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INVERSE OF FREQUENTLY HYPERCYCLIC OPERATORS

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Abstract We show that there exists an invertible frequently hypercyclic operator on $\ell^1(\mathbb{N})$ whose inverse is not frequently hypercyclic.

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1. Introduction

Given X a separable infinite-dimensional Banach space and T a continuous and linear operator on X, we can consider, for each vector $x \in X$, the set $\mathrm{Orb}(x,T) = \{T^n x : n \in \mathbb{N}\}$, which is called the orbit of x under the action of T. We denote by $\mathbb{N}$ the set $\{0,1,2,\ldots\}$. Linear dynamics is the theory studying the properties of such orbits. One of the basic notions in linear dynamics is hypercyclicity. An operator T is said to be hypercyclic if there exists a vector $x \in X$ such that $\mathrm{Orb}(x,T)$ is dense in X or, equivalently, such that for every nonempty open set $U \subset X$, the set $N_T(x,U) := \{n \in \mathbb{N} : T^n x \in U\}$ is nonempty (or, equivalently, infinite). Several important notions related to hypercyclicity have been introduced and deeply investigated during the last decades. We will mention some of them in this article, but for more information concerning linear dynamics, the reader can refer to two books [5, 11].

Though there is no hypercyclic operator in finite dimension, each separable infinite-dimensional Banach space supports a hypercyclic operator [1, 7]. We can wonder whether it is possible to require more on the sets $N_T(x,U)$. In 2004, Bayart and Grivaux [2, 3] introduced the notion of frequent hypercyclicity. An operator T is said to be frequently hypercyclic if there exists a vector $x \in X$ such that for every nonempty open set $U \subset X$, $\underline{\mathrm{dens}}(N_T(x,U)) > 0$. In these papers, Bayart and Grivaux gave sufficient conditions for frequent hypercyclicity and showed that there are simple frequently hypercyclic operators on each space $\ell^p(\mathbb{N})$ (with $1 \le p < \infty$). However, there exist separable infinite-dimensional Banach spaces supporting no frequently hypercyclic operator and even supporting no $\mathcal{U}$-frequently hypercyclic operator [15]. An operator T is said to be

$\mathcal{U}$-frequently hypercyclic if there exists a vector $x \in X$ such that for every nonempty open set $U \subset X$, $\overline{\text{dens}}(N_T(x,U)) > 0$.

Several open questions concerning frequent hypercyclicity posed in [3, 4] have been challenging for many years. One of them, recently solved, concerned the link between chaos and frequent hypercyclicity. We recall that an operator T is said to be chaotic if T is hypercyclic and possesses a dense set of periodic points. Indeed, Bayart and Grivaux [4] showed in 2007 that there exists a frequently hypercyclic weighted shift on c_0 that is not chaotic, and a chaotic operator that is not frequently hypercyclic was obtained in 2017 [13]. This last counterexample required the introduction of a new family of operators, called operators of C-type, which have been deeply investigated in [9].

The purpose of this article is to answer the following open question, which can be found in [3, 6, 10, 12]: Is the inverse of an invertible frequently hypercyclic operator also frequently hypercyclic?

It is well known that the inverse of an invertible hypercyclic operator is always hypercyclic. It is a direct consequence of Birkhoff transitivity theorem. For $\mathcal{U}$-frequent hypercyclicity, it was recently proved that it is not the case anymore; there exists an invertible $\mathcal{U}$-frequently hypercyclic operator on $\ell^p(\mathbb{N})$ (with $1 \leq p < \infty$) whose inverse is not $\mathcal{U}$-frequently hypercyclic [14]. This counterexample was obtained by considering a suitable operator of C-type. However, we know that if T is invertible and frequently hypercyclic then T^{-1} is $\mathcal{U}$-frequently hypercyclic [6]. This means that if we want to exhibit a frequently hypercyclic operator T whose inverse is not frequently hypercyclic then we first need to find a $\mathcal{U}$-frequently hypercyclic operator that is not frequently hypercyclic. Such operators exist [6, 9], but each known example is clearly not invertible. We will show in this article that there exists an invertible frequently hypercyclic operator on $\ell^1(\mathbb{N})$ whose inverse is not frequently hypercyclic by introducing a generalisation of operators of C-type.

2. Generalisation of operators of C-type

Operators of C-type are associated with four parameters v, w, φ and b, where

- $v = (v_n)_{n\geq 1}$ is a bounded sequence of nonzero complex numbers;
- $w = (w_j)_{j\geq 1}$ is a sequence of complex numbers that is both bounded and bounded below; that is, $0 < \inf_{k\geq 1} |w_k| \leq \sup_{k\geq 1} |w_k| < \infty$;
- φ is a map from $\mathbb{N}$ into itself, such that $\varphi(0) = 0$, $\varphi(n) < n$ for every $n \geq 1$, and the set $\varphi^{-1}(l) = \{n \geq 0 : \varphi(n) = l\}$ is infinite for every $l \geq 0$;
- $b = (b_n)_{n\geq 0}$ is a strictly increasing sequence of positive integers such that $b_0 = 0$ and $b_{n+1} - b_n$ is a multiple of $2(b_{\varphi(n)+1} - b_{\varphi(n)})$ for every $n \geq 1$.

These operators are then defined as follows.

Definition 2.1. For each $n \geq 0$, let $W_n = \prod_{b_n < j < b_{n+1}} w_j$. If $\inf_{n\geq 0} |W_n| > 0$, the *operator of* C*-type* $T_{v,w,\varphi,b}$ on $\ell^1(\mathbb{N})$ associated with the data v, w, φ and b given as above is defined by

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$$T_{v,w,\varphi,b}\, e_k = \begin{cases} w_{k+1}\, e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1),\ n \geq 0, \\ v_n\, e_{b_{\varphi(n)}} - W_n^{-1} e_{b_n} & \text{if } k = b_{n+1}-1,\ n \geq 1, \\ -W_0^{-1} e_0 & \text{if } k = b_1 - 1, \end{cases}$$

where $(e_k)_{k\geq 0}$ is the canonical basis of $\ell^1(\mathbb{N})$.

These operators have the nice property that every finite sequence is periodic. This means that for every $x \in c_{00} := \operatorname{span}\{e_k : k \geq 0\}$ there exists $m \geq 1$ such that $T^m_{v,w,\varphi,b} x = x$. More precisely, for every $k \in [b_n, b_{n+1})$, we have

$$T^{2(b_{n+1}-b_n)}_{v,w,\varphi,b} e_k = e_k \quad \text{(see [9, Lemma 6.4])}.$$

Periodic points are in general quite helpful for studying dynamical properties of an operator. For instance, a simple criterion for frequent hypercyclicity based on the behaviour of periodic points was given in [9, Theorem 5.35]. Unfortunately, if periodic points of an invertible operator T satisfy the conditions of this criterion then periodic points of T^{-1} (which are periodic points of T) will also satisfy these conditions. Therefore, we cannot use this criterion in order to establish the frequent hypercyclicity of our counterexample and it seems better to perturb these periodic points to prevent frequent hypercyclicity from being transmitted to its inverse.

For this reason, we will introduce a new family of operators that contains operators of C-type but also operators for which finite sequences are not periodic.

Definition 2.2. Let $R = (R_n)_{n\geq 0}$ be a sequence of nonzero complex numbers. If $\inf_{n\geq 0} |R_n| > 0$, the generalised C-type operator $T_{v,w,\varphi,b,R}$ on $\ell^1(\mathbb{N})$ associated with the data v, w, φ, b and R given as previously is defined by

$$T_{v,w,\varphi,b,R}\, e_k = \begin{cases} w_{k+1}\, e_{k+1} & \text{if } k \in [b_n, b_{n+1}-1),\ n \geq 0, \\ v_n\, e_{b_{\varphi(n)}} - R_n^{-1} e_{b_n} & \text{if } k = b_{n+1}-1,\ n \geq 1, \\ -R_0^{-1} e_0 & \text{if } k = b_1 - 1. \end{cases}$$

The operator $T_{v,w,\varphi,b,R}$ is thus an operator of C-type as soon as $R_n = W_n$ for every n. A direct consequence of this generalisation lies in the fact that the elements in c_{00} are in general not periodic for $T_{v,w,\varphi,b,R}$. However, because we want to deduce dynamical properties of generalised C-type operators by investigating the behaviour of orbits of finite sequences, we would like orbits of finite sequences to remain simple to study. For this reason, we first show that under some additional conditions, every vector e_k is an eigenvector for some power of $T_{v,w,\varphi,b,R}$. This is the purpose of the following lemma.

Lemma 2.3. *Let $W_n = \prod_{b_n<j<b_{n+1}} w_j$ and $R = (R_n)_{n\geq 0}$ with $\inf_{n\geq 0} |R_n| > 0$. If for every $n \geq 1$ we have*

$$R_n^{-1} W_n = (R^{-1}_{\varphi(n)} W_{\varphi(n)})^{\frac{b_{n+1}-b_n}{b_{\varphi(n)+1}-b_{\varphi(n)}}},$$

then for every $n \geq 0$, every $k \in [b_n, b_{n+1})$,

$$T^{2(b_{n+1}-b_n)}_{v,w,\varphi,b,R} e_k = (R_n^{-1} W_n)^2 e_k.$$

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Proof. We remark that it suffices to show that for every $n \geq 0$,

$$T_{v,w,\varphi,b,R}^{2(b_{n+1}-b_n)} e_{b_n} = (R_n^{-1} W_n)^2 e_{b_n}$$

because for every $k \in [b_n, b_{n+1})$, e_k is a multiple of $T^{k-b_n} e_{b_n}$.

