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PAPER

Consistent disjunctive sequent calculi and Scott domains

Longchun Wang[1,2] and Qingguo Li[1,*]

[1]School of Mathematics, Hunan University, Changsha 410086, China and [2]School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

[*]Corresponding author. Email: liqingguoli@aliyun.com

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Abstract

Based on the framework of disjunctive propositional logic, we first provide a syntactic representation for Scott domains. Precisely, we establish a category of consistent disjunctive sequent calculi with consequence relations, and show it is equivalent to that of Scott domains with Scott-continuous functions. Furthermore, we illustrate the approach to solving recursive domain equations by introducing some standard domain constructions, such as lifting and sums. The subsystems relation on consistent finitary disjunctive sequent calculi makes these domain constructions continuous. Solutions to recursive domain equations are given by constructing the least fixed point of a continuous function.

Keywords: Consistent disjunctive sequent calculus; Scott domain; categorical equivalence; fixed point; domain equation

1. Introduction

Domains theory provides a coherent framework for describing denotational semantics of functional programming languages. It has been an object of interest, since it was introduced by D. Scott and Ch. Strachey, on which significant progress has been made by mathematicians and computer scientists (Erné 2018; Goubault-Larrecq 2013; Ho et al. 2018; Yao 2016; Zhang 1992).

There is a long tradition of finding ways to represent various classes of domains in terms of logical languages, starting with Scott's seminal information systems (Scott 1982). An information system is a set of atomic formulae with a consistent predicates and a binary relation, which provides an elementary approach to presenting Scott domains. In Larsen and Winskel (1984), Larsen and Winskel showed how to use the concrete representation of information systems to advantage in solving recursive domain equations. Since then, many scholars have established several kinds of information systems for the representations of various domains (Hoofman 1993; Huang et al. 2015; Spreen et al. 2008; Vickers 2004; Wang and Li 2021; Wu et al. 2016).

In another development, Abramsky presented a complete logical system corresponding to Scott domains, and showed how a logical description can be usefully employed in denotational models (Abramsky 1987). His work is deliberately suggestive of logical semantics rather than logical syntax. An important difference from various information systems is that Abramsky's logic allows atomic formulae to be combined by conjunctive and disjunctive connectives. In Abramsky (1991), he extended the work to all SFP domains and led this program to fruition. Many researchers have tried to apply the Abramsky program to other classes of domains. For example, Jung, Kegelmann and Moshier (Jung et al. 1999) devised a coherent sequent calculus for strong proximity lattices. For a variety of results, see Jung (2013), Wang and Li (2020*a*).

The most relevant work for us is Chen and Jung's paper on a logical approach to stable domains (Chen and Jung 2006), in which they built a framework of disjunctive propositional logic and proved that the category of algebraic L-domains with stable functions can be described via its Lindenbaum algebra. This is particularly challenging to transfer Abramsky's idea to the world of stable domain theory.

The main contribution of this paper is to make a syntactic representation of Scott domain by developing the representation theory of disjunctive propositional logic. We show how a consistent disjunctive sequent calculus represents the Scott domain of its logical states. The category **SD** of Scott domains with Scott-continuous functions is an appropriate candidate for the denotational semantics of functional programming languages. We set up a category of consistent disjunctive sequent calculi with consequence relations, which is equivalent to the category **SD**. This demonstrates the capability of our approach in representing domains, and also exposes interrelationships and fundamental differences between algebraic L-domains and Scott domains on logical level. It shows that our method differs from that of Scott's information system and Abramsky's logical form. We neither rely on consistent predicates and atomic formulae to make inferences, nor focus on the Lindenbaum algebras of the logic.

Another most important part of our paper is to examine how to use the logical nature of consistent finitary disjunctive sequent calculi to solve recursive domains equations. This is based on a directed complete order $\trianglelefteq$ on consistent finitary disjunctive sequent calculi, following the ideas described in Abramsky (1991), Larsen and Winskel (1984). Some domain constructions such as lifting and sum are introduced. We choose these domain construction with a bit of care, not only because they can help us construct the consistent finitary disjunctive sequent calculi we need but also because they are continuous with respect to $\trianglelefteq$. In this way, the least solution to recursive domain equation can be reduced to the classical construction of the least fixed point of a continuous function.

The content is arranged as follows. Section 2 gives some basic notions and results on domains and the framework of disjunctive propositional logic. Section 3 introduces the notions of consistent disjunctive sequent calculi and logical states, and shows that each Scott domain is order isomorphic to the domain of logical states of some consistent disjunctive sequent calculus. Section 4 extends the relationship between Scott domains and consistent disjunctive sequent calculi to a categorical equivalence by defining appropriate morphisms between consistent disjunctive sequent calculi. Section 5 examines how to construct consistent disjunctive sequent calculi we need, and how to solve recursive domain equations.

2. Preliminaries

We first recall some notational conventions and basic notions needed for what follows. Those related to domain theory come from Gierz et al. (2003). For any set X, we use the symbols $A \sqsubseteq X$ to mean that A is a nonempty finite subset of X.

Let P be a poset and $X \subseteq P$. We write $\bigsqcup X$ for the least upper bound of X. We denote by $\downarrow X$ the down set $\{d \in P \mid (\exists x \in X) d \leq x\}$. Similarly, we denote by $\uparrow X$ the upper set $\{d \in P \mid (\exists x \in X) x \leq d\}$. If X is a singleton $\{x\}$, then we just write $\downarrow x$ or $\uparrow x$. X is said to be a *pairwise inconsistent* subset of P if $\uparrow x \cap \uparrow y = \emptyset$ for all $x \neq y \in X$. A nonempty subset D of P is said to be *directed* if every nonempty subset of D has a least upper bound in D. P is said to be a *complete lattice* if every subset of it has a least upper bound in P. The poset P is said to be *pointed* if it has a least element $\perp$. The poset P is said to be a *dcpo* if every directed subset D of it has a least upper bound.

Let P be a dcpo. An element $x \in P$ is called a *compact element* if for any directed subset D of P the relation $x \leq \bigsqcup D$ always implies the existence of some $d \in D$ such that $x \leq d$. We denote by $K(P)$ the set of compact elements of P and denote by $K^*(P)$ the set $K(P) - \{\perp\}$.

Definition 2.1. (1) *A dcpo P is said to be algebraic if for every element $x \in P$ there is a directed subset $D \subseteq K(P) \cap {\downarrow}x$ such that $x = \bigsqcup D$.*

(2) *An algebraic dcpo is called a Scott domain if any finite subset of it which are bounded above has a least upper bound.*

Example 2.1. Let X be a nonempty subset of real numbers. A partial map from X to X is a map $f : S \to X$, where dom(f), the domain of f, is a subset of X; here $S = \emptyset$ is allowed. The set of partial maps from X to itself is denoted by $(X \rightarrowtail X)$. We order $(X \rightarrowtail X)$ as follows: given $f, g \in (X \rightarrowtail X)$, define $f \leq g$ if and only if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(x) = g(x)$ for all x in dom(f). Then $(X \rightarrowtail X)$ is a Scott domain *(Davey and Priestley 2002)*. In addition,

- Every bounded finite directed set must be a Scott domain.
- $(\mathbb{N} \cup \{\omega\}, \leq)$ forms a Scott domain.
- $([0, 1], \leq)$ is not a Scott domain.

Definition 2.2. *Let P and Q be algebraic dcpos. A function $f : P \to Q$ is Scott-continuous if and only if for all directed subset D of P, we have*

$$f(\bigsqcup D) = \bigsqcup \{f(x) \mid x \in D\}.$$

In Chen and Jung (2006, Definition 2.1), Chen and Jung introduced a framework of disjunctive propositional logic for algebraic L-domains, in which a sequent is an object $\Gamma \vdash \varphi$ such that Γ is a finite set of formulae and φ is a single formula. As usual logic (Gallier 2015), the interpretation of a valid sequent $\Gamma \vdash \varphi$ is that the formula φ can be derived from the conjunction of the formulae in Γ.

Definition 2.3. (Chen and Jung 2006). *Let P be a set. Every element of P is called an atomic formula. Likewise, let $\mathcal{A}_P$ be a set of sequents of the form $p_1, p_2, \ldots, p_n \vdash F$ where the p_i are atomic formulae, and* F *is the syntactic constant for "false". Each element of $\mathcal{A}_P$ is called an atomic disjointness assumption, and the pair $(P, \mathcal{A}_P)$ is called a disjunctive basis.*

The class $\mathcal{L}(P)$ of formulae, and the class **T**(P) *of valid sequents are generated by mutual transfinite induction by the following rules:*

- *Disjunctive formulae*

$$\text{(At)}\ \frac{\phi \in P}{\phi \in \mathcal{L}(P)} \qquad \text{(Const)}\ \frac{}{\mathrm{T},\mathrm{F} \in \mathcal{L}(P)}$$

$$\text{(Conj)}\ \frac{\phi, \psi \in \mathcal{L}(P)}{\phi \wedge \psi \in \mathcal{L}(P)}$$

$$\text{(Disj)}\ \frac{\phi_i \in \mathcal{L}(P)(\mathit{all}\ i \in I) \qquad \phi_i, \phi_j \vdash \mathrm{F}(\mathit{all}\ i \neq j \in I)}{\dot{\bigvee}_{i \in I} \phi_i \in \mathcal{L}(P)}$$

- *Valid sequents*

$$\text{(Ax)}\ \frac{(\Gamma \vdash \mathrm{F}) \in \mathcal{A}_P}{\Gamma \vdash \mathrm{F}} \qquad \text{(Id)}\ \frac{\phi \in \mathcal{L}(P)}{\phi \vdash \phi}$$

$$\text{(Lwk)}\ \frac{\Gamma \vdash \psi \qquad \phi \in \mathcal{L}(P)}{\Gamma, \phi \vdash \psi} \qquad \text{(Cut)}\ \frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

$$\text{(LF)}\ \frac{\phi \in \mathcal{L}(P)}{\mathrm{F} \vdash \phi} \qquad \text{(RT)}\ \frac{}{\vdash \mathrm{T}}$$

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$$(\mathrm{L}\wedge)\ \frac{\Gamma, \phi, \psi \vdash \theta}{\Gamma, \phi \wedge \psi \vdash \theta} \qquad (\mathrm{R}\wedge)\ \frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \wedge \psi}$$

$$(\mathrm{L}\dot{\vee})\ \frac{\Gamma, \phi_i \vdash \theta(\mathit{all}\ i \in I) \qquad \phi_i, \phi_j \vdash \mathrm{F}(\mathit{all}\ i \neq j \in I)}{\Gamma, \dot{\bigvee}_{i\in I}\phi_i \vdash \theta}$$

$$(\mathrm{R}\dot{\vee})\ \frac{\Gamma \vdash \phi_{i_0}(\mathit{some}\ i_0 \in I) \qquad \phi_i, \phi_j \vdash \mathrm{F}(\mathit{all}\ i \neq j \in I)}{\Gamma \vdash \dot{\bigvee}_{i\in I}\phi_i}.$$

In this paper, we call the proof system of a disjunctive propositional logic a disjunctive sequent calculus and denote it by $(\mathcal{L}(P), \vdash)$. By Definition 2.3, it is easy to see that the class $\mathbf{T}(P)$ of valid sequents can be determined by a relation $\vdash$, namely

$$\text{a sequent } \Gamma \vdash \varphi \text{ is valid if and only if } (\Gamma, \varphi) \in \vdash. \tag{1}$$

In this case, we can verify a pair $(\mathcal{L}(P), \vdash)$ is a disjunctive sequent calculus by checking the class $\mathcal{L}(P)$ with the relation $\vdash$ satisfies all the rules defined in Definition 2.3.

Proposition 2.1. (Chen and Jung, 2006). *Suppose $(\mathcal{L}(P), \vdash)$ is a disjunctive sequence calculus.*

(1) *A sequent $\Gamma, \varphi, \psi \vdash \phi$ is valid if and only if $\Gamma, \varphi \wedge \psi \vdash \phi$ is a valid sequent.*
(2) *A sequent $\Gamma \vdash \varphi \wedge \psi$ is valid if and only if both $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$ are valid sequents.*
(3) *A sequent $\Gamma, \dot{\bigvee}_{i\in I}\phi_i \vdash \theta$ is valid if and only if $\phi_i, \phi_j \vdash \mathrm{F}$ and $\Gamma, \phi_i \vdash \theta$ are valid sequents for all $i \neq j \in I$.*

Lemma 2.1. *Suppose $(\mathcal{L}(P), \vdash)$ is a disjunctive sequence calculus.*

(1) Let $\varphi \vdash \phi$ be a valid sequent. Then the sequent $\varphi \vdash \psi$ is valid if and only if the sequent $\varphi \wedge \phi \vdash \psi$ is valid.
(2) If $\dot{\bigvee}_{i\in I}\phi_i \in \mathcal{L}(P)$, then $\phi_i \vdash \dot{\bigvee}_{i\in I}\phi_i$ is a valid sequent for every $i \in I$.

Proof. (1) It suffices to provides the following two derivations:

$$\dfrac{\dfrac{\varphi \vdash \psi}{\varphi, \phi \vdash \psi}\,(\mathrm{Lwk})}{\varphi \wedge \phi \vdash \psi}\,(\mathrm{L}\wedge) \qquad \dfrac{\varphi \wedge \phi \vdash \psi \qquad \dfrac{\varphi \vdash \phi \quad \varphi \vdash \varphi}{\varphi \vdash \varphi \wedge \phi}\,(\mathrm{R}\wedge)}{\varphi \vdash \psi}\,(\mathrm{Cut})$$

(2) By the rule (Id), the sequent $\dot{\bigvee}_{i\in I}\phi_i \vdash \dot{\bigvee}_{i\in I}\phi_i$ is valid. Then with part (3) of Proposition 2.1, the sequent $\phi_i \vdash \dot{\bigvee}_{i\in I}\phi_i$ is also valid for every $i \in I$. □

3. A Syntactic Representation of Scott Domains

In this section, we show how to use disjunctive sequent calculi to represent Scott domains. We begin by introducing some common notions.

Definition 3.1. *Suppose $(\mathcal{L}(P), \vdash)$ is a disjunctive sequent calculus.*

(1) A formula φ is said to be satisfiable if $\mathrm{T} \vdash \varphi$ and $\varphi \vdash \mathrm{F}$ are not valid sequents.
(2) A satisfiable formula built up from atomic formulae only by conjunctive connectives is called a simple conjunction.
(3) A satisfiable formula is said to be a flat formula if it has the form $\dot{\bigvee}_{i\in I}\mu_i$, where μ_i is a simple conjunction with $\mu_i, \mu_j \vdash \mathrm{F}$ is valid for every $i \neq j \in I$.

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We use $\mathcal{C}(P)$ and $\ell(P)$ to denote the sets of simple conjunctions and flat formulae, respectively.

Definition 3.2. *If the sequents $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are all valid, then we call φ and ψ are logically equivalent, and denote by $\varphi \approx \psi$.*

As in usual logic, we always assume that each atomic formula is satisfiable and two distinct atomic formulae are not logically equivalent.

Proposition 3.1. (Chen and Jung, 2006). *Let $(\mathcal{L}(P), \vdash)$ be a disjunctive sequent calculus. Then every satisfiable formula is logically equivalent to a flat formula.*

Definition 3.3. *Let $(\mathcal{L}(P), \vdash)$ be a disjunctive sequent calculus. A satisfiable formula φ is said to be irreducible provided, if $\dot{\bigvee}_{i\in I}\mu_i$ is a flat formula and $\varphi \vdash \dot{\bigvee}_{i\in I}\mu_i$ is a valid sequent then $\varphi \vdash \mu_{i_0}$ is a valid for some $i_0 \in I$.*

Definition 3.4. *A disjunctive sequent calculus $(\mathcal{L}(P), \vdash)$ is said to be consistent if, every simple conjunction in it is irreducible.*

Let $\mathcal{N}(P) = \ell(P) \cup \{\mathrm{T}\}$. For every $X \subseteq \mathcal{L}(P)$, we set

$$X[\vdash] = \{\varphi \in \mathcal{N}(P) \mid (\exists \Gamma \sqsubseteq X)\Gamma \vdash \varphi \in \mathbf{T}(P)\}. \tag{2}$$

Then for every pair of formulae $\varphi, \psi \in X[\vdash]$, there is some $\Gamma \sqsubseteq X$ such that $\Gamma \vdash \varphi \wedge \psi$ is a valid sequent by the rule (R$\wedge$). Moreover, by part (1) of Proposition 2.1, the sequent $\Gamma \vdash \varphi \wedge \psi$ is a valid sequent if and only if $\bigwedge \Gamma \vdash \varphi \wedge \psi$ is a valid sequent, where $\bigwedge \Gamma$ is the conjunction built up from all the formulae in Γ.

