and (10) yields the equilibrium condition

$$A(t) - K(t+1) = R(t-1)\theta[A(t-1) - K(t)]. \quad \textbf{(11)}$$

Since both $A(t)$ and $K(t+1)$ can be written as functions of interest rates, equation (11) is a $2n-3$ order difference equation in $R(t)$. We define a *stationary, competitive, perfect-foresight equilibrium* as any stationary sequence of values $\{R(t)\}_{t=-\infty}^{+\infty}$ such that equation (11) holds at every date t. We restrict attention to the class of stationary perfect-foresight equilibria for which aggregate asset holdings consist of positive holdings of both private capital and the outside government asset. The existence condition for this steady-state equilibrium [which is readily apparent from (11)] is a stationary value of $R = \lambda\psi\theta^{-1}$ such that $A(t) - K(t+1) > 0$ for all t. In this steady state, total asset holdings, $A(t)$, as well as the capital stock, $K(t)$, both grow at the gross rate of growth of output, $\lambda\psi$. We study a neighborhood of this steady-state equilibrium in the remainder of the paper.

2.7. Learning

We now relax the perfect-foresight assumption and consider the case in which agents learn using a least-squares learning rule. Let $\beta_t(t)$ denote the time t forecast of the gross inflation rate between time t and time $t+1$, so that the forecasted future price level, $P(t+1) = \beta_t(t)P(t)$. We imagine that agents estimate gross inflation by running a first-order autoregression on price data available through time $t-1$. The implied regression coefficient can be recursively updated according to the equation[2]

$$\beta_{t+1}(t+1) = \beta_t(t) + \gamma(t)\left\{\frac{\theta[A(t-1) - K(t)]}{A(t) - K(t+1)} - \beta_t(t)\right\}, \tag{12}$$

where the gain, $\gamma(t)$, also can be defined recursively as

$$\gamma(t+1) = \left(\gamma(t)^{-1}\left\{\frac{\theta[A(t-1) - K(t)]}{A(t) - K(t+1)}\right\}^{-2} + 1\right)^{-1}. \tag{13}$$

This least-squares specification for agent learning behavior is a standard choice in the macroeconomic learning literature.

The agents in this model live for many periods, and so, they need to forecast many periods into the future to decide how much to consume and save in for all the present period. The autoregression that we have specified implies that $\beta_{t-i}(t-i+j) = \beta_{t-i}(t-i)\ \forall\, i,\, j > 0$; that is, the agent extrapolates the predicted one-step-ahead inflation rate to all future periods for which a forecast is required. This feature of the first-order autoregression means that we can denote the forecast value $\beta_t(t+j)$ more simply by the date at which it is made; that is, we can denote $\beta_t(t+j)$ by $\beta(t)\ \forall\, j$. Of course, in the next period, a new piece of information will be available, namely, the price level $P(t)$, and so, a new forecast will be made that takes account of this additional information; this new forecast will then be extrapolated into the future by all agents who need to forecast one or more periods ahead.

Using this simplified notation, we can use the definition of the marginal product of capital, given in (4) to write the aggregate capital stock at time t under learning as

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$$K(t) = \lambda^{t-1}\left[\frac{\beta(t-1)^{-1}+\mu-1}{\alpha}\right]^{\frac{1}{\alpha-1}}\sum_{j=0}^{n-1}\psi^{t-j-1}e_{j+1}. \tag{14}$$

Notice that since $K(t)$ was determined by decisions made at time $t-1$, $K(t)$ is a function of $\beta(t-1)$. On the other hand, expected future values of the capital stock *at date t* depend on the current value of $\beta(t)$:

$$K(t+i) = \lambda^{t+i-1}\left[\frac{\beta(t)^{-1}+\mu-1}{\alpha}\right]^{\frac{1}{\alpha-1}}\sum_{j=0}^{n-1}\psi^{t+i-j-1}e_{j+1}, \tag{15}$$

for all $i > 0$. By the same logic, the wage rate at date t is given by

$$w(t) = (1-\alpha)\lambda^{t-1}\left[\frac{\beta(t-1)^{-1}+\mu-1}{\alpha}\right]^{\frac{\alpha}{\alpha-1}}, \tag{16}$$

and expected future wage rates at date t are given by

$$w(t+i) = (1-\alpha)\lambda^{(t+i-1)}\left[\frac{\beta(t)^{-1}+\mu-1}{\alpha}\right]^{\frac{\alpha}{\alpha-1}}, \tag{17}$$

for all $i > 0$. Accordingly, at time t, we can write all future expected wage rates as a function of next period's expected wage rate, $w(t+1)$:

$$w(t+i) = \lambda^{i-1}w(t+1), \tag{18}$$

for $i \geq 1$.

Although all agents form optimal consumption plans at time t on the basis of the inflation forecast $\beta(t)$, when agents lack perfect foresight, these inflation forecasts generally will be incorrect. Therefore, agents will want to *reoptimize* their consumption decisions for the remainder of their lives at every date, taking into account the new inflation forecast that is available at every date. For this reason, it is important to keep close track of the inflation forecasts that are being used in agents' consumption and savings decisions.

Let us begin by rewriting the consumption decision of the agent born at time t, using the simplified notation where $\beta_t(t+j) = \beta(t)$ for $j = 1, 2, \ldots n-2$. We have

$$c_t(t) = \frac{w(t)e_1 + \sum_{i=1}^{n-1} w(t+i)e_{i+1}\beta(t)^i}{\sum_{i=1}^{n} \delta^{\frac{i-1}{\rho}}\beta(t)^{\frac{(i-1)(\rho-1)}{\rho}}}. \tag{19}$$

The asset holdings of this agent can be written as

$$a_t(t) = w(t)e_1 - \frac{w(t)e_1 + \sum_{i=1}^{n-1} w(t+i)e_{i+1}\beta(t)^i}{\sum_{i=1}^{n} \delta^{\frac{i-1}{\rho}} \beta(t)^{\frac{(i-1)(\rho-1)}{\rho}}}. \tag{20}$$

Agents who were born prior to period t, in period $t-k$, $k=1, 2, \ldots, n-2$, have additional time t income from existing asset holdings brought over from the previous period. These agents make current consumption and savings decisions on the basis of the current forecast of inflation, $\beta(t)$. The fact that they use this new value for $\beta(t)$ implies that they reoptimize at date t, choosing a new consumption plan for the remaining $n-k$ periods of their lives. For these agents, we have

$$a_{t-k}(t) = w(t)e_{k+1} + R(t-1)a_{t-k}(t-1) \\ - \frac{w(t)e_{k+1} + \sum_{i=1}^{n-k-1} w(t+i)e_{k+i}\beta(t)^i + R(t-1)a_{t-k}(t-1)}{\sum_{i=1}^{n-k} \delta^{\frac{i-1}{\rho}} \beta(t)^{\frac{(i-1)(\rho-1)}{\rho}}}. \tag{21}$$

Define the terms

$$D_k(t) = \sum_{i=1}^{n-k} \delta^{\frac{i-1}{\rho}} \beta(t)^{\frac{(i-1)(\rho-1)}{\rho}} \tag{22}$$

and

$$W_k(t) = \frac{w(t)e_{k+1} + \sum_{i=1}^{n-k-1} w(t+i)e_{k+i+1}\beta(t)^i}{D_k(t)}. \tag{23}$$