It follows from the definition of $T_{v,w,\varphi,b,R}$ that

$$T_{v,w,\varphi,b,R}^{2(b_1-b_0)} e_{b_0} = (R_0^{-1} W_0)^2 e_{b_0}.$$

Let $N \geq 1$. Assume that we have $T_{v,w,\varphi,b,R}^{2(b_{n+1}-b_n)} e_{b_n} = (R_n^{-1} W_n)^2 e_{b_n}$ for every $n < N$. Then

$$T_{v,w,\varphi,b,R}^{b_{N+1}-b_N} e_{b_N} = -R_N^{-1} W_N e_{b_N} + v_N W_N e_{b_{\varphi(N)}},$$

and because $b_{N+1} - b_N$ is a multiple of $2(b_{\varphi(N)+1} - b_{\varphi(N)})$, we have by induction hypothesis

$$\begin{aligned} T_{v,w,\varphi,b,R}^{2(b_{N+1}-b_N)} e_{b_N} &= T_{v,w,\varphi,b,R}^{b_{N+1}-b_N} \left(-R_N^{-1} W_N e_{b_N} + v_N W_N e_{b_{\varphi(N)}}\right) \\ &= -R_N^{-1} W_N \left(-R_N^{-1} W_N e_{b_N} + v_N W_N e_{b_{\varphi(N)}}\right) \\ &\quad + v_N W_N \left((R_{\varphi(N)}^{-1} W_{\varphi(N)})^2\right)^{\frac{b_{N+1}-b_N}{2(b_{\varphi(N)+1}-b_{\varphi(N)})}} e_{b_{\varphi(N)}} \\ &= (R_N^{-1} W_N)^2 e_{b_N}. \end{aligned}$$

□

In particular, we remark that for operators of C-type, we get the previously mentioned result that $T_{v,w,\varphi,b}^{2(b_{n+1}-b_n)} e_k = e_k$ for every $k \in [b_n, b_{n+1})$ because $R_n = W_n$. We now need to know when a generalised C-type operator is invertible. In the paper [14], the invertibility of operators of C-type was obtained by requiring that the sequence (v_n) approximates zero sufficiently rapidly. Two adaptations will be necessary here. First, we will not consider operators of C-type but generalised C-type operators with $R_n = 1$ for every n and, secondly, we will need to consider a sequence (v_n) that takes infinitely often the same value in order to get the desired counterexample. Note that it is because of this last condition on v that we have to restrict ourselves to operators on $\ell^1(\mathbb{N})$.

Proposition 2.4. *Assume that $R_n = 1$ for every $n \geq 0$ and that*

$$\lim_{N\to\infty} \sup_{n\in\varphi^{-1}(N)} |v_n| = 0 \quad \textit{and} \quad \sup_{l\geq 1} \left(\sum_{m=0}^{m_l-1} \prod_{s=0}^{m} |v_{\varphi^s(l)}| \right) < \infty,$$

where $m_l = \min\{s \geq 0 : \varphi^s(l) = 0\}$. Then the generalised C-type operator $T_{v,w,\varphi,b,R}$ is invertible on $\ell^1(\mathbb{N})$ and

$$T_{v,w,\varphi,b,R}^{-1}\, e_k = \begin{cases} \frac{1}{w_k} e_{k-1} \ \textit{if}\ k \in (b_n, b_{n+1}),\ n \geq 0, \\ -\sum_{m=0}^{m_n-1} \left(\prod_{l=0}^{m} v_{\varphi^l(n)}\right) e_{b_{\varphi^{m+1}(n)+1}-1} - e_{b_{n+1}-1} \ \textit{if}\ k = b_n,\ n \geq 1, \\ -e_{b_1-1} \ \textit{if}\ k = 0. \end{cases}$$

Proof. We first prove that $T_{v,w,\varphi,b,R}$ is injective. Let $x=(x_k)_{k\geq 0}\in \ell^1(\mathbb{N})$ such that $T_{v,w,\varphi,b,R}x=0$ and $n\geq 0$. It follows that $x_k=0$ for every $k\in[b_n,b_{n+1}-1)$ and that

$$-x_{b_{n+1}-1}+\sum_{m\in\varphi^{-1}(n)\backslash\{0\}} v_m x_{b_{m+1}-1}=0.$$

Assume that there exists n_0 such that $|x_{b_{n_0+1}-1}|>\varepsilon>0$. Then we deduce that

$$\sum_{m\in\varphi^{-1}(n_0)\backslash\{0\}} |v_m||x_{b_{m+1}-1}|>\varepsilon$$

and thus that

$$\sum_{m\in\mathcal{N}_1}|x_{b_{m+1}-1}|>\frac{\varepsilon}{S_1},$$

where $\mathcal{N}_1=\varphi^{-1}(n_0)\backslash\{0\}$ and $S_1=\sup_{m\in\mathcal{N}_1}|v_m|$. By looking at $\sum_{n\in\mathcal{N}_1}x_{b_{n+1}-1}$, we deduce in the same way that

$$\sum_{n\in\mathcal{N}_1}\sum_{m\in\varphi^{-1}(n)} v_m x_{b_{m+1}-1}=\sum_{n\in\mathcal{N}_1}x_{b_{n+1}-1}\quad\text{and thus that}\quad \sum_{m\in\mathcal{N}_2}|x_{b_{m+1}-1}|>\frac{\varepsilon}{S_1S_2},$$

where $\mathcal{N}_2=\varphi^{-1}(\mathcal{N}_1)$ and $S_2=\sup_{m\in\mathcal{N}_2}|v_m|$. By repeating this argument, we get for every $k\geq 2$,

$$\sum_{m\in\mathcal{N}_k}|x_{b_{m+1}-1}|>\frac{\varepsilon}{\prod_{l=1}^k S_l},$$

where $\mathcal{N}_k=\varphi^{-1}(\mathcal{N}_{k-1})$ and $S_k=\sup_{m\in\mathcal{N}_k}|v_m|$. Therefore, because $\inf\mathcal{N}_k\geq k$ for every $k\geq 1$, we have by assumption $\lim_k S_k=0$ and it is then impossible that x belongs to $\ell^1(\mathbb{N})$.

The operator $T_{v,w,\varphi,b,R}$ is thus injective and we can compute that

$$T^{-1}_{v,w,\varphi,b,R}\,e_k=\begin{cases}\frac{1}{w_k}e_{k-1}\text{ if }k\in(b_n,b_{n+1}),\ n\geq 0,\\ -\sum_{m=0}^{m_n-1}\left(\prod_{l=0}^m v_{\varphi^l(n)}\right)e_{b_{\varphi^{m+1}(n)+1}-1}-e_{b_{n+1}-1}\text{ if }k=b_n,\ n\geq 1,\\ -e_{b_1-1}\text{ if }k=0.\end{cases}$$

We only show that for every $n\geq 1$,

$$T_{v,w,\varphi,b,R}\left(-\sum_{m=0}^{m_n-1}\left(\prod_{l=0}^m v_{\varphi^l(n)}\right)e_{b_{\varphi^{m+1}(n)+1}-1}-e_{b_{n+1}-1}\right)=e_{b_n}.$$

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Indeed, we have

$$\begin{aligned}
&T_{v,w,\varphi,b,R}\left(-\sum_{m=0}^{m_n-1}\Big(\prod_{l=0}^{m}v_{\varphi^l(n)}\Big)e_{b_{\varphi^{m+1}(n)+1}-1}-e_{b_{n+1}-1}\right)\\
&=\Big(\prod_{l=0}^{m_n-1}v_{\varphi^l(n)}\Big)e_0-\sum_{m=0}^{m_n-2}\Big(\prod_{l=0}^{m}v_{\varphi^l(n)}\Big)\Big(v_{\varphi^{m+1}(n)}e_{b_{\varphi^{m+2}(n)}}-e_{b_{\varphi^{m+1}(n)}}\Big)\\
&\quad-\big(v_ne_{b_{\varphi(n)}}-e_{b_n}\big)\\
&=-\sum_{m=0}^{m_n-2}\Big(\prod_{l=0}^{m+1}v_{\varphi^l(n)}\Big)e_{b_{\varphi^{m+2}(n)}}+\sum_{m=0}^{m_n-1}\Big(\prod_{l=0}^{m}v_{\varphi^l(n)}\Big)e_{b_{\varphi^{m+1}(n)}}-v_ne_{b_{\varphi(n)}}+e_{b_n}\\
&=e_{b_n}.
\end{aligned}$$

We now show that $T_{v,w,\varphi,b,R}$ is surjective. Let $z=(z_k)_{k\geq 0}\in\ell^1(\mathbb{N})$. It suffices to show that the sequence $(T^{-1}_{v,w,\varphi,b,R}P_{[0,b_{n+1})}z)_n$ is a Cauchy sequence where for every subset $I\subset\mathbb{N}$ we let $P_Iz=\sum_{k\in I}z_ke_k$.