Definition 3.5. *Suppose $(\mathcal{L}(P), \vdash)$ is a consistent disjunctive sequent calculus. A logical state of $(\mathcal{L}(P), \vdash)$ is a nonempty subset S of $\mathcal{N}(P)$ such that the following conditions hold:*

- **(S1)** $S[\vdash] \subseteq S$.
- **(S2)** *For every $\dot{\bigvee}_{i\in I}\mu_i \in S \cap \ell(P)$, there exists some $i_0 \in I$ such that $\mu_{i_0} \in S$.*
- **(S3)** *For every pair of formulae $\mu, \nu \in S \cap \mathcal{C}(P)$, the formula $\mu \wedge \nu \in S \cap \mathcal{C}(P)$.*

The following proposition gives some basic properties that will be used frequently.

Proposition 3.2. *Suppose $(\mathcal{L}(P), \vdash)$ is a consistent disjunctive sequent calculus.*

(1) The singleton $\{\mathrm{T}\}$ is a logical state.
(2) $\{\mu\}[\vdash]$ is a logical state for every $\mu \in \mathcal{C}(P)$.
(3) If $\{S_i \mid i \in I\}$ is a directed set of logical states, then $S = \bigcup\{S_i \mid i \in I\}$ is a logical state.
(4) If S is a logical state and $\Gamma \sqsubseteq S$, then there is some $\varphi \in S$ such that $\varphi \approx \bigwedge \Gamma$.

Proof. The proofs of (1) and (2) are trivial.

To prove (3), we need to check the set S satisfies conditions **(S1)**, (**S2**) and (**S3**).

For condition (**S1**), suppose that $\varphi \in S[\vdash]$. By Equation (2), we have some $\Gamma \sqsubseteq S$ such that $\Gamma \vdash \varphi$. Since the set $\{S_i \mid i \in I\}$ is directed, there is some $i_0 \in I$ such that $\Gamma \sqsubseteq S_{i_0}$. This implies that $\varphi \in S_{i_0}[\vdash]$. Note that S_{i_0} is a logical state, it follows that $\varphi \in S_{i_0} \subseteq S$. Thus $S[\vdash] \subseteq S$. For condition (**S2**), suppose that $\dot{\bigvee}_{j\in J}\mu_j \in S \cap \ell(P)$. Recall that $S = \bigcup\{S_i \mid i \in I\}$, there is some $i \in I$ such that $\dot{\bigvee}_{j\in J}\mu_j \in S_i \cap \mathcal{N}(P)$. As S_i is a logical state, we have some $j_i \in J$ such that $\mu_{j_i} \in S_i \subseteq S$. For condition (**S3**), suppose that $\mu, \nu \in S \cap \mathcal{C}(P)$. Since the set $\{S_i \mid i \in I\}$ is directed, the set $\{S_i \mid i \in I\} \cap \mathcal{C}(P) = \{S_i \cap \mathcal{C}(P) \mid i \in I\}$ is also directed. Then $\mu, \nu \in S_{i_0} \cap \mathcal{C}(P)$ for some $i_0 \in I$. Using condition (**S3**) for S_{i_0}, we know that the formula $\mu \wedge \nu \in S_{i_0} \subseteq S$.

(4) Since the case that $\bigwedge \Gamma \approx \mathrm{T}$ is clear, we now assume that $\bigwedge \Gamma$ is not logically equivalent to T. Let $\Gamma - \{\mathrm{T}\} = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$. For every $1 \leq j \leq n$, by part (2) of Lemma 2.1 and condition (**S2**),

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there is some $\nu_j \in S$ such that $\nu_j \vdash \varphi$ is a valid sequent. This implies that $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n \vdash \bigwedge \Gamma$ is a valid sequent. By condition (**S3**), the formula $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n \in \mathcal{C}(P)$, and hence $\bigwedge \Gamma$ is a satisfiable formula. By Proposition 3.1, there is a flat formula φ logically equivalent to $\bigwedge \Gamma$. Then $\Gamma \vdash \varphi$ is a valid sequent, and hence $\varphi \in S[\vdash] \subseteq S$. □

We denote by $|(\mathcal{L}(P), \vdash)|$ the collection of all the logical states of a consistent disjunctive sequent calculus $(\mathcal{L}(P), \vdash)$.

Proposition 3.3. *Let $(\mathcal{L}(P), \vdash)$ be a consistent disjunctive sequent calculus. If S is a logical state, then for every subset X of S the following set*

$$[X]_S = \bigcap \{W \in |(\mathcal{L}(P), \vdash)| \mid X \subseteq W \subseteq S\} \tag{3}$$

is also a logical state.

Proof. For any $\varphi \in [X]_S[\vdash]$, by Equation (2), there exists some $\Gamma \sqsubseteq [X]_S$ such that $\Gamma \vdash \varphi$ is a valid sequent. By Equation (3), $\Gamma \sqsubseteq W$ for all $W \in |(\mathcal{L}(P), \vdash)|$ with $X \subseteq W \subseteq S$. Thus $\varphi \in W[\vdash] \subseteq W$, and therefore $\varphi \in [X]_S$. So, $[X]_S$ satisfies condition (**S1**).

To show condition (**S2**), let $\dot{\bigvee}_{i \in I} \mu_i \in [X]_S \cap \ell(P)$. Then $\dot{\bigvee}_{i \in I} \mu_i \in W$ for every $W \in |(\mathcal{L}(P), \vdash)|$ with $X \subseteq W \subseteq S$. Thus $\mu_{i_W} \in W \cap \ell(P) \subseteq S$ for some some $i_W \in I$, and hence $\{\mu_{i_W} \mid W \in |(\mathcal{L}(P), \vdash)|, X \subseteq W \subseteq S\}$ is a subset of S. Note that $\mu_i, \mu_j \vdash \mathrm{F}$ are valid for all $i \neq j \in I$, using condition (**S3**) for the logical state S, it is easy to see that the set $\{\mu_{i_W} \mid W \in |(\mathcal{L}(P), \vdash)|, X \subseteq W \subseteq S\}$ must be a singleton, denoted by $\{\mu_{i_0}\}$. Consequently, there exists some $i_0 \in I$ such that $\mu_{i_0} \in [X]_S$.

The remainder is to show $[X]_S$ satisfies condition (**S3**). If $\mu, \nu \in [X]_S \cap \mathcal{C}(P)$, then $\mu, \nu \in W \cap \mathcal{C}(P)$ for every logical state W that satisfies $X \subseteq W \subseteq S$. Using condition (**S3**) for W, the formula $\mu \wedge \nu \in W \subseteq [X]_S$. □

The next theorem shows that, in addition to being a dcpo, the set of all logical states of a consistent disjunctive sequent calculus ordered by set inclusion are also algebraic and bounded completed.

Theorem 3.1. *Suppose $(\mathcal{L}(P), \vdash)$ is a consistent disjunctive sequent calculus. Then the poset $(|(\mathcal{L}(P), \vdash)|, \subseteq)$ is a Scott domain.*

Proof. For any directed subset $\{S_i \mid i \in I\}$ of $((\mathcal{L}(P), \vdash)|, \subseteq)$, by part (3) of Proposition 3.2, the union $\bigcup\{S_i \mid i \in I\}$ is a logical state. Then the least upper bound of the set $\{S_i \mid i \in I\}$ exists in $(|(\mathcal{L}(P), \vdash)|, \subseteq)$, and $\bigsqcup\{S_i \mid i \in I\} = \bigcup\{S_i \mid i \in I\}$. Therefore, with part (1) of Proposition 3.2, we know that $(|(\mathcal{L}(P), \vdash)|, \subseteq)$ forms a dcpo with a least element $\{\mathrm{T}\}$.

We now verify that $(|(\mathcal{L}(P), \vdash)|, \subseteq)$ is algebraic and bounded complete. By Definition 2.1, the proof can be divided into three steps.

Step 1: We prove that, if S is logical state S and $\Gamma \sqsubseteq S$ then the set $[\Gamma]_S$ is a compact element of the dcpo $(|(\mathcal{L}(P), \vdash)|, \subseteq)$.

By Proposition 3.3, the set $[\Gamma]_S$ is a logical state. Suppose $\{S_i \mid i \in I\}$ is a directed subset of the dcpo $(|(\mathcal{L}(P), \vdash)|, \subseteq)$ and $[\Gamma]_S \subseteq \bigcup\{S_i \mid i \in I\}$. As $\bigcup\{S_i \mid i \in I\}$ is a logical state, with Equation (3), it is trivial to check that $[\Gamma]_S = [\Gamma]_{\bigcup\{S_i \mid i \in I\}}$. Since $\Gamma \sqsubseteq \bigcup\{S_i \mid i \in I\}$, we have some $i_0 \in I$ such that $\Gamma \subseteq S_{i_0}$. Therefore, $[\Gamma]_S = [\Gamma]_{\bigcup\{S_i \mid i \in I\}} \subseteq S_{i_0}$.

Step 2: We prove that, for every logical state S, the set $\{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$ is directed and

$$S = \bigcup\{[\Gamma]_S \mid \Gamma \sqsubseteq S\}.$$

The set $\{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$ is not empty since $S \neq \emptyset$. For any $\Gamma_1, \Gamma_2 \sqsubseteq S$, we have $[\Gamma_1 \cup \Gamma_2]_S \in \{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$, and $[\Gamma_1]_S, [\Gamma_2]_S \subseteq [\Gamma_1 \cup \Gamma_2]_S$ by Equation (3). So, the set $\{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$ is directed. Because $[\Gamma]_S \subseteq S$ for all $\Gamma \sqsubseteq S$, it follows that $\bigcup\{[\Gamma]_S \mid \Gamma \sqsubseteq S\} \subseteq S$. Conversely, if $\varphi \in S$, then $\varphi \in$

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$[\{\varphi\}]_S \in \{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$. This means that $S \subseteq \bigcup\{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$. We have proven that $S = \bigcup\{[\Gamma]_S \mid \Gamma \sqsubseteq S\}$.

Step 3: We prove that any two logical states that are bounded above have a least upper bound.

Let S_1, S_2, and S_3 be logical states with $S_1, S_2 \subseteq S_3$. If one of $\{S_1, S_2\}$ equals to the set $\{\mathrm{T}\}$, then the other one is the least upper bound of $\{S_1, S_2\}$. If $\{\mathrm{T}\} \notin \{S_1, S_2\}$, then, by condition (**S2**), the sets $S_1 \cap \mathcal{C}(P)$ and $S_2 \cap \mathcal{C}(P)$ are not empty. In this case, we will check that

$$S_0 = \{\varphi \in \mathcal{N}(P) \mid \mu, \nu \in (S_1 \cup S_2) \cap \mathcal{C}(P), \mu \wedge \nu \vdash \varphi \in \mathbf{T}(P))\} \tag{4}$$

is the least upper bound of S_1 and S_2. For this, we first show S_0 is a logical state.

Suppose that $\varphi \in S_0[\vdash]$. Then there is some $\Gamma \sqsubseteq S_0$ such that $\Gamma \vdash \varphi$ is a valid sequent. Let $\Gamma = \{\varphi_1, \varphi_2, \cdots, \varphi\}$. For every $\varphi_i \in \Gamma$, there are some $\mu_i, \nu_i \in (S_1 \cup S_2) \cap \mathcal{C}(P)$ such that $\mu_i \wedge \nu_i \vdash \varphi_i$ is a valid sequent. Without lose of generality, we assume that $\mu_i \in S_1$ and $\nu_i \in S_2$ for all $i \in I$. Then $\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n \in S_1 \cap \mathcal{C}(P)$ and $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n \in S_2 \cap \mathcal{C}(P)$ by condition (**S3**). Put $\mu_0 = \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n$ and $\nu_0 = \nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n$. By part (1) of Proposition 2.1, we have $\mu_0 \wedge \nu_0 \vdash \bigwedge \Gamma$ is a valid sequent, and thus $\mu_0 \wedge \nu_0 \vdash \varphi$ is a valid sequent by the rule (Cut). So $\varphi \in S_0$. Condition (**S1**) follows.

For condition (**S2**), let $\dot{\bigvee}_{i \in I}\mu_i \in S_0 \cap \ell(P)$. Then there are $\nu_1, \nu_2 \in (S_1 \cup S_2) \cap \mathcal{C}(P)$ such that the sequent $\nu_1 \wedge \nu_2 \vdash \dot{\bigvee}_{i \in I}\mu_i$ is valid. Since $\nu_1, \nu_2 \in S_1 \cup S_2 \subseteq S_3$, the formula $\nu_1 \wedge \nu_2 \in S_3$. This implies that $\nu_1 \wedge \nu_2$ is an irreducible simple conjunction. So, there is some $i_0 \in I$ such that the sequent $\nu_1 \wedge \nu_2 \vdash \mu_{i_0}$ is valid, and thus $\mu_{i_0} \in S_0$ by Equation (4).

Assume that $\mu, \nu \in S_0 \cap \mathcal{C}(P)$. As we have seen in the above proof for condition (**S1**), there are some $\mu_1, \nu_1 \in (S_1 \cup S_2) \cap \mathcal{C}(P)$ such that $\mu_1 \wedge \nu_1 \vdash \mu \wedge \nu$ is a valid sequent. So, $\mu \wedge \nu \in S_0 \cap \mathcal{C}(P)$. This implies that S_0 satisfies condition (**S3**).

We next show $S_1, S_2 \subseteq S_0$. Let $\varphi \in S_1 \cup S_2$. If $\varphi = \mathrm{T}$, then clearly $\varphi \in S_0$. If $\varphi = \dot{\bigvee}_{i \in I}\mu_i \in \ell(P)$, then by part (2) of Lemma 2.1, $\mu_i \vdash \dot{\bigvee}_{i \in I}\mu_i$ are valid sequent for all $i \in I$. Using condition (**S2**) for $\dot{\bigvee}_{i \in I}\mu_i$, we have some $i_0 \in I$ such that $\mu_{i_0} \in S_1 \cup S_2$. So $\mu_{i_0} \wedge \mu_{i_0} \vdash \dot{\bigvee}_{i \in I}\mu_i$ is a valid sequent, which implies that $\varphi \in S_0$.

Finally, let $S_4 \in |(\mathcal{L}(P), \vdash)|$ and $S_1, S_2 \subseteq S_4$. For every $\varphi \in S_0$, there are some $\mu, \nu \in (S_1 \cup S_2) \cap \mathcal{C}(P)$ such that $\mu \wedge \nu \vdash \varphi$ is a valid sequent. Since $\mu, \nu \in (S_1 \cup S_2) \cap \mathcal{C}(P) \subseteq S_4$, it follows that $\nu \wedge \nu \in S_4$. Then $\varphi \in S_4[\vdash] \subseteq S_4$, and thus, $S_0 \subseteq S_4$. □

The above result has shown that each consistent disjunctive sequent calculus represents a Scott domain. Next, we will see that this is a complete characterization, that is, every Scott domain can be represented by a consistent disjunctive sequent calculus.

For every Scott domain $(D, \leq)$, the set

$$\mathcal{U}(D) = \{\uparrow A \mid A \text{ is a pairwise inconsistent subset of } K(D)\} \cup \{\emptyset\}. \tag{5}$$

is closed under finite intersections $\cap$ and arbitrary disjoint unions $\dot{\bigcup}$. This allows us to make the following definition.

Definition 3.6. *Suppose $(D, \leq)$ is a Scott domain and $P_D = \{\uparrow x \mid x \in K^*(D)\}$. Then a set $\mathcal{L}(P_D)$ of formulae is defined by induction as follows:*

1. *each element $\uparrow x$ of P_D is a formula in $\mathcal{L}(P_D)$, which is called an atomic formula,*
2. *the constant connectives* T *and* F *are formulae in $\mathcal{L}(P_D)$,*
3. *if φ and ψ are formulae in $\mathcal{L}(P_D)$, then $\varphi \wedge \psi$ is a formula in $\mathcal{L}(P_D)$,*
4. *if $\{\varphi_i \mid i \in I\}$ is a subset of formulae in $\mathcal{L}(P_D)$ such that $\widehat{\varphi_i} \cap \widehat{\varphi_j} = \emptyset$ for all $i \neq j \in I$, then $\dot{\bigvee}_{i \in I}\varphi_i$ is a formula in $\mathcal{L}(P_D)$, where $\widehat{\varphi}$ is the set replacing the connectives* F, T, $\wedge$ *and* $\dot{\bigvee}$ *in φ by* $\emptyset, D, \cap$ *and* $\dot{\bigcup}$*, respectively.*

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Proposition 3.4. *Suppose* $(D, \leq)$ *is a Scott domain and* $\mathcal{L}(P_D)$ *is the set of formulae defined in Definition 3.6. For every* $\Gamma = \{\psi_1, \psi_2, \cdots, \psi_n\} \sqsubseteq \mathcal{L}(P_D)$ *and* $\varphi \in \mathcal{L}(P_D)$*, define*

$$\Gamma \vdash \varphi \text{ is a valid sequent if and only if } \quad \widehat{\psi_1} \cap \widehat{\psi_2} \cap \cdots \cap \widehat{\psi_n} \subseteq \widehat{\varphi}. \tag{6}$$

Then the pair $(\mathcal{L}(P_D), \vdash)$ *is a consistent disjunctive sequent calculus.*

Proof. The set of atomic formulae has been defined in Definition 3.6, which is $\{\uparrow x \mid x \in K^*(D)\}$. Let $\uparrow x_1, \uparrow x_2, \cdots, \uparrow x_n$ be atomic formulae. By Equation (6), we know that the sequent $\uparrow x_1, \uparrow x_2, \cdots, \uparrow x_n \vdash \mathrm{F}$ is valid if and only if $\uparrow x_1 \cap \uparrow x_2 \cap \cdots \cap \uparrow x_n = \emptyset$. Then, it is trivial but tedious to check the set $\mathcal{L}(P_D)$ and the relation $\vdash$ satisfy all the rules of disjunctive formulae and valid sequents defined in Definition 2.3. Thus the pair $(\mathcal{L}(P_D), \vdash)$ is a disjunctive sequent calculus.