The expression for $W_k(t)$ can be written entirely as a function of $\beta(t-1)$, $\beta(t)$ by using our definition for the real wage rates because only $w(t)$ depends on $\beta(t-1)$:

$$W_k(t) = \frac{(1-\alpha)\lambda^{t-1}\left[\frac{\beta(t-1)^{-1}+\mu-1}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} e_{k+1}}{D_k(t)} \\ + \frac{\sum_{i=1}^{n-k-1} (1-\alpha)\lambda^{t+i-1}\left[\frac{\beta(t)^{-1}+\mu-1}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} e_{k+i+1}\beta(t)^i}{D_k(t)}. \tag{24}$$

Then, we can write

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$$a_t(t) = w(t)e_1 - W_0(t),$$
$$a_{t-k}(t) = w(t)e_{k+1} - W_k(t) + R(t-1)\left[1 - \frac{1}{D_k(t)}\right]a_{t-k}(t-1), \tag{25}$$

for $k = 1, \ldots, n-2$. It follows that

$$a_{t-k}(t-k) = w(t-k)e_1 - W_0(t-k) \tag{26}$$

and

$$a_{t-k}(t-j) = w(t-j)e_{k-j+1} - W_{k-j}(t-j) + R(t-j-1)\left[1 - \frac{1}{D_{k-j}(t-j)}\right]a_{t-k}(t-j-1) \tag{27}$$

for all $j < k$.

Using the above definitions, and noting that $A(t) = \sum_{i=0}^{n-2} \psi^{t-i-1} a_{t-i}(t)$, we deduce an expression for aggregate asset holdings at date t as a function of expectations, β, formed at time t and at dates in the past, and of past realized rates of return R:

$$\begin{aligned} A(t) &= \sum_{i=0}^{n-2} \psi^{t-i-1}[w(t)e_{i+1} - W_i(t)] \\ &+ R(t-1)\sum_{i=0}^{n-3} \psi^{t-i-2}[w(t-1)e_{i+1} - W_i(t-1)]\left[1 - \frac{1}{D_{i+1}(t)}\right] \\ &+ R(t-1)R(t-2)\sum_{i=0}^{n-4} \psi^{t-i-3}[w(t-2)e_{i+1} - W_i(t-2)] \\ &\times \prod_{j=1}^{2}\left[1 - \frac{1}{D_{i+j}(t+j-2)}\right] \\ &+ R(t-1)R(t-2)R(t-3)\sum_{i=0}^{n-5} \psi^{t-i-4}[w(t-3)e_{i+1} - W_i(t-3)] \\ &\times \prod_{j=1}^{3}\left[1 - \frac{1}{D_{i+j}(t+j-3)}\right] \\ &+ \cdots + \prod_{j=1}^{n-2} R(t-j)\psi^{t-n+1}[w(t-n+2)e_1 - W_0(t-n+2)] \\ &\times \prod_{j=1}^{n-2}\left[1 - \frac{1}{D_j(t+j-n+2)}\right]. \end{aligned} \tag{28}$$

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Next, we note that the equilibrium condition (11) can be written more generally as

$$R(t-\ell-1) = \frac{A(t-\ell) - K(t-\ell+1)}{\theta[A(t-\ell-1) - K(t-\ell)]}, \tag{29}$$

for $\ell = 0, 1, \ldots, n-3$. Using (29), we can substitute out for $R(t-\ell-1)$ in the above expression for $A(t)$. Collecting terms in $A(t)$, we have a recursive expression for aggregate asset holdings at time t:

$$A(t) = \frac{\sum_{i=0}^{n-2} \psi^{t-i-1}[w(t)e_{i+1} - W_i(t)]}{1 - \sum_{i=1}^{n-2}\sum_{j=0}^{n-i-2} \psi^{t-j-i-1} \frac{[w(t-i)e_{j+1} - W_j(t-i)]}{\theta^i[A(t-i) - K(t-i+1)]} \prod_{k=1}^{i}\left[1 - \frac{1}{D_{j+k}(t+k-i)}\right]}$$
$$- \frac{\sum_{i=1}^{n-2}\sum_{j=0}^{n-i-2} \psi^{t-j-i-1}\left[\frac{K(t+1)[w(t-i)e_{j+1} - W_j(t-i)]}{\theta^i[A(t-i) - K(t-i+1)]}\right] \prod_{k=1}^{i}\left[1 - \frac{1}{D_{j+k}(t+k-i)}\right]}{1 - \sum_{i=1}^{n-2}\sum_{j=0}^{n-i-2} \psi^{t-j-i-1} \frac{[w(t-i)e_{j+1} - W_j(t-i)]}{\theta^i[A(t-i) - K(t-i+1)]} \prod_{k=1}^{i}\left[1 - \frac{1}{D_{j+k}(t+k-i)}\right]}. \tag{30}$$

This expression for aggregate asset holding under learning is quite useful because it depends only on expectations formed at time t and earlier and on past values of aggregate asset holdings. We can therefore combine equation (30) with equations (12) and (13) to define a dynamic system under learning. By substituting appropriately, we can create a first-order nonlinear system in which $\beta(t+1)$, $\gamma(t+1)$, and $A(t)$ are all functions of past values of these same three variables. We seek to understand the dynamic behavior of this system in a neighborhood of the steady state where $\beta = \theta(\lambda\psi)^{-1}$, $\gamma = 1 - [\theta/(\lambda\psi)]^{-2}$, and $A = \bar{A}$ (a complicated expression we do not display here), under the existence condition $A - K > 0$.

2.8. Learning Equilibria

The steady state of the system under learning coincides with the steady state of the system under perfect foresight. We now consider the local stability of this steady state under learning. In particular, we show that the parameter space for which the steady state exists can be divided into two regions, one in which the steady state is locally stable under learning, and another in which it is locally unstable under learning. In the parameter regions where instability arises, we later show (numerically) that the system under learning possesses complicated limiting dynamics (strange attractors), which we refer to as the learning equilibria of the model following Bullard (1994) and Grandmont (1998). We can discuss the local stability of the steady state under learning and show where learning equilibria are likely to exist most simply in the $n = 2$ period version of the model; the analysis for higher values of n is similar.

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Consider the version of the two-period model in which the productivity profile $\{e_1, e_2\} = \{1, 0\}$ and in which five of the seven deep parameters $\delta, \rho, \lambda, \psi$, and μ are all set equal to 1, leaving the two parameters α and θ free to vary. This much-simplified version of the model merely serves to facilitate our illustration of stability analysis as will become clear below. In this simple case, under least-squares learning, the aggregate capital stock and asset holdings are given by

$$K(t) = [\alpha\beta(t-1)]^{\frac{1}{1-\alpha}}, \tag{30}$$

$$A(t) = \frac{1-\alpha}{2} K(t)^{\alpha}. \tag{31}$$

The steady state where both inside and outside assets are held is one in which $\beta(t) = \theta$ for all t, such that $A - K > 0$. Keeping $\theta \geq 1$, this steady-state existence condition can be written as $1 \leq \theta \leq (1-\alpha)/2\alpha$, with $\alpha \in (0, 1)$. Note that in the limiting case in which $\theta = 1$, this condition implies that $\alpha < 1/3$. Let us define the set of free deep parameters for which the steady state exists in this special case as $\mathcal{E} = \{\theta, \alpha \mid \theta \in [1, (1-\alpha)/2\alpha], \alpha \in (0, 1/3)\}$.

