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PAPER

Quasi-Nelson algebras and fragments

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Abstract

The variety of quasi-Nelson algebras (QNAs) has been recently introduced and characterised in several equivalent ways: among others, as (1) the class of bounded commutative integral (but non-necessarily involutive) residuated lattices satisfying the Nelson identity, as well as (2) the class of (0, 1)-congruence orderable commutative integral residuated lattices. Logically, QNAs are the algebraic counterpart of quasi-Nelson logic, which is the (algebraisable) extension of the substructural logic $\mathcal{FL}_{ew}$ (Full Lambek calculus with Exchange and Weakening) by the Nelson axiom. In the present paper, we collect virtually all the results that are currently known on QNAs, including solutions to certain questions left open in earlier publications. Furthermore, we extend our study to some subreducts of QNAs, that is, classes of algebras corresponding to fragments of the algebraic language obtained by eliding either the implication or the lattice operations.

Keywords: (quasi-)Nelson algebras; (quasi-)Kleene algebras; weakly pseudo-complemented; twist-structures

1. Introduction

Constructive logic with strong negation $\mathcal{N}$ is a well-known system of non-classical logic that was introduced by Nelson (1949) more than 70 years ago and has been studied from an algebraic point of view for more than four decades (see Rasiowa 1974). As the name suggests, one of the salient features of $\mathcal{N}$ is the *strong negation* connective ($\sim$), which satisfies the De Morgan laws as well as the double negation axiom ($\sim\sim\phi \Rightarrow \phi$). Another one is the *Nelson axiom*:

$$((p \Rightarrow (p \Rightarrow q)) \wedge (\sim q \Rightarrow (\sim q \Rightarrow \sim p))) \Rightarrow (p \Rightarrow q) \qquad \text{(Nelson-Ax)}$$

whose algebraic counterpart is the *Nelson identity*[1]:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y. \qquad \text{(Nelson-Id)}$$

It is well known that $\mathcal{N}$ can be viewed either (1) as a conservative language expansion (by the strong negation connective) of the negation-free fragment of intuitionistic logic, or (2) as an axiomatic strengthening of the substructural logic $\mathcal{FL}_{ew}$, that is, the *Full Lambek Calculus* ($\mathcal{FL}$) enriched with the rules of *exchange* (e) and *weakening* (w). From an algebraic point of view, $\mathcal{FL}_{ew}$ can be viewed as the logic of all *commutative integral bounded residuated lattices* (abbreviated CIBRLs; see the next section for all unexplained terminology). Adding the double negation axiom to $\mathcal{FL}_{ew}$, one obtains the logic of all *involutive* CIBRLs. Further adding (Nelson-Ax) to involutive $\mathscr{FL}_{ew}$, one obtains precisely $\mathscr{N}$. One may thus ask: what logic/algebraic models result from adding to $\mathscr{FL}_{ew}$ just (Nelson-Ax)?

The above question has been considered in some depth only very recently, in the papers Rivieccio and Spinks (2019, 2020). The corresponding logic has been dubbed *quasi-Nelson logic* ($\mathscr{QN}$), and its algebraic counterpart has been characterised in several equivalent ways; namely, as:

(1) *Quasi-Nelson residuated lattices* (QNRLs), that is, precisely, models of $\mathscr{FL}_{ew}$ plus (Nelson-Ax);
(2) *Non-involutive twist-algebras*, that is, special binary products of Heyting algebras which generalise the well-known Vakarelov–Fidel–Sendlewski twist-algebra construction for representing Nelson algebras;
(3) *Quasi-Nelson algebras* (QNAs), that is, a non-involutive weakening of Nelson algebras, presented à la Rasiowa (Rasiowa, 1974, Section V.1);
(4) ($\mathbf{0}, \mathbf{1}$)-*congruence orderable CIBRLs*.

Regarding (1), it is worth noting that, as a class of (commutative, integral and bounded) residuated lattices, QNRL is quite specific. In particular, every quasi-Nelson residuated lattice has a distributive lattice reduct and is three-potent. All these properties are implied (in the setting of CIBRLs) by just (Nelson-Ax).

(2) mirrors a distinctive feature of Nelson algebras, which can be stated as follows: every Nelson algebra is uniquely determined by a pair $\langle \mathbf{H}, \nabla \rangle$, where $\mathbf{H}$ is a Heyting algebra and ∇ a (special) lattice filter of $\mathbf{H}$. It was proved in Rivieccio and Spinks (2019, 2020) that this result generalises to QNAs, though in this case one needs to consider not one but two Heyting algebras, related by two back-and-forth maps (see Subsection 2.3 for details). This representation, as we shall illustrate, is very powerful and allows one to import a number of useful results from the structure theory of Heyting algebras.

