Combinatorics, Probability and Computing (1998) 7, 121–127. Printed in the United Kingdom

On the Number of Incidences Between Points and Curves

JÁNOS PACH[1]† and MICHA SHARIR[2]†

[1,2] Courant Institute of Mathematical Sciences,
New York University, New York, NY 10012, USA

[1] Department of Computer Science, City College,
CUNY, New York, NY, USA

[2] School of Mathematical Sciences, Tel Aviv University,
Tel Aviv 69978, Israel

(e-mail: [1]pach@cims6.cims.nyu.edu
[2]sharir@math.tau.ac.il)

Received 26 April 1996; revised 10 February 1997

We apply an idea of Székely to prove a general upper bound on the number of incidences between a set of m points and a set of n 'well-behaved' curves in the plane.

1. Introduction

Let Γ be a given class of simple curves in the plane. We say that Γ has k *degrees of freedom and multiplicity-type* s if

(i) for any k points there are at most s curves of Γ passing through all of them, and
(ii) any pair of curves from Γ intersect in at most s points.

For example, the classes of all straight lines, all unit circles, all circles, and all graphs of the form $y = p(x)$, where p is a polynomial of degree d, have 2, 2, 3, and $d+1$ degrees of freedom, respectively, and have multiplicity type 1, 2, 2, and d, respectively.

Given a finite set P of points and a finite set C of curves, we define $I(P, C)$ to be the number of incidences between them.

† Work on this paper by both authors has been supported by NSF Grant CCR-94-24398. Work by János Pach was also supported by Grant OTKA-T-020914 and by a CUNY Research Award. Work by Micha Sharir was also supported by NSF Grant CCR-93-11127, by a Max-Planck Research Award, and by grants from the U.S.-Israeli Binational Science Foundation, and the G.I.F., the German-Israeli Foundation for Scientific Research and Development.

Theorem 1.1. *Let P be a set of m points and let C be a set of n simple curves all lying in the plane. If C has k degrees of freedom and multiplicity-type s, then*

$$I(P,C) \leqslant c(k,s)\left(m^{k/(2k-1)}n^{(2k-2)/(2k-1)} + m + n\right), \tag{1.1}$$

where $c(k,s)$ is a positive constant that depends on k and s.

This result was first formulated in [7] under the additional constraint that the elements of C are algebraic curves defined in terms of k real parameters. In that case, it can be obtained by a fairly straightforward extension of the proof technique in Clarkson, Edelsbrunner, Guibas, Sharir & Welzl [3], where this bound was established for the classes of lines, unit circles, and arbitrary circles. For lines and unit circles, the first (quite involved) proofs of Theorem 1.1 were obtained by Szemerédi and Trotter [9], and Spencer, Szemerédi and Trotter [10]. Recently, Székely [8] discovered a very elegant proof that works in the special case $k = 2$ (and any constant multiplicity type). His argument is based on a simple lower bound on the number of crossings in a graph drawing (see Lemma 2.1). The aim of this note is to generalize Székely's idea to establish Theorem 1.1 for every k and s.

Note that the cruder bound

$$I(P,C) \leqslant s^{1/k}(m-k+1)n^{1-1/k} + (k-1)n \tag{1.2}$$

is an immediate corollary of an old result of Kővári, Sós, and Turán [4] in extremal graph theory (see also Canham [2], Pach and Agarwal [6]). This follows from the easy observation that the bipartite graph $H \subseteq P \times C$, whose edges represent the incidences between P and C, does not contain $K_{k,s+1}$ as a subgraph.

2. Proof of Theorem 1.1

We require the following generalization of a result of Ajtai, Chvátal, Newborn and Szemerédi [1] and Leighton [5], due to Székely [8].

Lemma 2.1. *Let H be a multigraph drawn in the plane, with maximal edge-multiplicity M. If $|E(H)| \geqslant 5|V(H)|M$, then the number of crossings between the edges of H is at least*

$$\frac{|E(H)|^3}{100|V(H)|^2M}.$$

Now let P and C be as in the statement of Theorem 1.1, and put $I = I(P,C)$. Assume, without loss of generality, that every curve in C is a simple *open* curve (*i.e.* homeomorphic to a segment). Let $d(p)$ denote the number of curves in C passing through $p \in P$, so that $I = \sum_{p\in P} d(p)$.