The system under least-squares learning consists of equations (12) and (13) in this simple case [because $A(t)$ is not recursive unless $n \geq 3$]. Substituting the definitions for $K(t)$ and $A(t)$ given in (31) and (32) into (12) and (13), we can write the two-equation, least-squares learning system entirely in terms of present and past values of β and γ. For stability analysis, it is useful to have a first-order representation of this dynamical system. When $n = 2$, we can write the system under least-squares learning in the following first-order form as

$$\begin{aligned}
\beta(t+1) &= f[\beta(t), \beta(t-1), \beta(t-2), \gamma(t)],\\
\beta(t) &= \beta(t),\\
\beta(t-1) &= \beta(t-1),\\
\gamma(t+1) &= g[\beta(t), \beta(t-1), \beta(t-2), \gamma(t)].
\end{aligned} \tag{32}$$

Letting $\varphi(t) = [\beta(t+1), \beta(t), \beta(t-1), \gamma(t+1)]$, we can express the dynamical system under learning more compactly as $\varphi(t) = M[\varphi(t-1)]$, where M is defined by the right-hand side of the system (33). The steady state where both inside and outside assets are held occurs at $\overline{\varphi} = M(\overline{\varphi})$, where $\overline{\varphi} = [\theta, \theta, \theta, 1 - \theta^{-2}]'$. The stability of the system under learning can be assessed by linearizing the system and evaluating the resulting Jacobian matrix at a steady state. Despite the simplicity of the two-period special case that we consider here, the analytic eigenvalues are quite complicated. Therefore, we pursue the following numerical approach to conduct our stability analysis.

Recall that, in the special version of the two-period model that we are considering, $\mathcal{E}$ describes the set of free parameters for which the steady state where both inside and outside assets are held exists. We therefore consider a grid of values for α and θ that covers this entire set. For each (α, θ) pair in $\mathcal{E}$, we calculate the

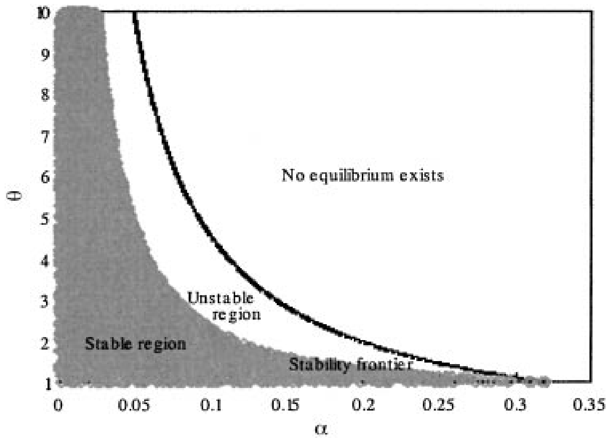

FIGURE 1. Stability frontier for the simplified, two-period economy. A diamond is plotted if a randomly selected (α, θ) pair is associated with a set of eigenvalues whose maximum modulus is less than one, creating the gray area in the figure. Thus, when α and θ are sufficiently small, the system will be locally stable under least-squares learning. The region where learning equilibria may exist occurs when the modulus of the largest eigenvalue crosses the unit circle, which lies in a neighborhood along the border between the stable and unstable regions, labeled the stability frontier.

maximum of the modulus of the four numerically determined eigenvalues for the linearized system evaluated at the steady state. Local stability of the system under least-squares learning requires that all four eigenvalues have modulus less than unity. If the maximum modulus is less than unity, we plot a diamond in Figure 1. We see that these diamonds all lie in the southwest region of the parameter space and represent only a fraction of the region of the parameter space in which an equilibrium with inside and outside assets exists. Hence, we find that there are parameter regions in which the equilibrium is stable and there are regions in which it is unstable under least-squares learning, and there is a well-defined "stability frontier" between these two regions, as noted in Figure 1. This figure indicates that as α increases toward the upper bound of $1/3$, the range of θ values under which the least-squares learning system is locally stable shrinks steadily. Similarly, as θ increases beyond the lower bound of 1, the range of α values for which the least-squares learning system is locally stable also shrinks.[3] Learning equilibria, if they exist, will arise in a region of the parameter space with nonzero measure that lies along the border between the stable and unstable regions of the parameter space.[4] Bullard (1994) has shown that, for two-period-endowment overlapping generations economies similar to the production economy studied here, learning equilibria correspond to the existence of a Hopf bifurcation in the learning-system dynamics. These bifurcations occur in a neighborhood that is very close to the stability frontier of the system under least-squares learning. For the production economy examined in this paper, the dimension of the dynamical system under learning is larger than in the endowment economy, and as a consequence, it is not possible to establish the existence of learning equilibria analytically. Instead, we use a straightforward numerical simulation strategy to find these equilibria in a more complicated version of our model (as set forth below).

Our computational strategy can be described briefly as follows: Suppose the system under learning is initialized at the steady state. If we then perturb the system by adding a small shock to the steady-state value of β, we can observe the resulting response of the dynamical system. If the system returns to the steady state, then it is locally stable under learning; otherwise, the steady state is locally unstable. In the latter case, if the system does not explode—that is, if the dynamics remain bounded—then we have reason to believe that we have found a learning equilibrium. If the forecast errors are stationary and have not diminished to zero, as is the case throughout this paper, then the learning equilibrium is not a perfect-foresight equilibrium.[5]

3. COMPUTATIONAL STRATEGY

3.1. Overview

We now turn to the question of whether the learning equilibria discussed in the preceding section can be isolated in a more realistic context, where agents live for more than two periods. The multiperiod version of the model allows us to interpret each period as a length of time that is conducive to a more realistic calibration of the model, as will become clear later. Our strategy is to conduct an evolutionary search over a well-defined parameter space, in an attempt to locate parameter regions in which the implied dynamics under learning are complicated and generate data that match several aspects of the actual data collected on the U.S. economy over the past 100 years.

3.2. Model Parameterization

We interpret larger values of n as allowing individuals to update their consumption and savings plans more frequently over their lifetimes. We consider a version of the model in which agents live for $n = 11$ periods. Assuming that a typical individual's productive lifetime is approximately 55 years, we can interpret each period in our model as comprising an interval of 5-years. Thus, although we continue to refer to "periods" in model time, the reader should bear in mind that each period in model time consists of 5-year increments in real time. The parameters of our model are chosen with this interpretation in mind.[6] The 11-period model is computationally feasible and captures the essential insight of modeling learning in a multiperiod context, namely, that agents frequently reoptimize over their lifetimes, taking into account new information that was unavailable to them when they were younger. Such reoptimization by individual agents is not possible in the standard two-period environment.

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The lifetime sequence of labor productivity coefficients, $\{e_i\}_{i=1}^{11}$, is assumed to be the same for all agents. The 11-period productivity profile we chose is based on data from Hansen (1993) and is hump-shaped. Hansen reports relative hourly earnings from two different samples for seven different age-gender groupings. We use weighted averages of the age-gender groupings and then fit a polynomial function, which provides us with a smooth endowment profile for $\{e_i\}_{i=1}^{11}$. Since we use a model with inelastic labor supply, we need to impose some retirement period on the agents. We do this by setting $e_i = 0$ for the last few periods of life.[7]