(3) refers to another peculiarity of algebras of Nelson logic: as observed earlier, they can be singled out either as a subvariety of residuated lattices (this insight being, however, relatively recent: see Spinks and Veroff 2008a,b) or as a variety of Kleene algebras (see Definition 15) enriched with a (non-residuated) implication (known as *weak implication*) with a similar behaviour to that of some relevance logic implications. This result, too, carries over to the non-involutive setting: QNAs may be viewed as 'non-involutive Kleene algebras' (see Definition 15) enriched with a relevance-type weak implication.

The property mentioned in (4) was introduced in Spinks et al. (2019) as a generalisation of the *congruence orderability* property of Idziak et al. (2009); see Subsection 2.2 for the technical definitions. As observed in Spinks et al. (2019), among involutive CIBRLs, Nelson residuated lattices are precisely the ($\mathbf{0}, \mathbf{1}$)-congruence orderable ones. Thus, in the context of involutive CIBRLs, ($\mathbf{0}, \mathbf{1}$)-congruence orderability may be regarded as an abstract congruence-theoretic counterpart of the identity (Nelson-Id). This result was generalised in Rivieccio and Spinks (2020) showing that the class of ($\mathbf{0}, \mathbf{1}$)-congruence orderable (non-necessarily involutive) CIBRLs is precisely the variety of QNRLs.

The above considerations should have given the reader an idea of the interest (intrinsic and extrinsic) in the class of QNAs/QNRLs, and of the motivations that led us to study them. The present paper is a further contribution towards a more satisfactory understanding of this class of algebras. We shall give a survey of the main results from Rivieccio and Spinks (2019, 2020), as well as some from Rivieccio (2020a,b), which deal with subreducts of QNAs obtained by eliding the implications from the algebraic language. We shall then focus on a fragment of the language that has not been considered before, corresponding to the 'algebraisable core' of quasi-Nelson logic; the results on this fragment and the corresponding algebras are entirely new.

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2. Quasi-Nelson Logic, Algebras and Residuated Lattices

Let us begin by introducing QNAs, the algebras in the full language (we refer the reader to Rivieccio and Spinks 2019, 2020 for further details and proofs; see also Galatos et al. 2007 for all unexplained algebraic and logical terminology). The most convenient way to do so is to take the substructural logic point of view, starting from the notion of residuated lattice.

Definition 1. *A CIBRL is an algebra* $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0\rangle$ *such that:*

(i) $\langle A; *, 1\rangle$ *is a commutative monoid,* *(Mon)*
(ii) $\langle A; \wedge, \vee, 0, 1\rangle$ *is a bounded lattice (with order* $\leq$*),* *(Lat)*
(iii) $a * b \leq c$ *iff* $a \leq b \Rightarrow c$ *for all* $a, b, c \in A$. *(Res)*

The class of CIBRLs is equationally definable (a variety). A slightly more general class (that we will not need much in the present context) is that of (non-necessarily lower-bounded) commutative integral residuated lattices (CIRLs), which differ from CIBRLs because the constant 0 is not included in the algebraic signature. CIRLs are the algebraic counterpart of the logic called $\mathscr{FL}_{ew}$, which is the extension of the Full Lambek Calculus $\mathscr{FL}$ obtained by adding the rules of *exchange* (e) and *weakening* (w). This entails that CIBRLs are the algebraic counterpart of the expansion of $\mathscr{FL}_{ew}$ by a propositional constant (usually denoted by $\perp$ or 0) which is interpreted as the least element on the algebras.

If the logical/algebraic signature includes a constant symbol 0 (as will always be assumed in the present paper), then one can define a *negation* connective by $\sim p := p \Rightarrow 0$, to which corresponds a similarly defined negation operator on every algebraic model. This allows us to write the axiom (Nelson-Ax) mentioned in the Introduction as well as its *alter ego* on algebraic models, the identity (Nelson-Id).

The papers Rivieccio and Spinks (2019, 2020) and Liang and Nascimento (2019) concern the logic obtained by extending $\mathscr{FL}_{ew}$ (including a 0 constant) with the addition of the Nelson axiom. We dubbed this logic *quasi-Nelson logic* ($\mathscr{QN}$), and the corresponding algebras *QNAs* or *QNRLs*. Further adding the double negation axiom ($\sim\sim p \Rightarrow p$) to $\mathscr{QN}$, one obtains Nelson's logic $\mathscr{N}$, whose algebraic counterpart is the variety of Nelson algebras.

2.1 QNRLs and QNAs

Definition 2. *A* quasi-Nelson residuated lattice *is a CIBRL that satisfies the identity* (Nelson-Id). *A* Nelson residuated lattice *(or* Nelson algebra*) is a quasi-Nelson residuated lattice that also satisfies the involutive identity* $\sim\sim x \approx x$.

Every Heyting algebra satisfies the identity (Nelson-Id) and is therefore an example of a quasi-Nelson residuated lattice (by contrast, the only examples of Heyting algebras which are also Nelson algebras are the Boolean algebras). The class of QNRLs can thus be viewed as a common generalisation of Heyting algebras and Nelson residuated lattices.