Let $N>n$. We have

$$\begin{aligned}
&\|T^{-1}_{v,w,\varphi,b,R}P_{[0,b_{N+1})}z-T^{-1}_{v,w,\varphi,b,R}P_{[0,b_{n+1})}z\|\\
&\quad=\Big\|\sum_{k=b_{n+1}}^{b_{N+1}-1}z_kT^{-1}_{v,w,\varphi,b,R}e_k\Big\|\\
&\quad\leq\sum_{l=n+1}^{N}\sum_{k=b_l+1}^{b_{l+1}-1}\frac{1}{|w_k|}|z_k|+\sum_{l=n+1}^{N}\left(\sum_{m=0}^{m_l-1}\prod_{s=0}^{m}|v_{\varphi^s(l)}|\right)|z_{b_l}|+\sum_{l=n+1}^{N}|z_{b_l}|\\
&\quad\leq\left(\frac{1}{\inf_k|w_k|}+\sup_{l\geq 1}\Big(\sum_{m=0}^{m_l-1}\prod_{s=0}^{m}|v_{\varphi^s(l)}|\Big)+1\right)\|P_{[0,b_{N+1})}z-P_{[0,b_{n+1})}z\|.
\end{aligned}$$

We conclude that the sequence $(T^{-1}_{v,w,\varphi,b,R}P_{[0,b_{n+1})}z)_n$ is a Cauchy sequence because $\left(\frac{1}{\inf_k|w_k|}+\sup_{l\geq1}\left(\sum_{m=0}^{m_l-1}\prod_{s=0}^{m}|v_{\varphi^s(l)}|\right)\right)<\infty$ and thus $T_{v,w,\varphi,b,R}$ is surjective. Finally, it follows from the open mapping theorem that the inverse of $T_{v,w,\varphi,b,R}$ is continuous. □

We can remark that a generalised C-type operator $T_{v,w,\varphi,b,R}$ with $R_n=1$ is therefore invertible if the sequence $(\sup_{n\in\varphi^{-1}(N)}|v_n|)_N$ decreases sufficiently rapidly.

Corollary 2.5. *Assume that $R_n=1$ for every $n\geq 0$. If $\sup_{n\in\varphi^{-1}(m)}|v_n|\leq\frac{1}{2^m}$ for every $m\geq 0$, then the generalised C-type operator $T_{v,w,\varphi,b,R}$ is invertible.*

Proof. Let $m_l=\min\{s\geq 0:\varphi^s(l)=0\}$ as defined in Proposition 2.4. We have

$$\sup_{n\in\varphi^{-1}(N)}|v_n|\leq\frac{1}{2^N}\xrightarrow[N\to\infty]{}0,$$

and because for every $l \geq 1$, every $0 \leq m \leq m_l$, we have $\varphi^{m_l - m}(l) \geq m$, it follows that

$$\sum_{m=0}^{m_l-1} \prod_{s=0}^{m} |v_{\varphi^s(l)}| \leq \sum_{m=0}^{m_l-1} \frac{1}{2^{\varphi^{m+1}(l)}} \leq \sum_{m=0}^{m_l-1} \frac{1}{2^{m_l-m-1}} \leq 2.$$

We can then deduce from Proposition 2.4 that $T_{v,w,\varphi,b,R}$ is invertible. □

3. A frequently hypercyclic operator whose inverse is not frequently hypercyclic

Let $n_0 = 0$ and $n_k = 2^{k-1}$ for every $k \geq 1$. We will consider a generalised C-type operator $\tilde{T} = T_{v,w,\varphi,b,R}$ with the following parameters:

- for every $n \in [n_k, n_{k+1})$, $\varphi(n) = n - n_k$;
- for every m, every $n \in \varphi^{-1}(m)$, $v_n = 2^{-\tau_m}$;
- for every $k \geq 0$, every $n \in [n_k, n_{k+1})$, every $i \in (b_n, b_{n+1})$,

$$w_i = \begin{cases} \frac{1}{2} & \text{if } \; b_n < i \leq b_n + \eta_n \\ 1 & \text{if } \; b_n + \eta_n < i < b_{n+1} - 2\delta_n \\ \frac{1}{2} & \text{if } \; b_{n+1} - 2\delta_n \leq i < b_{n+1} - \delta_n \\ 2 & \text{if } \; b_{n+1} - \delta_n \leq i < b_{n+1} \end{cases}$$

- for every $n \geq 0$, $R_n = 1$,

where $(\tau_m)_{m \geq 0}$ is a strictly increasing sequence of positive integers and for every $k \geq 0$, for every $n \in [n_k, n_{k+1})$,

$$\delta_n = \delta^{(k)}, \quad \eta_n = \eta^{(k)} \quad \text{and} \quad b_{n+1} - b_n = \Delta^{(k)},$$

where $(\delta^{(k)})_{k \geq 0}$, $(\eta^{(k)})_{k \geq 0}$ and $(\Delta^{(k)})_{k \geq 0}$ are three increasing sequences of positive integers satisfying for every $k \geq 0$,

$$2\delta^{(k)} + \eta^{(k)} < \Delta^{(k)}, \quad \Delta^{(k+1)} \text{ is a multiple of } 2\Delta^{(k)} \quad \text{and} \quad \frac{\eta^{(k)}}{\Delta^{(k)}} = \frac{\eta^{(0)}}{\Delta^{(0)}}.$$

From now on we will denote by $\tilde{T}$ this operator that depends on the four parameters $(\tau_m)_{m \geq 0}$, $(\delta^{(k)})_{k \geq 0}$, $(\eta^{(k)})_{k \geq 0}$ and $(\Delta^{(k)})_{k \geq 0}$, and we will show that under convenient conditions on these parameters $\tilde{T}$ is an invertible frequently hypercyclic operator on $\ell^1(\mathbb{N})$ whose inverse is not frequently hypercyclic. More precisely, we will see that if we let $\Delta^{(k)} = 8^{k+1}$, $\eta^{(k)} = \delta^{(k)} = 8^k$ and if $(\tau_n)_n$ grows sufficiently rapidly, then $\tilde{T}$ is an invertible frequently hypercyclic operator on $\ell^1(\mathbb{N})$ whose inverse is not frequently hypercyclic.

Observe that because the sequence $(\tau_n)_n$ is strictly increasing, it follows from Corollary 2.5 that $\tilde{T}$ is already an invertible operator. Moreover, each finite sequence is an eigenvector for some power of $\tilde{T}$.

Proposition 3.1. *For every $n \geq 0$, every $x \in \operatorname{span}\{e_k : k < b_{n+1}\}$,*

$$\tilde{T}^{2(b_{n+1}-b_n)} x = 2^{-2\eta_n} x.$$

Proof. By definition of (w_j), we have $W_n = 2^{-\eta_n}$ for every $n \geq 0$. For every $n \geq 1$, let $n \in [n_K, n_{K+1})$ for some $K \geq 1$, and $\varphi(n) \in [n_k, n_{k+1})$ for some $k \geq 0$. We get for every $n \geq 1$,

$$\begin{aligned}(R_{\varphi_n}^{-1} W_{\varphi(n)})^{\frac{b_{n+1}-b_n}{b_{\varphi(n)+1}-b_{\varphi(n)}}} &= (2^{-\eta_{\varphi(n)}})^{\frac{b_{n+1}-b_n}{b_{\varphi(n)+1}-b_{\varphi(n)}}} \\ &= (2^{-\eta^{(k)}})^{\frac{\Delta^{(K)}}{\Delta^{(k)}}} = 2^{-\eta^{(K)}} = R_n^{-1} W_n\end{aligned}$$

because $(\frac{\eta^{(k)}}{\Delta^{(k)}})_k$ is a constant sequence.

It follows from Lemma 2.3 that for every $n \geq 0$, every $j \in [b_n, b_{n+1})$,

$$\tilde{T}^{2(b_{n+1}-b_n)} e_j = 2^{-2\eta_n} e_j.$$

Finally, if $x = \sum_{m=0}^{n} \sum_{j=b_m}^{b_{m+1}-1} x_j e_j$ and $n \in [n_K, n_{K+1})$ for some $K \geq 0$, we have, by using the fact that $\Delta^{(k+1)}$ is a multiple of $2\Delta^{(k)}$ for every $k \geq 0$,

$$\begin{aligned}\tilde{T}^{2(b_{n+1}-b_n)} x &= \sum_{k=0}^{K-1} \sum_{m=n_k}^{n_{k+1}-1} \sum_{j=b_m}^{b_{m+1}-1} (2^{-2\eta^{(k)}})^{\frac{2\Delta^{(K)}}{2\Delta^{(k)}}} x_j e_j + \sum_{m=n_K}^{n} \sum_{j=b_m}^{b_{m+1}-1} 2^{-2\eta^{(K)}} x_j e_j \\ &= \sum_{k=0}^{K-1} \sum_{m=n_k}^{n_{k+1}-1} \sum_{j=b_m}^{b_{m+1}-1} 2^{-2\eta^{(K)}} x_j e_j + \sum_{m=n_K}^{n} \sum_{j=b_m}^{b_{m+1}-1} 2^{-2\eta^{(K)}} x_j e_j \\ &= 2^{-2\eta^{(K)}} x = 2^{-2\eta_n} x.\end{aligned}$$

□

In order to show that $\tilde{T}$ is in fact frequently hypercyclic, we begin by stating the following technical lemma.