Our strategy is to keep n and $\{e_i\}$ fixed and conduct our search over the remaining parameters of the model, which mainly govern tastes and technology. As suggested by the two-period example, changes in any of these parameter values can cause the dynamics of the system under learning to undergo phase changes. In principle, we want to allow these parameters to vary across the entire domain in order to have the best chance of finding a parameter vector that best achieves the goals of our search. However, we also want to search in a reasonable parameter range, in part as a way to speed up the search process. Accordingly, we restrict the range over which we allow variation in each of the parameters of the deep-parameter vector. Table 1 provides the parameter ranges that we adopted for each of the seven model parameters. Our choices for the θ parameter, which governs the growth rate of the outside asset, are based on the following considerations: To ensure that the capital stock is positive, we require that the steady-state gross inflation rate $\theta/\lambda\psi < 1/(1-\mu)$. On the other hand, we also require that the steady-state gross inflation rate satisfies $\theta/\lambda\psi > 1$. Since the implied restrictions on θ depend on the choices made for λ, ψ, and μ, we chose a value for θ after choices for these other three parameters had been made. In particular, for given values of λ, ψ, and μ, we set

$$\theta = \lambda\psi + \theta_{\text{pct}}[1/(1-\mu) - 1]\,\lambda\psi,$$

where $\theta_{\text{pct}} \in (0, 1)$. This formula for θ ensures that the above restrictions always hold. In searching for parameterizations of our model, we chose values for θ_{pct} rather than θ, and then obtained a value for θ using the preceding formula. Thus, in Table 1, we provide the ranges for θ_{pct} rather than for θ. Although θ_{pct} can take

TABLE 1. Ranges for model parameters[a]

Model parameter	1-year range	5-year range
δ	0.90–1.10	0.59–1.61
ρ	1.01–10.0	same
α	0.15–0.30	same
μ	0.05–1.00	0.22–1.00
λ	1.0025–1.0300	1.0126–1.159
ψ	1.001–1.020	1.005–1.040
θ_{pct}	0.00001–0.50	same

[a]The allowed parameter ranges are expressed in annual terms in the second column and in 5-year terms in the third column.

on values between zero and one, we found by experimenting with our system that lower values tended to be the most relevant, and so, for much of our analysis, we restricted θ_{pct} to relatively low values in order to speed up our search.

3.3. Data Targets

An artificial economy is a tuple $\{\delta, \rho, \alpha, \mu, \lambda, \psi, \theta_{\text{pct}}\}$ in our framework, which we sometimes refer to as a *candidate vector*. We wish to choose values for these vectors such that the implied behavior of the dynamic system under learning has aspects that match corresponding aspects of the U.S. data. Before we can assess the properties of the simulated data associated with a candidate parameter vector, this vector must meet two necessary conditions: (1) the parameterization must be such that there exists a steady state where both inside and outside assets are held, and (2) there is persistent volatility in the simulated asset returns, that is, the system has achieved some kind of complicated attractor. The first objective was achieved by calculating the values of A and K at the steady state where $R = \beta^{-1} = \lambda\psi/\theta$ for each candidate vector of parameter values. A steady state with inside and outside assets exists if $A - K > 0$. The second objective was achieved by checking whether the time path for real returns following a initial displacement from the steady state was asymptotically converging toward the steady state, or exhibiting explosive behavior; in either of these two cases, the objective of persistent volatility was judged to be unsatisfied.

If these two necessary conditions were met, we moved on to assessing the candidate vector's performance on dynamics. For this purpose, we chose to identify the returns to capital in the model economy with the volatile returns to equity in the U.S. economy. The time-series evidence that we use to assess the performance of our model is based on Robert Shiller's updated version of the data set used by Grossman and Shiller (1981), which provides annual data on stock prices and dividends as well as per-capita consumption for the United States from 1890 to 1997. This is the standard data set used in the macrofinance literature. Using these data, we constructed real stock returns (which include real dividends) and real per-capita consumption growth rates at 5-year, nonoverlapping frequencies, covering the period 1890–1994, for a total of 21 observations. The four statistics from these data that we will use to assess the performance of our model are presented in Table 2.[8] We chose these particular statistics because they capture several important features of the data that effectively characterize what is commonly called excess volatility. In particular, the standard deviation of real stock returns is more than six times the standard deviation in real per-capita consumption growth. Furthermore, the first-order serial correlation of real per-capita consumption growth is somewhat negative whereas the contemporaneous correlation between real stock returns and per-capita consumption growth is somewhat positive. The picture painted by these statistics is one in which fundamental factors, typically taken to be represented by real per-capita consumption growth, do not vary as much as real stock returns, and there is apparently

Table 2. U.S. data on returns to capital and per-capita consumption growth, 1890–1994[a]

Statistic	Value
Standard deviation of real stock returns	55.2
Standard deviation of real per-capita consumption growth	8.75
First-order serial correlation of real per-capita consumption growth	−0.24
Contemporaneous correlation between real stock returns and per-capita consumption growth	0.24

[a] We use 5-year, nonoverlapping time intervals.

very little in the way of a relationship between these two variables.[9] The four statistics in Table 2 serve as targets for the 11-period model that we seek to calibrate.

The last of our seven objectives concerns the endogenously determined forecast errors, defined for the contemporaneous case by

$$e(t) = \beta(t)^{-1} - R(t).$$

The hallmark of learning equilibria is that these forecast errors do not tend to zero as time tends to $\pm\infty$.[10] A reasonable restriction to place on the learning equilibria that we isolate through our parameter search is that the forecast errors associated with these equilibria are not *systematic*; instead, they are sufficiently random that agents will be led to conclude that their linear least-squares forecasting model is consistent with the world in which they live. We operationalized this by considering the correlations between $e(t)$ and $e(t-j)$, where $j = 1, 2, \ldots, 10$.[11] If the forecasts are sufficiently random, then forecast errors at 1–10 lags should be uncorrelated with one another. We note that, since our learning specification is a first-order autoregression, our objective regarding forecast errors is somewhat more rigorous than one might expect least-squares learners to adopt. For each parameterization, we calculated the correlation between $e(t)$ and $e(t-j)$ using the 21-observation sample that we drew from the artificial time series generated by the model. We focused on the maximum of these 10 correlation coefficients in absolute value, setting a target of zero for this maximum.

Given our two necessary conditions and our five targets, and given the seven restricted parameter value ranges of Table 2, we developed and implemented a genetic algorithm to conduct the search for a parameter vector that could come as close as possible to meeting our objectives and data targets. A genetic algorithm is a population-based, stochastic, directed search algorithm that incorporates basic principles of population genetics.[12] The algorithm works on a population of “strings.” Each string encodes one candidate solution to some well-specified problem. In our application, each string is a seven-element parameter vector for our model. At the beginning of every “generation,” all strings are evaluated according to some fitness criterion. In our application the fitness criterion is how close the

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candidate system came to meeting all our objectives and data targets. Following the principle of survival of the fittest, strings with a higher fitness level have a greater probability of advancing to the next generation of candidate solutions. The strings that are selected to be retained in the population of candidate solutions undergo, with some fixed probabilities, naturally occurring genetic operations of crossover and mutation, which serve to advance the search for increasingly higher fitness levels. We chose to use a genetic algorithm to conduct our search for a parameterization of our model because these algorithms are known from the artificial intelligence literature to be efficient searchers of large and rugged landscapes, such as the 11-period model we are considering here. Indeed, Holland (1975) has shown that genetic algorithms optimize on the trade-off between exploration of new solutions and exploitation of the best solutions discovered in the past. A more detailed discussion of this search algorithm is given in the Appendix.

4. MAIN FINDINGS

4.1. Dynamics of Artificial Economies

The results that we report are based on searches of the parameter space using a genetic algorithm. Despite our use of a rather sophisticated technology, we found this search to be a difficult one, in the sense that separate searches beginning with randomly chosen candidate parameter vectors tended to end up in different portions of the parameter space.[13] As a countermeasure, we chose to search the parameter space a number of times and here we report results from several of the interesting economies that we found. We believe that these results fulfill our primary purpose, which is to demonstrate that learning equilibria provide an empirically plausible explanation for the observed excess volatility of financial markets, even though there may exist economies that match the data even more closely than those reported here within the same parameter space. We begin by examining the dynamics of these artificial economies, and then we turn to a discussion of the individual parameter vectors.