Now we have to show that every simple conjunction in $(\mathcal{L}(P_D), \vdash)$ is irreducible. Let $\uparrow x_1 \wedge \uparrow x_2 \wedge \cdots \wedge \uparrow x_n$ be a simple conjunction, where $\{x_1, x_2, \cdots, x_n\} \subseteq K^*(D)$. Then $\uparrow x_1, \uparrow x_2, \cdots, \uparrow x_n \vdash \mathrm{F}$ is not a valid sequent, and thus the set $\uparrow x_1 \cap \uparrow x_2 \cap \cdots \cap \uparrow x_n \neq \emptyset$. Since D is a Scott domain, it follows that $\bigsqcup\{x_1, x_2, \cdots, x_n\} \in D$. Put $\bigsqcup\{x_1, x_2, \cdots, x_n\} = d$. Hence $d \in K^*(D)$ and the formula $\uparrow x_1 \wedge \uparrow x_2 \wedge \cdots \wedge \uparrow x_n$ is logically equivalent to the formula $\uparrow d$. Assume that $\uparrow x_1 \wedge \uparrow x_2 \wedge \cdots \wedge \uparrow x_n \vdash \dot{\bigvee}_{i \in I} \mu_i$ is a valid sequent, where $\dot{\bigvee}_{i \in I} \mu_i \in \ell(P_D)$. By Equation (6), we have $\uparrow d = \widehat{\uparrow d} = \widehat{\uparrow x_1} \cap \widehat{\uparrow x_2} \cap \cdots \cap \widehat{\uparrow x_n} \subseteq \dot{\bigcup}_{i \in I} \widehat{\mu_i}$. This implies that there is some $i_0 \in I$ such that $d \in \widehat{\mu_{i_0}}$. Thus the sequent $\uparrow x_1 \wedge \uparrow x_2 \wedge \cdots \wedge \uparrow x_n \vdash \mu_{i_0}$ is valid. □

In the sequel, we use $(\mathcal{L}(P_D), \vdash)$ to denote the consistent disjunctive sequent calculus defined by Proposition 3.4.

Lemma 3.1. *Suppose* $(D, \leq)$ *is a Scott domain. A nonempty subset* S *of* $\mathcal{N}(P_D)$ *is a logical state of* $(\mathcal{L}(P_D), \vdash)$ *if and only if*

$$S = \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d \in \widehat{\varphi}\},$$

for some $d \in D$*, where* $\widehat{\varphi}$ *is defined in Definition 3.6.*

Proof. Let S be a logical state of $(\mathcal{L}(P_D), \vdash)$. We have to look for an element $d_S \in D$ such that $S = \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\}$.

If $S = \{\mathrm{T}\}$, then $S = \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} = D\}$. Taking $\perp = d_S$, we have $S = \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\}$. For the case that $S \neq \{\mathrm{T}\}$, the process is divided into three stages.

(1) For every $\psi \in S$ with $\psi \neq \mathrm{T}$, there exists some pairwise inconsistent subset A of $K^*(D)$ such that $\widehat{\psi} = \dot{\bigcup}_{a \in A} \uparrow a$. Then $\dot{\bigvee}_{a \in A} \uparrow a \in S$. Thus $\uparrow a_0 \in S$ for some $a_0 \in A$ by condition (**S2**). So, $\{\uparrow a \mid \uparrow a \in S\} \neq \emptyset$ and $\bigcap\{\widehat{\varphi} \in \mathcal{U}(D) \mid \varphi \in S\} = \bigcap\{\uparrow a \mid \uparrow a \in S\}$.

(2) We claim that $\{a \in K^*(D) \mid \uparrow a \in S\}$ is a directed set of D. In fact, we have shown the set $\{a \in K^*(D) \mid \uparrow a \in S\}$ is not empty. For every $a_1, a_2 \in \{a \in K^*(D) \mid \uparrow a \in S\}$, by condition (**S3**), $\uparrow a_1 \wedge \uparrow a_2 \in S$. This implies $\uparrow a_1 \cap \uparrow a_2 \neq \emptyset$. Since D is a Scott domain, $\uparrow a_1 \cap \uparrow a_2 = \uparrow b$ for some $b \in K^*(D)$. Therefor, $b \in \{a \in K^*(D) \mid \uparrow a \in S\}$ and $a_1, a_2 \leq b$.

(3) Since $\{a \in K^*(D) \mid \uparrow a \in S\}$ is directed, $\bigsqcup\{a \in K^*(D) \mid \uparrow a \in S\} \in D$. Set

$$d_S = \bigsqcup\{a \in K^*(D) \mid \uparrow a \in S\}. \tag{7}$$

Then $\bigcap\{\uparrow a \mid \uparrow a \in S\} = \uparrow d_S$. Now we show $S = \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\}$.

To prove $S \subseteq \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\}$, let $\psi \in S$. If $\psi = \mathrm{T}$, then $\widehat{\psi} = D$. Thus $\psi \in \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\}$. If $\psi \neq \mathrm{T}$, then ψ is of the form $\dot{\bigvee}_{a \in A} \uparrow a$, where A is a pairwise inconsistent subset of $K^*(D)$. This implies that $\uparrow a_0 \in S$ for some $a_0 \in A$, and hence $d_S \in \uparrow a_0 \subseteq \dot{\bigcup}_{a \in A} \uparrow a = \widehat{\psi}$. So we also have $\psi \in \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\}$.

To prove $\{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_S \in \widehat{\varphi}\} \subseteq S$, let $\varphi \in \mathcal{N}(P_D)$ with $d_S \in \widehat{\varphi} \in \mathcal{U}(D)$. If $\varphi = \mathrm{T}$, then $\varphi \in S$. If $\varphi \neq \mathrm{T}$, then $d_S \in \widehat{\varphi}$ implies that $d_S \in \uparrow d \subseteq \widehat{\varphi}$ for some $d \in K^*(D)$. Since $d_S = \bigsqcup\{a \mid \uparrow a \in S\}$ and $\{a \mid \uparrow a \in S\}$ is directed, by the definition of compact elements of a dcpo, there is some

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$a_0 \in \{a \mid \uparrow a \in S\}$ such that $d_S \leq a_0$. So $d \leq a_0$, and hence $\uparrow a_0 \subseteq \widehat{\varphi}$. This means that $\uparrow a_0 \vdash \varphi$ is a valid sequent. By condition (**S1**), it follows that $\varphi \in S$.

For the converse implication, given a $d \in D$, let $S = \{\varphi \in \mathcal{L}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d \in \widehat{\varphi}\}$. We need to prove S is a logical state of $(\mathcal{L}(P_D), \vdash)$.

Assume that $\psi \in S[\vdash]$. By Equation (2), there is some nonempty finite subset Γ of S such that the sequent $\Gamma \vdash \psi$ is valid. Because of the definition of S, it follows that $d \in \widehat{\varphi}$ for all $\varphi \in \Gamma$. By Equation (6), we have $d \in \widehat{\psi}$. This implies that $\psi \in S$ and hence $S[\vdash] \subseteq S$. Therefore, the set S satisfies condition (**S1**).

For condition (**S2**), if $\dot{\bigvee}_{i \in I} \mu_i$ is a flat formula in S, then $d \in \dot{\bigcup}_{i \in I} \widehat{\mu_i}$. This impiles that there is some $i_0 \in I$ such that $d \in \widehat{\mu_{i_0}}$ and $\widehat{\mu_{i_0}} \in \mathcal{U}(D)$. Therefore, $\mu_{i_0} \in S$.

Assume that $\mu, \nu \in S \cap \mathcal{C}(P)$. Then $d \in \widehat{\mu} \cap \widehat{\nu}$ and thus $\widehat{\mu} \cap \widehat{\nu}$ is not the empty set. By Equation (6), the sequent $\mu \wedge \nu \vdash \mathrm{F}$ is not valid. Condition (**S3**) follows. □

Theorem 3.2. *If $(D, \leq)$ is a Scott domain, then it is order isomorphic to the Scott domains of logical states of some consistent disjunctive sequent calculus.*

Proof. Proposition 3.4 has constructed a consistent disjunctive sequent calculus $(\mathcal{L}(P_D), \vdash)$ associated with the Scott domain $(D, \leq)$. Define a function from D to $|(\mathcal{L}(P_D), \vdash)|$ as follows:

$$f(d) = \{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d \in \widehat{\varphi}\}.$$

By Lemma 3.1, the set $\{\varphi \in \mathcal{N}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d \in \widehat{\varphi}\}$ is an element of $|(\mathcal{L}(P_D), \vdash)|$. So, the function f is well defined. It is surjection because $f(d_S) = S$ for every $S \in |(\mathcal{L}(P_D), \vdash)|$, where d_S is defined by Equation (7). Furthermore, it is trivial that

$$d_1 \leq d_2 \text{ if and only if } \{\varphi \in \mathcal{L}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_1 \in \widehat{\varphi}\} \subseteq \{\varphi \in \mathcal{L}(P_D) \mid \widehat{\varphi} \in \mathcal{U}(D), d_2 \in \widehat{\varphi}\},$$

for all $d_1, d_2 \in D$. As a consequence, the function f is an order isomorphism from $(D, \leq)$ to $(|(\mathcal{L}(P_D), \vdash)|, \subseteq)$, which implies that $(D, \leq)$ is isomorphic to $(|(\mathcal{L}(P_D), \vdash)|, \subseteq)$. □

4. A Categorical Equivalence

In this section, we introduce morphisms between consistent disjunctive sequent calculi, called consequence relations, which induce a category equivalent to **SD**.

For convenience, we next write a consistent disjunctive sequent calculus with subscripts.

Definition 4.1. *Suppose $\mathbb{P} = (\mathcal{L}(P), \vdash_P)$ and $\mathbb{Q} = (\mathcal{L}(Q), \vdash_Q)$ are consistent disjunctive sequent calculi. A consequence relation from $\mathbb{P}$ to $\mathbb{Q}$, denoted by $\Theta : \mathbb{P} \to \mathbb{Q}$, is a binary relation $\Theta \subseteq \mathcal{C}(P) \times \mathcal{L}(Q)$ that satisfies*

- **(C1)** *if $(\nu, \psi) \in \Theta$, $\mu \in \mathcal{C}(P)$ and $\mu \vdash_P \nu$ is a valid sequent, then $(\mu, \psi) \in \Theta$.*
- **(C2)** *if $(\mu, \varphi_1), (\mu, \varphi_2) \in \Theta$ and $\varphi_1 \wedge \varphi_2 \vdash_Q \psi$ is a valid sequent, then $(\mu, \psi) \in \Theta$.*
- **(C3)** *if $(\mu, \psi) \in \Theta$ and $\psi \in \ell(Q)$, then $(\mu, \nu) \in \Theta$ and $\nu \vdash_Q \psi$ is a valid sequent for some $\nu \in \mathcal{C}(Q)$.*
- **(C4)** *$(\mu, \mathrm{F}) \notin \Theta$ for every $\mu \in \mathcal{C}(P)$.*

Intuitively, a consequence relation expresses how some formula in one consistent disjunctive sequent calculus entails formulae in another.

Proposition 4.1. *Consistent disjunctive sequent calculi with consequence relations form a category, written as* **CDC**.

Proof. Let $\Theta : \mathbb{P} \to \mathbb{Q}$ and $\Theta' : \mathbb{Q} \to \mathbb{R}$ be consequence relations. Define a relation $\Theta' \circ \Theta \subseteq \mathcal{C}(P) \times \mathcal{L}(R)$ by

$$(\mu, \varphi) \in \Theta' \circ \Theta \Leftrightarrow (\exists \nu \in \mathcal{C}(Q))((\mu, \nu) \in \Theta, (\nu, \varphi) \in \Theta'). \tag{8}$$

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Then it is easy but tedious to check that $\Theta' \circ \Theta$ satisfies the axioms for a consequence relation. The composition $\circ$ is clearly associative, the proof being identical to that for a traditional relation composition. Conditions (**C1**) and (**C2**) ensure that the identity morphism is given by a relation $\mathrm{id}_P \subseteq \mathcal{C}(P) \times \mathcal{L}(P)$, defined as follows:

$$(\mu, \varphi) \in \mathrm{id}_P \Leftrightarrow \mu \vdash_P \varphi \in \mathbf{T}(P). \tag{9}$$

We complete the proof. □

Given a consequence relation $\Theta : \mathbb{P} \to \mathbb{Q}$ and a subset X of $\mathcal{L}(P)$, set

$$\Theta[X] = \{\varphi \in \mathcal{N}(Q) \mid (\exists \mu \in X \cap \mathcal{C}(P))(\mu, \varphi) \in \Theta\}. \tag{10}$$

Proposition 4.2. *Let $\Theta : \mathbb{P} \to \mathbb{Q}$ be a consequence relation.*

(1) *If S is a logical state of $\mathbb{P}$, then $\Theta[S]$ is a logical state of $\mathbb{Q}$.*
(2) *If $\mu \in \mathcal{C}(P)$, then $\Theta[\{\mu\}] = \Theta[\{\mu\}[\vdash_P]]$. Moreover, $(\mu, \psi) \in \Theta$ if and only if there is some $\phi \in \Theta[\{\mu\}]$ such that $\psi \approx \phi$.*

Proof. (1) We first show $(\Theta[S])[\vdash_Q] \subseteq \Theta[S]$. To this end, let $\varphi \in (\Theta[S])[\vdash_Q]$. According to Equations (2) and (10), there are $\mu \in S \cap \mathcal{C}(P)$ and $\psi \in \mathcal{L}(Q)$ such that $(\mu, \psi) \in \Theta$ and the sequent $\psi \vdash_Q \varphi$ is valid. Using condition (**C2**), it follows that $(\mu, \varphi) \in \Theta$, and thus $\varphi \in \Theta[S]$.

Next, let $\dot{\bigvee}_{i \in I} \mu_i \in \Theta[S] \cap \ell(P)$. Then there is some $\mu \in S \cap \mathcal{C}(P)$ such that $(\mu, \dot{\bigvee}_{i \in I} \mu_i) \in \Theta$. Using condition (**C3**), we have some $\nu \in \mathcal{C}(Q)$ such that $(\mu, \nu) \in \Theta$ and the sequent $\nu \vdash_Q \dot{\bigvee}_{i \in I} \mu_i$ is valid. Thus there is some $i_0 \in I$ such that the sequent $\nu \vdash_Q \mu_{i_0}$ is valid. Using condition (**C2**) again, it follows that $(\mu, \mu_{i_0}) \in \Theta$, and hence $\mu_{i_0} \in \Theta[S]$.

Finally, let $\mu, \nu \in \Theta[S] \cap \mathcal{C}(Q)$. By Equation (10), there are $\mu_1, \nu_1 \in S \cap \mathcal{C}(P)$ such that $(\mu_1, \mu), (\nu_1, \nu) \in \Theta$. Since $\mu_1, \nu_1 \in S$, it follows that $\mu_0 \vdash_P \mu_1$ and $\mu_0 \vdash_P \nu_1$ are all valid sequents for some $\mu_0 \in S \cap \mathcal{C}(P)$. Then $(\mu_0, \mu), (\mu_0, \nu) \in \Theta$, and hence $(\mu_0, \mu \wedge \nu) \in \Theta$. With conditions (**C2**) and (**C4**), it is easy to see that the sequent $\mu \wedge \nu$ is not logically equivalent to F. So, $\mu \wedge \nu \in \Theta[S] \cap \mathcal{C}(Q)$, as we required.