Lemma 3.2.

1. *For every $y \in c_{00}$, there exists k_0 such that for every $k \geq k_0$,*

$$\|\tilde{T}^k y\| \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}} k}.$$

2. *For every $K_0 \geq 0$, every $N \geq 1$, there exists $K \geq K_0$ such that for every $n \in [n_K, n_K + N)$, every $k \geq 0$,*

$$\|\tilde{T}^k e_{b_n}\| \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}} k}.$$

Proof. For proof of statement (1), let $K \geq 0$, $y = \sum_{l=0}^{b_{n_{K+1}}-1} y_l e_l$ and $C = \sup_{k < 2\Delta^{(K)}} \|\tilde{T}^k\|$. If we consider a positive integer L and $k \in [2L\Delta^{(K)}, 2(L+1)\Delta^{(K)})$, then by Proposition 3.1 we get

$$\begin{aligned}\|\tilde{T}^k y\| &\leq \|\tilde{T}^{k-2L\Delta^{(K)}}\| \|\tilde{T}^{2L\Delta^{(K)}} y\| \leq C 2^{-2L\eta^{(K)}} \|y\| \\ &\leq C 2^{-\frac{k-2\Delta^{(K)}}{\Delta^{(K)}} \eta^{(K)}} \|y\| \leq C 2^{-\frac{\eta^{(0)}}{\Delta^{(0)}} k + 2\eta^{(K)}} \|y\|\end{aligned}$$

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because $\frac{\eta^{(K)}}{\Delta^{(K)}} = \frac{\eta^{(0)}}{\Delta^{(0)}}$. We can then deduce that there exists k_0 such that for every $k \geq k_0$,

$$\|\tilde{T}^k y\| \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}k}$$

because

$$\frac{C2^{-\frac{\eta^{(0)}}{\Delta^{(0)}}k+2\eta^{(K)}}\|y\|}{2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}k}} = C2^{-\frac{2\eta^{(0)}}{3\Delta^{(0)}}k+2\eta^{(K)}}\|y\| \xrightarrow[k\to\infty]{} 0.$$

For proof of statement (2), let $K_0 \geq 0$, $N \geq 1$ and $C = \sup_{m\leq N}\sup_{j\in[0,2(b_{m+1}-b_m))}\|\tilde{T}^j e_{b_m}\|$. We consider $K \geq K_0$ such that $n_{K+1} - n_K > N$ and $2^{\frac{\eta^{(K)}}{3}} > C+1$. Let $n \in [n_K, n_K + N)$ and $k \geq 0$. If $k < \Delta^{(K)}$, then by definition of (w_i),

$$\|\tilde{T}^k e_{b_n}\| \leq 2^{-\min\{k,\eta^{(K)}\}} \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}k}$$

because $\frac{\eta^{(0)}}{3\Delta^{(0)}} \leq 1$ and $\frac{\eta^{(0)}}{3\Delta^{(0)}}\Delta^{(K)} = \frac{\eta^{(K)}}{3}$. On the other hand, if $k \in [\Delta^{(K)}, 2\Delta^{(K)})$, because $\varphi(n) = n - n_K \leq N$ and $\tilde{T}^{2(b_{m+1}-b_m)}e_{b_m} = e^{-2\eta_m}e_{b_m}$ for every $m \geq 0$, we have, by definition of $\tilde{T}$,

$$\begin{aligned}\|\tilde{T}^k e_{b_n}\| &\leq 2^{-\eta^{(K)}}\|\tilde{T}^{k-\Delta^{(K)}}e_{b_n}\| + |v_n|2^{-\eta^{(K)}}\|\tilde{T}^{k-\Delta^{(K)}}e_{b_{\varphi(n)}}\|\\ &\leq 2^{-\eta^{(K)}}2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}(k-\Delta^{(K)})} + C2^{-\eta^{(K)}}\\ &\leq (C+1)2^{-\eta^{(K)}} \leq 2^{-\frac{2\eta^{(K)}}{3}} \leq 2^{-\frac{2\eta^{(0)}}{3\Delta^{(0)}}\Delta^{(K)}} \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}k}.\end{aligned}$$

Finally, for every $L \geq 1$, every $k \in [2L\Delta^{(K)}, 2(L+1)\Delta^{(K)})$, we get

$$\begin{aligned}\|\tilde{T}^k e_{b_n}\| &= 2^{-2L\eta^{(K)}}\|\tilde{T}^{k-2L\Delta^{(K)}}e_{b_n}\|\\ &\leq 2^{-2L\eta^{(K)}}2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}(k-2L\Delta^{(K)})}\\ &\leq 2^{-2L\Delta^{(K)}\frac{\eta^{(0)}}{\Delta^{(0)}}}2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}(k-2L\Delta^{(K)})} \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}k}.\end{aligned}$$

□

We are now able to construct a frequently hypercyclic vector for $\tilde{T}$.

Proposition 3.3. *$\tilde{T}$ is a frequently hypercyclic operator on $\ell^1(\mathbb{N})$.*

Proof. Let $(y^{(j)})_{j\geq 1}$ be a dense sequence in $\ell^1(\mathbb{N})$ with $\deg(y^{(j)}) < b_{n_{j+1}}$; that is, $y^{(j)} = \sum_{k=0}^{b_{n_{j+1}}-1} y_k^{(j)} e_k$. Let $A(s,l)$ be sets of positive lower density such that for every $j \in A(s,l)$, every $j' \in A(s',l')$, if $j \neq j'$ then $|j-j'| \geq s+s'$ and $\min A(s,l) \geq l$. The construction of such sets can be found in [8, Lemma 2.5].

We select four sequences of integers $(N_j)_{j\geq 1}$, $(s_j)_{j\geq 1}$, $(l_j)_{j\geq 1}$ and (k_m) with $m \in \bigcup_{j\geq 1} A(s_j,l_j)$ such that for every $j \geq 1$, every $m \in A(s_j,l_j)$,

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(a) N_j is sufficiently large so that

$$\|y^{(j)}\| \frac{2^{-\left(2N_j\Delta^{(j)}-(2N_j+1)\eta^{(j)}\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \leq \frac{1}{2^j}.$$

(b) s_j is sufficiently large so that $s_j > (2N_j+1)\Delta^{(j)}$,

$$\|y^{(j)}\| \frac{2^{-\left(s_j-(2N_j+1)\Delta^{(j)}-(2N_j+1)\eta^{(j)}-1\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \leq \frac{1}{2^j}$$

and

$$\|y^{(j)}\| \frac{2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}s_j+(2N_j+1)\eta^{(j)}+1}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \leq \frac{1}{2^j},$$

and so that for every $r < n_{j+1}$ and every $n \geq s_j - (2N_j+1)\Delta^{(j)}$,

$$\|\tilde{T}^n e_{b_r}\| \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}n}.$$

This last condition can be guaranteed thanks to Lemma 3.2.

(c) l_j is sufficiently large so that $l_j > (2N_j+1)\Delta^{(j)}$ and

$$\|y^{(j)}\| \frac{2^{-\left(l_j-(2N_j+1)\Delta^{(j)}-(2N_j+1)\eta^{(j)}-2\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \leq \frac{1}{2^j}.$$

(d) k_m is sufficiently large so that $n_{k_m} + n_{j+1} < n_{k_m+1}$, $\eta^{(k_m)} \geq (2N_j+1)\Delta^{(j)}$ and $\delta^{(k_m)} \geq m$ and so that for every $N \in [n_{k_m}, n_{k_m} + n_{j+1})$, every $n \geq 0$,

$$\|\tilde{T}^n e_{b_N}\| \leq 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}n},$$

which can be guaranteed thanks to Lemma 3.2.

We then let $x = \sum_{j\geq 1}\sum_{m\in A(s_j,l_j)} x^{(m)}$, where

$$x^{(m)} = \sum_{n=0}^{n_{j+1}-1}\sum_{i=b_n}^{b_{n+1}-1} y_i^{(j)} \frac{2^{-\left(m-i+b_n-2N_j\Delta^{(j)}-2N_j\eta^{(j)}-1\right)}}{v_{n_{k_m}+n}\left(\prod_{t=b_n+1}^{i} w_t\right)} e_{b_{n_{k_m}+n+1}-(m-i+b_n-2N_j\Delta^{(j)})}.$$

The rest of the proof consists in showing that x belongs to $\ell^1(\mathbb{N})$ and that for every $m \in A(s_j,l_j)$, $\|\tilde{T}^m x - y^{(j)}\| < \varepsilon_j$ for some sequence $(\varepsilon_j)_j$ tending toward 0. Because $(y^{(j)})_j$ is a dense sequence in $\ell^1(\mathbb{N})$ and each set $A(s_j,l_j)$ has a positive lower density, it will then follow that x is a frequently hypercyclic vector for $\tilde{T}$.

