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FITTING SUBGROUP AND NILPOTENT RESIDUAL OF FIXED POINTS

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Abstract

Let q be a prime and let A be an elementary abelian group of order at least q^3 acting by automorphisms on a finite q'-group G. We prove that if $|\gamma_\infty(C_G(a))| \leq m$ for any $a \in A^\#$, then the order of $\gamma_\infty(G)$ is m-bounded. If $F(C_G(a))$ has index at most m in $C_G(a)$ for any $a \in A^\#$, then the index of $F_2(G)$ is m-bounded.

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1. Introduction

Suppose that a finite group A acts by automorphisms on a finite group G. The action is coprime if the groups A and G have coprime orders. We denote by $C_G(A)$ the set

$$\{g \in G \mid g^a = g \text{ for all } a \in A\},$$

the centraliser of A in G (the fixed-point subgroup). In what follows, we denote by $A^\#$ the set of nontrivial elements of A. It is known that centralisers of coprime automorphisms have a strong influence on the structure of G.

Ward showed that if A is an elementary abelian q-group of rank at least three and if $C_G(a)$ is nilpotent for any $a \in A^\#$, then the group G is nilpotent [11]. Later, the second author showed that if, under these hypotheses, $C_G(a)$ is nilpotent of class at most c for any $a \in A^\#$, then the group G is nilpotent with (c, q)-bounded nilpotency class [8]. Throughout the paper, we use the expression '$(a, b, \ldots)$-bounded' to abbreviate 'bounded from above in terms of $a, b, \ldots$ only'. Subsequently, the above result was extended to the case where A is not necessarily abelian. Namely, it was shown in [3] that if A is a finite group of prime exponent q and order at least q^3 acting on a finite q'-group G in such a manner that $C_G(a)$ is nilpotent of class at most c for any $a \in A^\#$, then G is nilpotent with class bounded solely in terms of c and q. Many other results illustrating the influence of centralisers of automorphisms on the structure of G can be found in [7].

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In this article we address the case where A is an elementary abelian q-group of rank at least three and $C_G(a)$ is 'almost' nilpotent for any $a \in A^\#$. Recall that the nilpotent residual of a finite group G is the intersection of all terms of the lower central series of G. This will be denoted by $\gamma_\infty(G)$. One of the results obtained in [2] is that if A and G are as above and $\gamma_\infty(C_G(a))$ has order at most m for any $a \in A^\#$, then the order of $\gamma_\infty(G)$ is (m, q)-bounded. The purpose of this article is to obtain a better result by showing that the order of $\gamma_\infty(G)$ is m-bounded and, in particular, that the order of $\gamma_\infty(G)$ can be bounded by a number that is independent of the order of A.

Theorem 1.1. *Let q be a prime and let m be a positive integer. Let A be an elementary abelian group of order at least q^3 acting by automorphisms on a finite q'-group G. Assume that $|\gamma_\infty(C_G(a))| \leq m$ for any $a \in A^\#$. Then $|\gamma_\infty(G)|$ is m-bounded.*

Further, suppose that the Fitting subgroup of $C_G(a)$ has index at most m in $C_G(a)$ for any $a \in A^\#$. It was shown in [9] that under this assumption the index of the Fitting subgroup of G is (m, q)-bounded. In view of Theorem 1.1, it is natural to conjecture that, in fact, the index of the Fitting subgroup of G can be bounded in terms of m alone. We have not been able to confirm this. Our next result should be regarded as evidence in favour of the conjecture. Recall that the second Fitting subgroup $F_2(G)$ of a finite group G is defined as the inverse image of $F(G/F(G))$, that is, $F_2(G)/F(G) = F(G/F(G))$. Here $F(G)$ stands for the Fitting subgroup of G.

Theorem 1.2. *Let q be a prime and let m be a positive integer. Let A be an elementary abelian group of order at least q^3 acting by automorphisms on a finite q'-group G. Assume that $F(C_G(a))$ has index at most m in $C_G(a)$ for any $a \in A^\#$. Then the index of $F_2(G)$ is m-bounded.*

In the next section, we give some lemmas that will be used in the proofs of the above results. Section 3 deals with the proof of Theorem 1.2. In Section 4, we prove Theorem 1.1.