(2) For every $\mu \in \mathcal{C}(P)$, since $\mu \vdash_P \mu$ is a valid sequent, it follows that $\{\mu\} \subseteq \{\mu\}[\vdash_P]$. Then by Equation (10), it is clear that $\Theta[\{\mu\}] \subseteq \Theta[\{\mu\}[\vdash_P]]$. Conversely, let $\varphi \in \Theta[\{\mu\}[\vdash_P]]$. Then there is some $\nu \in \{\mu\}[\vdash_P] \cap \mathcal{C}(P)$ such that $(\nu, \varphi) \in \Theta$. Note that $\nu \in \{\mu\}[\vdash_P]$, it follows that $\mu \vdash_P \nu$ is a valid sequent. By condition (**C1**), we have $(\mu, \varphi) \in \Theta$. This implies that $\varphi \in \Theta[\{\mu\}]$, so $\{\mu\}[\vdash_P] \cap \mathcal{C}(P) \subseteq \Theta[\{\mu\}]$.

If $(\mu, \psi) \in \Theta$, then ψ is not logically equivalent to F. This shows that $\psi \approx \phi$ for some $\phi \in \mathcal{N}(Q)$. Thus $(\mu, \phi) \in \Theta$ by condition (**C2**), and hence $\phi \in \Theta[\{\mu\}]$. Conversely, if $\phi \in \Theta[\{\mu\}]$ and $\psi \approx \phi$, then clearly $(\mu, \psi) \in \Theta$ by Equation (10) and condition (**C2**). □

A correspondence between consequence relations from $\mathbb{P}$ to $\mathbb{Q}$ and Scott-continuous functions from $|\mathbb{P}|$ to $|\mathbb{Q}|$ is shown by the following theorem.

Theorem 4.1. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are consistent disjunctive sequent calculi.*

(1) *If $\Theta : \mathbb{P} \to \mathbb{Q}$ is a consequence relation, then the function $f_\Theta : |\mathbb{P}| \to |\mathbb{Q}|$ defined by*

$$f_\Theta(S) = \Theta[S] \tag{11}$$

is Scott-continuous.

(2) *If $f : |\mathbb{P}| \to |\mathbb{Q}|$ is a Scott-continuous function, then the relation $\Theta_f \subseteq \mathcal{C}(P) \times \mathcal{L}(Q)$ defined by*

$$(\mu, \psi) \in \Theta_f \Leftrightarrow \psi \approx \varphi \text{ for some } \varphi \in f(\{\mu\}[\vdash_P]) \tag{12}$$

is a consequence relation from $\mathbb{P}$ to $\mathbb{Q}$.

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(3) *Moreover,* $\Theta_{f_\Theta} = \Theta$ *and* $f_{\Theta_f} = f$.

Proof. (1) The function f_Θ is well defined by part (1) of Proposition 4.2, and the function f_Θ is clearly monotonic by Equation (10).

Let $\{S_i \mid i \in I\}$ be a directed subset of $|\mathbb{P}|$. Then $\bigcup_{i\in I} S_i \in |\mathbb{P}|$ by part (3) of Proposition 3.2. It is clear that $\bigcup_{i\in I} f_\Theta(S_i) \subseteq f_\Theta(\bigcup_{i\in I} S_i)$, since the function f_Θ is monotonic. To prove the function f_Θ is Scott-continuous, it suffices to show that $f_\Theta(\bigcup_{i\in I} S_i) \subseteq \bigcup_{i\in I} f_\Theta(S_i)$. If $\varphi \in f_\Theta(\bigcup_{i\in I} S_i) = \Theta[\bigcup_{i\in I} S_i]$, then there exists some $\mu \in \bigcup_{i\in I} S_i \cap \mathcal{C}(P)$ such that $(\mu, \varphi) \in \Theta$. From $\mu \in \bigcup_{i\in I} S_i$, it follows that $\mu \in S_{i_0}$ for some $i_0 \in I$. Thus $\varphi \in f_\Theta(S_{i_0})$, and therefore, $f_\Theta(\bigcup_{i\in I} S_i) \subseteq \bigcup_{i\in I} f_\Theta(S_i)$.

(2) We need to show that the relation Θ_f satisfies conditions (**C1**–**C4**).

For condition (**C1**), if $\nu \in \mathcal{C}(P)$, $(\nu, \psi) \in \Theta_f$ and the sequent $\mu \vdash_P \nu$ is valid, then $\psi \approx \varphi$ for some $\varphi \in f(\{\nu\}[\vdash_P])$. Since $\mu \vdash_P \nu$ is a valid sequent, it follows that $\{\nu\}[\vdash_P] \subseteq \{\mu\}[\vdash_P]$. Note that f is monotone, we have $\varphi \in f(\{\mu\}[\vdash_P])$. This implies that $(\mu, \psi) \in \Theta_f$.

For condition (**C2**), if $(\mu, \varphi_1), (\mu, \varphi_2) \in \Theta_f$ and $\varphi_1 \wedge \varphi_2 \vdash_Q \psi$ is a valid sequent, then $\varphi_1 \approx \psi_1$ and $\varphi_2 \approx \psi_2$ for some $\psi_1, \psi_2 \in f(\{\mu\}[\vdash_P])$. Since $f(\{\mu\}[\vdash_P])$ is a logical state, there is some $\phi \in f(\{\mu\}[\vdash_P])$ such that $\psi_1 \wedge \psi_2 \vdash_Q \phi$ is valid and $\phi \approx \psi$. Therefore, $(\mu, \psi) \in \Theta_f$.

For condition (**C3**), if $(\mu, \psi) \in \Theta_f$ and $\varphi \in \ell(P)$, then it is easy to see that $\psi \in f(\{\mu\}[\vdash_P])$. Put $\varphi = \dot{\bigvee}_{i\in I} \mu_i$, where $\mu_i \in \mathcal{C}(Q)$ for every $i \in I$. Thus $\mu_i \vdash_Q \psi$ is a valid sequent for every $i \in I$. Note that $f(\{\mu\}[\vdash_P])$ is a logical state, it follows that $\mu_{i_0} \in f(\{\mu\}[\vdash_P])$ for some $i_0 \in I$. Let $\mu_{i_0} = \nu$. Then we get some $\nu \in \mathcal{C}(Q)$ such that $(\mu, \nu) \in \Theta$ and $\nu \vdash_Q \psi$ is valid.

For condition (**C4**), if $\mu \in \mathcal{C}(P)$, then $f(\{\mu\}[\vdash_P])$ is a logical state of $\mathbb{Q}$. Thus we have $f(\{\mu\}[\vdash_P]) \subseteq \mathcal{N}(P)$. This means that F is not an element of $f(\{\mu\}[\vdash_P])$, and so $(\mu, \mathrm{F}) \notin \Theta_f$.

(3) If $\mu \in \mathcal{C}(P)$ and $\varphi \in \mathcal{L}(Q)$, then

$$\begin{aligned}
(\mu, \varphi) \in \Theta_{f_\Theta} &\Leftrightarrow \varphi \approx \psi \text{ for some } \psi \in f_\Theta(\{\mu\}[\vdash_P]) \\
&\Leftrightarrow \varphi \approx \psi \text{ for some } \psi \in \Theta[\{\mu\}[\vdash_P]] \\
&\Leftrightarrow (\mu, \varphi) \in \Theta \text{ and } \mu \vdash_P \nu \in \mathbf{T}(P)) \text{ for some } \nu \in \mathcal{C}(Q) \\
&\Leftrightarrow (\mu, \varphi) \in \Theta.
\end{aligned}$$

This implies that $\Theta_{f_\Theta} = \Theta$.

If $S \in |\mathbb{P}|$, then we have

$$\begin{aligned}
f_{\Theta_f}(S) &= \Theta_f[S] \\
&= \{\varphi \in \mathcal{N}(Q) \mid (\exists \mu \in S \cap \mathcal{C}(P))(\mu, \varphi) \in \Theta_f\} \\
&= \{\varphi \in \mathcal{N}(Q) \mid (\exists \mu \in S \cap \mathcal{C}(P))\varphi \in f(\{\mu\}[\vdash_P])\} \\
&= \bigcup \{f(\{\mu\}[\vdash_P]) \mid \mu \in S \cap \mathcal{C}(P)\} \\
&= f(\bigcup \{\{\mu\}[\vdash_P] \mid \mu \in S \cap \mathcal{C}(P)\} \\
&= f(S).
\end{aligned}$$

This shows that $f_{\Theta_f} = f$. □

Theorem 4.2. **CDC** *and* **SD** *are categorically equivalent.*

Proof. Let $\mathcal{G} : \mathbf{CDC} \to \mathbf{SD}$ be the functor which maps every consistent disjunctive sequent calculi $\mathbb{P}$ to $(|\mathbb{P}|, \subseteq)$ and every consequence relation $\Theta : \mathbb{P} \to \mathbb{Q}$ to f_Θ, where f_Θ is defined by Equation (11).

That each Scott domain is isomorphic to the Scott domain of logical states of some consistent disjunctive sequent calculus is shown by Theorem 3.2. It remains to verify that the functor $\mathcal{G}$ is full and faithful.

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If $\mathbb{P}$ and $\mathbb{Q}$ are two consistent disjunctive sequent calculi and $f : |\mathbb{P}| \to |\mathbb{Q}|$ is a Scott-continuous function, then the relation Θ_f defined by Equation (12) is a consequence relation from $\mathbb{P}$ to $\mathbb{Q}$. By the definition of the functor $\mathcal{G}$ and Theorem 4.1, we have $\mathcal{G}(\Theta_f) = f_{\Theta_f} = f$. So the functor $\mathcal{G}$ is full.

Let $\Theta_1 : \mathbb{P} \to \mathbb{Q}$ and $\Theta_2 : \mathbb{P} \to \mathbb{Q}$ be two consequence relations. If $\mathcal{G}(\Theta_1) = \mathcal{G}(\Theta_1)$ then $f_{\Theta_1} = f_{\Theta_2}$, where f_{Θ_1} and f_{Θ_2} are defined in Theorem 4.1. For every $\mu \in \mathcal{C}(\mathbb{P})$, we have

$$\begin{aligned}
(\mu, \varphi) \in \Theta_1 &\Leftrightarrow \phi \approx \varphi \text{ for some } \phi \in \Theta_1[\{\mu\}] \\
&\Leftrightarrow \phi \approx \varphi \text{ for some } \phi \in \Theta_1[\{\mu\}[\vdash_P]] \\
&\Leftrightarrow \phi \approx \varphi \text{ for some } \phi \in f_{\Theta_1}\{\mu\}[\vdash_P] \\
&\Leftrightarrow \phi \approx \varphi \text{ for some } \phi \in f_{\Theta_2}\{\mu\}[\vdash_P] \\
&\Leftrightarrow \phi \approx \varphi \text{ for some } \phi \in \Theta_2[\{\mu\}[\vdash_P]] \\
&\Leftrightarrow (\mu, \varphi) \in \Theta_2
\end{aligned}$$

This implies that $\Theta_1 = \Theta_2$, and thus the functor $\mathcal{G}$ is faithful. □

As previously described, we have known what a consistent finitary disjunctive sequent calculus is and how it represents the Scott domain of its logical states. From a categorical point of view, the category **SD** is essentially the same as the category **CDC**.

In Wang and Li (2020*b*), another subclass of disjunctive propositional logics was introduced and studied, which are called finitary disjunctive sequent calculi. The only difference between disjunctive sequent calculi and finitary disjunctive sequent calculi is that all the rules of generating formulae and valid sequents are specific for the binary connective $\dot{\vee}$ rather than arbitrary disjunctive connective $\dot{\bigvee}$.

Definition 4.2. (Wang and Li, 2020*b*) A finitary disjunctive sequent calculus is a disjunctive propositional logic in which the rules (Disj), (L$\dot{\vee}$) and (R$\dot{\vee}$) are replaced, respectively, by the following rules:

$$(Disj^*)\ \frac{\phi_1, \phi_2 \in \mathcal{L}(P) \qquad \phi_1, \phi_2 \vdash F}{\phi_1 \dot{\vee} \phi_2 \in \mathcal{L}(P)}$$

$$(L\dot{\vee}^*)\ \frac{\Gamma, \phi_1 \vdash \theta \qquad \Gamma, \phi_2 \vdash \theta \qquad \phi_1, \phi_2 \vdash F}{\Gamma, \phi_1 \dot{\vee} \phi_2 \vdash \theta}$$

$$(R\dot{\vee}^*)\ \frac{\Gamma \vdash \phi_1 \qquad \phi_1, \phi_2 \vdash F}{\Gamma \vdash \phi_1 \dot{\vee} \phi_2}.$$

The notions of flat formulae, irreducible simple conjunctions, and flat formulae in a finitary disjunctive sequent calculus are almost identical to those in a disjunctive sequent calculus, the only change being the substitution of binary disjunctive connectives for arbitrary disjunctive connectives. Then we can define a consistent finitary disjunctive sequent calculus as same as a consistent disjunctive sequent calculus.

Definition 4.3. *A finitary disjunctive sequent calculus* $(\mathcal{L}(P), \vdash)$ *is said to be consistent if, every simple conjunction in it is irreducible.*

It is not difficult to see that the collection of consistent finitary disjunctive sequent calculi with consequence relations forms a category **CFDC**, which is a full subcategory of the category **CDC**. Similar to the process of representing the category of **SD** by the category **CDC**, we can show the following theorem.

Theorem 4.3. *The categories* **CFDC** *and* **SD** *are categorically equivalent.*

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So far, we have provided two logical representations for Scott domains in the framework of disjunctive propositional logic, and before that, we concentrated more on the case of consistent disjunctive sequent calculi. The case of consistent finitary disjunctive sequent calculi is slightly more simpler than that of disjunctive sequent calculi, because it only needs to deal with less formulae. For convenience in application, we turn our attention to consistent finitary disjunctive sequent calculi in the sequel.

5. Constructing Domains

In this section, we study how to construct the consistent finitary disjunctive sequent calculi that we need for modeling the semantics of programming languages, and then we examine how to solve some recursive domain equations.

5.1 A directed complete order

We now use the symbol $\mathcal{CFDC}$ to denote the collection of all consistent finitary disjunctive sequent calculi. We first consider a partial order on $\mathcal{CFDC}$, which captures an intuition, that of one consistent finitary disjunctive sequent calculus being a subsystem of another.

Definition 5.1. *Let $\mathbb{P} = (\mathcal{L}(P), \vdash_P)$ and $\mathbb{Q} = (\mathcal{L}(Q), \vdash_Q)$ be consistent finitary disjunctive sequent calculi. We say that $\mathbb{P}$ is a subsystem of $\mathbb{Q}$, symbols by $\mathbb{P} \trianglelefteq \mathbb{Q}$, provided*

- (**u1**) *if $p \in P$ is an atomic formula in $\mathbb{P}$, then p is also an atomic formula in $\mathbb{Q}$;*
- (**u2**) *if $\Gamma \vdash_P \varphi$ is an atomic disjointness assumption in $\mathbb{P}$, then $\Gamma \vdash_Q \varphi$ is also an atomic disjointness assumption in $\mathbb{Q}$;*
- (**u3**) *if $\Gamma \vdash_Q \varphi$ is a valid sequent in $\mathbb{Q}$, where $\Gamma \sqsubseteq \mathcal{L}(P)$ and $\varphi \in \mathcal{L}(P)$, then $\Gamma \vdash_P \varphi$ is a valid sequent in $\mathbb{P}$.*

If $\mathbb{P} \trianglelefteq \mathbb{Q}$, then it is easy to see that $\mathcal{L}(P) \subseteq \mathcal{L}(Q)$ and $\mathbf{T}(P) \subseteq \mathbf{T}(Q)$ by Definition 2.3. Thus the relation $\trianglelefteq$ on the collection $\mathcal{CFDC}$ is a partial order, and there is a least consistent finitary disjunctive sequent calculus $\mathbb{O}$, the unique one with the empty set $\emptyset$ as atomic formulae. Moreover, condition (**u3**) tells us that the relationship between $\mathbb{P}$'s formulae in $\mathbb{Q}$ is the same as in $\mathbb{P}$.

By the following theorem, we will see that the collection $\mathcal{CFDC}$ is directed complete with respect to $\trianglelefteq$. This is enough for our theory since in that case the standard theory of fixed points of continuous function is to be available.

Theorem 5.1. *All of consistent finitary disjunctive sequent calculi $\mathcal{CFDC}$ under the relation $\trianglelefteq$ is directed complete.*

Proof. Let $\wp = \{\mathbb{P}_i \mid \mathbb{P}_i = (\mathcal{L}(P_i), \vdash_{P_i}), i \in I\}$ be a directed subset of consistent finitary disjunctive sequent calculi with respect to the partial order $\trianglelefteq$. Set

(1) $P = \bigcup_i P_i$,
(2) $\mathcal{A}_P = \bigcup_i \mathcal{A}_{P_i}$,
(3) $\mathcal{L}(P) = \bigcup_i \mathcal{L}(P_i)$,
(4) $\mathbf{T}(P) = \bigcup_i \mathbf{T}(P_i)$.