An observation that is central to the present study is that, within (quasi-)Nelson logic, the second implication connective $\rightarrow$ can be defined by $p \rightarrow q := p \Rightarrow (p \Rightarrow q)$. This is indeed the implication connective originally taken as primitive by D. Nelson (followed by H. Rasiowa) in defining his logic. Traditionally, $\rightarrow$ is called *weak implication*, while $\Rightarrow$ is the *strong* one. Taking the former as primitive, the strong implication can be recovered by defining $p \Rightarrow q := (p \rightarrow q) \wedge (\sim q \rightarrow \sim p)$; the monoid connective $*$ is also definable as $p * q := \sim (p \Rightarrow \sim q)$. Indeed, one of the main results in the theory of Nelson logic/algebras (which, as shown in Rivieccio and Spinks 2019, 2020, extends to quasi-Nelson without drastic changes) is that the presentation over the language $\{\wedge, \vee, *, \Rightarrow, \sim, 0, 1\}$ of Definition 2 ('Nelson residuated lattices') is equivalent, on the

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level of both logic and algebras, to the one over the language $\{\wedge, \vee, \to, \sim, 0, 1\}$, corresponding to the original denomination of 'Nelson algebras'.

Definition 3. *A* QNA *is an algebra* $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, 0, 1\rangle$ *of type* $\langle 2, 2, 2, 1, 0, 0\rangle$ *satisfying the following properties:*

(QN1) The reduct $\langle A; \wedge, \vee, 0, 1\rangle$ *is a bounded distributive lattice (with order* $\leq$*).*
(QN2) The relation $\preceq$ *on* A *defined for all* $a, b \in A$ *by* $a \preceq b$ *iff* $a \to b = 1$ *is a quasi-order on* A*.*
(QN3) The relation $\equiv := \preceq \cap (\preceq)^{-1}$ *is a congruence on the reduct* $\langle A; \wedge, \vee, \to, 0, 1\rangle$ *and the quotient algebra* $\mathbf{A}_+ = \langle A; \wedge, \vee, \to, 0, 1\rangle/\equiv$ *is a Heyting algebra.*
(QN4) For all $a, b \in A$*, it holds that* $\sim(a \to b) \equiv \sim\sim(a \wedge \sim b)$*.*
(QN5) For all $a, b \in A$*, it holds that* $a \leq b$ *iff* $a \preceq b$ *and* $\sim b \preceq \sim a$*.*
(QN6) For all $a, b \in A$*,*
(QN6.1) $\sim\sim(\sim a \to \sim b) \equiv \sim a \to \sim b$
(QN6.2) $\sim(a \vee b) \equiv \sim a \wedge \sim b$
(QN6.3) $\sim\sim a \wedge \sim\sim b \equiv \sim\sim(a \wedge b)$
(QN6.4) $\sim a \equiv \sim\sim\sim a$
(QN6.5) $a \preceq \sim\sim a$
(QN6.6) $a \wedge \sim a \preceq 0$*.*

The above definition is obviously a generalisation of Rasiowa's presentation of Nelson algebras (Rasiowa 1974, Chapter V, p. 68) as well as of Odintsov's definition of **N3**-lattices (Odintsov 2003, Definition 5.1). Like these classes of algebras, QNAs also form a variety (this is a consequence of the term equivalence with QNRLs: see Subsection 2.3).

Let $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, 0, 1\rangle$ be a QNA. Upon defining the operations:

$$x \Rightarrow y := (x \to y) \wedge (\sim y \to \sim x) \qquad x * y := x \wedge y \wedge \sim(x \Rightarrow \sim y)$$

one verifies that $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a Nelson residuated lattice in the sense of Definition 2 (Rivieccio and Spinks 2019, Proposition 2.5); notice that the term for the monoid operation $*$ does not coincide but generalises the corresponding one for Nelson algebras. Conversely, to every quasi-Nelson residuated lattice $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$, one can associate a QNA $\langle A; \wedge, \vee, \to, \sim, 0, 1\rangle$ by defining $x \to y := x \Rightarrow (x \Rightarrow y)$; see Rivieccio and Spinks (2019, Theorem 2.6). These constructions yield a term equivalence between the class of QNRLs and the class of QNAs.

The above results summarise the equivalence mentioned earlier between (1) QNRLs and (3) QNAs. We next look at the equivalence of either of these presentations with (4), leaving (2) for the last part of the section.