We classify the points of P as follows, noting that the average number of curves passing through a point of P is I/m. Define

$$P^* = \left\{ p \in P \;\middle|\; d(p) \leqslant \frac{I}{2m} \right\},$$

and, for $j \geqslant 0$,

$$P_j = \left\{ p \in P \;\middle|\; \frac{2^{j-1}I}{m} < d(p) \leqslant \frac{2^j I}{m} \right\}.$$

Put $m_j = |P_j|$, $d_j = 2^{j-1}I/m$, and $I_j = I(P_j, C)$. Note that

$$\frac{I_j}{2} \leqslant m_j d_j < I_j.$$

This also implies that

$$m_j < \frac{m}{2^{j-1}}.$$

First, we have

$$I(P^*, C) \leqslant \frac{I(P,C)}{2},$$

so that

$$I(P,C) \leqslant 2I(P \setminus P^*, C).$$

Next, put

$$J_1 = \left\{ j \geqslant 0 \;\middle|\; m_j^k < \frac{n}{2^j} \right\},$$
$$J_2 = \left\{ j \geqslant 0 \;\middle|\; m_j^k \geqslant \frac{n}{2^j} \right\}.$$

Since

$$\sum_{j \in J_1} m_j < \sum_{j=0}^{\infty} n^{1/k} 2^{-j/k} = \frac{2^{1/k}}{2^{1/k}-1} n^{1/k},$$

we have, by (1.2),

$$I\left(\bigcup_{j \in J_1} P_j, C \right) \leqslant \left(\frac{(2s)^{1/k}}{2^{1/k}-1} + (k-1) \right) n. \tag{2.1}$$

We next bound $I(\bigcup_{j \in J_2} P_j, C) = \sum_{j \in J_2} I_j$. We consider each P_j, for $j \in J_2$, separately, and fix for now such an index j.

Taking $c(k,s) \geqslant k-1$, we may assume that every curve in C contains at least k points of P_j. Otherwise, we can remove the n_1 curves containing fewer than k points of P_j per curve, apply Theorem 1.1 to P_j and the remaining curves, and observe that the removed curves contribute at most $(k-1)n_1$ to I_j. This readily implies that (1.1) holds for P_j and C.

If two points $p, q \in P_j$ are separated by at most $k-2$ points of P_j along a curve $\gamma \in C$, we connect them by the piece of γ lying between them. Denote the resulting graph drawing by G_j. Note that some edges contained in the same curve may overlap, and the same pair of vertices can be connected in G_j by several edges (along different curves γ). Let H_j denote the graph obtained from G_j by erasing every edge whose multiplicity exceeds $Ad_j^{1-1/(k-1)}$, where $A > 0$ is a sufficiently large constant to be specified later. Clearly,

$$I_j - n \leqslant |E(G_j)| \leqslant (k-1)I_j.$$

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Our goal is to show that the number of edges of $H_j \subseteq G_j$ is also at least $c'I_j$, for an appropriate constant c', and then to apply Lemma 2.1 to H_j.

For every $p, q \in P_j$, let $E_p(q)$ denote the set of edges in G_j connecting p and q and write $E_p = \bigcup_{q \in P_j} E_p(q)$. We have

$$(k-1)d_j \leqslant |E_p| = d_{G_j}(p) \leqslant 4(k-1)d_j.$$

Consider the set $E_{p,q}$ consisting of all edges pr (including $r = q$) that belong to some curve containing an element of $E_p(q)$. See Figure 1 for an illustration.