The set $\mathbf{T}(P)$ determines a relation $\vdash$ by Equation (1). We first prove that $\mathbb{P} = (\mathcal{L}(P), \vdash)$ is a finitary disjunctive sequent calculus with a disjunctive basis $(P, \mathcal{A}_P)$. It is easy to verify the rules for a finitary disjunctive sequent calculus: consider the rule (*Disj**) for illustration.

If $\phi_1, \phi_2 \in \mathcal{L}(P)$ such that $\phi_1, \phi_2 \vdash \mathrm{F}$ is a valid sequent, then $\phi_1 \in \mathcal{L}(P_1)$, $\phi_2 \in \mathcal{L}(P_2)$ and the sequent $\phi_1, \phi_2 \vdash_{P_3} \mathrm{F}$ is valid for some $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3 \in \wp$. Since $\wp$ is directed, there is a $\mathbb{P}_4 \in \wp$ such

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that $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$ are all subsystems of $\mathbb{P}_4$. Thus $\phi_1 \in \mathcal{L}(P_4)$, $\phi_2 \in \mathcal{L}(P_4)$ and $\phi_1, \phi_2 \vdash_{P_4} \mathrm{F}$ is a valid sequent in $\mathbb{P}_4$. Using the rule $(Disj^*)$ in the consistent finitary disjunctive sequent calculus $\mathbb{P}_4$, we have $\phi_1 \dot{\vee} \phi_2 \in \mathcal{L}(P_4)$, and thus $\phi_1 \dot{\vee} \phi_2 \in \mathcal{L}(P)$, as we required.

Next, we show that the finitary disjunctive sequent calculus $\mathbb{P}$ is consistent. Assume that μ is a simple conjunction and $\mu \vdash \phi_1 \dot{\vee} \phi_2$ is a valid sequent in $\mathbb{P}$. Note that $\wp$ is directed, it follows that $\mu \vdash_{P_i} \phi_1 \dot{\vee} \phi_2$ is a valid sequent in $\mathbb{P}_i$ for some $i \in I$. Since $\mathbb{P}_i$ is consistent, either $\mu \vdash \phi_1$ or $\mu \vdash \phi_2$ is a valid sequent in $\mathbb{P}_i$. Thus one of the sequents $\mu \vdash \phi_1$ and $\mu \vdash \phi_2$ is valid in $\mathbb{P}$.

So the pair $(\mathcal{L}(P), \vdash)$ is a well-defined consistent finitary disjunctive sequent calculus. The remainder is to show that it is the least upper bound of $\wp$.

For a given $i \in I$, we claim that $\mathbb{P}_i = (\mathcal{L}(P_i), \vdash_{P_i})$ is a subsystem of $\mathbb{P} = (\mathcal{L}(P), \vdash)$.

In fact, it is trivial that $P_i \subseteq P$ and $\mathcal{A}_{P_i} \subseteq \mathcal{A}_P$ since $P = \bigcup_i P_i$, and $\mathcal{A}_P = \bigcup_i \mathcal{A}_{P_i}$. That is, conditions (**u1**) and (**u2**) are satisfied. It only remains to show condition (**u3**). Let $\Gamma \vdash \varphi$ be a valid sequent in $(\mathcal{L}(P), \vdash)$, where $\Gamma \sqsubseteq \mathcal{L}(P_i)$ and $\varphi \in \mathcal{L}(P_i)$. Since $\mathbf{T}(P) = \bigcup_i \mathbf{T}(P_i)$, there is some $j \in I$ such that the sequent $\Gamma \vdash_{P_j} \varphi$ is valid in $\mathbb{P}_j$. The directness of $\wp$ implies that $\mathbb{P}_i \trianglelefteq \mathbb{P}_k$ and $\mathbb{P}_j \trianglelefteq \mathbb{P}_k$ for some $k \in I$. Using condition (**u2**) for $\mathbb{P}_j \trianglelefteq \mathbb{P}_k$, it follows that the sequent $\Gamma \vdash_{P_k} \varphi$ is valid in $\mathbb{P}_k$. Thus the sequent $\Gamma \vdash_{P_i} \varphi$ in valid in $\mathbb{P}_i$ using condition (**u3**) for $\mathbb{P}_i \trianglelefteq \mathbb{P}_k$.

This complete the proof that $\mathbb{P}_i \trianglelefteq \mathbb{P}$ for every $i \in I$, which means that $\mathbb{P}$ is an upper bound for $\wp$. If $\mathbb{Q} = (\mathcal{L}(Q), \vdash_Q)$ is another then

$$\bigcup_i P_i = P \subseteq Q, \bigcup_i \mathcal{L}(P_i) = \mathcal{L}(P) \subseteq \mathcal{L}(Q), \text{ and } \bigcup_i \mathbf{T}(P_i) = \mathbf{T}(P) \subseteq \mathbf{T}(Q).$$

Suppose that $\Gamma \vdash_Q \varphi$ is a valid sequent in $\mathbb{Q}$, where $\Gamma \sqsubseteq \mathcal{L}(P)$ and $\varphi \in \mathcal{L}(P)$. Since $\wp$ is directed, there is some $j \in I$ such that $\Gamma \sqsubseteq \mathcal{L}(P_j)$ and $\varphi \in \mathcal{L}(P_j)$. From $\mathbb{P}_j \trianglelefteq \mathbb{Q}$ and $\Gamma \vdash_Q \varphi$ is a valid sequent in $\mathbb{Q}$, it follows that $\Gamma \vdash_{P_j} \varphi$ is a valid sequent in $\mathbb{P}_j$, and hence $\Gamma \vdash_P \varphi$ is a valid sequent in $\mathbb{P}_j$. This establishes that $\mathbb{P} \trianglelefteq \mathbb{Q}$, which means that $\mathbb{P}$ is the least upper bound of $\wp$. Thus the partial order of consistent finitary disjunctive sequent calculi is directed complete. □

The subsystem relation $\trianglelefteq$ can be extended to $(n+1)$-tuples of consistent finitary disjunctive sequent calculi. Let n be a natural number. We denote by $\mathcal{CFDC}^{n+1}$ all the $(n+1)$-tuples $(\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n)$ of consistent finitary disjunctive sequent calculi. For $(\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n)$ and $(\mathbb{Q}_0, \mathbb{Q}_1, \cdots, \mathbb{Q}_n)$ in $\mathcal{CFDC}^{n+1}$, we define

$$(\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n) \trianglelefteq (\mathbb{Q}_0, \mathbb{Q}_1, \cdots, \mathbb{Q}_n)$$

if and only if $\mathbb{P}_i \trianglelefteq \mathbb{Q}_i$ for all $0 \le i \le n$. Then the relation $\trianglelefteq$ is a partial order on $\mathcal{CFDC}^n$ with a least element $(\mathbb{O}, \mathbb{O}, \cdots, \mathbb{O})$.

We write $\overrightarrow{\mathbb{P}}$ for a $(n+1)$-tuple $(\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_n)$. The proof of the following proposition is analogous to that of Theorem 5.1 and will be omitted.

Proposition 5.1. *The collection $\mathcal{CFDC}^{n+1}$ under the relation $\trianglelefteq$ is also directed complete, where the least upper bound of a directed set of some $(n+1)$-tuples consistent finitary disjunctive sequent calculi is the $(n+1)$-tuples of consistent finitary disjunctive sequent calculi consisting of the least upper bounds on every component.*

We now define the notion of continuous operations on consistent finitary disjunctive sequent calculi as same as that on usual posets.

Definition 5.2. *Let $\mathfrak{F} : \mathcal{CFDC}^{n+1} \to \mathcal{CFDC}^{m+1}$ be an operation on consistent finitary disjunctive sequent calculi.*

(1) *The operation $\mathfrak{F}$ is said to be monotonic if $\mathfrak{F}(\overrightarrow{\mathbb{P}}) \trianglelefteq \mathfrak{F}(\overrightarrow{\mathbb{Q}})$ for all $\overrightarrow{\mathbb{P}} \trianglelefteq \overrightarrow{\mathbb{Q}} \in \mathcal{CFDC}^{n+1}$.*

(2) *The operation* $\mathfrak{F}$ *is said to be continuous if it is monotonic and for all directed subclass* $\{\overrightarrow{\mathbb{P}_i} \mid i \in I\}$ *of* $\mathcal{CFDC}^{n+1}$,

$$\bigsqcup_{i\in I}\mathfrak{F}(\overrightarrow{\mathbb{P}_i}) = \mathfrak{F}(\bigsqcup_{i\in I}\overrightarrow{\mathbb{P}_i}).$$

Lemma 5.1. *Let* $\mathfrak{F}$ *and* $\mathfrak{G}$ *be respectively* $(n+1)$ *to* $(m+1)$ *and* $(m+1)$ *to* $(k+1)$ *operations on consistent finitary disjunctive sequent calculi.*

(1) *If* $\mathfrak{F}$ *and* $\mathfrak{G}$ *are both continuous, then so is their composition* $\mathfrak{G} \circ \mathfrak{F}$.
(2) $\mathfrak{F}(\mathbb{P}_0, \mathbb{P}_2, \cdots, \mathbb{P}_n)$ *is continuous if and only if it is continuous in each component of its argument separately.*
(3) $\mathfrak{F}(\overrightarrow{\mathbb{P}}) = (\mathfrak{F}_0(\overrightarrow{\mathbb{P}}), \mathfrak{F}_1(\overrightarrow{\mathbb{P}}), \cdots, \mathfrak{F}_n(\overrightarrow{\mathbb{P}}))$ *is continuous if and only if each of its component operators* $\mathfrak{F}_i$ *is.*

Proof. The proof is essentially the same as those for Scott-continuous functions on posets. □

We will see that all the domain constructions we defined later are continuous. Thus we know that all compositions of them are continuous.

5.2 Constructions

In this subsection, we study how some basic domain constructors on consistent finitary disjunctive sequent calculi are defined, whose effects on the underlying Scott domains of logical states are what we want.

One of basic but useful examples of domain constructors is lifting.

Definition 5.3. *Let* $\mathbb{P} = (\mathcal{L}(P), \vdash_P)$ *be a consistent finitary disjunctive sequent calculus and* $\flat \notin \mathcal{L}(P)$. *Set* $P^\flat = P \cup \{\flat\}$ *and*

$$\mathcal{A}_{P^\flat} = \{p_1, p_2, \cdots, p_n \vdash_{P^\flat} \mathrm{F} \mid p_1, p_2, \cdots, p_n \vdash \mathrm{F} \in \mathcal{A}_P\} \cup \{p \vdash_{P^\flat} \flat \mid p \in P\}.$$

Then the finitary disjunctive sequent calculus $\mathbb{P}^\flat = (\mathcal{L}(P^\flat), \vdash_{P^\flat})$ *with* $(P^\flat, \mathcal{A}_{P^\flat})$ *as disjunctive basis is said to be a lifting of* $\mathbb{P}$ *if the atomic formula* $\flat$ *is irreducible.*

Lemma 5.2. *Suppose* $\mathbb{P}^\flat$ *is the lifting of a consistent finitary disjunctive sequent calculus* $\mathbb{P}$. *Then* p *is logically equivalent to* $p \wedge \flat$ *for every* $p \in P$.

Proof. Let $p \in P$. Since the sequent $p \vdash_{P^\flat} \flat$ is valid, by the rules (Id) and (R $\wedge$), the sequent $p \vdash_{P^\flat} p \wedge \flat$ is valid. By the rules (Id), (Lwk) and (L $\wedge$), the sequent $p \wedge \flat \vdash_{P^\flat} p$ is also valid. So, p is logically equivalent to $p \wedge \flat$. □

Theorem 5.2. *If a finitary disjunctive sequent calculus* $\mathbb{P}$ *is consistent then so is its lifting* $\mathbb{P}^\flat$, *and furthermore,*

$$|\mathbb{P}^\flat| = \{\{\flat\}[\vdash_{P^\flat}]\} \cup \{S[\vdash_{P^\flat}] \mid S \in |\mathbb{P}|\}. \tag{13}$$

Proof. To show the finitary disjunctive sequent calculus $\mathbb{P}^\flat$ is consistent, we have to check that μ is irreducible for every simple conjunctions μ in $\mathbb{P}^\flat$.

If μ is a simple conjunction built up only by $\flat$, then $\mu \approx \flat$, and Thus μ is irreducible. For the case that μ is a simple conjunction with some $p \in P$ as a component, let $\mu \vdash_{P^\flat} \dot{\bigvee}_{i\in I}\mu_i$ be a valid sequent in $\mathbb{P}^\flat$, where $\dot{\bigvee}_{i\in I}\mu_i$ is a flat formula in $\mathbb{P}^\flat$. To verify μ is irreducible, we look for some $i_0 \in I$ such that $\mu \vdash_{P^\flat} \mu_{i_0}$ is valid by considering the following two cases.

(1) If $\dot{\bigvee}_{i\in I}\mu_i$ is built up only by $\flat$, then it is of the form $\flat \wedge \flat \wedge \cdots \wedge \flat$ since $\flat \dot{\vee} \flat$ is not a well-defined formula in $\mathbb{P}^\flat$. That is, the index set I is a singleton $\{i\}$. Thus $\mu \vdash_{P^\flat} \mu_i$ is valid.

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(2) If there is some $p \in P$ occurred in $\dot{\bigvee}_{i\in I}\mu_i$, then by Lemma 5.2, all occurrences of $\flat$ can be dropped from $\dot{\bigvee}_{i\in I}\mu_i$ and the resulting formula, denoted by $\dot{\bigvee}_{i\in I}\underline{\mu_i}$, is logically equivalent to $\dot{\bigvee}_{i\in I}\mu_i$. Similarly, $\underline{\mu}$ is logically equivalent to μ. This implies that the sequent $\underline{\mu} \vdash_P \dot{\bigvee}_{i\in I}\underline{\mu_i}$ is valid in $\mathbb{P}$. Note that $\underline{\mu}$ is an irreducible simple conjunction in $\mathbb{P}$, there is some $i_0 \in I$ such that $\underline{\mu} \vdash_P \underline{\mu_{i_0}}$ is valid. Thus $\underline{\mu} \vdash_{P^\flat} \mu_{i_0}$ is valid in $\mathbb{P}^\flat$, because $\mu \approx \underline{\mu}$ and $\dot{\bigvee}_{i\in I}\mu_i \approx \dot{\bigvee}_{i\in I}\underline{\mu_i}$.

For the second part, we first show that $\{\{p^\flat\}[\vdash_{P^\flat}]\} \cup \{S[\vdash_\flat] \mid S \in |\mathbb{P}|\} \subseteq |\mathbb{P}^\flat|$. By part (2) of Proposition 3.2, we have $\{\flat\}[\vdash_{P^\flat}] \in |\mathbb{P}^\flat|$. So it suffices to show that $S[\vdash_{P^\flat}] \in |\mathbb{P}^\flat|$ for every $S \in |\mathbb{P}|$.

Assume that $\varphi \in (S[\vdash_{P^\flat}])[\vdash_{P^\flat}]$. By Equation (2), there is some $\Gamma \sqsubseteq S[\vdash_{P^\flat}]$ such that $\Gamma \vdash_{P^\flat} \varphi$ is a valid sequent in $\mathbb{P}^\flat$. Then by the remark below Equation (2), we have some $\Delta \sqsubseteq S$ such that $\Delta \vdash_P \bigwedge \Gamma$ is a valid sequent in $\mathbb{P}$. Thus the sequent $\Delta \vdash_{P^\flat} \bigwedge \Gamma$ is valid in $\mathbb{P}^\flat$. By part (1) of Proposition 2.1 and the rule (Cut), the sequent $\Delta \vdash_{P^\flat} \varphi$ is valid in $\mathbb{P}^\flat$, and hence $(S[\vdash_{P^\flat}])[\vdash_{P^\flat}] \subseteq S[\vdash_{P^\flat}]$. So $S[\vdash_{P^\flat}]$ satisfies condition (**S1**).