2.2 $(\mathbf{0}, \mathbf{1})$*-congruence orderable CIBRLs*

Given an arbitrary algebra $\mathbf{A}$ and elements $a, b \in A$, we denote by $\Theta^{\mathbf{A}}(a, b)$ the principal congruence generated by $\{a, b\}$ in $\mathbf{A}$. Following Idziak et al. (2009), we say that an algebra $\mathbf{A}$ with a constant term $\mathbf{c}$ is $\mathbf{c}$-*congruence orderable* if, for all $a, b \in A$,

$$\Theta^{\mathbf{A}}(c, a) = \Theta^{\mathbf{A}}(c, b) \quad \text{implies} \quad a = b.$$

A class of algebras K with a constant term $\mathbf{c}$ is $\mathbf{c}$-*congruence orderable* if every member of K is $\mathbf{c}$-congruence orderable. Boolean and Heyting algebras are $\mathbf{1}$-congruence orderable, as are all their subreducts that include $\mathbf{1}$ in the language. Boolean algebras, in fact, are *simultaneously* $\mathbf{c}$-congruence orderable for $\mathbf{c} \in \{\mathbf{0}, \mathbf{1}\}$.

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Nelson algebras (thus, a fortiori, QNAs) are as a class neither **1**-congruence orderable nor **0**-congruence orderable. Indeed, a Nelson algebra **A** is **1**-congruence orderable iff **A** is **0**-congruence orderable iff **A** is a Boolean algebra (Rivieccio and Spinks 2019, Corollary 5.10). This observation led us to introduce in Spinks et al. (2019) the following more relaxed congruence condition.

Recall that for an algebra **A**, the elements $a, b \in A$ are *residually distinct* if $\Theta^{\mathbf{A}}(a, b) = A \times A$. We say that an algebra **A** with constant terms **c**, **d** realising residually distinct elements $c, d \in A$ is $(\mathbf{c}, \mathbf{d})$-*congruence orderable* if, for all $a, b \in A$,

$$\Theta^{\mathbf{A}}(c, a) = \Theta^{\mathbf{A}}(c, b) \text{ and } \Theta^{\mathbf{A}}(d, a) = \Theta^{\mathbf{A}}(d, b) \quad \text{implies} \quad a = b.$$

Likewise, a class K with constant terms **c**, **d** is said to be $(\mathbf{c}, \mathbf{d})$-*congruence orderable* if every member of K is $(\mathbf{c}, \mathbf{d})$-congruence orderable.

Nelson algebras are $(\mathbf{0}, \mathbf{1})$-congruence orderable in the above sense: in fact, Nelson algebras are precisely *the* $(\mathbf{0}, \mathbf{1})$-congruence orderable compatibly involutive CIBRLs (Spinks et al. 2019, Corollary 7.2). Moving to the setting of non-necessarily involutive CIBRLs, we have the following:

Proposition 4 (Spinks et al. 2019, Propositions 5.8 and 5.9). *Let* **A** *be a quasi-Nelson residuated lattice.*

(i) **A** *is* **1**-*congruence orderable iff* **A** *is a Heyting algebra.*
(ii) **A** *is* **0**-*congruence orderable iff* **A** *is a Boolean algebra.*

Proposition 5 (Rivieccio and Spinks 2020, Propositions 16). *A CIBRL* **A** *is* $(\mathbf{0}, \mathbf{1})$-*congruence orderable if and only if* **A** *is a quasi-Nelson residuated lattice.*

2.3 Twist-algebras

We conclude the section with the fourth equivalent presentation of QNAs/QNRLs as twist-algebras. Recall that a Heyting algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, 0, 1\rangle$ is a CIBRL in which every element is idempotent, that is, $a * a = a$ for all $a \in H$. Thus, the monoid operation coincides with the lattice meet and is usually omitted from the algebraic signature. The set $D(\mathbf{H})$ of *dense elements* of a Heyting algebra **H** can be defined in any of the following ways:

$$D(\mathbf{H}) := \{a \vee {\sim} a : a \in H\} = \{a \in H : {\sim} a = 0\} = \{a \vee (a \to b) : a, b \in H\}.$$

A lattice filter F of **H** is *dense* if $D(\mathbf{H}) \subseteq F$.

Definition 6. *Let* $\mathbf{H}_+ = \langle H_+; \wedge_+, \vee_+, \to_+, 0_+, 1_+\rangle$ *and* $\mathbf{H}_- = \langle H_-; \wedge_-, \vee_-, \to_-, 0_-, 1_-\rangle$ *be Heyting algebras (with orders* $\leq_+$ *and* $\leq_-$*), let* $\nabla \subseteq H_+$ *be a dense filter, and let* $n\colon H_+ \to H_-$ *and* $p\colon H_- \to H_+$ *be maps satisfying the following conditions*[2]*:*

(i) n *is a bounded lattice homomorphism,*
(ii) p *preserves finite meets and both lattice bounds,*
(iii) $n \circ p = Id_{H_-}$ *and* $Id_{H_+} \leq_+ p \circ n$.