We report five illustrative best-of-generation strings—the ones with the best fitness values—from searches that we conducted and, for simplicity, we refer to these five cases as economies 1, 2, 3, 4, and 5. In Figure 2, we display one view of the attracting sets of these economies. These are the limiting dynamics of systems simulated for 1,000 periods following a small shock. We have plotted the deviation of the real return to capital from its mean at time t against the same deviation at time $t-1$ in order to give a two-dimensional view of the limiting dynamics. It is clear from the figure that Economy 2 has particularly simple dynamics, a three-period cycle. Economies 1 and 5 follow motion on relatively complicated closed curves, whereas Economies 3 and 4 possess more complicated attracting sets.

In Table 3, we report the time-series statistics associated with each of these five artificial economies, as compared to the U.S. data. Economies 1 and 2 are

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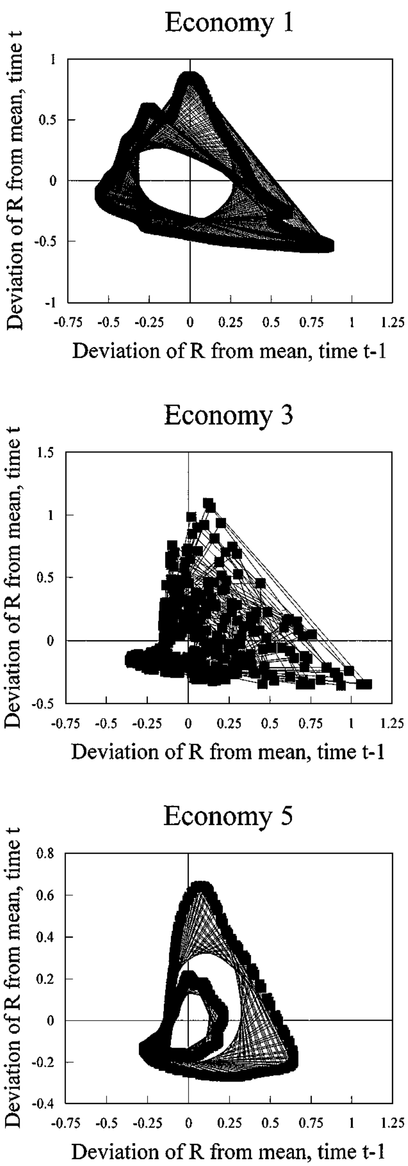

FIGURE 2. Attracting sets for five artificial economies viewed in two dimensions. These figures plot the deviation in the real rate of return to capital (R) from its mean value at time t against the same deviation at time $t-1$, for each of the five artificial economies.

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Table 3. Data statistics associated with artificial economies compared to those from the U.S. data, in terms of 5-year time periods

Statistic	U.S. data	Artificial economy[a] 1	2	3	4	5
Standard deviation of returns to capital	55.2	43.8	55.2	24.6	26.5	20.3
Standard deviation of per-capita consumption growth	8.75	8.63	21.9	8.17	18.9	8.29
Serial correlation of per-capita consumption growth	−0.24	−0.32	−0.53	−0.61	−0.47	−0.29
Contemporaneous correlation between returns to capital and per-capita consumption growth	0.24	0.95	0.99	0.43	0.26	.20
Maximum correlation coefficient, forecast errors at 1 to 10 lags	—	0.92	1.00	0.75	0.63	0.68

[a] We used a sample size of 21 for the artificial economies, which is the number of data points in the actual data.

what we call *high-volatility economies*, because the returns to capital in these cases are about as volatile as in the U.S. data, in fact exactly so in the case of Economy 2. We refer to Economies 3, 4, and 5 as *low-volatility economies*. If we consider the standard deviation of per-capita consumption growth, we see that for Economies 1, 3, and 5, the volatility of this variable is close to the U.S. data, and in most cases substantially less than the volatility of returns to capital. These statistics show that this general-equilibrium model under least-squares learning can capture an essential feature of the excess volatility phenomenon, namely the much greater variation in real returns to capital compared with underlying fundamentals as captured by per-capita consumption growth rates. However, in terms of relative volatility (the ratio of the standard deviation of returns to that of consumption growth) in these two variables, our model comes up a little short. None of the artificial economies has a relative volatility of more than about 5, and most are considerably smaller, whereas the relative volatility in the U.S. data is about 6.3. Thus, although this model generates excess volatility, it does not generate enough excess volatility to precisely match our chosen benchmark data set.

The first-order serial correlation of per-capita consumption growth is negative in the U.S. data, and is also negative in each of the five artificial economies. In two cases, Economies 1 and 5, this correlation is relatively close to the U.S. data. The contemporaneous correlation between the returns to capital and per-capita consumption growth is somewhat positive, 0.24, in the U.S. data. The high-volatility economies that we reporthave far too high a correlation on this dimension to match the U.S. data. The low-volatility economies do better, with Economy 4 coming particularly close to target.

Our volatile learning equilibria are driven by expectational errors. Agents are using first-order autoregressions to predict the future, and they make mistakes that do not vanish asymptotically. In the last row of Table 3, we report the maximum correlation for forecast errors at 1 to 10 lags. This statistic provides one means of assessing the randomness of the forecast errors in our model. For the two high-volatility economies, this maximum correlation is very high, and in the case of Economy 2, it is actually 1.0. Recall that Economy 2 follows a three-period cycle, so that every third forecast error is exactly the same. The other economies follow more complicated trajectories, with forecast errors that are not perfectly correlated within 10 lags. It appears that the low-volatility economies have forecast errors that are not obviously systematic, so that agents in these economies would be unable to distinguish these errors from random errors within the 21-observation sample that we consider.

We can pursue this matter further and ask whether agents who were actually situated in our model would adopt this same view. Suppose that agents were concerned about possible misspecification of their least-squares regression model and used a simple diagnostic test, such as the Durbin–Watson test statistic, to detect for the presence of serially correlated errors.[14] Using our sample of 21 end-of-run forecast errors, we calculated the Durbin–Watson test statistic for each of our five artificial economies. This statistic, d, was 2.61, 2.99, 1.60, 1.54, and 1.50, respectively, for Economies 1, 2, 3, 4, and 5. A plot of the associated forecast errors for Economies 1, 3, 4, and 5 (Economy 2 has perfectly systematic errors and has been omitted) is given in Figure 3. With 21 observations and 1 regressor, the lower and upper bounds at the 5% significance level are $d_L = 1.132$ and $d_U = 1.420$.[15] Let us suppose that the agents first test a null hypothesis of no serial correlation against the alternative of positive serial correlation. For Economies 1–5, the calculated Durbin–Watson statistics all lie above the upper bound and, therefore, agents would not reject the null hypothesis of no positive serial correlation. If agents tested for negative serial correlation, the corresponding test statistic values would be $4 - d$, or 1.39, 1.01, 2.40, 2.46, and 2.50 for economies 1, 2, 3, 4, and 5, respectively. For Economy 2—the perfectly cyclical economy—the null hypothesis of no negative serial correlation is rejected as $1.01 < d_L$, and for Economy 1, the test is inconclusive. For the three low-volatility economies, however, a null hypothesis of no negative serial correlation cannot be rejected, and so, in these three cases, agents would conclude that there was little evidence of serially correlated errors.