2. Preliminaries

If A is a group of automorphisms of a group G, the subgroup generated by elements of the form $g^{-1}g^\alpha$ with $g \in G$ and $\alpha \in A$ is denoted by $[G, A]$. The subgroup $[G, A]$ is an A-invariant normal subgroup in G. Our first lemma is a collection of well-known facts on coprime actions (see, for example, [5]). Throughout the paper, we will use it without explicit references.

Lemma 2.1. *Let A be a group of automorphisms of a finite group G with $(|G|, |A|) = 1$. Then:*

(i) $G = [G, A]C_G(A)$;

(ii) $[G, A, A] = [G, A]$;

(iii) *A leaves invariant some Sylow p-subgroup of G for each prime $p \in \pi(G)$;*

(iv) *$C_{G/N}(A) = C_G(A)N/N$ for any A-invariant normal subgroup N of G;*

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(v) *if A is a noncyclic elementary abelian group and $A_1, \dots, A_s$ are the maximal subgroups in A, then $G = \langle C_G(A_1), \dots, C_G(A_s)\rangle$ and, furthermore, if G is nilpotent, then $G = \prod_i C_G(A_i)$.*

The following lemma was proved in [10]. The case where the group G is soluble was established in Goldschmidt [4, Lemma 2.1].

Lemma 2.2. *Let G be a finite group acted on by a finite group A such that $(|A|, |G|) = 1$. Then $[G, A]$ is generated by all nilpotent subgroups T such that $T = [T, A]$.*

Lemma 2.3. *Let q be a prime and let A be an elementary abelian group of order at least q^2 acting by automorphisms on a finite q'-group G. Let $A_1, \dots, A_s$ be the subgroups of index q in A. Then $[G, A]$ is generated by the subgroups $[C_G(A_i), A]$.*

Proof. If G is abelian, the result is immediate from Lemma 2.1(v) since the subgroups $C_G(A_i)$ are A-invariant. If G is nilpotent, the result can be obtained by considering the action of A on the abelian group $G/\Phi(G)$. Finally, the general case follows from the nilpotent case and Lemma 2.2. □

The following lemma is an application of the three subgroup lemma.

Lemma 2.4. *Let A be a group of automorphisms of a finite group G and let N be a normal subgroup of G contained in $C_G(A)$. Then $[[G, A], N] = 1$. In particular, if $G = [G, A]$, then $N \leq Z(G)$.*

Proof. Indeed, by the hypotheses, $[N, G, A] = [A, N, G] = 1$. Thus $[G, A, N] = 1$ and the lemma follows. □

In the next lemma, we will employ the fact that if A is any coprime group of automorphisms of a finite simple group, then A is cyclic (see, for example, [6]). We denote by $R(H)$ the soluble radical of a finite group H, that is, the largest normal soluble subgroup of H.

Theorem 2.5. *Let q be a prime and let m be a positive integer such that $m < q$. Let A be an elementary abelian group of order q^2 acting on a finite q'-group G in such a way that the index of $R(C_G(a))$ in $C_G(a)$ is at most m for any $a \in A^{\#}$. Then $[G, A]$ is soluble.*

Proof. We argue by contradiction. Choose a counterexample G of minimal order. Then $G = [G, A]$ and $R(G) = 1$. Suppose that G contains a proper normal A-invariant subgroup N. Since $[N, A]$ is subnormal, we conclude that $[N, A] = 1$ and so $N = C_N(A)$. Then by Lemma 2.4, N is central and, in view of $R(G) = 1$, we have a contradiction.

Hence G has no proper normal A-invariant subgroups and so $G = S_1 \times \cdots S_l$, where S_i are isomorphic nonabelian simple subgroups transitively permuted by A. We will prove that, under these assumptions, G has order at most m.

If $l = 1$, then G is a simple group and so $G = C_G(a)$ for some $a \in A^{\#}$. In this case, we conclude that G has order at most m by the hypotheses. Suppose, therefore, that $l \neq 1$ and so $l = q$, or $l = q^2$.