The quasi-Nelson twist-algebra $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla) = \langle A; \wedge, \vee, \sim, \to, 0, 1\rangle$ *has universe:*

$$A := \{\langle a_+, a_-\rangle \in H_+ \times H_- : a_+ \vee_+ p(a_-) \in \nabla,\ a_+ \wedge_+ p(a_-) = 0_+\}$$

and operations defined as follows. For all $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in H_+ \times H_-$,

$$1 := \langle 1_+, 0_-\rangle,$$
$$0 := \langle 0_+, 1_-\rangle,$$
$${\sim}\langle a_+, a_-\rangle := \langle p(a_-), n(a_+)\rangle,$$

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$$\langle a_+, a_- \rangle \wedge \langle b_+, b_- \rangle := \langle a_+ \wedge_+ b_+, a_- \vee_- b_- \rangle,$$
$$\langle a_+, a_- \rangle \vee \langle b_+, b_- \rangle := \langle a_+ \vee_+ b_+, a_- \wedge_- b_- \rangle,$$
$$\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle := \langle a_+ \to_+ b_+, n(a_+) \wedge_- b_- \rangle.$$

In what follows we shall sometimes, for short, use the expression 'QN twist-algebra' instead of 'quasi-Nelson twist-algebra'. It is not obvious that the set A is closed under the above-defined operations (see Rivieccio and Spinks 2020, Proposition 9 for a proof). On the other hand, checking that every QN twist-algebra $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla)$ is a QNA is straightforward. This entails that $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla)$ can also be viewed as a quasi-Nelson residuated lattice; one can then check that the residuated operations are given, for all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in H_+ \times H_-$, by:

$$\langle a_+, a_- \rangle \Rightarrow \langle b_+, b_- \rangle = \langle (a_+ \to_+ b_+) \wedge_+ (p(b_-) \to_+ p(a_-)), n(a_+) \wedge_- b_- \rangle$$
$$\langle a_+, a_- \rangle * \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, (n(a_+) \to_- b_-) \wedge_- (n(b_+) \to_- a_-) \rangle.$$

Also observe that the lattice order $\leq$ on $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla)$ is given, for all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in H_+ \times H_-$, by $\langle a_+, a_- \rangle \leq \langle b_+, b_- \rangle$ if and only if ($a_+ \leq_+ b_+$ and $b_- \leq_- a_-$).

We proceed to show that every QNA arises in the above-described way. Recall that, for every QNA $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, 0, 1 \rangle$, the relation $\equiv$ introduced in Definition 3 is a congruence on the reduct $\langle A; \wedge, \vee, \to, 0, 1 \rangle$, and the quotient $\mathbf{A}_+ := \langle A; \wedge, \vee, \to, 0, 1 \rangle / \equiv$ is a Heyting algebra. Consider the set $F(A) := \{a \in A : \sim a \leq a\}$. Defining:

$$\nabla_{\mathbf{A}} := \{[a] : a \equiv b \text{ for some } b \in F(A)\},$$

where $[a]$ is the equivalence class of a by $\equiv$, we have that $\nabla_{\mathbf{A}}$ is a lattice filter of $\mathbf{A}_+$ such that $D(\mathbf{A}_+) \subseteq \nabla_+$.

To obtain the second Heyting algebra factor, consider the set $\sim A = \{\sim a : a \in A\}$. Notice that, for all $a, b \in A$, if $\sim a \equiv b$, then $b \in \sim A$. Indeed, recall that, by item (QN3.4), we have $\sim a \equiv \sim\sim\sim a$ and likewise $\sim b \equiv \sim\sim\sim b$. Then, assuming $\sim a \equiv b$, we have $\sim\sim\sim a \equiv \sim\sim b$ (see Lemma 22.viii). Hence, $b \equiv \sim a \equiv \sim\sim\sim a \equiv \sim\sim b$. By transitivity, $b \equiv \sim\sim b$. We can then use item (QN3) of Definition 3 to conclude $b = \sim\sim b$. Hence, $b \in \sim A$, as claimed. This means that $\{b \in A : \sim a \equiv b\} = \{b \in \sim A : \sim a \equiv b\}$ for all $a \in A$. We define A_- as the set $\sim A/\!\equiv := \{[\sim a] : a \in A\}$, having then that $A_- \subseteq A_+$. The set A_- is closed under all the operations of $\mathbf{A}_+$ except, potentially, the join. We then have a quotient $\langle A_-; \wedge_-, \to_-, 0_-, 1_- \rangle$ that we can enrich with a join given, for all $a, b \in A$, by $[\sim a] \vee_- [\sim b] := [\sim (a \wedge b)]$. We let $\mathbf{A}_- := \langle A_-; \wedge_-, \vee_-, \to_-, 0_-, 1_- \rangle$. Lastly, we define maps $n_{\mathbf{A}} : A_+ \to A_-$ and $p_{\mathbf{A}} : A_- \to A_+$ as follows: $n_{\mathbf{A}}([a]) = [\sim\sim a]$ and $p_{\mathbf{A}}([\sim a]) = [\sim a]$. Then the tuple $\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}} \rangle$ satisfies the required properties for defining a QN twist-algebra $\mathbf{Tw}\langle A_+, A_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}} \rangle$.