For condition (**S2**), we have the following implications:

$$\begin{aligned}
&\dot{\bigvee}_{i\in I}\mu_i \in S[\vdash_{P^\flat}] \cap \ell(P)\\
&\Rightarrow \Gamma \vdash_{P^\flat} \dot{\bigvee}_{i\in I}\mu_i \in \mathbf{T}(P^\flat) \text{ for some } \Gamma \sqsubseteq S \cap \ell(P)\\
&\Rightarrow \dot{\bigvee}_{j\in J}\mu_j \vdash_{P^\flat} \dot{\bigvee}_{i\in I}\mu_i \in \mathbf{T}(P^\flat) \text{ for some } \dot{\bigvee}_{j\in J}\mu_j \in S \cap \ell(P)\\
&\Rightarrow \dot{\bigvee}_{j\in J}\mu_j \vdash_{P^\flat} \dot{\bigvee}_{i\in I}\mu_i \in \mathbf{T}(P^\flat) \text{ and } \mu_{j_0} \in S \cap \mathcal{C}(P) \text{ for some } j_0 \in J\\
&\Rightarrow \mu_{j_0} \vdash_{P^\flat} \dot{\bigvee}_{i\in I}\mu_i \in \mathbf{T}(P^\flat) \text{ and } \mu_{j_0} \in S \cap \mathcal{C}(P)\\
&\Rightarrow \mu_{j_0} \vdash_{P^\flat} \mu_{i_0} \in \mathbf{T}(P^\flat) \text{ for some } i_0 \in I\\
&\Rightarrow \mu_{i_0} \in S[\vdash_{P^\flat}].
\end{aligned}$$

To check condition (**S3**), let $\mu, \nu \in S[\vdash_{P^\flat}] \cap \mathcal{C}(P)$. Then there is some $\Gamma \sqsubseteq S$ such that $\Gamma \vdash_{P^\flat} \mu \wedge \nu$ is a valid sequent. By part (4) of Proposition 3.2, we have some $\varphi \in S$ such that $\varphi \approx \bigwedge \Gamma$. Then $\varphi \vdash_{P^\flat} \mu \wedge \nu$ is a valid sequent. Since φ is not logically equivalent to the constant F, the formula $\mu \wedge \nu$ is a flat formula, and thus $\mu \wedge \nu \in S[\vdash] \subseteq S$.

Next we show $|\mathbb{P}^\flat| \subseteq \{\{\flat\}[\vdash_{P^\flat}]\} \cup \{S[\vdash_{P^\flat}] \mid S \in |\mathbb{P}|\}$. Assume that $W \neq \{\flat\}[\vdash_{P^\flat}]$ is a logical state of $\mathbb{P}^\flat$. Set

$$S_W = \left\{\dot{\bigvee}_{i\in I}\underline{\mu_i} \in \ell(P) \mid \dot{\bigvee}_{i\in I}\mu_i \in W\right\} \cup \{\mathrm{T}\}.$$

Since $\dot{\bigvee}_{i\in I}\underline{\mu_i}$ is logically equivalent to $\dot{\bigvee}_{i\in I}\mu_i$, we have $W = W[\vdash_{P^\flat}] = S_W[\vdash_{P^\flat}]$. So we need only to show $S_W \in |\mathbb{P}|$.

To prove S_W satisfies condition (**S1**), let $\varphi \in S_W[\vdash_P]$. If $\varphi = \mathrm{T}$, then clearly $\varphi \in S_W$. If $\varphi \neq \mathrm{T}$, then $\varphi \in S_W[\vdash_P] \cap \ell(P)$. Setting $\varphi = \dot{\bigvee}_{i\in I}\mu_i$, by Equation (2), there are some $\dot{\bigvee}_{i\in I}\underline{\mu_{1_i}}, \dot{\bigvee}_{i\in I}\underline{\mu_{2_i}}, \cdots, \dot{\bigvee}_{i\in I}\underline{\mu_{n_i}} \in S_W$ such that $\dot{\bigvee}_{i\in I}\underline{\mu_{1_i}}, \dot{\bigvee}_{i\in I}\underline{\mu_{2_i}}, \cdots, \dot{\bigvee}_{i\in I}\underline{\mu_{n_i}} \vdash_P \dot{\bigvee}_{i\in I}\mu_i$ is a valid sequent in $\mathbb{P}$. Because $\dot{\bigvee}_{i\in I}\underline{\mu_{j_i}} \approx \dot{\bigvee}_{i\in I}\mu_{j_i}$ in $\mathbb{P}^\flat$ for all $j = 1, 2, \cdots, n$, the sequent $\dot{\bigvee}_{i\in I}\mu_{1_i}, \dot{\bigvee}_{i\in I}\mu_{2_i}, \cdots, \dot{\bigvee}_{i\in I}\mu_{n_i} \vdash_{P^\flat} \dot{\bigvee}_{i\in I}\mu_i$ is valid in $\mathbb{P}^\flat$. This implies $\dot{\bigvee}_{i\in I}\mu_i \in W$, since $W[\vdash_{P^\flat}] \subseteq W$ and $\{\dot{\bigvee}_{i\in I}\mu_{1_i}, \dot{\bigvee}_{i\in I}\mu_{2_i}, \cdots, \dot{\bigvee}_{i\in I}\mu_{n_i}\} \sqsubseteq W$. Note that $\dot{\bigvee}_{i\in I}\underline{\mu_i}$ coincides with $\dot{\bigvee}_{i\in I}\mu_i$, it follows that $\dot{\bigvee}_{i\in I}\mu_i \in S_W$. Thus $S_W[\vdash_P] \subseteq S_W$.

Let $\dot{\bigvee}_{j\in J}\underline{\mu_j} \in S_W \cap \ell(P)$. Then $\dot{\bigvee}_{j\in J}\mu_j \in W$. This implies that there is some $j_0 \in J$ such that $\mu_{j_0} \in W$, and hence $\underline{\mu_{j_0}} \in S_W$. Condition (**S2**) follows.

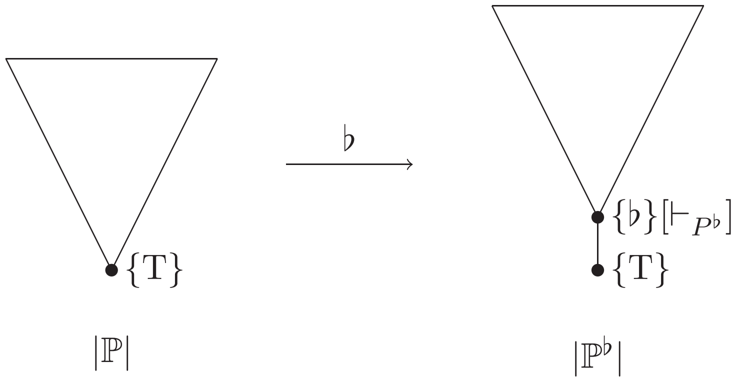

Figure 1. The relationship between $|\mathbb{P}^\flat|$ and $|\mathbb{P}|$.

For condition (**S3**), let $\underline{\mu}, \underline{\nu} \in S_W \cap \mathcal{C}(P)$, where $\mu, \nu \in W$. Since W is a logical state, we have $\mu \wedge \nu \in W$, which means that $\underline{\mu} \wedge \underline{\nu} \in S_W$. □

It is easy to see that $\mathbb{P}$ is a subsystem of its lifting $\mathbb{P}^\flat$. From Equation (13), we can see that the elements of $|\mathbb{P}^\flat|$ are correspondence with those of $|\mathbb{P}|$ plus an extra element. The following proposition says that they have more close relationship.

Proposition 5.2. *Let $\mathbb{P}^\flat$ be the lifting of a consistent finitary disjunctive sequent calculus $\mathbb{P}$.*

(1) *For every $S \in |\mathbb{P}|$, if $S \neq \{\mathrm{T}\}$ then $\{\flat\}[\vdash_{P^\flat}]$ is a proper subset of $S[\vdash_{P^\flat}]$.*
(2) *If $S_1 \subseteq S_2 \in |\mathbb{P}|$ then $S_1[\vdash_{P^\flat}] \subseteq S_2[\vdash_{P^\flat}]$. Moreover, $S_1[\vdash_{P^\flat}] = S_2[\vdash_{P^\flat}]$ if and only if $S_1 = S_2$.*

Proof. (1) Assume that $S \neq \{\mathrm{T}\}$ is a logical state of $\mathbb{P}$. Then there is a flat formula $\dot{\bigvee}_{i\in I}\mu_i$ belongs to S, where μ_i are simple conjunctions in $\mathbb{P}$ for all $i \in I$. By condition (**S2**), there is some $i_0 \in I$ such that $\mu_{i_0} \in S$. Since $p \vdash_{P^\flat} \flat$ is a valid sequent for every $p \in P$, we can easy to see that the sequent $\mu_{i_0} \vdash_{P^\flat} \flat$ is valid. Thus $\flat \in S[\vdash_{P^\flat}]$, which implies that $\{\flat\}[\vdash_{P^\flat}] \subseteq S[\vdash_{P^\flat}]$.

For the above simple conjunction μ_{i_0}, it is built up only by some atomic formulae in P. Note that $\flat \vdash_{P^\flat} p$ is not a valid sequent in $\mathbb{P}^\flat$ for every $p \in P$, it follows that the sequent $\flat \vdash_{P^\flat} \mu_{i_0}$ is not valid. So $\mu_{i_0} \notin \{\flat\}[\vdash_{P^\flat}]$, which implies that $\{\flat\}[\vdash_{P^\flat}] \neq S[\vdash_{P^\flat}]$.

(2) Let $S_1, S_2 \in |\mathbb{P}|$. By Equation (2), it is clear that $S_1[\vdash_{P^\flat}] \subseteq S_2[\vdash_{P^\flat}]$ whenever $S_1 \subseteq S_2$. Thus for the case of $S_1 = S_2$, we have $S_1[\vdash_{P^\flat}] = S_2[\vdash_{P^\flat}]$. For the converse implication, assume that $S_1[\vdash_{P^\flat}] = S_2[\vdash_{P^\flat}]$. We now show that $S_1 = S_2$. Suppose not, without lose of generality, we assume that there is some flat formula $\varphi \in S_1$ but $\varphi \notin S_2$. Then $\varphi \in S_1 \subseteq S_1[\vdash_{P^\flat}] = S_2[\vdash_{P^\flat}]$, and hence there is some $\Gamma \sqsubseteq S_2$ such that $\Gamma \vdash_{P^\flat} \varphi$ is a valid sequent in $\mathbb{P}^\flat$. This implies that the sequent $\Gamma \vdash_P \varphi$ is valid in $\mathbb{P}$. Thus $\varphi \in S_2[\vdash_P] \subseteq S_2$, which is a contradiction. □

The lifting construction we defined can induce the usual lifting construction on Scott domains. Combining Equation (13) with Proposition 5.2, we know that $|\mathbb{P}^\flat|$ has essentially the same structures as $|\mathbb{P}|$ except that $|\mathbb{P}^\flat|$ has been added a now element, see Figure 1.

There are two classical symmetrical approaches to summing two pointed posets A and B so that the result poset is pointed. The first way is to identity the two bottoms of A and B and keeps all other elements as they are. This is often called a coalesced sum and denoted as $A \oplus B$.

Now we construct the coalesced sum of two consistent finitary disjunctive sequent calculi.

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Definition 5.4. *Let* $\mathbb{P} = (\mathcal{L}(P), \vdash_P)$ *and* $\mathbb{Q} = (\mathcal{L}(Q), \vdash_Q)$ *be consistent finitary disjunctive sequent calculi, where* $\mathcal{L}(P) \cap \mathcal{L}(Q) = \{\mathrm{T}, \mathrm{F}\}$. *Set* $P \oplus Q = P \cup Q$ *and*

$$\begin{aligned}\mathcal{A}_{P\oplus Q} =&\{p_1, p_2, \cdots, p_n \vdash_\oplus \mathrm{F} \mid p_1, p_2, \cdots, p_n \vdash_P \mathrm{F} \in \mathcal{A}_P\}\\ &\cup \{q_1, q_2, \cdots, q_m \vdash_\oplus \mathrm{F} \mid q_1, q_2, \cdots, q_m \vdash_Q \mathrm{F} \in \mathcal{A}_Q\}\\ &\cup \{p, q \vdash_\oplus \mathrm{F} \mid p \in P, q \in Q\}.\end{aligned} \tag{14}$$

Then the finitary disjunctive sequent calculus $\mathbb{P} \oplus \mathbb{Q} = (\mathcal{L}(P \oplus Q), \vdash_\oplus)$ *with* $(P \oplus Q, \mathcal{A}_{P\oplus Q})$ *as disjunctive basis is said to be the coalesced sum of* $\mathbb{P}$ *and* $\mathbb{Q}$ *if* μ *is irreducible in* $\mathbb{P} \oplus \mathbb{Q}$ *for every* $\mu \in \mathcal{C}(P) \cup \mathcal{C}(Q)$.

Lemma 5.3. *Let* $\mathbb{P} \oplus \mathbb{Q}$ *be the coalesced sum of* $\mathbb{P}$ *and* $\mathbb{Q}$.

(1) $\mu \in \mathcal{C}(P \oplus Q)$ *if and only if* $\mu \in \mathcal{C}(P)$ *or* $\mu \in \mathcal{C}(Q)$.
(2) *If* $\Gamma \vdash_P \varphi$ *is a valid sequent in* $\mathbb{P}$, *then* $\Gamma \vdash_\oplus \varphi$ *is a valid sequent in* $\mathbb{P} \oplus \mathbb{Q}$.

Proof. (1) Clearly, $\mu \in \mathcal{C}(P) \cup \mathcal{C}(Q)$ implies $\mu \in \mathcal{C}(P \oplus Q)$. Conversely, assume that $\mu \in \mathcal{C}(P \oplus Q)$. If $\mu \notin \mathcal{C}(P) \cup \mathcal{C}(Q)$, then there are some $p \in P$ and $q \in Q$ occurred in μ. Since $p, q \vdash_\oplus \mathrm{F}$ is a valid sequent in $\mathbb{P} \oplus \mathbb{Q}$, it follows that $\mu \vdash_\oplus \mathrm{F}$ is also a valid sequent in $\mathbb{P} \oplus \mathbb{Q}$ by the rule (Lwk). This contradicts to the fact that μ is a satisfiable formula.

(2) The sequent $\Gamma \vdash_\oplus \varphi$ is clearly valid in $\mathbb{P} \oplus \mathbb{Q}$, since it can derived by exactly the same manner as the valid sequent $\Gamma \vdash_P \varphi$ derived in $\mathbb{P}$. □

The next result justifies the construction $\oplus$ on consistent finitary disjunctive sequent calculi.

Theorem 5.3. *If* $\mathbb{P}$ *and* $\mathbb{Q}$ *are consistent finitary disjunctive sequent calculi then so is* $\mathbb{P} \oplus \mathbb{Q}$, *and furthermore:*

$$|\mathbb{P} \oplus \mathbb{Q}| = \{S[\vdash_\oplus] \mid S \in |\mathbb{P}| \cup |\mathbb{Q}|\}. \tag{15}$$

Proof. Let $\mathbb{P} \oplus \mathbb{Q}$ be the coalesced sum of $\mathbb{P}$ and $\mathbb{Q}$. By Lemma 5.4, we know that each of simple conjunction in $\mathbb{P} \oplus \mathbb{Q}$ in irreducible. Therefore, $\mathbb{P} \oplus \mathbb{Q}$ is consistent.

To verify Equation (15), we first verify $\{S[\vdash_\oplus] \mid S \in |\mathbb{P}| \cup |\mathbb{Q}|\} \subseteq |\mathbb{P} \oplus \mathbb{Q}|$ by checking $S[\vdash_\oplus]$ satisfies all the conditions for a logical state, where $S \in |\mathbb{P}| \cup |\mathbb{Q}|$. Without lose of generality, we assume that $S \in |\mathbb{P}|$.

For condition (**S1**), let $\varphi \in (S[\vdash_\oplus])[\vdash_\oplus]$. Then it is easy to see that $\varphi \in S[\vdash_\oplus]$ by Equation (2) and the rule (Cut). That is, $(S[\vdash_\oplus])[\vdash_\oplus] \subseteq S[\vdash_\oplus]$.

For condition (**S2**), assume that $\dot{\bigvee}_{i\in I}\mu_i \in S[\vdash_\oplus] \cap \ell(P \oplus Q)$. By Equation (2) and part (4) of Proposition 3.2, there is some $\dot{\bigvee}_{j\in J}\nu_j \in S \cap \ell(P)$ such that $\dot{\bigvee}_{j\in J}\nu_j \vdash_\oplus \dot{\bigvee}_{i\in I}\mu_i$ is valid. Using condition (**S2**) for $\dot{\bigvee}_{j\in J}\nu_j \in S \cap \ell(P)$, there is some $j_0 \in J$ such that $\nu_{j_0} \in S$. Thus $\nu_{j_0} \vdash_\oplus \dot{\bigvee}_{i\in I}\mu_i$ is a valid sequent by part (3) of Proposition 2.1. Since the simple conjunction ν_{j_0} is irreducible, there is some $i_0 \in I$ such that $\nu_{j_0} \vdash_\oplus \mu_{i_0}$ is a valid sequent. This implies that $\mu_{i_0} \in S[\vdash_\oplus]$.

For condition (**S3**), assume that $\mu, \nu \in S[\vdash_\oplus] \cap \mathcal{C}(P \oplus Q)$. Then there is some $\Gamma \sqsubseteq S$ such that $\Gamma \vdash_\oplus \mu \wedge \nu$ is a valid sequent. Since S is a logical state of $\mathbb{P}$, it follows that $\bigwedge \Gamma$ is not logically equivalent to F in $\mathbb{P}$, so it is not in $\mathbb{P} \oplus \mathbb{Q}$. This implies that $\mu \wedge \nu$ is a simple conjunction in $\mathbb{P} \oplus \mathbb{Q}$, and thus $\mu \wedge \nu \in S[\vdash_\oplus] \cap \mathcal{C}(P \oplus Q)$.