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In the first case, $G = S \times S^a \times \cdots \times S^{a^{q-1}}$ for some $a \in A$ and there exists $b \in A$ such that $S^b = S$. Here $S = S_1$. We see that $C_G(a)$ is the 'diagonal' of the direct product. In particular, $C_G(a) \cong S$ is a simple group and so $C_G(a)$ is of order at most m. Since $m < q$ and b leaves $C_G(a)$ invariant, we conclude that $C_G(a) \leq C_G(b)$. Combining this with the fact that b stabilises all simple factors, we deduce that b acts trivially on G. It follows that $|G| \leq m$.

Finally, suppose that G is a product of q^2 simple factors that are transitively permuted by A. For each $a \in A$, we see that $C_G(a)$ is a product of q 'diagonal' subgroups. In particular, $C_G(a)$ contains a direct product of q nonabelian simple groups. This is a contradiction since $[C_G(a) : R(C_G(a))]$ is at most m and $m < q$.

This proves that G has order at most m. Then, of course, A acts trivially on G. We conclude that $[G, A] = 1$. This is a contradiction and completes the proof. □

3. Proof of Theorem 1.2

Assume the hypothesis of Theorem 1.2. Thus, A is an elementary abelian group of order at least q^3 acting on a finite q'-group G in such a manner that $F(C_G(a))$ has index at most m in $C_G(a)$ for any $a \in A^{\#}$. We wish to show that $F_2(G)$ has m-bounded index in G. It is clear that A contains a subgroup of order q^3. Thus, replacing A by such a subgroup, if necessary, we may assume that A has order q^3. In what follows, $A_1, \ldots, A_s$ denote the subgroups of index q in A.

It was proved in [9, 2.11] that, under this hypothesis, the subgroup $F(G)$ has (q, m)-bounded index in G. Hence, if $q \leq m$, the subgroup $F(G)$ (and, consequently, $F_2(G)$) has m-bounded index. We will therefore assume that $q > m$. In this case, A acts trivially on $C_G(a)/F(C_G(a))$ for any $a \in A^{\#}$. Consequently, $[C_G(a), A] \leq F(C_G(a))$ for any $a \in A^{\#}$.

Observe that $\langle [C_G(A_i), A], [C_G(A_j), A] \rangle$ is nilpotent for any $1 \leq i, j \leq s$. This is because the intersection $A_i \cap A_j$ contains a nontrivial element a and the subgroups $[C_G(A_i), A]$ and $[C_G(A_j), A]$ are both contained in the nilpotent subgroup $[C_G(a), A]$.

Lemma 3.1. *The subgroup* $[G, A]$ *is nilpotent.*

Proof. We argue by contradiction. Suppose G is a counterexample of minimal possible order. By Lemma 2.5, the subgroup $[G, A]$ is soluble. Let V be a minimal A-invariant normal subgroup of G. Then V is an elementary abelian p-group and G/V is an r-group for some primes $p \neq r$. Write $G = VH$, where H is an A-invariant Sylow r-subgroup such that $H = [H, A]$. From Lemma 2.3, H is generated by the subgroups $[C_H(A_i), A]$. Thus, H centralises $[V, A]$ since $[C_V(A_i), A]$ and $[C_H(A_j), A]$ have coprime order for each $1 \leq i, j \leq s$. Hence $[V, A] \leq Z(G)$, and by the minimality we conclude that $[V, A] = 1$ and $V = C_V(A)$. But then, by Lemma 2.4, $V \leq Z(G)$ since V is a normal subgroup and $G = [G, A]$. This is a contradiction and the lemma is proved. □

We can now easily complete the proof of Theorem 1.2. By the above lemma, A acts trivially on the quotient $G/F(G)$. Therefore $G = F(G)C_G(A)$. This shows that $F(C_G(A)) \leq F_2(G)$. Since the index of $F(C_G(A))$ in $C_G(A)$ is at most m, the result follows.

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4. Proof of Theorem 1.1

We say that a finite group G is metanilpotent if $\gamma_\infty(G) \leq F(G)$.

The following elementary lemma will be useful (for the proof, see, for example, [1, Lemma 2.4]).

