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STABLE SOLUTIONS TO THE STATIC CHOQUARD EQUATION

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Abstract

This paper is concerned with the static Choquard equation

$$-\Delta u = \Big(\frac{1}{|x|^{N-\alpha}} * |u|^p\Big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N, p > 2$ and $\max\{0, N-4\} < \alpha < N$. We prove that if $u \in C^1(\mathbb{R}^N)$ is a stable weak solution of the equation, then $u \equiv 0$. This phenomenon is quite different from that of the local Lane–Emden equation, where such a result only holds for low exponents in high dimensions. Our result is the first Liouville theorem for Choquard-type equations with supercritical exponents and $\alpha \neq 2$.

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1. Introduction

We study the static Choquard equation

$$-\Delta u = \Big(\frac{1}{|x|^{N-\alpha}} * |u|^p\Big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N, p > 2$ and $0 < \alpha < N$. If $p \le (N+\alpha)/(N-2)$, the energy functional associated to (1.1) can be defined for any $u \in H^1(\mathbb{R}^N)$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.$$

In such circumstances, problems of type (1.1) have a variational structure and can be treated by using variational methods (see, for example, [8, 20]). This paper is also concerned with the supercritical case $p > (N+\alpha)/(N-2)$. Therefore, throughout the paper, we study local solutions of (1.1) in the following weak sense.

DEFINITION 1.1. We call $u \in C^1(\mathbb{R}^N)$ a *(weak) solution* of (1.1) if

$$I'(u)\varphi := \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi\,dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}u(x)\varphi(x)|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy = 0 \tag{1.2}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$. Moreover, a solution u of (1.1) is called *stable* if

$$\begin{aligned}\langle I''(u)\varphi, \varphi\rangle := \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx - (p-1)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}\varphi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\ -p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}u(x)\varphi(x)|u(y)|^{p-2}u(y)\varphi(y)}{|x-y|^{N-\alpha}}\,dx\,dy \geq 0\end{aligned} \tag{1.3}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$.

Equation (1.1) belongs to a class of generalised Choquard equations

$$-\Delta u + V(x)u = \Big(\frac{1}{|x|^{N-\alpha}} * |u|^p\Big)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

The Choquard equations arise naturally in various branches of mathematical physics, such as quantum mechanics, Hartree–Fock theory, the physics of multiple-particle systems and the physics of laser beams (see [16, 19, 22]). Several variational techniques have been developed to deal with these equations. For example, the papers by Lieb [15], Lions [17], Ma–Zhao [18] and Moroz–Van Schaftingen [20] discuss the existence, symmetry and uniqueness of solutions of (1.4). A survey on the mathematical treatment of Choquard-type equations can be found in the recent review paper [21].

In this paper, we study the Liouville theorem on nonexistence of nontrivial solutions of (1.1). Numerous attempts have been made to establish such theorems for several types of equations after the pioneering paper of Gidas–Spruck [7], where they proved that the Lane–Emden equation,

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

has no positive classical solution in the subcritical case with $1 < p < (N+2)/(N-2)$ ($= +\infty$ if $N = 2$). For the static Choquard equation (1.1), the Liouville theorem is known to hold for positive classical solutions in the subcritical range, that is, $1 < p < (N+\alpha)/(N-2)$, and such a result is sharp (see [12, 14, 24]).

Analogous results could be asked for sign-changing solutions belonging to some particular classes, such as the stable one. From a physical point of view, a system is in a stable state if it can recover from perturbations. Hence, a small change will not prevent the system from returning to equilibrium. From that intuition, stable solutions are those for which the energy of the system attains a local minimum. For more physical background and motivation on stable solutions, we refer to the monograph [5] by Dupaigne.

In his celebrated paper, Farina [6] showed that the Lane–Emden equation (1.5) has no nontrivial stable classical solution if $1 < p < p_{JL}$. Here,

$$p_{JL} = \begin{cases} +\infty & \text{if } N \leq 10, \\ \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11 \end{cases}$$

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is called the Joseph–Lundgren exponent [10]. The Joseph–Lundgren exponent is always greater than the Sobolev one, that is, $p_{JL} > (N+2)(N-2)$. Farina's work showed that the Joseph–Lundgren exponent divides the range of nonexistence and existence of stable solutions, a role similar to the Sobolev exponent for positive solutions. Farina's results have been extended to p-Laplace equations in [1, 3, 13], Kirchhoff equations in [9], bi-harmonic equations in [4] and Grushin equations in [11]. For all these equations, the Joseph–Lundgren exponents for Liouville's theorem for stable solutions have been explicitly computed and are finite in high dimensions.