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Let $j \geq 1$ and $m \in A(s_j,l_j)$. We have

$$\begin{aligned}\|x^{(m)}\| &\leq \sum_{n=0}^{n_{j+1}-1}\sum_{i=b_n}^{b_{n+1}-1} |y_i^{(j)}| \frac{2^{-\left(m-i+b_n-2N_j\Delta^{(j)}-2N_j\eta^{(j)}-1\right)}}{|v_{n_{k_m}+n}|\left(\prod_{t=b_n+1}^{i}|w_t|\right)}\\ &\leq \|y^{(j)}\| \frac{2^{-\left(m-(2N_j+1)\Delta^{(j)}-2N_j\eta^{(j)}-1\right)}}{\inf\{|v_{n_{k_m}+n}| : n < n_{j+1}\}2^{-\eta^{(j)}}}\\ &\leq \|y^{(j)}\| \frac{2^{-\left(m-(2N_j+1)\Delta^{(j)}-(2N_j+1)\eta^{(j)}-1\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}}.\end{aligned}$$

Moreover, because $\min A(s,l) \geq l$, it follows from (c) that

$$\begin{aligned}&\sum_{j\geq 1}\sum_{m\in A(s_j,l_j)} \|y^{(j)}\| \frac{2^{-\left(m-(2N_j+1)\Delta^{(j)}-(2N_j+1)\eta^{(j)}-1\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}}\\ &\quad \leq \sum_{j\geq 1} \|y^{(j)}\| \frac{2^{-\left(l_j-(2N_j+1)\Delta^{(j)}-(2N_j+1)\eta^{(j)}-2\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \leq 1.\end{aligned}$$

We can thus deduce that x is well defined and belongs to $\ell^1(\mathbb{N})$.

Let $J \geq 1$ and $M \in A(s_J,l_J)$. In order to estimate $\|\tilde{T}^M x - y^{(J)}\|$, we compute the elements $\tilde{T}^M x^{(m)}$ with $m \in \bigcup_{j\geq 1} A(s_j,l_j)$ by dividing our study into three cases: $m = M$, $m > M$ and $m < M$. Let $j \geq 1$ and $m \in A(s_j,l_j)$.

Case 1. ($m = M$). Let $n < n_{J+1}$ and $i \in [b_n,b_{n+1})$. We have

$$\begin{aligned}&\tilde{T}^{M-i+b_n-2N_J\Delta^{(J)}} e_{b_{n_{k_M}+n+1}-(M-i+b_n-2N_J\Delta^{(J)})}\\ &\quad = 2^{M-i+b_n-2N_J\Delta^{(J)}-1}(v_{n_{k_M}+n}e_{b_n} - e_{b_{n_{k_M}+n}})\end{aligned}$$

because $1 \leq M - i + b_n - 2N_J\Delta^{(J)} \leq \delta^{(k_M)}$ by (c) and (d) and because $n_{k_M} + n \in [n_{k_M}, n_{k_M+1})$ by (d). Moreover, because $(2N_J+1)\Delta^{(J)} \leq \eta^{(k_M)}$ by (d), we have by Proposition 3.1

$$\begin{aligned}&\tilde{T}^{M-i+b_n} e_{b_{n_{k_M}+n+1}-(M-i+b_n-2N_J\Delta^{(J)})}\\ &\quad = 2^{M-i+b_n-2N_J\Delta^{(J)}-2N_J\eta^{(J)}-1} v_{n_{k_M}+n}e_{b_n}\\ &\qquad - 2^{M-i+b_n-4N_J\Delta^{(J)}-1} e_{b_{n_{k_M}+n}+2N_J\Delta^{(J)}}\end{aligned}$$

and

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$$\begin{aligned}&\tilde{T}^M e_{b_{n_{k_M}+n+1-(M-i+b_n-2N_J\Delta^{(J)})}}\\&\quad = 2^{M-i+b_n-2N_J\Delta^{(J)}-2N_J\eta^{(J)}-1}\left(\prod_{t=b_n+1}^{i} w_t\right) v_{n_{k_M}+n} e_i\\&\qquad - 2^{M-2i+2b_n-4N_J\Delta^{(J)}-1} e_{b_{n_{k_M}+n}+2N_J\Delta^{(J)}+i-b_n}.\end{aligned}$$

It then follows from (a) that

$$\begin{aligned}&\|\tilde{T}^M x^{(M)} - y^{(J)}\|\\&\quad \le \sum_{n=0}^{n_{J+1}-1}\sum_{i=b_n}^{b_{n+1}-1} |y_i^{(J)}| \frac{2^{-\left(M-i+b_n-2N_J\Delta^{(J)}-2N_J\eta^{(J)}-1\right)}}{|v_{n_{k_M}+n}|\left(\prod_{t=b_n+1}^{i}|w_t|\right)} 2^{M-2i+2b_n-4N_J\Delta^{(J)}-1}\\&\quad \le \|y^{(J)}\| \frac{2^{-\left(2N_J\Delta^{(J)}-(2N_J+1)\eta^{(J)}\right)}}{\inf\{2^{-\tau_n} : n < n_{J+1}\}} \le \frac{1}{2^J}.\end{aligned}$$

Case 2. $(m > M)$. Let $n < n_{j+1}$ and $i \in [b_n, b_{n+1})$. By the properties of $(A(s,l))$, and by (b) we have

$$m - i + b_n - 2N_j\Delta^{(j)} \ge m - (2N_j+1)\Delta^{(j)} > m - s_j \ge M,$$

and by (d) we have

$$m - i + b_n - 2N_j\Delta^{(j)} \le m \le \delta^{(k_m)}.$$

It then follows that

$$\tilde{T}^M e_{b_{n_{k_m}+n+1-(m-i+b_n-2N_j\Delta^{(j)})}} = 2^M e_{b_{n_{k_m}+n+1+M-(m-i+b_n-2N_j\Delta^{(j)})}}$$

and by using (b), we get

$$\begin{aligned}&\|\tilde{T}^M x^{(m)}\|\\&\quad \le \sum_{n=0}^{n_{j+1}-1}\sum_{i=b_n}^{b_{n+1}-1} |y_i^{(j)}| \frac{2^{-\left(m-i+b_n-2N_j\Delta^{(j)}-2N_j\eta^{(j)}-1\right)}}{|v_{n_{k_m}+n}|\left(\prod_{t=b_n+1}^{i}|w_t|\right)} 2^M\\&\quad \le \frac{\|y^{(j)}\|}{2^{m-M-s_j}} \frac{2^{-\left(s_j-(2N_j+1)\Delta^{(j)}-(2N_j+1)\eta^{(j)}-1\right)}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \le \frac{1}{2^{m-M-s_j+j}}.\end{aligned}$$

Case 3. $(m < M)$. Let $n < n_{j+1}$ and $i \in [b_n, b_{n+1})$. Since $m - i + b_n - 2N_j\Delta^{(j)} \le m \le \delta^{(k_m)}$ by (d), we have

$$\begin{aligned}&\tilde{T}^{m-i+b_n-2N_j\Delta^{(j)}} e_{b_{n_{k_m}+n+1-(m-i+b_n-2N_j\Delta^{(j)})}}\\&\quad = 2^{m-i+b_n-2N_j\Delta^{(j)}-1} v_{n_{k_m}+n} e_{b_n} - 2^{m-i+b_n-2N_j\Delta^{(j)}-1} e_{b_{n_{k_m}+n}},\end{aligned}$$

and because $M - (m - i + b_n - 2N_j\Delta^{(j)}) \ge M - m \ge s_j$, we deduce from (b) and (d) that

$$\begin{aligned}
&\|\tilde{T}^M e_{b_{n_{k_m}+n+1}-(m-i+b_n-2N_j\Delta^{(j)})}\| \\
&\quad \leq \frac{2^{m-i+b_n-2N_j\Delta^{(j)}-1}|v_{n_{k_m}+n}|}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m)}} + \frac{2^{m-i+b_n-2N_j\Delta^{(j)}-1}}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m)}} \\
&\quad \leq \frac{2^{m-i+b_n-2N_j\Delta^{(j)}}}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m)}}.
\end{aligned}$$

We can then deduce from (b) that

$$\begin{aligned}
&\|\tilde{T}^M x^{(m)}\| \\
&\quad \leq \sum_{n=0}^{n_{j+1}-1}\sum_{i=b_n}^{b_{n+1}-1} |y_i^{(j)}| \frac{2^{-\left(m-i+b_n-2N_j\Delta^{(j)}-2N_j\eta^{(j)}-1\right)}}{|v_{n_{k_m}+n}|\left(\prod_{t=b_n+1}^{i}|w_t|\right)} \frac{2^{m-i+b_n-2N_j\Delta^{(j)}}}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m)}} \\
&\quad \leq \|y^{(j)}\| \frac{2^{(2N_j+1)\eta^{(j)}+1}}{\inf\{2^{-\tau_n} : n < n_{j+1}\} 2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m)}} \\
&\quad \leq \frac{\|y^{(j)}\|}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m-s_j)}} \frac{2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}s_j+(2N_j+1)\eta^{(j)}+1}}{\inf\{2^{-\tau_n} : n < n_{j+1}\}} \\
&\quad \leq \frac{1}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m-s_j)+j}}.
\end{aligned}$$