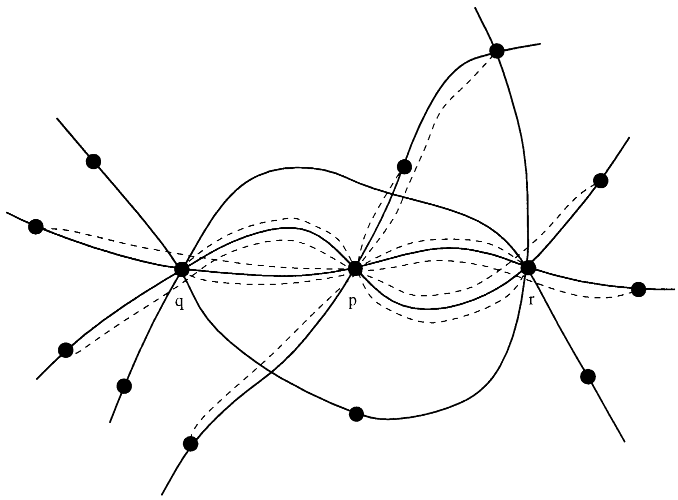

Figure 1 The neighbourhood of p in the graph G_j (with $k = 3$).

Let $R_p = \{r \in P_j \mid |E_p(r)| > Ad_j^{1-1/(k-1)}\}$. Note that all the edges connecting p to $r \in R_p$ are erased from H_j. We have

$$|R_p| \leqslant \frac{d_{G_j}(p)}{Ad_j^{1-1/(k-1)}} \leqslant \frac{4(k-1)d_j}{Ad_j^{1-1/(k-1)}} < \frac{4k}{A} d_j^{1/(k-1)}.$$

If $R_p = \emptyset$, then $d_{H_j}(p) = d_{G_j}(p)$. Suppose that R_p is nonempty. Let $q \neq p \in P_j$. If $q \notin R_p$, then all the edges in $E_p(q)$ remain in H_j. Otherwise, every curve containing an edge e from $E_p(q)$ contains at least $k-1$ edges $pr \in E_{p,q} \subseteq E(G_j)$. We want to charge e to one such edge pr that remains in H_j. We say that e is *good* if there is at least one such edge. If e is bad, then the curve γ containing e passes through p and through at least $k-1$ distinct points of R_p, and in this case γ contains at most $2(k-1)$ bad edges. However, there are at most s curves passing through p and through any fixed $(k-1)$-tuple of R_p. This implies that the number of bad edges is at most

$$2(k-1)s\binom{|R_p|}{k-1} < \frac{2s(k-1)\cdot|R_p|^{k-1}}{(k-1)!} < \frac{2s}{(k-2)!}\left(\frac{4k}{A}\right)^{k-1} d_j < \frac{1}{2}(k-1)d_j,$$

if A is chosen sufficiently large. We have thus shown that more than half of the edges in E_p are good, so each of them can charge an edge in $E_p \cap E(H_j)$. Using the fact that the

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same edge pr cannot belong to more than $2k-2$ different sets $E_{p,q}$, we obtain

$$d_{H_j}(p) \geqslant \frac{1}{2(2k-2)}|E_p| = \frac{1}{4(k-1)}d_{G_j}(p) \geqslant \frac{d_j}{4},$$

implying that

$$|E(H_j)| \geqslant m_j(d_j/4)/2 \geqslant I_j/16.$$

Every point of a curve $\gamma \in C$ belongs to the relative interior of at most $\binom{k}{2}$ edges of H_j lying in γ. This implies that there is a subgraph $H'_j \subseteq H_j$ with $|E(H'_j)| \geqslant |E(H_j)|/\binom{k}{2}$ so that no two edges of H'_j overlap.

Let J_3 be the set of indices $j \in J_2$ such that $|E(H'_j)| \geqslant 5|V(H'_j)|M$, with $M = Ad_j^{1-1/(k-1)}$. For $j \in J_2 \backslash J_3$, we have $I_j \leqslant 16\binom{k}{2}|E(H'_j)| \leqslant 80Am_jd_j^{1-1/(k-1)}$. Using the fact that $d_j \leqslant I_j/m_j$, we get $I_j \leqslant 80Am_j^{1/(k-1)}I_j^{1-1/(k-1)}$, or $I_j \leqslant (80A)^{k-1}m_j$. Hence