We note further that the system we are examining is completely deterministic, an extreme assumption that we made in part for tractability but also to keep the nature of our results clear. If we were to add a reasonable amount of noise to the system, for instance by making output subject to stochastic shocks, the forecast errors generated by our system might appear to be even more complicated to the agents in our model.

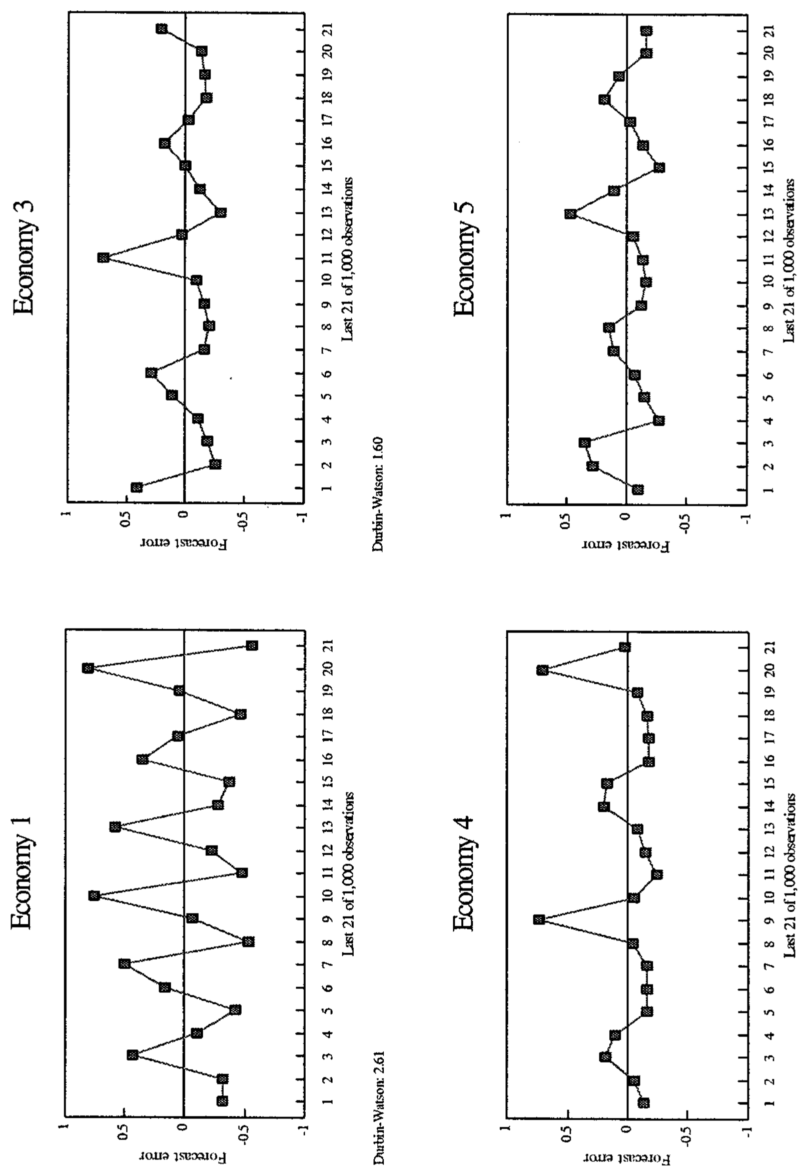

FIGURE 3. Forecast errors. We plot the last 21 of 1,000 observations for Economies 1, 3, 4, and 5. In each case, the Durbin–Watson statistic is given as if calculated with knowledge of the entire sample.

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Table 4. Parameter vectors for artificial economies reported in terms of 5-year time periods

Economy	δ	ρ	α	μ	λ	ψ	θ_{pct}
1	1.176	2.809	0.1849	0.8940	1.031	1.019	0.01815
2	1.752	3.850	0.1827	0.8630	1.139	1.021	0.02515
3	1.392	1.292	0.1553	0.7507	1.140	1.073	0.07121
4	1.571	1.036	0.1518	0.8033	1.132	1.081	0.08228
5	1.418	1.022	0.1511	0.6616	1.181	1.105	0.08898

4.2. Other Characteristics of the Artificial Economies

We turn next to an evaluation of other aspects of the artificial economies, based on the best-of-generation parameter vectors at the end of our searches. These parameter vectors, for which growth rates should be interpreted in terms of 5-year time periods, are listed in Table 4. Of course, we have already tried to use our seven parameters to meet existence and volatility objectives, as well as five data-based targets, and to ask the model to perform well on a number of additional dimensions is pushing the envelope somewhat. A better approach would be to consider models with more realistic features in order to try to address more aspects of the data. Nevertheless, we think that our economies fall short of a completely convincing demonstration of the existence of empirically plausible learning equilibria, mainly because our economies involve capital-share parameters that are too low, and depreciation rates that are too high.[16] However, we stress that we are only taking a first step in this paper as a means of illustrating the potential of our approach, and that in many respects our parameter vectors are quite reasonable.

In evaluating the parameter vectors presented in Table 4, we want to think in terms of modern developed economies, even though we restricted our comparisons to U.S. data in the preceding section. The rate of technological change, λ, and the rate of population growth, ψ, are expressed in gross rates over 5-year intervals. Thus, the *annual* rate of technological change across the five economies ranges from about 0.5% to 3.4%. These values are within the range that one might expect for modern developed economies.[17] The rate of population growth, on an annual basis, ranges from about 0.4% to about 2%. Again, these values are within the range of observed values for modern economies. The value for θ_{pct} has no direct interpretation, but can be related to the inflation rates observed for these economies. The steady-state inflation rates for the five economies range from about 1.1% up to about 22%, which is consistent with average inflation rates observed in OECD economies during the postwar era. Economy 1 has a steady-state inflation rate of 4%, the U.S. postwar average.

Regarding preferences, Table 4 reveals that the curvature parameter, ρ, which can be interpreted as the coefficient of relative risk aversion, ranged from about 1—logarithmic preferences—up to 3.85. These values are consistent with those used in much macroeconomic research. The discount factor, δ, was consistently above unity. On an annual basis, the rate of time preference, as conventionally measured, ranged from −3.2% to −8.6%. As is well known, in overlapping generations models, this parameter plays a quite different role from that played in representative-agent models and, in particular, there is no requirement that it take on a positive value. Negative values are consistent with a number of empirical estimates, including those of Hurd (1989), who estimated the difference between the real interest rate facing agents and the rate of time preference as 4.1%. Typically, the real interest rate is taken as the rate on short-term government debt, which (in the postwar U.S. data) averages about 1%, or as the after-tax rate, which averages about 0%. These figures imply an estimate of the rate of time preference of around −3 or −4%. Hurd's (1989) alternative estimate of the difference between the rate of time preference and the real interest rate was even larger at 6.1%.

The least satisfactory aspect of our artificial economies is that capital's share of output is too low to be consistent with estimates of the capital share for the U.S. economy, and the depreciation rate is too large. Most capital share calibrations in the real-business-cycle literature use values of one-third or higher, based principally on inclusion of consumer durables in the measure of the capital stock. Even if one does not include consumer durables in the measured capital stock, capital's share of output is around one-fourth in the postwar U.S. data, higher than in any of our five economies. Estimates of the annual depreciation rate in the U.S. data range from about 4 to 12%. In contrast, the five economies listed in Table 4 have annual depreciation rates ranging from a minimum of more than 19% up to 36%. These depreciation rates are outside the range of U.S. experience. We leave it as a challenge for future research to develop models with learning equilibria that can perform more satisfactorily on these dimensions.