Lemma 4.1. *Let G be a metanilpotent finite group. Let P be a Sylow p-subgroup of $\gamma_\infty(G)$ and let H be a Hall p'-subgroup of G. Then $P = [P, H]$.*

Let us now assume the hypothesis of Theorem 1.1. Thus A is an elementary abelian group of order at least q^3 acting on a finite q'-group G in such a manner that $\gamma_\infty(C_G(a))$ has order at most m for any $a \in A^\#$. We wish to show that $\gamma_\infty(G)$ has m-bounded order. Replacing A by a subgroup, if necessary, we may assume that A has order q^3. Since $\gamma_\infty(C_G(a))$ has order at most m, we obtain that $F(C_G(a))$ has index at most $m!$ (see, for example, [7, 2.4.5]). By [2, Theorem 1.1], $\gamma_\infty(G)$ has (q, m)-bounded order. Without loss of generality, we will assume that $m! < q$. In particular, $[G, A]$ is nilpotent by Lemma 3.1.

Lemma 4.2. *If G is soluble, then $\gamma_\infty(G) = \gamma_\infty(C_G(A))$.*

Proof. We will use induction on the Fitting height h of G.

Suppose first that G is metanilpotent. Let P be a Sylow p-subgroup of $\gamma_\infty(G)$ and H be a Hall A-invariant p'-subgroup of G. By Lemma 4.1, we have $\gamma_\infty(G) = [P, H] = P$. It is sufficient to show that $P \leq \gamma_\infty(C_G(A))$. Therefore, without loss of generality, we assume that $G = PH$. With this in mind, observe that $\gamma_\infty(C_G(a)) = [C_P(a), C_H(a)]$ for any $a \in A^\#$.

We will prove that $P = [C_P(A), C_H(A)]$. Note that A acts trivially on $\gamma_\infty(C_G(a))$ for any $a \in A^\#$ since $m < q$. Hence $\gamma_\infty(C_G(a)) \leq C_P(A)$ for any $a \in A^\#$. Let $a, b \in A$. We have $[\gamma_\infty(C_G(a)), C_H(b)] \leq [C_P(A), C_H(b)] \leq \gamma_\infty(C_G(b))$. Let us show that $P = C_P(A)$.

First, assume that P is abelian. Observe that the subgroup $N = \prod_{a \in A^\#} \gamma_\infty(C_G(a))$ is normal in G. Since N is A-invariant, we obtain that A acts on G/N in such a way that $C_G(a)$ is nilpotent for any $a \in A^\#$. Thus G/N is nilpotent by [11]. Therefore $P = \prod_{a \in A^\#} \gamma_\infty(C_G(a))$. In particular, $P = C_P(A)$.

Now suppose that P is not abelian. Consider the action of A on $G/\Phi(P)$. By the above, $P/\Phi(P) = C_P(A)\Phi(P)/\Phi(P)$, which implies that $P = C_P(A)$.

Since $P = C_P(A)$ is a normal subgroup of G, by Lemma 2.4 we deduce that $[H, A]$ centralises P. Therefore $P = [C_P(A), C_H(A)]$ since $H = [H, A]C_H(A)$. This completes the proof for metanilpotent groups.

If G is soluble and has Fitting height $h > 2$, we consider the quotient group $G/\gamma_\infty(F_2(G))$, which has Fitting height $h - 1$. Clearly, $\gamma_\infty(F_2(G)) \leq \gamma_\infty(G)$. Hence, we deduce that $\gamma_\infty(G) = \gamma_\infty(C_G(A))$. □

Recall that, under our assumptions, $[G, A]$ is nilpotent and $C_G(A)$ has a normal nilpotent subgroup of index at most $m!$. Let R be the soluble radical of G. Since $G = [G, A]C_G(A)$, the index of R in G is at most $m!$. Lemma 4.2 shows that the order of $\gamma_\infty(R)$ is at most m. We pass to the quotient $G/\gamma_\infty(R)$ and, without loss of generality,

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assume that R is nilpotent. If $G = R$, we have nothing to prove. Therefore assume that $R < G$ and use induction on the index of R in G. Since $[G, A] \leq R$, it follows that each subgroup of G containing R is A-invariant. If T is any proper normal subgroup of G containing R, by induction the order of $\gamma_\infty(T)$ is m-bounded and the theorem follows. Hence we can assume that G/R is a nonabelian simple group. We know that G/R is isomorphic to a quotient of $C_G(A)$ and so, being simple, G/R has order at most m.