Liouville's theorem for stable solutions of Choquard-type equations is not fully understood, although there are some recent partial results. In [14], Lei established a Liouville theorem for positive stable classical solutions of (1.1) in the case $\alpha = 2 < N$. Lei's method relies heavily on a comparison property deduced from an equivalent semilinear elliptic system and hence cannot be extended to sign-changing stable solutions or to the case $\alpha \neq 2$. Zhao [25] proved a Liouville theorem for finite Morse index classical solutions, which includes the stable ones. However, Zhao's result only holds for the subcritical case $2 < p < (N+\alpha)/(N-2)$.

This paper is the first attempt to establish an optimal Liouville theorem for stable solutions of (1.1) in the case that $\alpha \neq 2$ and p may be a supercritical exponent. Our main result is the following theorem.

Theorem 1.2. *Assume that* $N, p > 2$ *and* $\max\{0, N-4\} < \alpha < N$. *If* $u \in C^1(\mathbb{R}^N)$ *is a stable weak solution of (1.1), then* $u \equiv 0$.

Remark 1.3. Our theorem agrees with [14, Theorem 1.8] in the case $\alpha = 2$ with the corresponding dimensions $N = 3, 4, 5$. Let us emphasise that our result is new even in this case because the stable solutions are assumed to be positive and classical in [14].

Remark 1.4. Our result reveals that the Joseph–Lundgren exponent for the static Choquard equation with $\max\{0, N-4\} < \alpha < N$ is infinite in any dimension. This phenomenon is very different from that of the local equations mentioned above, where the Joseph–Lundgren exponents are finite in high dimensions.

Remark 1.5. In [6], Farina proved a Liouville theorem for stable classical solutions of (1.5). Here, we adapt some ideas in [3] to deal with stable solutions of (1.1) which only belong to the $C^1(\mathbb{R}^N)$ class. Nevertheless, when $p = (N+\alpha)/(N-2)$ and in some circumstances, solutions in $H^1(\mathbb{R}^N)$ or positive solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ of (1.1) are known to be classical ones (see [2, 8]). Similar Liouville-type results for stable $H^1_{\text{loc}}(\mathbb{R}^N)$ solutions of Hénon equations were obtained in [23]. However, it is not clear how to adapt the techniques in [23] to study (1.1) due to the presence of a nonlocal term in the right-hand side of (1.1).

The next section is devoted to the proof of our main result. Throughout the paper, we use C to denote various positive constants. At times, we append subscripts to C to specify its dependence on the subscript parameters. We also denote by B_R the ball with centre 0 and radius $R > 0$. Our proof is inspired by the method of energy estimates from the work of Farina and his collaborators [3, 6].

2. Proof of the main result

We begin with the following technical lemma.

LEMMA 2.1. *If* $N, p > 2$ *and* $\max\{0, N-4\} < \alpha < N$, *then there exists* $\gamma \geq 1$ *such that*

$$\max\left\{\frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^2 + 1, \frac{\gamma+3}{2}\right\} < p < \frac{1}{2}\left[(\gamma+1)\frac{N+\alpha}{N-2} - (\gamma-1)\right]. \tag{2.1}$$

PROOF. For each $\gamma > 0$, we define

$$f(\gamma) = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^2 + 1, \quad g(\gamma) = \frac{\gamma+3}{2}, \quad h(\gamma) = \frac{1}{2}\left[(\gamma+1)\frac{N+\alpha}{N-2} - (\gamma-1)\right].$$

From $\alpha > N-4$, we deduce that $h(\gamma) > g(\gamma)$. By direct computations, $h(\gamma) > f(\gamma)$ if $\gamma > (N-2)/(6+2\alpha-N)$. Since $\alpha > N-4$, we see that $(N-2)/(6+2\alpha-N) < 1$. Hence, the interval $(\max\{f(\gamma), g(\gamma)\}, h(\gamma))$ is not empty for all $\gamma \geq 1$. Moreover,

$$f(1) = g(1) = 2 \quad \text{and} \quad \lim_{\gamma\to+\infty} \max\{f(\gamma), g(\gamma)\} = +\infty.$$

Therefore, the conclusion follows immediately from the continuity of f, g, h. □

PROOF OF THEOREM 1.2. Choose $\gamma \geq 1$ satisfying (2.1) in Lemma 2.1 and suppose $\psi \in C_c^1(\mathbb{R}^N)$ is a nonnegative function.