In conclusion, thanks to properties of the sets $A(s,l)$, we have for every $J \geq 1$, every $M \in A(s_J, l_J)$,

$$\begin{aligned}
&\|T^M x - y^{(J)}\| \\
&\leq \frac{1}{2^J} + \sum_{j\geq 1}\sum_{\substack{m\in A(s_j,l_j)\\ m>M}} \frac{1}{2^{m-M-s_j+j}} + \sum_{j\geq 1}\sum_{\substack{m\in A(s_j,l_j)\\ m<M}} \frac{1}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m-s_j)+j}} \\
&\leq \frac{1}{2^J} + \sum_{j\geq 1}\sum_{m\geq M+s_j+s_J} \frac{1}{2^{m-M-s_j+j}} + \sum_{j\geq 1}\sum_{m\leq M-s_j-s_J} \frac{1}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}(M-m-s_j)+j}} \\
&\leq \frac{1}{2^J} + \sum_{j\geq 1}\sum_{m\geq s_J} \frac{1}{2^{m+j}} + \sum_{j\geq 1}\sum_{m\geq s_J} \frac{1}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}m+j}} \\
&\leq \frac{1}{2^J} + \frac{1}{2^{s_J-1}} + \frac{1}{2^{\frac{\eta^{(0)}}{3\Delta^{(0)}}s_J}}\left(\sum_{m\geq 0} 2^{-\frac{\eta^{(0)}}{3\Delta^{(0)}}m}\right) \xrightarrow[J\to\infty]{} 0.
\end{aligned}$$

We conclude that x is a frequently hypercyclic vector for $\tilde{T}$. □

It remains to show that $\tilde{T}^{-1}$ is not frequently hypercyclic under convenient conditions on the sequence (τ_m). The proof of this fact will rely on the study of dynamical behaviours of finite sequences under the action of $\tilde{T}^{-1}$. Therefore, we begin by a technical lemma

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concerning finite sequences and given a vector $x \in \ell^1(\mathbb{N})$, we use for every $n \geq 0$ and for every $I \subset \mathbb{N}$ the following notations:

$$P_n x := \sum_{k=b_n}^{b_{n+1}-1} x_k e_k \quad \text{and} \quad P_I x = \sum_{n \in I} P_n x.$$

Lemma 3.4. *Let $x \in \ell^1(\mathbb{N})$. The following conditions are satisfied:*

1. $\|\tilde{T}^{-1}x\| \leq 2\|x\|$.
2. *For every $l \geq 0$, every $s \geq 1$, every $n \in \varphi^{-s}(l)\backslash\{0\}$, every $j \geq 0$,*

$$\|P_l \tilde{T}^{-j} P_n x\| \leq 2^{j-\tau_{l+s-1}} \|P_n x\|.$$

In particular, for every $l \geq 0$, every $s \geq 1$, every $j \geq 0$,

$$\|P_l \tilde{T}^{-j} P_{\varphi^{-s}(l)\backslash\{0\}} x\| \leq 2^{j-\tau_{l+s-1}} \|P_{\varphi^{-s}(l)\backslash\{0\}} x\|.$$

3. *For every $k \geq 0$, every $l \in [n_k, n_{k+1})$, every $j \geq 0$,*

$$\|P_l \tilde{T}^{-j} P_l x\| \geq 2^{\eta^{(k)} \left\lfloor \frac{j}{\Delta^{(k)}} \right\rfloor - \delta^{(k)}} \|P_l x\|.$$

In particular, for every $l \geq 0$, if $P_l x \neq 0$, then

$$\lim_{j \to \infty} \|P_l \tilde{T}^{-j} P_l x\| = \infty.$$

Proof. Let $x \in \ell^1(\mathbb{N})$. For proof of statement (1), we have by Proposition 2.4

$$\begin{aligned}
\|\tilde{T}^{-1}x\| &\leq \sum_{n\geq 0} \sum_{k=b_n+1}^{b_{n+1}-1} |x_k| \|\tilde{T}^{-1}e_k\| + |x_{b_0}| \|\tilde{T}^{-1}e_{b_0}\| + \sum_{n\geq 1} |x_{b_n}| \|\tilde{T}^{-1}e_{b_n}\| \\
&\leq \sum_{n\geq 0} \sum_{k=b_n+1}^{b_{n+1}-1} \frac{|x_k|}{|w_k|} + |x_{b_0}| + \sum_{n\geq 1} |x_{b_n}| \Big(1 + \sum_{m=0}^{m_n-1} \big(\prod_{l=0}^{m} |v_{\varphi^l(n)}|\big)\Big) \\
&\leq \sum_{n\geq 0} \sum_{k=b_n+1}^{b_{n+1}-1} 2|x_k| + |x_{b_0}| + \sum_{n\geq 1} |x_{b_n}| \Big(1 + \sum_{m=0}^{m_n-1} \frac{1}{2^{\tau_{\varphi(n)}+m}}\Big)
\end{aligned}$$

because $\tau_j \geq 1$ for every $j \geq 0$. Finally, because $\sum_{m=0}^{m_n-1} \frac{1}{2^{\tau_{\varphi(n)}+m}} \leq \frac{1}{2^{\tau_{\varphi(n)}-1}} \leq 1$, we conclude that $\|\tilde{T}^{-1}x\| \leq 2\|x\|$.

For proof of statement (2), let $l \geq 0$, $s \geq 1$, $n \in \varphi^{-s}(l)\backslash\{0\}$, $j \geq 0$ and $k \in [b_n, b_{n+1})$. If $j \leq k - b_n$, we have $\|P_l \tilde{T}^{-j} e_k\| = 0$ and if $k - b_n < j \leq k + b_{n+1} - 2b_n$, then

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$$\begin{aligned}
&\|P_l\tilde{T}^{-j}e_k\| \\
&\le \left(\prod_{t=b_n+1}^{k}|w_t|\right)^{-1}\left\|P_l\tilde{T}^{-(j-k+b_n-1)}\left(-e_{b_{n+1}-1}-\sum_{m=0}^{m_n-1}\left(\prod_{l=0}^{m}v_{\varphi^l(n)}\right)e_{b_{\varphi^{m+1}(n)+1}-1}\right)\right\| \\
&\le 2^{k-b_n}\left\|\tilde{T}^{-(j-k+b_n-1)}\left(\sum_{m=0}^{m_n-1}\left(\prod_{l=0}^{m}v_{\varphi^l(n)}\right)e_{b_{\varphi^{m+1}(n)+1}-1}\right)\right\| \\
&\le |v_n|2^{j-1}\sum_{m=0}^{m_n-1}\left(\prod_{l=1}^{m}|v_{\varphi^l(n)}|\right)\le |v_n|2^j.
\end{aligned}$$

In addition, if $k+b_{n+1}-2b_n<j<2(b_{n+1}-b_n)$, then $\|P_l\tilde{T}^{-j}e_k\|=0$. Because $\tilde{T}^{2(b_{n+1}-b_n)}e_k=2^{-2\eta_n}e_k$, we have $\tilde{T}^{-2(b_{n+1}-b_n)}e_k=2^{2\eta_n}e_k$ and because $2^{2\eta_n}\le 2^{2(b_{n+1}-b_n)}$, we can deduce that if $j\ge 2(b_{n+1}-b_n)$, then $\|P_l\tilde{T}^{-j}e_k\|\le|v_n|2^j$. Therefore, we can write

$$\|P_l\tilde{T}^{-j}P_nx\|\le\sum_{k=b_n}^{b_{n+1}-1}|x_k|\|P_l\tilde{T}^{-j}e_k\|\le\sum_{k=b_n}^{b_{n+1}-1}|x_k||v_n|2^j\le 2^{j-\tau_{l+s-1}}\|P_nx\|$$

because $(\tau_m)_m$ is increasing and if $n\in\varphi^{-s}(l)\backslash\{0\}$, then $\varphi(n)\ge l+s-1$.

For proof of statement (3), let $k\ge0$ and $l\in[n_k,n_{k+1})$. We can remark that $P_l\tilde{T}^{-\Delta^{(k)}}P_lx=-2^{\eta^{(k)}}P_lx$ and it then suffices to prove that for every $j<\Delta^{(k)}$, $\|P_l\tilde{T}^{-j}P_lx\|\ge2^{-\delta^{(k)}}\|P_lx\|$. Let $j<\Delta^{(k)}$ and $m\in[b_l,b_{l+1})$.