Proposition 7 (Rivieccio and Spinks 2020, Proposition 10). *Every QNA* $\mathbf{A}$ *is isomorphic to the quasi-Nelson twist-algebra* $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}} \rangle$ *through the map* ι *given by* $\iota(a) = \langle [a], [\sim a] \rangle$ *for all* $a \in A$.

We have thus established the announced equivalence of the four presentations, which we summarise below.

Theorem 1 (Rivieccio and Spinks 2020, Theorem 4). *The following classes of algebras are equivalent in the sense discussed above.*[3]

(i) QNRLs (Definition 2).
(ii) Quasi-Nelson twist-algebras (Definition 6).
(iii) QNAs (Definition 3).
(iv) $(\mathbf{0}, \mathbf{1})$*-congruence orderable CIBRLs.*

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3. Subvarieties and Fragments of Quasi-Nelson

3.1 Subvarieties

One of the advantages of twist constructions of the type of Definition 6 is that one can easily strengthen/relax the conditions (on algebras or maps), obtaining more special/more general classes of algebras. For instance, the involutive QNAs (i.e. Nelson algebras) correspond precisely to twist-algebras $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n, p, \nabla \rangle$ such that $\mathbf{A}_+ \cong \mathbf{A}_-$ through the mutually inverse maps n and p. Other subvarieties of QNAs can be characterised in a similar way, as shown below.

Proposition 8 (Rivieccio and Spinks 2020, Proposition 11). *Let $\mathbf{A} = Tw\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla \rangle$ be a quasi-Nelson twist-algebra.*

(i) $\mathbf{A} \models \sim\sim x \ll x$ *(i.e. $\mathbf{A}$ is a Nelson algebra) iff $\mathbf{H}_+ \cong \mathbf{H}_-$ via mutually inverse Heyting algebra isomorphisms n and p.*
(ii) $\mathbf{A} \models x \to \sim\sim y \ll \sim\sim (x \to y)$ *iff n preserves the Heyting implication.*
(iii) $\mathbf{A} \models x \wedge \sim x \approx 0$ *iff* $\mathbf{A} \models x \wedge \sim (x \wedge y) \approx x \wedge \sim y$ *iff* $n[\nabla] = \{1_-\}$ *iff* $\langle A; \wedge, \vee, \to, \sim, 0, 1 \rangle = \langle A; \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ *is a Heyting algebra iff* $h\colon \mathbf{H}_+ \cong \mathbf{A}$, *where* $h(a_+) := \langle a_+, n(a_+) \to_- 0_- \rangle$ *for all* $a_+ \in H_+$.
(iv) $\mathbf{A} \models \sim x \vee \sim\sim x \approx 1$ *iff* $p(n(a_+)) \vee_+ p(a_-) = 1_+$ *for all* $\langle a_+, a_- \rangle \in A$.
(v) $\mathbf{A} \models \sim (x \wedge y) \approx \sim x \vee \sim y$ *iff* $\mathbf{A} \models \sim (x \wedge y) \ll \sim x \vee \sim y$ *iff p preserves finite joins.*
(vi) $\mathbf{A} \models x \vee \sim x \approx 1$ *iff* $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, 0, 1 \rangle = \langle A; \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$ *is a Boolean algebra iff* $\nabla = \{1_+\}$.
(vii) $\mathbf{A} \models x \wedge \sim (\sim x \to 0) \approx 0$ *iff $\mathbf{H}_-$ is a Boolean algebra.*
(viii) $\mathbf{A} \models (x \Rightarrow y) \vee (y \Rightarrow x) \approx 1$ *iff $\mathbf{H}_+$ and $\mathbf{H}_-$ are Gödel algebras; see Rivieccio et al. (2020a, Proposition 4.2).*

In the preceding proposition (and henceforth), the expression $\alpha \ll \beta$ is an abbreviation of the identity $\alpha \wedge \beta \approx \alpha$. The second identity on item (iii) is precisely saying that the $\{\to\}$-free reduct of $\mathbf{A}$ is a pseudo-complemented lattice (more on this below), while item (iv) is saying that it is a Stone algebra; both varieties are well-known generalisations of Boolean algebras (see e.g. Sankappanavar 1987, Definition 2.1 for further background and more fine-grained classifications).