Applying (1.2) with $\varphi = |u|^{\gamma-1}u\psi^2$, since

$$\nabla\varphi = \gamma|u|^{\gamma-1}\psi^2\nabla u + 2|u|^{\gamma-1}u\psi\nabla\psi,$$

for any positive number $\varepsilon < \gamma$,

$$\begin{aligned}
\gamma\int_{\mathbb{R}^N}|u|^{\gamma-1}\psi^2|\nabla u|^2\,dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+\gamma-1}\psi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
= -2\int_{\mathbb{R}^N}|u|^{\gamma-1}u\psi\nabla u\cdot\nabla\psi\,dx \\
\leq 2\int_{\mathbb{R}^N}|u|^{\gamma}\psi|\nabla u||\nabla\psi|\,dx \\
\leq \varepsilon\int_{\mathbb{R}^N}|u|^{\gamma-1}\psi^2|\nabla u|^2\,dx + C_\varepsilon\int_{\mathbb{R}^N}|u|^{\gamma+1}|\nabla\psi|^2\,dx.
\end{aligned}$$

That is,

$$\begin{aligned}
(\gamma-\varepsilon)\int_{\mathbb{R}^N}|u|^{\gamma-1}\psi^2|\nabla u|^2\,dx \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+\gamma-1}\psi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
+ C_\varepsilon\int_{\mathbb{R}^N}|u|^{\gamma+1}|\nabla\psi|^2\,dx.
\end{aligned} \tag{2.2}$$

Applying (1.3) with $\varphi = |u|^{(\gamma-1)/2}u\psi$, since

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$$\nabla\varphi = \frac{\gamma+1}{2}|u|^{(\gamma-1)/2}\psi\nabla u + |u|^{(\gamma-1)/2}u\nabla\psi,$$

we have

$$\begin{aligned}(p-1)&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+\gamma-1}\psi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\&\quad+p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+(\gamma-1)/2}\psi(x)|u(y)|^{p+(\gamma-1)/2}\psi(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\&\le\int_{\mathbb{R}^N}\left|\frac{\gamma+1}{2}|u|^{(\gamma-1)/2}\psi\nabla u+|u|^{(\gamma-1)/2}u\nabla\psi\right|^2dx\\&\le(1+\varepsilon)\Big(\frac{\gamma+1}{2}\Big)^2\int_{\mathbb{R}^N}|u|^{\gamma-1}\psi^2|\nabla u|^2\,dx+C_\varepsilon\int_{\mathbb{R}^N}|u|^{\gamma+1}|\nabla\psi|^2\,dx,\end{aligned}\tag{2.3}$$

where we have used the inequality $|A+B|^2\le(1+\varepsilon)|A|^2+C_\varepsilon|B|^2$ in the last estimation.

Combining (2.2) and (2.3),

$$\begin{aligned}(p-1)&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+\gamma-1}\psi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\&\quad+p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+(\gamma-1)/2}\psi(x)|u(y)|^{p+(\gamma-1)/2}\psi(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\&\le\frac{1+\varepsilon}{\gamma-\varepsilon}\Big(\frac{\gamma+1}{2}\Big)^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+\gamma-1}\psi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy+C_\varepsilon\int_{\mathbb{R}^N}|u|^{\gamma+1}|\nabla\psi|^2\,dx.\end{aligned}$$

That is,

$$\begin{aligned}\Big[p-1-\frac{1+\varepsilon}{\gamma-\varepsilon}\Big(\frac{\gamma+1}{2}\Big)^2\Big]&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+\gamma-1}\psi(x)^2|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\&\quad+p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p+(\gamma-1)/2}\psi(x)|u(y)|^{p+(\gamma-1)/2}\psi(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\&\le C_\varepsilon\int_{\mathbb{R}^N}|u|^{\gamma+1}|\nabla\psi|^2\,dx.\end{aligned}\tag{2.4}$$