- If $0\le j\le m-b_l$, we have $\|P_l\tilde{T}^{-j}P_le_m\|=\prod_{t=m-j+1}^{m}|w_t|^{-1}\ge2^{-\delta^{(k)}}$.
- If $m-b_l<j<\Delta^{(k)}$, we have

  $$\|P_l\tilde{T}^{-j}P_le_m\|=\left(\prod_{t=b_l+1}^{m}|w_t|^{-1}\right)\left(\prod_{t=b_{l+1}-j+m-b_l+1}^{b_{l+1}-1}|w_t|^{-1}\right),$$

  and because $b_{l+1}-j+m-b_l+1>m$, each weight is taken at most one time. We can then deduce from the definition of $(w_i)_i$ that

  $$\|P_l\tilde{T}^{-j}P_le_m\|\ge2^{-\delta^{(k)}}.$$ □

We can now prove that if the sequence (τ_m) grows sufficiently rapidly, then $\tilde{T}^{-1}$ is not frequently hypercyclic.

Proposition 3.5. *Let $S_l=\sum_{l'\le l}(2(b_{l'+1}-b_{l'})+l')$ for every $l\ge0$ and let $(J_l)_{l\ge0}$ be a sequence of positive integers such that for every $j\ge J_l$, every $x\in\ell^1(\mathbb{N})$, $\|P_l\tilde{T}^{-j}P_lx\|\ge 2^{S_l}\|P_lx\|$. If for every $l\ge0$,*

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$$\tau_l \geq S_l + 2\eta_l + \delta_l + 2l + 3 \quad \textit{and} \quad \frac{J_l}{\tau_l - l - S_l - \delta_l - 3} \leq \frac{1}{2^l},$$

then $\tilde{T}^{-1}$ *is not frequently hypercyclic.*

Proof. Let $S_l = \sum_{l' \leq l}(2(b_{l'+1} - b_{l'}) + l')$ for every $l \geq 0$ and let $(J_l)_{l\geq 0}$ be a sequence of positive integers such that for every $j \geq J_l$, every $x \in \ell^1(\mathbb{N})$,

$$\|P_l \tilde{T}^{-j} P_l x\| \geq 2^{S_l} \|P_l x\|.$$

Because the set of frequently hypercyclic vectors is always included in the set of hypercyclic vectors, it suffices to show that if x is a hypercyclic vector for $\tilde{T}^{-1}$, then x cannot be frequently hypercyclic for this operator.

Let $x \in \ell^1(\mathbb{N})$ be a hypercyclic vector for $\tilde{T}^{-1}$. We can already remark that for every $j \geq 0$, every $n \geq 0$, we have

$$\|\tilde{T}^{-j}x\| \geq \|P_n \tilde{T}^{-j} x\| \geq \|P_n \tilde{T}^{-j} P_n x\| - \sum_{s\geq 1} \|P_n \tilde{T}^{-j} P_{\varphi^{-s}(n)\setminus\{0\}} x\|.$$

In particular, if $P_n x \neq 0$, it follows that the set $\{j \geq 0 : \sum_{s\geq 1} \|P_n \tilde{T}^{-j} P_{\varphi^{-s}(n)\setminus\{0\}} x\| > \frac{\|P_n \tilde{T}^{-j} P_n x\|}{4}\}$ is nonempty because

$$\sum_{s\geq 1} \|P_n \tilde{T}^{-j} P_{\varphi^{-s}(n)\setminus\{0\}} x\| \geq \|P_n \tilde{T}^{-j} P_n x\| - \|P_n \tilde{T}^{-j} x\|,$$

$\|P_n \tilde{T}^{-j} P_n x\|$ tends toward ∞ as $j \to \infty$ (Lemma 3.4 (3)) and for every J there exists $j \geq J$ such that $\|\tilde{T}^{-j}x\| \leq 1$ because x is hypercyclic.

The goal of this proof will be to show that there exist an increasing sequence $(l_m)_{m\geq 0}$ and a sequence $(j_m)_{m\geq 1}$ tending toward infinity such that for every $m \geq 1$,

$$\frac{\#\{j < j_m : \|\tilde{T}^{-j}x\| \geq \frac{3}{4}\|P_{l_0}x\|\}}{j_m} \geq 1 - \frac{1}{2^{l_{m-1}}}.$$

It will then follow that x is not frequently hypercyclic for $\tilde{T}^{-1}$ because $\|P_{l_0}x\| > 0$ and thus that $\tilde{T}^{-1}$ is not frequently hypercyclic.

To this end, we construct by induction three sequences of positive integers $(l_m)_{m\geq 0}$, $(j_m)_{m\geq 1}$ and $(s_m)_{m\geq 1}$. Because x is hypercyclic for $\tilde{T}^{-1}$, $x - P_0 x \neq 0$ and we can consider $l_0 \geq 1$ such that $\|P_{l_0}x\| \geq \frac{1}{2^{l_0}}\|x - P_0 x\|$. Assume now that $(l_t)_{0\leq t\leq m-1}$ and $(j_t)_{1\leq t\leq m-1}$ have been chosen. We let

$$j_m = \min\{j \geq 0 : \sum_{s\geq 1} \|P_{l_{m-1}} \tilde{T}^{-j} P_{\varphi^{-s}(l_{m-1})} x\| > \frac{\|P_{l_{m-1}} \tilde{T}^{-j} P_{l_{m-1}} x\|}{4}\}$$

and by definition of j_m, there then exist $s_m \geq 1$ such that

$$\|P_{l_{m-1}} \tilde{T}^{-j_m} P_{\varphi^{-s_m}(l_{m-1})} x\| > \frac{\|P_{l_{m-1}} \tilde{T}^{-j_m} P_{l_{m-1}} x\|}{2^{s_m+2}}$$

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and $l_m \in \varphi^{-s_m}(l_{m-1})$ such that

$$\|P_{l_{m-1}}\tilde{T}^{-j_m}P_{l_m}x\| > \frac{\|P_{l_{m-1}}\tilde{T}^{-j_m}P_{l_{m-1}}x\|}{2^{l_m+s_m+3}}.$$

Because $P_{l_0}x \neq 0$, the integer j_1 is well defined. Moreover, we can show that each integer j_m with $m \geq 2$ is also well defined because we can compute by induction that for every $m \geq 0$,

$$2^{S_{l_m}}\|P_{l_m}x\| \geq \|P_{l_0}x\|.$$

Indeed, assume that $2^{S_{l_{m-1}}}\|P_{l_{m-1}}x\| \geq \|P_{l_0}x\|$. Let k_{m-1} and k_m such that $l_{m-1} \in [n_{k_{m-1}}, n_{k_{m-1}+1})$ and $l_m \in [n_{k_m}, n_{k_m+1})$. It follows from Lemma 3.4 (3) that

$$\|P_{l_{m-1}}\tilde{T}^{-j_m}P_{l_{m-1}}x\| \geq 2^{\eta^{(k_{m-1})}\left\lfloor\frac{j_m}{\Delta^{(k_{m-1})}}\right\rfloor - \delta^{(k_{m-1})}}\|P_{l_{m-1}}x\|.$$

On the other hand, because $\tilde{T}^{-\left\lfloor\frac{j_m}{2\Delta^{(k_m)}}\right\rfloor 2\Delta^{(k_m)}}P_{l_m}x = 2^{2\eta^{(k_m)}\left\lfloor\frac{j_m}{2\Delta^{(k_m)}}\right\rfloor}P_{l_m}x$, it follows from Lemma 3.4 (2) that

$$\begin{aligned}\|P_{l_{m-1}}\tilde{T}^{-j_m}P_{l_m}x\| &\leq 2^{2\eta^{(k_m)}\left\lfloor\frac{j_m}{2\Delta^{(k_m)}}\right\rfloor}\sup_{0\leq j<2\Delta^{(k_m)}}\|P_{l_{m-1}}\tilde{T}^{-j}P_{l_m}x\| \\ &\leq 2^{2\eta^{(k_m)}\frac{j_m}{2\Delta^{(k_m)}}}2^{2\Delta^{(k_m)}-\tau_{l_{m-1}+s_m-1}}\|P_{l_m}x\|.\end{aligned}$$

Because $\tau_{l_{m-1}+s_m-1} \geq 2\eta_{l_{m-1}+s_m-1} + \delta_{l_{m-1}+s_m-1} + s_m + 3$, we deduce from three previous inequalities that

$$\begin{aligned}\|P_{l_m}x\| &\geq \frac{2^{2\eta^{(k_{m-1})}\left(\frac{j_m}{\Delta^{(k_{m-1})}}-1\right)-\delta^{(k_{m-1})}}}{2^{2\eta^{(k_m)}\frac{j_m}{2\Delta^{(k_m)}}}2^{2\Delta^{(k_m)}-\tau_{l_{m-1}+s_m-1}}2^{l_m+s_m+3}}\|P_{l_{m-1}}x\| \\ &= \frac{2^{\tau_{l_{m-1}+s_m-1}}}{2^{2\eta^{(k_{m-1})}+\delta^{(k_{m-1})}+2\Delta^{(k_m)}+l_m+s_m+3}}\|P_{l_{m-1}}x\| \quad \text{since } \frac{\eta^{(k_{m-1})}}{\Delta^{(k_{m-1})}} = \frac{\eta^{(k_m)}}{\Delta^{(k_m)}} \\ &\geq \frac{1}{2^{2\Delta^{(k_m)}+l_m}}\|P_{l_{m-1}}x\| \\ &\geq \frac{1}{2^{2\Delta^{(k_m)}+l_m}}2^{-S_{l_{m-1}}}\|P_{l_0}x\| \\ &\geq 2^{-S_{l_m}}\|P_{l_0}x\| \qquad (\text{by definition of } S_l).\end{aligned}$$