4.3. Remark on Interpretation

So far, we have behaved like econometricians in locating a best-fit parameter vector for our model, and interpreting that fit. However, there remains an important unanswered question concering our approach: What is it that drives economies to the region of the parameter space where learning equilibria exist, e.g. the stability frontier outlined in Figure 1? A satisfactory answer to this question lies beyond the scope of our analysis because we have not attempted to model an endogenous process that would keep these economies on that frontier. Instead, we have appealed to the data and have argued that the equilibria we describe are the ones that provide the closest match. Nevertheless, we give here a heuristic argument as to why we might observe economies in the volatile region as opposed to, say, in the stable region where the steady state would obtain.

Suppose that all parameters are given by nature with the exception of the policy parameter θ, which is set by the government so as to maximize government revenue before setting other explicit taxes. In this scenario, the government moves θ to as large a value as possible. All else equal, larger values of θ will tend to move our systems closer to the stability frontier, and will tend to make observed dynamics

https://doi.org/10.1017/S1365100501019071 Published online by Cambridge University Press

more volatile. A government pursuing such a policy would push θ higher but would stop near the stability frontier. At that point, revenue would be maximized on average, but would be volatile. If the government tried to raise still more revenue, the dynamics would collapse (the system would move too far into the unstable region). On the other hand, a lower value of θ would produce lower average revenue. Thus, one might expect certain types of politico-economic equilibria to exist on the stability frontier, and not to exist elsewhere in the parameter space. Although we have not attempted to construct such equilibria formally, we think that our heuristic argument is at least suggestive on this score.

5. CONCLUSIONS

Observed levels of volatility in markets where expectations seem to play a large role, such as capital markets, have long been a puzzle for economists. The data we use reveal that the standard deviation in annual returns to equity in the United States over the past century is around 18%. Changes in fundamentals, on the other hand, do not seem to be nearly so pronounced. For instance, the postwar quarterly standard deviation of technological change as measured by the Solow residual is only about 2.4% [Ríos-Rull (1996)] and the standard deviation of annual aggregate per-capita consumption growth over the past century is only about 3%. Most other processes fundamental to economic value, such as demographics, government policy, or preferences, are usually viewed either as constant or as slowly changing, low-volatility variables. If economic fundamentals are not changing very much from year to year, why are capital asset returns so volatile? Questions such as this have led economists to debate whether the data in such markets are consistent with fundamental factors, or whether observed returns are instead consistently deviating from the returns one might expect on the basis of fundamentals alone.

We have explored the feasibility of the latter answer to this question—that observed returns are not closely related to changes in fundamentals, and are instead mainly driven by changes in investor sentiment. In our framework, the deviations of returns from those suggested by fundamental factors arise from the fact that agents must learn about the environment in which they operate. The result is a system in which expectational error is a driving force behind economic volatility. To illustrate our ideas most vividly (and to keep the analysis relatively simple), we have set up an environment that is very stark: There is no uncertainty whatsoever in the underlying model. In the absence of learning, the steady-state return to capital would be completely pinned down by unchanging fundamentals, and therefore would exhibit a standard deviation of zero. We have shown that a calibration of our model with learning exists that can deliver time-series data that match some of the essential features of the U.S. data on real stock returns and per-capita consumption growth. In particular, our model comes close to capturing the observed excess volatility in the data.

As we have emphasized, the learning equilibria we isolate do not provide a perfect fit to the features of the data that characterize excess volatility in capital markets. Our results must therefore be regarded as a first step in an effort to provide a learning equilibrium explanation of the excess volatility phenomenon. We think that further research on quantitative learning equilibria (e.g., as an explanation of business cycles or exchange-rate fluctuations) is warranted, and we believe that we have suggested a reasonable approach in pursuing this line of research.

NOTES

1. Our decision to include a stock of unbacked government liabilities that grows at a constant rate was made in order to keep the model very close to the class of economies studied by Bullard (1994) in which learning equilibria were isolated. By studying a related class of economies, we had some confidence that the excess volatility equilibria we sought to isolate would actually exist in the version of the model we consider here with capital accumulation and production—otherwise we would be left with little guidance as to where and how to find such learning equilibria.

2. See, e.g., Ljung and Söderström (1983) for a derivation of the recursive least-squares updating equations used here.

3. In an effort to provide a clear diagram, we have not included the entire feasible parameter set in Figure 1.

4. Later, in Section 4.3, we give some heuristic reasoning on forces that we think would drive our system to this region of the parameter space.

5. That is, we can easily differentiate between learning equilibria and perfect-foresight cycles. Perfect-foresight cycles are unlikely to exist when, as is the case here, preferences are close to being logarithmic and output is generated by a Cobb–Douglas production technology.

6. Ideally, we would like to consider a 55-period version of our model so that each period can be interpreted as a single year. However, we have found that our search strategy for the 55-period version of the model is not feasible given our current computational resources.

7. In some economies, we set the productivity profile to zero in the last three periods, whereas in others we set it to zero during the last two periods. We did not find any qualitative differences in the results that were dependent on this feature of the model.

8. These same statistics, calculated at 1-year horizons using our data set, are very similar to those reported by Campbell (1999) for annual U.S. data from 1890 to 1992.

9. In our model, there is a sharper definition of fundamental factors, namely preferences, technology, and government policy. We have specified all of these factors to be constant or growing at a constant rate, and not subject to stochastic shocks, so that from this perspective the fundamentals of our model are unchanging over time. Nevertheless, as long as the context is clear, we refer interchangeably to "the fundamentals" of the model economy as either the implied sequence of consumption growth rates or as the unchanging preferences and technology.

10. Hence, learning equilibria may be easily distinguished from perfect-foresight cycles.

11. Agents only live for 11 periods, and so, we consider all of the correlations between forecast errors that occur over agents' lifetimes.

12. See, e.g., Michalewicz (1994) for an introduction to genetic algorithms.

13. In part, this may reflect our inability (because of resource constraints) to allow the genetic algorithm to search for a sufficiently long period of time. How long such a search might take is unknown, however.

14. Bray and Savin (1986) used a Durbin–Watson test as a guide to misspecification in an econometric learning model.

15. The lower bound for the Durbin–Watson test statistic when there is no intercept term in the regression (as in our case) comes from the tables reported by Farebrother (1980). The upper bound is not affected by the absence of an intercept term.

16. When we tried to limit our genetic algorithm search to relatively high-capital share, low-depreciation cases we found that it was difficult or impossible to meet our existence check for equilibria in which agents hold both inside and outside assets.

17. We note that, in the model, these rates are constant, but an econometrician evaluating the rate of technological progress would have to estimate it from the volatile data produced by these economies, and these estimates would be subject to some uncertainty. This same caveat also applies to the other parameters of the model.

18. We also experimented with further penalizing strings that yielded values for θ_j outside the target range $[\underline{\theta}_j, \overline{\theta}_j]$ by adding a quadratic term to the penalty point function, when θ_{ij} values were outside these bounds. However, this modification appeared to make little difference for our results, and so, we adopted the simpler, linear penalty point mechanism described earlier.

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APPENDIX: THE SEARCH ALGORITHM

The search algorithm proceeds in the following sequence of steps: First, a population of N parameter vectors, or strings, is randomly initialized. We typically set $N = 30$, 50, or 100, based on suggestions from the literature on genetic algorithms. Each string has seven elements—the seven parameters of our model: δ, ρ, α, μ, λ, ψ, θ. Denote each of the seven elements of string s_i by ϕ_{ij}, so that ϕ_{i1} is the value for the parameter δ of string s_i. The initial parameter values for all N strings were drawn with uniform probability from the parameter ranges specified in Table 2.