Next, we show $|\mathbb{P} \oplus \mathbb{Q}| \subseteq \{S[\vdash_\oplus] \mid S \in |\mathbb{P}| \cup |\mathbb{Q}|\}$. Let $W \in |\mathbb{P} \oplus \mathbb{Q}|$. If $W = \{\mathrm{T}\}$, then $W \in \{S[\vdash_\oplus] \mid S \in |\mathbb{P}| \cup |\mathbb{Q}|\}$. Now we assume that $W \neq \{\mathrm{T}\}$ and set

$$W_0 = \{\mu \in \mathcal{C}(P \oplus Q) \mid \mu \in W\}.$$

Since $W \neq \{\mathrm{T}\}$, by condition (**S2**), the set $W_0 \neq \emptyset$. We claim that $W_0 \subseteq \mathcal{C}(P)$ or $W_0 \subseteq \mathcal{C}(Q)$. In fact, suppose not, that is, there are some $\mu_1 \in \mathcal{C}(P) \cap W_0$ and $\mu_2 \in \mathcal{C}(Q) \cap W_0$. Note that $p, q \vdash_\oplus \mathrm{F}$ is a valid sequent for every $p \in P$ and $q \in Q$, it follows that the sequent $\mu_1, \mu_2 \vdash_\oplus \mathrm{F}$ is valid by the rule (Lwk). This implies that $\mathrm{F} \in W$ by condition (S1), which is a contradiction. Without lose of generality, we assume that $W_0 \subseteq \mathcal{C}(P)$. Set

$$S_W = W_0[\vdash_P].$$

The remainder is to show that S_W is a logical state of $\mathbb{P}$ and $W = S_W[\vdash_\oplus]$.

By Equation (2) and the rule (Cut), it is not difficult to see that

$$S_W[\vdash_P] = (W_0[\vdash_P])[\vdash_P] \subseteq W_0[\vdash_P] = S_W.$$

That is, S_W satisfies condition (**S1**). For condition (**S2**), assume that $\dot{\bigvee}_{i\in I}\mu_i \in S_W \cap \ell(P)$. Then there are some $\nu_1, \nu_2, \cdots, \nu_n \in W_0$ such that $\nu_1, \nu_2, \cdots, \nu_n \vdash_P \dot{\bigvee}_{i\in I}\mu_i$ is a valid sequent in $\mathbb{P}$. Thus $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n \in W_0$ and the sequent $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n \vdash_P \dot{\bigvee}_{i\in I}\mu_i$ is valid in $\mathbb{P}$. This implies that there is some $i_0 \in I$ such that $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n \vdash_P \mu_{i_0}$ is a valid sequent in $\mathbb{P}$ since $\nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_n$ is irreducible. Therefore, $\mu_{i_0} \in S_W$. To prove condition (**S3**), let $\mu, \nu \in S_W$. Then there is some $\Gamma \subseteq W_0$ such that $\bigwedge \Gamma \vdash_P \mu \wedge \nu$ is a valid sequent. Note that $\Gamma \sqsubseteq W \cap \mathcal{C}(P \oplus Q)$, it follows that $\bigwedge \Gamma \in W \cap \mathcal{C}(P \oplus Q) \subseteq \mathcal{C}(P)$, and hence $\mu \wedge \nu$ is not logically equivalent to F. This shows that $\mu \wedge \nu \in W_0[\vdash_P] = S_W$.

Let $\varphi \in W$. If $\varphi = \mathrm{T}$, then clearly $\varphi \in S_W[\vdash_\oplus]$. If $\varphi = \dot{\bigvee}_{i\in I}\mu_i \in W \cap \ell(P \oplus Q)$. Then there is some $i_0 \in I$ such that $\mu_{i_0} \in W_0$. Thus the sequent $\mu_{i_0} \vdash_P \dot{\bigvee}_{i\in I}\mu_i$ is valid in $\mathbb{P}$. This implies that the sequent $\mu_{i_0} \vdash_\oplus \dot{\bigvee}_{i\in I}\mu_i$ is valid in $\mathbb{P} \oplus \mathbb{Q}$. So $\varphi = \dot{\bigvee}_{i\in I}\mu_i \in S_W[\vdash_\oplus]$, and hence $W \subseteq S_W[\vdash_\oplus]$. Conversely, let $\psi \in S_W[\vdash_\oplus]$. If $\psi = \mathrm{T}$, then $\psi \in W$. If $\psi = \dot{\bigvee}_{j\in J}\mu_j \in S_W[\vdash_\oplus] \cap \ell(P \oplus Q)$, then there is some $\mu \in W_0$ such that $\mu \vdash_\oplus \dot{\bigvee}_{j\in J}\mu_j$ is valid in $\mathbb{P} \oplus \mathbb{Q}$. Note that $\mu \in W$, it follows that $\dot{\bigvee}_{j\in J}\mu_j \in W[\vdash_\oplus] \subseteq W$. So $S_W[\vdash_\oplus] \subseteq W$. □

Proposition 5.3. *Let $\mathbb{P} \oplus \mathbb{Q}$ be the coalesced sum of $\mathbb{P}$ and $\mathbb{Q}$.*

(1) *If $S_1 \in |\mathbb{P}|$ and $S_2 \in |\mathbb{Q}|$, then $S_1[\vdash_\oplus] \cap S_2[\vdash_\oplus] = \{\mathrm{T}\}$.*
(2) *If $S_1, S_2 \in |\mathbb{P}|$ (or $S_1, S_2 \in |\mathbb{Q}|$) with $S_1 \subseteq S_2$, then $S_1[\vdash_\oplus] \subseteq S_2[\vdash_\oplus]$. Moreover, $S_1[\vdash_\oplus] = S_2[\vdash_\oplus]$ if and only if $S_1 = S_2$.*

Proof. (1) Assume that $S_1 \in |\mathbb{P}|$ and $S_2 \in |\mathbb{Q}|$. By Equation (2), we have $\mathrm{T} \in S_1[\vdash_\oplus] \cap S_2[\vdash_\oplus]$. If there is a flat formula $\varphi \in S_1[\vdash_\oplus] \cap S_2[\vdash_\oplus]$, then there are flat formulae $\dot{\bigvee}_{i\in I}\mu_i \in S_1$ and $\dot{\bigvee}_{j\in J}\nu_j \in S_2$ such that both $\dot{\bigvee}_{i\in I}\mu_i \vdash_\oplus \varphi$ and $\dot{\bigvee}_{j\in J}\nu_j \vdash_\oplus \varphi$ are valid. With part (3) of Proposition 2.1 and condition (**S2**), there are two simple conjunctions $\mu_{i_0} \in S_1$ and $\nu_{j_0} \in S_2$ such that $\mu_{i_0} \vdash_\oplus \varphi$ and $\nu_{j_0} \vdash_\oplus \varphi$ are valid, which means that $\mu_{i_0} \wedge \nu_{j_0} \vdash_\oplus \varphi$. Because of the fact that $p, q \vdash_\oplus \mathrm{F}$ is valid for all $p \in P$ and $q \in Q$, we know $\mu_{i_0} \wedge \nu_{j_0} \vdash_\oplus \mathrm{F}$ is valid, a contradiction. So $S_1[\vdash_\oplus] \cap S_2[\vdash_\oplus] = \{\mathrm{T}\}$.

(2) Similar to the proof of part (2) of Proposition 5.2. □

By Equation (15) and Proposition 5.3, clearly the map $f : |\mathbb{P}| \oplus |\mathbb{Q}| \to |\mathbb{P} \oplus \mathbb{Q}|$ defined by

$$f(S) = S[\vdash_\oplus]$$

is an order isomorphism, where $|\mathbb{P}| \oplus |\mathbb{Q}|$ is the coalesced sum of two poset. Then we can easy obtain the structure $|\mathbb{P} \oplus \mathbb{Q}|$ based on the structures $|\mathbb{P}|$ and $|\mathbb{Q}|$, see Figure 2:

The other way of summing two pointed posets A and B is to add a new bottom such that the summands are lifted above it separately. This is often called a separated sum and denoted as $A \uplus B$.

The separated sum on consistent finitary disjunctive sequent calculi can be defined in terms of lifting and coalesced sum.

Definition 5.5. *The separated sum $\mathbb{P} \uplus \mathbb{Q}$ of two consistent finitary disjunctive sequent calculi is $\mathbb{P}^{\flat_1} \oplus \mathbb{Q}^{\flat_2}$.*

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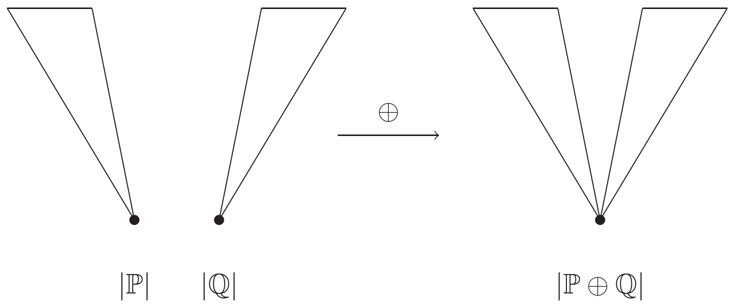

Figure 2. Coalesced sum.

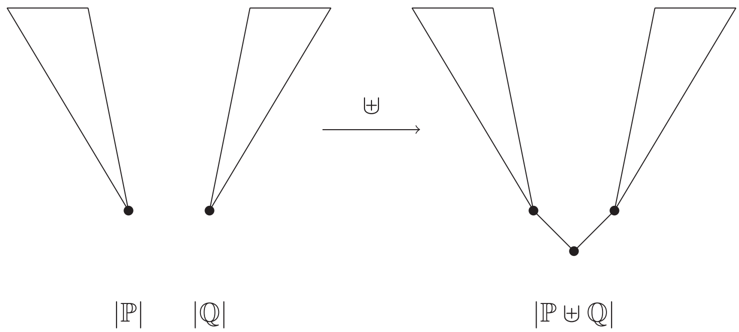

Figure 3. Separated sum.

As a directly consequence of Theorems 5.2 and 5.3, we have the following theorem, which justifies the separated sum.

Theorem 5.4. *If $\mathbb{P}$ and $\mathbb{Q}$ are consistent finitary disjunctive sequent calculi then so is $\mathbb{P} \uplus \mathbb{Q}$, and furthermore:*

$$\begin{aligned} |\mathbb{P} \uplus \mathbb{Q}| = &\{(\{\flat_1\}[\vdash_{P^{\flat_1}}])[\vdash_\oplus]\} \cup \{(S[\vdash_{P^{\flat_1}}])[\vdash_\oplus] \mid S \in |\mathbb{P}|\} \\ &\cup \{(\{\flat_2\}[\vdash_{P^{\flat_2}}])[\vdash_\oplus]\} \cup \{(S[\vdash_{P^{\flat_2}}])[\vdash_\oplus] \mid S \in |\mathbb{Q}|\}. \end{aligned} \tag{16}$$

If $\mathbb{P}$ and $\mathbb{Q}$ are two consistent finitary disjunctive sequent calculi, then we get a Scott domain $|\mathbb{P} \uplus \mathbb{Q}|$ associated with the separated sum of them. Analogous to the case of coalesced sum, we can see that the Scott domain $|\mathbb{P} \uplus \mathbb{Q}|$ is order isomorphic to the separated sum $|\mathbb{P}| \uplus |\mathbb{Q}|$ of two posets. The structure of $|\mathbb{P} \uplus \mathbb{Q}|$ can be seen in Figure 3:

5.3 Solving recursive domain equations

A recursive domain equation like $\mathbb{P} = \mathbb{P}^\flat$ states every solution to this domain equation has a natural domain structural interpretation: it is actually equal to its lifted version. In this subsection, we will show that this equation, as well as other similar recursive domain equations, has a solution.

The following theorem is a very powerful and important observation, which allows us to solve recursive domain equations.

Theorem 5.5. *If $\mathfrak{F}$ is a continuous unary domain construction, then the consistent finitary disjunctive sequent calculus $\bigsqcup_{n\geq 0} \mathfrak{F}^n(\mathbb{O})$ is the least fixed point of $\mathfrak{F}$.*

Proof. The proof of this result is identical to those for Scott-continuous functions over pointed dcpos, for example, see Davey and Priestley (2002, 8.15 CPO fixpoint Theorem I). □

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There is still one important step we should to take before solving recursive domain equations, namely to justify the domain constructions we defined are all continuous.

Lemma 5.4. *The operation* $\flat : \mathbb{P} \mapsto \mathbb{P}^\flat$ *is monotonic on consistent finitary disjunctive sequent calculi ordered by* $\trianglelefteq$.

Proof. Let $\mathbb{P} \trianglelefteq \mathbb{Q}$. We verify that $\mathbb{P}^\flat \trianglelefteq \mathbb{Q}^\flat$ by checking the conditions of Definition 5.1

Since $P \subseteq Q$, we have $P^\flat = P \cup \{\flat\} \subseteq \cup\{\flat\} = Q^\flat$. Condition (**u1**) follows.

For condition (**u2**), let $\Gamma \vdash_{P^\flat} \varphi$ be a disjointness assumption on $\mathcal{A}_{P^\flat}$. If the sequent $\Gamma \vdash_{P^\flat} \varphi$ is of the form $p_1, p_2, \cdots, p_n \vdash_{P^\flat} \mathrm{F} \in \mathcal{A}_{P^\flat}$, then we have the following implications:

$$\begin{aligned} p_1, p_2, \cdots, p_n \vdash_{P^\flat} \mathrm{F} \in \mathcal{A}_{P^\flat} &\Rightarrow p_1, p_2, \cdots, p_n \vdash_P \mathrm{F} \in \mathcal{A}_P \\ &\Rightarrow p_1, p_2, \cdots, p_n \vdash_Q \mathrm{F} \in \mathcal{A}_Q \\ &\Rightarrow p_1, p_2, \cdots, p_n \vdash_{Q^\flat} \mathrm{F} \in \mathcal{A}_{Q^\flat}. \end{aligned}$$

If $\Gamma \vdash_{P^\flat} \varphi$ is of the form $p \vdash_{P^\flat} \flat$ then clearly $p \vdash_{Q^\flat} \flat \in \mathcal{A}_{Q^\flat}$, since $p \in P \subseteq Q \subseteq Q^\flat$.

For condition (**u3**), let $\Gamma \vdash_{Q^\flat} \varphi$ be a valid sequent in $\mathbb{Q}^\flat$, where $\Gamma \sqsubseteq \mathcal{L}(P^\flat)$ and $\varphi \in \mathcal{L}(P^\flat)$. Then we have to show that $\Gamma \vdash_{P^\flat} \varphi$ is valid in $\mathbb{P}^\flat$. By part (1) of Proposition 2.1, it suffices to check $\bigwedge \Gamma \vdash_{P^\flat} \varphi$ is valid in $\mathbb{P}^\flat$. We consider the following situations:

Case 1: $\bigwedge \Gamma \approx \mathrm{F}$ in $\mathbb{P}^\flat$. It is clear that $\bigwedge \Gamma \vdash_{P^\flat} \varphi$ is valid in $\mathbb{P}^\flat$ by the rule (LF).

Case 2: $\bigwedge \Gamma \approx \mathrm{T}$ in $\mathbb{P}^\flat$. Then $\bigwedge \Gamma \approx \mathrm{T}$ in $\mathbb{Q}^\flat$. Since $\Gamma \vdash_{Q^\flat} \varphi$ is valid in $\mathbb{Q}^\flat$, the formula φ must be logically equivalent to T in $\mathbb{Q}^\flat$, and then φ is logically equivalent to T in $\mathbb{P}^\flat$. So $\bigwedge \Gamma \vdash_{P^\flat} \varphi$ is valid in $\mathbb{P}^\flat$.

Case 3: $\bigwedge \Gamma \approx \dot{\bigvee}_{i \in I} \mu_i$ in $\mathbb{P}^\flat$, where $\dot{\bigvee}_{i \in I} \mu_i$ is a flat formula in which there is some $p \in P$ occurred. In this case, φ is not logically equivalent to F.

If φ is logically equivalent to T or $\flat$, then it is easy to see $\dot{\bigvee}_{i \in I} \underline{\mu_i} \vdash_P \varphi$ is valid in $\mathbb{P}^\flat$, where $\dot{\bigvee}_{i \in I} \underline{\mu_i}$ is defined as that in the proof for Theorem 5.2. Note that $\dot{\bigvee}_{i \in I} \underline{\mu_i} \approx \dot{\bigvee}_{i \in I} \mu_i$ in $\mathbb{P}^\flat$, it follows that $\bigwedge \Gamma \vdash_P \varphi$ is valid in $\mathbb{P}^\flat$.