The sequences $(l_m)_{m\geq 0}$, $(j_m)_{m\geq 1}$ and $(s_m)_{m\geq 1}$ are thus well defined and we can now show that $(l_m)_{m\geq 0}$ is an increasing sequence, $(j_m)_{m\geq 1}$ tends to infinity and for every $m \geq 1$,

$$\frac{\#\{j < j_m : \|\tilde{T}^{-j}x\| \geq \frac{3}{4}\|P_{l_0}x\|\}}{j_m} \geq 1 - \frac{1}{2^{l_{m-1}}}.$$

The sequence $(l_m)_{m\geq 1}$ is increasing because $l_m \in \varphi^{-s_m}(l_{m-1})$ and $s_m \geq 1$. On the other hand, we have

$$j_m > \tau_{l_{m-1}+s_m-1} - l_0 - S_{l_{m-1}} - \delta_{l_{m-1}} - s_m - 2$$

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because for every $j \le \tau_{l_{m-1}+s_m-1} - l_0 - S_{l_{m-1}} - \delta_{l_{m-1}} - s_m - 2$, we have by Lemma 3.4 (2) and Lemma 3.4 (3),

$$\begin{aligned}\|P_{l_{m-1}}\tilde{T}^{-j}P_{\varphi^{-s_m}(l_{m-1})}x\| &\le 2^{j-\tau_{l_{m-1}+s_m-1}}\|P_{\varphi^{-s_m}(l_{m-1})}x\| \\ &\le 2^{j-\tau_{l_{m-1}+s_m-1}}\|x - P_0x\| \\ &\le 2^{j-\tau_{l_{m-1}+s_m-1}+l_0}\|P_{l_0}x\| \\ &\le 2^{j-\tau_{l_{m-1}+s_m-1}+l_0+S_{l_{m-1}}}\|P_{l_{m-1}}x\| \\ &\le 2^{j-\tau_{l_{m-1}+s_m-1}+l_0+S_{l_{m-1}}+\delta_{l_{m-1}}}\|P_{l_{m-1}}\tilde{T}^{-j}P_{l_{m-1}}x\| \\ &\le \frac{\|P_{l_{m-1}}\tilde{T}^{-j}P_{l_{m-1}}x\|}{2^{s_m+2}}.\end{aligned}$$

In particular, because $(S_j)_j$, $(\delta_j)_j$ and $(l_j)_j$ are increasing, we have

$$\begin{aligned}j_m &\ge S_{l_{m-1}+s_m-1} + \delta_{l_{m-1}+s_m-1} + 2l_{m-1} + 2s_m + 1 - l_0 - S_{l_{m-1}} - \delta_{l_{m-1}} - s_m - 2 \\ &\ge l_{m-1}\end{aligned}$$

and the sequence $(j_m)_{m\ge1}$ therefore tends toward infinity. By assumption, we also have $\|P_{l_{m-1}}\tilde{T}^{-j}P_{l_{m-1}}x\| \ge 2^{S_{l_{m-1}}}\|P_{l_{m-1}}x\|$ for every $j \ge J_{l_{m-1}}$ and

$$\begin{aligned}\frac{j_m - J_{l_{m-1}}}{j_m} &\ge 1 - \frac{J_{l_{m-1}}}{\tau_{l_{m-1}+s_m-1} - l_0 - S_{l_{m-1}} - \delta_{l_{m-1}} - s_m - 2} \\ &\ge 1 - \frac{J_{l_{m-1}}}{\tau_{l_{m-1}+s_m-1} - l_{m-1} - S_{l_{m-1}} - \delta_{l_{m-1}} - s_m - 2} \\ &\ge 1 - \frac{J_{l_{m-1}}}{\tau_{l_{m-1}} - l_{m-1} - S_{l_{m-1}} - \delta_{l_{m-1}} - 3} \\ &\ge 1 - \frac{1}{2^{l_{m-1}}}\end{aligned}$$

because $\tau_{l_{m-1}+s_m-1} \ge \tau_{l_{m-1}} + s_m - 1$. We can then conclude that

$$\frac{\#\{j < j_m : \|\tilde{T}^{-j}x\| \ge \frac{3}{4}\|P_{l_0}x\|\}}{j_m} \ge 1 - \frac{1}{2^{l_{m-1}}}$$

because for every $J_{l_{m-1}} \le j < j_m$, by definition of j_m,

$$\begin{aligned}\|\tilde{T}^{-j}x\| &\ge \|P_{l_{m-1}}\tilde{T}^{-j}P_{l_{m-1}}x\| - \sum_{s=1}^{\infty}\|P_{l_{m-1}}\tilde{T}^{-j}P_{\varphi^{-s}(l_{m-1})}x\| \\ &\ge \frac{3}{4}\|P_{l_{m-1}}\tilde{T}^{-j}P_{l_{m-1}}x\| \\ &\ge \frac{3}{4}2^{S_{l_{m-1}}}\|P_{l_{m-1}}x\| \\ &\ge \frac{3}{4}\|P_{l_0}x\|.\end{aligned}$$

It follows that no hypercyclic vector for $\tilde{T}^{-1}$ is frequently hypercyclic and thus $\tilde{T}^{-1}$ is not frequently hypercyclic. □

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Thanks to different results of this section, we can thus exhibit the first example of an invertible frequently hypercyclic operator whose inverse is not frequently hypercyclic.

Theorem 3.6. *There exists an invertible frequently hypercyclic operator T on $\ell^1(\mathbb{N})$ such that T^{-1} is not frequently hypercyclic.*

Proof. Let $n_0 = 0$ and $n_k = 2^{k-1}$ for every $k \geq 1$. For every $k \geq 0$, we let

$$\Delta^{(k)} = 8^{k+1} \quad \text{and} \quad \eta^{(k)} = \delta^{(k)} = 8^k$$

so that

$$2\delta^{(k)} + \eta^{(k)} < \Delta^{(k)}, \quad \Delta^{(k+1)} \text{ is a multiple of } 2\Delta^{(k)} \quad \text{and} \quad \frac{\eta^{(k)}}{\Delta^{(k)}} = \frac{\eta^{(0)}}{\Delta^{(0)}}.$$

Let $(\tau_n)_{n\geq 0}$ be an increasing sequence of positive integers and T the generalised C-type operator $T_{v,w,\varphi,b,R}$ such that

- for every $n \in [n_k, n_{k+1})$, $\varphi(n) = n - n_k$;
- for every m, every $n \in \varphi^{-1}(m)$, $v_n = 2^{-\tau_m}$;
- for every $k \geq 0$, every $n \in [n_k, n_{k+1})$, every $i \in (b_n, b_{n+1})$,

$$w_i = \begin{cases} \frac{1}{2} & \text{if } b_n < i \leq b_n + \eta_n \\ 1 & \text{if } b_n + \eta_n < i < b_{n+1} - 2\delta_n \\ \frac{1}{2} & \text{if } b_{n+1} - 2\delta_n \leq i < b_{n+1} - \delta_n \\ 2 & \text{if } b_{n+1} - \delta_n \leq i < b_{n+1} \end{cases}$$

- for every $n \geq 0$, $R_n = 1$;

and such that for every $k \geq 0$, every $n \in [n_k, n_{k+1})$,

$$\delta_n = \delta^{(k)}, \quad \eta_n = \eta^{(k)} \quad \text{and} \quad b_{n+1} - b_n = \Delta^{(k)}.$$

The operator T is well defined, invertible (Corollary 2.5) and frequently hypercyclic (Proposition 3.3) for any choice of $(\tau_n)_{n\geq 0}$. Moreover, for every $x \in \ell^1(\mathbb{N})$, every $l \geq 0$, the sequence $(P_l \tilde{T}^{-j} P_l x)_{j\geq 0}$ does not depend on $(\tau_n)_{n\geq 0}$. Let $S_l = \sum_{l' \leq l}(2(b_{l'+1} - b_{l'}) + l')$ for every $l \geq 0$. By using Lemma 3.4 (3), we can thus find a sequence $(J_l)_{l\geq 0}$ such that for every increasing sequence $(\tau_n)_{n\geq 0}$, every $j \geq J_l$ and every $x \in \ell^1(\mathbb{N})$,

$$\|P_l T^{-j} P_l x\| \geq 2^{S_l} \|P_l x\|.$$

By choosing for $(\tau_n)_{n\geq 0}$ a rapidly increasing sequence so that for every $l \geq 0$,

$$\tau_l \geq S_l + 2\eta_l + \delta_l + 2l + 3 \quad \text{and} \quad \frac{J_l}{\tau_l - l - S_l - \delta_l - 3} \leq \frac{1}{2^l},$$

we can then deduce from Proposition 3.5 that T^{-1} is not frequently hypercyclic. □

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