Since

$$\lim_{\varepsilon\to0^+}\Big[p-1-\frac{1+\varepsilon}{\gamma-\varepsilon}\Big(\frac{\gamma+1}{2}\Big)^2\Big]=p-1-\frac{1}{\gamma}\Big(\frac{\gamma+1}{2}\Big)^2>0$$

thanks to (2.1) in Lemma 2.1, we can fix $\varepsilon>0$ sufficiently small such that

$$p-1-\frac{1+\varepsilon}{\gamma-\varepsilon}\Big(\frac{\gamma+1}{2}\Big)^2>0.$$

We also choose $\psi=\eta_R^m$, where

$$\eta_R=1\text{ in }B_R,\quad\eta_R=0\text{ in }\mathbb{R}^N\setminus B_{2R},\quad0\le\eta_R\le1\quad\text{and}\quad|\nabla\eta_R|\le\frac{C}{R}\text{ in }\mathbb{R}^N$$

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and m is a positive integer such that $(m-1)(2p+\gamma-1)/(\gamma+1) \geq m$. Then (2.4) yields

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p+(\gamma-1)/2}\eta_R(x)^m|u(y)|^{p+(\gamma-1)/2}\eta_R(y)^m}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad\leq C\int_{\mathbb{R}^N} |u|^{\gamma+1}\eta_R^{2m-2}|\nabla\eta_R|^2\,dx \\
&\quad\leq C\Big(\int_{\mathbb{R}^N} \big(|u|^{\gamma+1}\eta_R^{2m-2}\big)^{(2p+\gamma-1)/2(\gamma+1)}\,dx\Big)^{2(\gamma+1)/(2p+\gamma-1)} \\
&\qquad\times\Big(\int_{\mathbb{R}^N} \big(|\nabla\eta_R|^2\big)^{(2p+\gamma-1)/(2p-\gamma-3)}\,dx\Big)^{(2p-\gamma-3)/(2p+\gamma-1)} \\
&\quad\leq C\Big(\int_{\mathbb{R}^N} |u|^{p+(\gamma-1)/2}\eta_R^m\,dx\Big)^{2(\gamma+1)/(2p+\gamma-1)} \\
&\qquad\times\Big(\int_{\mathbb{R}^N} |\nabla\eta_R|^{2(2p+\gamma-1)/(2p-\gamma-3)}\,dx\Big)^{(2p-\gamma-3)/(2p+\gamma-1)}.
\end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned}
\Big(\int_{\mathbb{R}^N} |u|^{p+(\gamma-1)/2}\eta_R^m\,dx\Big)^2 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)|^{p+(\gamma-1)/2}\eta_R(x)^m|u(y)|^{p+(\gamma-1)/2}\eta_R(y)^m\,dx\,dy \\
&\leq CR^{N-\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p+(\gamma-1)/2}\eta_R(x)^m|u(y)|^{p+(\gamma-1)/2}\eta_R(y)^m}{|x-y|^{N-\alpha}}\,dx\,dy.
\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5),

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p+(\gamma-1)/2}\eta_R(x)^m|u(y)|^{p+(\gamma-1)/2}\eta_R(y)^m}{|x-y|^{N-\alpha}}\,dx\,dy\Big)^{2(p-1)/(2p+\gamma-1)} \\
&\leq CR^{(N-\alpha)(\gamma+1)/(2p+\gamma-1)}\Big(\int_{\mathbb{R}^N} |\nabla\eta_R|^{2(2p+\gamma-1)/(2p-\gamma-3)}\,dx\Big)^{(2p-\gamma-3)/(2p+\gamma-1)} \\
&\leq CR^{(N-\alpha)(\gamma+1)/(2p+\gamma-1)+(N-(2(2p+\gamma-1))/(2p-\gamma-3))(2p-\gamma-3)/(2p+\gamma-1)} \\
&= CR^{N-2-(N+\alpha)(\gamma+1)/(2p+\gamma-1)}.
\end{aligned}$$

Since $N-2-(N+\alpha)(\gamma+1)/(2p+\gamma-1) < 0$ thanks to (2.1), we may let $R \to +\infty$ to get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p+(\gamma-1)/2}|u(y)|^{p+(\gamma-1)/2}}{|x-y|^{N-\alpha}}\,dx\,dy = 0.$$

That is, $u \equiv 0$. This completes the proof of Theorem 1.2. □

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