If φ is logically equivalent to a flat formula $\dot{\bigvee}_{j \in J} \nu_j$ in $\mathbb{P}^\flat$ and there is some $p \in P$ occurred in $\dot{\bigvee}_{j \in J} \nu_j$, then we have the following implications:

$$\begin{aligned} \Gamma \vdash_{Q^\flat} \varphi \text{ is valid in } \mathbb{Q}^\flat &\Rightarrow \bigwedge \Gamma \vdash_{Q^\flat} \varphi \text{ is valid in } \mathbb{Q}^\flat \\ &\Rightarrow \dot{\bigvee}_{i \in I} \mu_i \vdash_{Q^\flat} \dot{\bigvee}_{j \in J} \nu_j \text{ is valid in } \mathbb{Q}^\flat \\ &\Rightarrow \dot{\bigvee}_{i \in I} \underline{\mu_i} \vdash_Q \dot{\bigvee}_{j \in J} \underline{\nu_j} \text{ is valid in } \mathbb{Q} \\ &\Rightarrow \dot{\bigvee}_{i \in I} \underline{\mu_i} \vdash_P \dot{\bigvee}_{j \in J} \underline{\nu_j} \text{ is valid in } \mathbb{P} \\ &\Rightarrow \dot{\bigvee}_{i \in I} \mu_i \vdash_{P^\flat} \dot{\bigvee}_{j \in J} \nu_j \text{ is valid in } \mathbb{P}^\flat \\ &\Rightarrow \bigwedge \Gamma \vdash_{P^\flat} \varphi \text{ is valid in } \mathbb{P}^\flat. \end{aligned}$$

Case 4: $\bigwedge \Gamma \approx \dot{\bigvee}_{i \in I} \mu_i$ in $\mathbb{P}^\flat$, where $\dot{\bigvee}_{i \in I} \mu_i$ is a flat formula built up only by $\flat$. Clearly, the formula φ is not logically equivalent to F. Furthermore, we claim that φ is not logically equivalent to a flat formula $\dot{\bigvee}_{j \in J} \nu_j$ in which some $p \in P$ is occurred. Otherwise, we would see that the sequent $\flat \vdash_{Q^\flat} p$ is valid in $\mathbb{Q}^\flat$, a contradiction. So we have $\varphi \approx \mathrm{T}$ or $\varphi \approx \flat$, and then the sequent $\bigwedge \Gamma \vdash_{P^\flat} \varphi$ is valid in $\mathbb{P}^\flat$.

In conclusion, the operation $\flat : \mathbb{P} \mapsto \mathbb{P}^\flat$ is monotonic. $\square$

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Theorem 5.6. *The operation $\flat : \mathbb{P} \mapsto \mathbb{P}^\flat$ is continuous on consistent finitary disjunctive sequent calculi.*

Proof. Let $\{\mathbb{P}_i \mid i \in I\}$ be a directed subset of $\mathcal{CFDC}$. Then $\{\mathbb{P}_i^\flat \mid i \in I\}$ is also a directed subset of $\mathcal{CFDC}$. By Theorem 5.1, $\bigsqcup_{i\in I} \mathbb{P}_i$ and $\bigsqcup_{i\in I} \mathbb{P}_i^\flat$ are members of $\mathcal{CFDC}$. Since the operation $\flat : \mathbb{P} \mapsto \mathbb{P}^\flat$ is monotonic, it is clear that $\bigsqcup_{i\in I} \mathbb{P}_i^\flat \trianglelefteq (\bigsqcup_{i\in I} \mathbb{P}_i)^\flat$. To show the operation $\flat : \mathbb{P} \mapsto \mathbb{P}^\flat$ is continuous, we need only to check

$$(\bigsqcup_{i\in I} \mathbb{P}_i)^\flat \trianglelefteq \bigsqcup_{i\in I} \mathbb{P}_i^\flat.$$

For this, let $(\mathcal{L}(R), \vdash_R) = (\bigsqcup_{i\in I} \mathbb{P}_i)^\flat$ and $(\mathcal{L}(S), \vdash_S) = \bigsqcup_{i\in I} \mathbb{P}_i^\flat$. We divide our proof into three steps.

First, since $R = \bigcup_{i\in I} P_i \cup \{\flat\}$ is the set of atomic formulae of $(\mathcal{L}(R), \vdash_R)$ and $S = \bigcup_{i\in I} (P_i \cup \{\flat\})$ is the set of atomic formulae of $(\mathcal{L}(S), \vdash_S)$, we have $R = S$.

Second, assume that $\Gamma \vdash_R \varphi$ is an atomic disjointness assumption in $(\mathcal{L}(R), \vdash_R)$, then there are two issues to consider.

Case 1: $\Gamma \vdash_R \varphi$ is of the form $p_1, p_2, \cdots, p_n \vdash_R \mathrm{F}$. By the following implications:

$$\begin{aligned} p_1, p_2, \cdots, p_n \vdash_R \mathrm{F} \in \mathcal{A}_R &\Rightarrow p_1, p_2, \cdots, p_n \vdash_{P_i} \mathrm{F} \in \mathcal{A}_{P_i} \text{ for some } i \in I \\ &\Rightarrow p_1, p_2, \cdots, p_n \vdash_{P_i^\flat} \mathrm{F} \in \mathcal{A}_{P_i^\flat} \\ &\Rightarrow p_1, p_2, \cdots, p_n \vdash_S \mathrm{F} \in \mathcal{A}_S, \end{aligned}$$

we know that $\Gamma \vdash_S \varphi$ is an atomic disjointness assumption in $(\mathcal{L}(S), \vdash_S)$.

Case 2: $\Gamma \vdash_R \varphi$ is of the form $r \vdash_R \flat$ for some $r \in R - \{\flat\}$. Since $R = S$, we have $r \in S - \{\flat\}$, and hence there is some $j \in I$ such that $r \in \mathbb{P}_j$. This implies that the sequent $r \vdash_{P_j} \flat$ is valid in $\mathbb{P}_i^\flat$, and hence $r \vdash_S \flat$ is valid in $(\mathcal{L}(S), \vdash_S)$.

Finally, let $\Gamma \vdash_S \varphi$ be a valid sequent in $(\mathcal{L}(S), \vdash_S) = \bigsqcup_{i\in I} \mathbb{P}_i^\flat$, where $\Gamma \sqsubseteq \mathcal{L}(R)$ and $\varphi \in \mathcal{L}(R)$. Since $\mathbf{T}(S) = \bigcup_{i\in I} \mathbf{T}(P_i^\flat)$, it follows that $\Gamma \vdash_{P_{i_0}^\flat} \varphi$ is valid in $\mathbb{P}_{i_0}^\flat$ for some $i_0 \in I$. Note that $\mathbb{P}_{i_0} \trianglelefteq \bigsqcup_{i\in I} \mathbb{P}_i$ and the operation $\flat$ is monotonic, we have $\mathbb{P}_{i_0}^\flat \trianglelefteq (\bigsqcup_{i\in I} \mathbb{P}_i)^\flat$. So the sequent $\Gamma \vdash_R \varphi$ is valid in $(\mathcal{L}(R), \vdash_R) = (\bigsqcup_{i\in I} \mathbb{P}_i)^\flat$ □

Lemma 5.5. *The operation $\oplus : (\mathbb{P}, \mathbb{Q}) \mapsto \mathbb{P} \oplus \mathbb{Q}$ is monotonic.*

Proof. Let $(\mathbb{P}_1, \mathbb{Q}_1) \trianglelefteq (\mathbb{P}_2, \mathbb{Q}_2)$. Then $\mathbb{P}_1 \trianglelefteq \mathbb{P}_2$ and $\mathbb{Q}_1 \trianglelefteq \mathbb{Q}_2$, which implies that $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$. For convenience, we set $\mathbb{P}_1 \oplus \mathbb{Q}_1 = (\mathcal{L}(R), \vdash_R)$ and $\mathbb{P}_2 \oplus \mathbb{Q}_2 = (\mathcal{L}(S), \vdash_S)$. Since $R = P_1 \cup Q_1$ and $S = P_2 \cup Q_2$, we have $R \subseteq S$. This implies that each atomic formula in $\mathbb{P}_1 \oplus \mathbb{Q}_1$ is an atomic formula in $\mathbb{P}_2 \oplus \mathbb{Q}_2$.

Assume that $p_1, p_2, \cdots, p_n \vdash_R \mathrm{F}$ is an atomic disjointness assumption in $\mathbb{P}_1 \oplus \mathbb{Q}_1$. Then we have $p_1, p_2, \cdots, p_n \vdash_{P_1} \mathrm{F} \in \mathcal{A}_{P_1}$ or $p_1, p_2, \cdots, p_n \vdash_{Q_1} \mathrm{F} \in \mathcal{A}_{Q_1}$ or $p_1, p_2, \cdots, p_n \vdash_S \mathrm{F}$ is of the form $p, q \vdash_R \mathrm{F}$ for some $p \in P_1$ and $q \in Q_1$, by Equation (14). For the first two cases, since $\mathcal{A}_{P_1} \subseteq \mathcal{A}_{P_2}$ and $\mathcal{A}_{Q_1} \subseteq \mathcal{A}_{Q_2}$, we know that $p_1, p_2, \cdots, p_n \vdash_R \mathrm{F}$ is an atomic disjointness assumption in $\mathbb{P}_2 \oplus \mathbb{Q}_2$. For the third case, note that $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$, it follows that $p \in P_2$ and $q \in Q_2$, and hence $p, q \vdash_S \mathrm{F}$ is an atomic disjointness assumption in $\mathbb{P}_2 \oplus \mathbb{Q}_2$.

Assume that $\Gamma \vdash_S \varphi$ is a valid sequent in $(\mathcal{L}(S), \vdash_S)$, where $\Gamma \sqsubseteq \mathcal{L}(R)$ and $\varphi \in \mathcal{L}(S)$. We show that the sequent $\Gamma \vdash_S \varphi$ is valid in $(\mathcal{L}(R), \vdash_R)$ by induction.

For the base step, assume that the sequent $\Gamma \vdash_S \varphi$ is an atomic disjointness assumption $p_1, p_2, \cdots, p_n \vdash_S \mathrm{F} \in \mathcal{A}_S$, where $p_1, p_2, \cdots, p_n \in S$. If $p_1, p_2, \cdots, p_n \vdash_{P_2} \mathrm{F} \in \mathcal{A}_{P_2}$ or $p_1, p_2, \cdots, p_n \vdash_{Q_2} \mathrm{F} \in \mathcal{A}_{Q_2}$, using condition **(u3)** for $\mathbb{P}_1 \trianglelefteq \mathbb{P}_2$ or $\mathbb{P}_1 \trianglelefteq \mathbb{P}_2$ respectively, we have $p_1, p_2, \cdots, p_n \vdash_{P_2} \mathrm{F}$ is valid in $\imath_2$ or $p_1, p_2, \cdots, p_n \vdash_{Q_2} \mathrm{F}$ in $\mathbb{Q}_2$. In this two case, the sequent $\Gamma \vdash_R \varphi$ is valid in $\mathbb{P}_1 \oplus \mathbb{Q}_1$. If $\Gamma \vdash_S \varphi$ is of the form $p, q \vdash_S \mathrm{F}$, where $p \in P_1$ and $q \in Q_1$, then

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$p, q \vdash_R \mathrm{F}$ is an atomic disjointness assumption in $\mathbb{P}_1 \oplus \mathbb{Q}_1$. In this case, we also know that the sequent $\Gamma \vdash_R \varphi$ is valid.

For the inductive step, we only check the case that the valid sequent $\Gamma \vdash_S \varphi$ is derived from two valid sequent $\Gamma \vdash_S \phi$ and $\phi \vdash_S \varphi$ by the rule (*Cut*). By inductive hypothesis, we know the sequent $\Gamma \vdash_R \phi$ and $\phi \vdash_R \varphi$ are valid in $(\mathcal{L}(R), \vdash_R)$. So by the rule (*Cut*), the sequent $\Gamma \vdash_R \varphi$ is valid in $(\mathcal{L}(R), \vdash_R)$. □

The proofs for the results in the next lemma are similar to that of Lemma 5.5.

Lemma 5.6. *Let $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ be consistent finitary disjunctive sequent calculi.*

(1) $\mathbb{P} \trianglelefteq \mathbb{P} \oplus \mathbb{Q}$.
(2) *If $\mathbb{P} \trianglelefteq \mathbb{R}$ and $\mathbb{Q} \trianglelefteq \mathbb{R}$, then $\mathbb{P} \oplus \mathbb{Q} \trianglelefteq \mathbb{R}$.*
(3) *If $\mathbb{P} \trianglelefteq \mathbb{R}$ and $\mathbb{Q} \trianglelefteq \mathbb{S}$, then $\mathbb{P} \oplus \mathbb{Q} \trianglelefteq \mathbb{R} \oplus \mathbb{S}$.*

Theorem 5.7. *The operation $\oplus : (\mathbb{P}, \mathbb{Q}) \mapsto \mathbb{P} \oplus \mathbb{Q}$ is continuous on consistent finitary disjunctive sequent calculi.*

Proof. Let $\{(\mathbb{P}_i, \mathbb{Q}_i) \in \mathcal{CFDC}^2 \mid i \in I\}$ be a directed set and $\mathbb{P}_i \oplus \mathbb{Q}_i$ is the coalesced sum of $\mathbb{P}_i$ and $\mathbb{Q}_i$ for every $i \in I$. Then by Lemma 5.5, the set $\{\mathbb{P}_i \oplus \mathbb{Q}_i \mid i \in I\}$ is directed. We must to show that

$$\bigsqcup{}_{i\in I}\mathbb{P}_i \oplus \bigsqcup{}_{i\in I}\mathbb{Q}_i = \bigsqcup{}_{i\in I}(\mathbb{P}_i \oplus \mathbb{Q}_i).$$

But this follows easily from Lemmas 5.5 and 5.6. □

Theorem 5.8. *The operations $\uplus$ is continuous.*

Proof. Since the separated sum $\uplus$ is defined by composing the two continuous operations of lifting and coalesced sum, by Lemma 5.1, $\uplus$ is continuous. □

Example 5.1. For the least consistent finitary disjunctive sequent calculus $\mathbb{O}$ whose atomic formulae set is the empty set, when we lift $\mathbb{O}$, we get a consistent finitary disjunctive sequent calculus $\mathbb{O}_1$ whose atomic formulae set is the singleton $\{\flat_1\}$ and whose atomic disjointness assumptions set is the empty set.

And when we lift $\mathbb{O}_1$, we get another consistent finitary disjunctive sequent calculus $\mathbb{O}_2$ whose atomic formulae set is the set $\{\flat_1, \flat_2\}$ and whose atomic disjointness assumptions set has only one valid sequent $\flat_1 \vdash_2 \flat_2$. Now we repeat this lifting n times to obtain $\mathbb{O}_n$, the resulting consistent finitary disjunctive sequent calculus has the n atomic formulae $\{\flat_1, \flat_2, \cdots, \flat_n\}$ and has the set $\{\flat_i \vdash_j \flat_j \mid 0 \leq i < j \leq n\}$ as its atomic disjointness assumptions.

By Theorem 5.5, the consistent finitary disjunctive sequent calculi $\mathbb{O}_n$ are exactly the iterates used in solving the following recursive domain equation:

$$\mathbb{P} = \mathbb{P}_\flat.$$

That is, the limit $\mathbb{O}_\omega = \bigsqcup_{n\geq 0} \mathbb{O}_n$ is the least solution, where $\mathbb{O}_0 = \mathbb{O}$.

Example 5.2. Let $\mathbb{P}$ be a fixed consistent finitary disjunctive sequent calculus. Then the least solution to the recursive domain equation

$$\mathbb{Q} = \mathbb{P} \oplus \mathbb{Q}$$

is given by $\bigsqcup_{n\geq 1} \mathbb{Q}_n$, where $\mathbb{Q}_0 = \mathbb{O}$ and $\mathbb{Q}_{n+1} = \mathbb{P} \oplus \mathbb{Q}_n$.

6. Conclusion

We have provided a logical syntactic representation of Scott domains by developing the theory of disjunctive propositional logic. Because the categories **CDC** and **SD** are equivalent, a rather

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abstract category is represented by a concrete logical syntactic category. We have also worked to produce some constructions on consistent finitary disjunctive sequent calculi. This induced the corresponding usual constructions on Scott domains. As an application, we have built up solutions to recursive domain equations by constructing a fixed point on a directed complete partial order.

The results that we presented fully reveal that the category **CDC** is a right kind of logic for **SD**, which is what we intend to achieve in this paper. However, it is natural to ask if the category **CDC** possesses much more interesting properties. For example, if such a category always has limits? So the category **CDC** indeed deserves much deeper study in the future.

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