$$\sum_{j \in J_2 \backslash J_3} I_j \leqslant (80A)^{k-1}m. \tag{2.2}$$

Assuming now that $j \in J_3$, we can apply Lemma 2.1 to H'_j and obtain that the number of crossings in H'_j is at least

$$\frac{c'I_j^3}{m_j^2(I_j/m_j)^{1-1/(k-1)}},$$

for an appropriate constant c'. On the other hand, since any two curves have at most s intersection points, this number cannot exceed $\binom{n}{2}s$. Comparing the last two bounds, and adding the term $(k-1)n$ to account for curves passing through fewer than k points of P_j (see the above analysis), we obtain

$$\begin{aligned} I_j &\leqslant c''\left(m_j^{k/(2k-1)}n^{(2k-2)/(2k-1)} + n\right) \\ &\leqslant c''\left(2^{-(j-1)k/(2k-1)}m^{k/(2k-1)}n^{(2k-2)/(2k-1)} + n\right), \end{aligned}$$

for an appropriate constant c''.

Since $j \in J_2$ (*i.e.* $m_j^k \geqslant n/2^j$), we have

$$\begin{aligned} n &= n^{1/(2k-1)}n^{(2k-2)/(2k-1)} \leqslant 2^{j/(2k-1)}m_j^{k/(2k-1)}n^{(2k-2)/(2k-1)} \\ &\leqslant 2 \cdot 2^{-j(k-1)/(2k-1)}m^{k/(2k-1)}n^{(2k-2)/(2k-1)}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j \in J_3} I_j &\leqslant c''m^{k/(2k-1)}n^{(2k-2)/(2k-1)} \cdot \sum_{j \geqslant 0}\left(2^{-(j-1)k/(2k-1)} + 2 \cdot 2^{-j(k-1)/(2k-1)}\right) \\ &\leqslant c'''m^{k/(2k-1)}n^{(2k-2)/(2k-1)}, \end{aligned} \tag{2.3}$$

for yet another constant c'''.

Combining (2.1), (2.2), and (2.3), we obtain that (1.1) holds for an appropriate choice of $c(k,s)$. This completes the proof of Theorem 1.1. □

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3. The complexity of many cells in an arrangement of curves

Let C be a family of n simple curves cutting the plane into finitely many *cells*. A maximal connected piece of the boundary of a cell that belongs to the same curve of C is called a *side* of the cell. We say that C has *property* $P_{k,s}$, if no two members of C have more than s points in common, and for every k-tuple of pairwise disjoint connected open sets there are at most s curves in C touching all of them at distinct points. A slight modification of the above argument yields the following.

Proposition 3.1. *Let C be any family of n curves in the plane with property $P_{k,s}$. Then the total number of sides of m distinct cells determined by C is at most*

$$c'(k,s)\left(m^{k/(2k-1)}n^{(2k-2)/(2k-1)}+m+n\right),$$

where $c'(k,s)$ is a positive constant depending on k and s.

This bound is known to be tight for families of lines, as shown by Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [3].

If no two members of C have l points in common, then C has property $P_{l+2,s}$ for some s depending only on l. Moreover, if all members of C are unbounded in both directions (*i.e.* pass through the 'point at infinity'), or they are closed and $l > 2$, then property $P_{l,s}$ holds for a suitably large s.

We sketch a proof of this claim only for the case of closed curves and $l > 2$; the other cases can be treated in a similar manner. Suppose that C does not have property $P_{l,s}$ for $s = 2^l l!$. Then there exist pairwise disjoint connected open sets $S_1, \ldots, S_l$ and $s+1$ distinct curves in C, each of which touches all the sets S_i at distinct points. By the pigeonhole principle, there exist two curves γ, γ' in C that touch all the sets S_i in the same (clockwise or counterclockwise) order and on the same (exterior or interior) side. In this case, it is easily verified that, for each i, γ' must intersect γ between its (first) points of incidence with the closure of S_i and with the closure of $S_{i+1 \bmod l}$. This contradiction implies that C has property $P_{l,s}$ for the above value of s.

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