The identity on item (viii) corresponds to the the so-called *pre-linearity* property, which is well known in the fuzzy logic literature. A preliminary study of pre-linearity in the context of QNAs is contained in the recent paper Rivieccio et al. (2020a), from which we wish to cite three results. (1) Pre-linear Nelson algebras can be shown to coincide with a variety known in the fuzzy literature as *nilpotent minimum algebras*. (2) Pre-linear QNAs can be viewed as a subvariety of *weak nilpotent minimum algebras* (see Esteva and Godo 2001, where these classes of algebras are introduced as the algebraic counterparts of axiomatic extensions of the well-known *monoidal t-norm logic*). (3) In the setting of non-necessarily involutive CIBRLs, the identity $(x * x) \vee (x \wedge \sim x) \approx x$ is implied but does not imply (Nelson-Id): this solves a conjecture formulated in the concluding section of Rivieccio and Spinks (2020).

Another subvariety of QNAs, which has not been singled out before, arises from another question left open in Rivieccio and Spinks (2020, Remark 5). Given that Heyting algebras and Nelson algebras are both special subclasses of QNAs, one may ask whether the latter class is the least (quasi)variety that contains both. The logical counterpart to this question is the following: is quasi-Nelson logic (viewed as a consequence relation) the intersection of intuitionistic and Nelson logic?

The answer is negative, as we now proceed to show. Consider the identity:

$$(x \Rightarrow (x * x)) \vee (\sim\sim y \Rightarrow y) \approx 1. \tag{HN}$$

Observe that every Heyting algebra $\mathbf{H}$ satisfies (HN), because $a \Rightarrow (a * a) = 1$ for all $a \in A$. Similarly, every Nelson algebra $\mathbf{N}$ satisfies (HN), because $\sim\sim a \Rightarrow a = 1$ for all $a \in N$.

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Proposition 9. *Let* $\mathbf{A}$ *be a subdirectly irreducible QNA. The following are equivalent:*

(i) $\mathbf{A}$ *satisfies* (HN).
(ii) $\mathbf{A}$ *is either a Heyting or a Nelson algebra.*

Proof. In the light of the above considerations, the only non-trivial implication is from (i) to (ii). Let then $\mathbf{A}$ be a subdirectly irreducible QNA. By contraposition, suppose $\mathbf{A}$ is neither Heyting nor Nelson. Then there must be elements $a, b \in A$ such that $a \neq a * a$ (thus $a * a < a$) and $b \neq \sim\sim b$ (thus $b < \sim\sim b$). This means that $a \Rightarrow a * a < 1$ and $\sim\sim b \Rightarrow b < 1$. By Rivieccio and Spinks (2020, Corollary 3)., $\mathbf{A}$ has a unique co-atom $c < 1$. Thus, the preceding considerations imply $a \Rightarrow a * a \leq c$ and $\sim\sim b \Rightarrow b \leq c$, therefore $(a \Rightarrow (a * a)) \vee (\sim\sim b \Rightarrow b) \leq c$. So $(a \Rightarrow (a * a)) \vee (\sim\sim b \Rightarrow b) \neq 1$. □

Let us denote by $\mathbb{H}$ the variety of Heyting algebras and by $\mathbb{N}$ the variety of Nelson algebras. Then $\mathbb{H} \vee \mathbb{N}$ denotes the join of these two varieties, that is, the variety generated by $\mathbb{H} \cup \mathbb{N}$. Obviously $\mathbb{H} \vee \mathbb{N} \subseteq \mathbb{QN}$, and the preceding result entails that the inclusion is proper.[4]

Corollary 10. $\mathbb{H} \vee \mathbb{N}$ *is the subvariety of* $\mathbb{QN}$ *axiomatised (relatively to* $\mathbb{QN}$*) by* (HN).

Before we proceed to illustrate how the twist construction can be used to consider more general classes of algebras (corresponding to fragments of the algebraic language of QNAs), let us see another application of the strategy mentioned in the preceding section: namely, the possibility of importing results from the theory of Heyting algebras into the quasi-Nelson setting.

3.2 Congruences, subdirectly irreducibles and directly indecomposables

Proposition 11 (Rivieccio and Spinks 2020, Proposition 8). *Let* $\mathbf{A} = Tw\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla \rangle$ *be a quasi-Nelson twist-algebra. The lattice Con(*$\mathbf{A}$*) of congruences of* $\mathbf{A}$ *is isomorphic to the lattice Con(*$\mathbf{H}_+$*) of congruences of* $\mathbf{H}_+$ *via the maps* $(.)_+$ *and* $(.)^{\bowtie}$ *defined as follows:*

(i) For $\theta \in Con(\mathbf{A})$ *and* $a_+, b_+ \in H_+$, $\langle a_+, b_+ \rangle \in \theta_+$ *if and only if there are* $a_-, b_- \in H_-$ *such that* $\langle a_+ \to_+ b_+, a_- \rangle, \langle b_+ \to_+ a_+, b_- \rangle \in A$ *and* $\langle \langle a_+ \to_+ b_+, a_- \rangle, \langle 1_+, 0_- \rangle \rangle$, $\langle \langle b_+ \to_+ a_+, b_- \rangle, \langle 1_+, 0_- \rangle \rangle \in \theta$.
(ii) For $\eta \in Con(\mathbf{H}_+)$ *and* $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in A$, $\langle \langle a_+, a_- \rangle, \langle b_+, b_- \rangle \rangle \in \eta^{\bowtie}$ *if and only if* $\langle a_+, b_+ \rangle, \langle p(a_-), p(b_-) \rangle \in \eta$.