Second, the values of A and K at the steady state where $R = \lambda\psi/\theta$ are calculated for each parameter string. If $A - K \leq 0$, then the parameter vector is assigned a large number of *penalty points* and no further calculations are made for this string. If, on the other hand, $A - K > 0$, so that the steady state of interest exists, the string is assigned zero penalty points for this step in the algorithm. This step constitutes our existence check. In our search algorithm, high fitness is associated with an absence of penalty points, and so, a string with a large number of penalty points is not likely to remain long in the population of candidate solutions, as will become clear below.

If a string passes the existence check, the next step is to simulate the system for *maxit* iterations using the candidate parameter vector. We found that we could get effective simulations by setting the number of iterations as low as 250, but we generally used higher values such as 300, 500, or 1,000. The system is initialized at the steady state and then the initial value $\beta(0)$ is perturbed by a small amount. If the resulting dynamic path for β is determined according to simple criteria to be explosive or convergent, the

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simulation is stopped (so as to save time) and the candidate string is again assigned a number of penalty points that are inversely related to the number of periods the simulation was continued before being terminated. If the checks for explosive or convergent behavior are satisfied, the model is simulated for *maxit* periods and earns zero penalty points. This check fulfills our objective of having persistently volatile behavior in asset returns.

If a string passes both the existence and persistent volatility checks, we then take the last 21 observations from the *maxit* simulated observations on rates of return, per-capita consumption amounts, and forecast errors for this string and we use this sample of 21 observations to construct the statistics that we will then compare with our target values and ranges. In particular, we calcualte the standard deviation of returns, the standard deviation of the growth rates of per-capita consumption, the serial correlation of per-capita consumption growth, and the correlation between returns to capital and per-capita consumption growth. Finally, we calculate the maximum correlation coefficient of the forecast errors at 1–10 lags. We limit our sample to 21 observations because that is the number of observations from the U.S. data that we have on nonoverlapping 5-year returns and 5-year growth rates of per-capita consumption.

The fitness assigned to a string that passes both the existence and volatility checks is zero. This fitness value may then be altered further according to how well the string performs with respect to the five statistical targets. Each of the five data statistics gets an equal weight in further altering the fitness of a string. Let s_{it} denote the candidate parameter vector i at generation t, and let θ_{ijt} denote each of the five data statistics ($j = 1, \ldots, 5$) that result from simulating the economy with parameter string s_{it} and analyzing the final 21 observations. Denote the target value for each θ_j by $\hat{\theta}_j$ and the upper and lower bounds by $\bar{\theta}_j$ and $\underline{\theta}_j$. Then, the fitness of string s_{it} is given by

$$F(\boldsymbol{s_{it}}) = \sum_{j=1}^{5} p_{ijt},$$

where

$$p_{ijt} = \begin{cases} (\theta_{ijt} - \hat{\theta}_j)/(\underline{\theta}_j - \hat{\theta}_j) & \text{if} \quad \theta_{ijt} < \hat{\theta}_j, \\ (\theta_{ijt} - \hat{\theta}_j)/(\bar{\theta}_j - \hat{\theta}_j) & \text{otherwise.} \end{cases}$$

Given this fitness definition, parameter vectors that come closest to generating data that match the desired targets will have lower fitness values and a vector that delivers an exact fit on all five targets will have a fitness of zero.[18] Once fitness values have been determined for all N strings in the population, we apply genetic operators that constitute the heart of the genetic algorithm.

First, a selection tournament is held. Two strings are randomly chosen with replacement from the population of strings, and their fitness values are compared. The string with the better fitness value wins the tournament and a copy of this string is placed in the next "generation," $G(t+1)$, of candidate strings. This binary selection tournament is conducted $N-1$ more times so that $G(t+1)$ consists of a population of N strings. The purpose of this selection operation is to direct the search process toward increasingly fit strings. The remaining operations of the genetic algorithm, crossover and mutation, inject the population with new, untried parameter vectors, so as to advance the search for highly fit strings.

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These operators work on the strings in $G(t+1)$, the strings that have won a selection tournament.

The crossover operation is conducted as follows. The N strings in $G(t+1)$ are considered two at a time, $s_{i,t+1}$, $s_{i+1,t+1}$. With probability p^c, crossover is performed on two vectors; otherwise, crossover is not performed. If crossover is performed, we use one of three methods with equal probability. Each of these methods has been shown to have certain strengths in tackling difficult, nonlinear search spaces and therefore we chose to use all three methods to conduct our search. The first method is single-point crossover in which an integer I is chosen uniformly from the set $[1, \ldots, 5]$. The two strings are then cut at integer I and the elements of the two strings, ϕ_{ij} and $\phi_{i+1,j}$ to the right of this cut point, that is, $j > I$, are then swapped. The second method, shuffle crossover, involves draws from a binomial distribution. If the jth draw is a one, then the jth elements of the two strings, ϕ_{ij} and $\phi_{i+1,j}$ are swapped; otherwise, the jth elements are not swapped. In the third and final method, arthmetic crossover, a random real number a $\in [0, 1]$ is chosen and this number is used to create two new vectors that are linear combinations of the original two strings: $as_{i,t+1} + (1-a)s_{i+1,t+1}$ and $as_{i+1,t+1} + (1-a)s_{i,t+1}$.

Mutation is performed on the strings in $G(t+1)$ following the application of the crossover operation [i.e., on the recombined strings of $G(t+1)$]. The mutation operator makes use of the upper and lower bounds for each of the seven parameter elements of a string, $\bar{\phi}_j$, $\underline{\phi}_j$, that we specified in Table 2, and is applied with probability p^m to every element of every string of $G(t+1)$. If mutation is to be performed on element $\phi_{i,j,t+1}$, then two real numbers, r_1 and r_2, are drawn from [0,1]. The new, mutated value of $\phi_{i,j,t+1}$ is given by

$$\phi'_{i,j,t+1} = \begin{cases} \phi_{i,j,t+1} + \left(\bar{\phi}_j - \phi_{i,j,t+1}\right)\left[1 - r_2^{\left(1-\frac{t}{T}\right)^b}\right] & \text{if} \quad r_1 > 0.5, \\ \phi_{i,j,t+1} - \left(\phi_{i,j,t+1} - \underline{\phi}_j\right)\left[1 - r_2^{\left(1-\frac{t}{T}\right)^b}\right] & \text{if} \quad r_1 < 0.5, \end{cases}$$

where b is a parameter governing the degree to which the mutation operation is nonuniform. This mutation operation is such that the probability of choosing a new parameter element far from the existing value diminishes as $t \to T$, where T is the maximum number of generations. The purpose of this nonuniform mutation operation is to ensure that, with the passage of time (i.e., following many generations), the genetic algorithm samples more intensively from the neighborhood of existing parameter values because, in the latter stages of a search (close to time T), the parameter vectors should be approaching the optimum.

Following crossover and mutation, the strings of $G(t+1)$ are once again evaluated for their fitness, which involves simulating the economy implied by each $s_{i,t+1} \in G(t+1)$. The genetic algorithm selection tournament was conducted anew, followed by another application of the crossover and mutation operators on the winners. This process was repeatedly conducted for some maximum number of generations, which we typically set to 500. The genetic-algorithm parameter values that we used were $p^c = 0.95$, $p^m = 0.20$, and $b = 2$.