As usual, given a set of primes π, we write $O_\pi(U)$ to denote the maximal normal π-subgroup of a finite group U. Let $\pi = \pi(m!)$ be the set of primes at most m. Let $N = O_{\pi'}(G)$. Our assumptions imply that G/N is a π-group and $N \leq F(G)$. Thus, by the Schur–Zassenhaus theorem [5, Theorem 6.2.1], the group G has an A-invariant π-subgroup K such that $G = NK$. Let $K_0 = O_\pi(G)$.

Suppose that $K_0 = 1$. Then G is a semidirect product of N by $K = C_K(A)$. For an automorphism $a \in A^\#$, observe that $[C_N(a), K] \leq \gamma_\infty(C_G(a))$ since $C_N(a)$ and K have coprime order. On the one hand, being a subgroup of $\gamma_\infty(C_G(a))$, the subgroup $[C_N(a), K]$ must be a π-group. On the other hand, being a subgroup of N, the subgroup $[C_N(a), K]$ must be a π'-group. We conclude that $[C_N(a), K] = 1$ for each $a \in A^\#$. Since N is a product of all such centralisers $C_N(a)$, it follows that $[N, K] = 1$. Since $K_0 = 1$ and K is a π-group, we deduce that $K = 1$ and so $G = N$ is a nilpotent group.

In general, K_0 does not have to be trivial. However, considering the quotient G/K_0 and taking into account the above paragraph, we deduce that $G = N \times K$. In particular, $\gamma_\infty(G) = \gamma_\infty(K)$ and so, without loss of generality, we can assume that G is a π-group. It follows that the number of prime divisors of $|R|$ is m-bounded and we can use induction on this number. It will be convenient to prove our theorem first under the additional assumption that $G = G'$.

Suppose that R is an p-group for some prime $p \in \pi$. Note that if s is a prime that is different from p and H is an A-invariant Sylow s-subgroup of G, then, in view of Lemma 4.2, we have $\gamma_\infty(RH) \leq \gamma_\infty(C_G(A))$ because RH is soluble. We will require the following observation about finite simple groups (for the proof, see, for example, [2, Lemma 3.2]).

Lemma 4.3. *Let D be a nonabelian finite simple group and let p be a prime. There exists a prime s that is different from p such that D is generated by two Sylow s-subgroups.*

In view of Lemma 4.3 and the fact that G/R is simple, we deduce that G/R is generated by the image of two Sylow s-subgroups H_1 and H_2, where s is a prime that is different from p. Both subgroups RH_1 and RH_2 are soluble and A-invariant since $[G, A] \leq R$. Therefore both $[R, H_1]$ and $[R, H_2]$ are contained in $\gamma_\infty(C_G(A))$.

Let $H = \langle H_1, H_2 \rangle$. Thus $G = RH$. Since $G = G'$, it is clear that $G = [R, H]H$ and $[R, G] = [R, H]$. We have $[R, H] = [R, H_1][R, H_2]$ and therefore the order of $[R, H]$ is m-bounded. Passing to the quotient $G/[R, G]$, we can assume that $R = Z(G)$. So we are in the situation where $G/Z(G)$ has order at most m. By a theorem of Schur, the order of G' is m-bounded as well (see, for example, [7, 2.4.1]). Taking into account that $G = G'$, we conclude that the order of G is m-bounded.

Now suppose that $\pi(R) = \{p_1, \ldots, p_t\}$, where $t \geq 2$. For each $i = 1, \ldots, t$, consider the quotient $G/O_{p_i'}(G)$. The above paragraph shows that the order of $G/O_{p_i'}(G)$ is m-bounded. Since t also is m-bounded, the result follows.

Thus, in the case where $G = G'$, the theorem is proved. Let us now deal with the case where $G \neq G'$. Let $G^{(l)}$ be the last term of the derived series of G. The previous paragraph shows that $|G^{(l)}|$ is m-bounded. Consequently, $|\gamma_\infty(G)|$ is m-bounded since $G/G^{(l)}$ is soluble and $G^{(l)} \leq \gamma_\infty(G)$. The proof is now complete.

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