Proposition 12. *Let* $\mathbf{A} = Tw\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla \rangle$ *be a quasi-Nelson twist-algebra.*

(i) $\mathbf{A}$ *is subdirectly irreducible iff* $\mathbf{H}_+$ *is a subdirectly irreducible Heyting algebra.*
(ii) $\mathbf{A}$ *is directly indecomposable iff* $\mathbf{H}_+$ *is a directly indecomposable Heyting algebra.*

Proof. Item (i) is just an application of Proposition 11. Regarding (ii), observe that $\mathbf{A}$ is not directly indecomposable iff there are non-trivial factor congruences $\theta, \theta' \in \mathrm{Con}(\mathbf{A})$. If this is the case, then $\theta_+, \theta'_+ \in \mathrm{Con}(\mathbf{H}_+)$ are non-trivial factor congruences of $\mathbf{H}_+$. Indeed, this follows from Proposition 11 together with the observation that $\mathbf{H}_+$, as a residuated lattice, is congruence-permutable (Galatos et al. 2007, p. 94). By the same token, $\mathbf{A}$ is congruence-permutable as well. Then, if $\eta_1, \eta_2 \in \mathrm{Con}(\mathbf{H}_+)$ are non-trivial factor congruences, then $\eta_1^{\bowtie}, \eta_2^{\bowtie} \in \mathrm{Con}(\mathbf{A})$ are non-trivial factor congruences. □

3.3 Fragments as twist-algebras

As mentioned earlier, it is easy to relax some of the conditions in Definition 6 so as to obtain classes of twist-algebras in a more reduced algebraic language. For instance, one can consider structures $Tw(\mathbf{L}_+, \mathbf{L}_-, n, p, \nabla)$ where $\mathbf{L}_+$ and $\mathbf{L}_-$ are 'implication-free Heyting algebras' (i.e. pseudo-complemented distributive lattices). What is interesting is that the twist construction actually allows us to characterise certain subreducts of QNAs, as we now proceed to explain.

Let $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a quasi-Nelson residuated lattice. Recall that the negation $\sim$ can be defined by $\sim x := x \Rightarrow 0$, and the weak implication $\rightarrow$ by $x \rightarrow y := x \Rightarrow (x \Rightarrow y)$. In turn, the weak implication can be used to introduce a second negation operation (denoted $\neg$) given by $\neg x := x \rightarrow 0$. Thus, one obtains an enriched algebra $\langle A; \wedge, \vee, *, \Rightarrow, \rightarrow, \sim, \neg, 0, 1\rangle$, which is obviously term equivalent to the original $\mathbf{A}$. If we now elide the monoid operation as well as both implications from the algebraic signature, we obtain an algebra $\langle A; \wedge, \vee, \sim, \neg, 0, 1\rangle$ which may not allow us to recover all the original operations of $\mathbf{A}$. Abstractly, we can consider the class of all $\langle \wedge, \vee, \sim, \neg, 0, 1\rangle$-subalgebras of QNRLs: is this class a quasivariety or even a variety, and if so can we give a (quasi-)equational presentation for it?

In the involutive case, the above question was addressed by Sendlewski (1991), who showed that the class of $\langle \wedge, \vee, \sim, \neg, 0, 1\rangle$-subreducts of Nelson algebras is indeed a variety; he dubbed its members *weakly pseudo-complemented Kleene algebras*. The recent paper Rivieccio (2020a) shows that Sendlewski's techniques can be extended without essential changes so as to obtain a characterisation of the $\langle \wedge, \vee, \sim, \neg, 0, 1\rangle$-subreducts of QNAs. This class of algebras (dubbed *weakly pseudo-complemented quasi-Kleene algebras*) is also a variety, and its members are precisely the twist-algebras over pseudo-complemented distributive lattices, as we now proceed to explain.

Recall that *pseudo-complemented distributive lattices* (also called *distributive p-algebras* or simply *p-lattices*) are algebras $\mathbf{L} = \langle L; \wedge, \vee, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ that are precisely the $\{\wedge, \vee, \neg, 0, 1\}$-subreducts of Heyting algebras; see Balbes and Dwinger (1974, Chapter VIII), Sankappanavar (1987). One can axiomatise p-lattices by requiring $\langle L; \wedge, \vee, 0, 1\rangle$ to be a bounded distributive lattice (with order $\leq$, bottom 0 and top 1) satisfying the following property: for all $a, b \in L$,

$$\text{(P)} \qquad a \leq \neg b \quad \text{if and only if} \quad a \wedge b = 0. \qquad \text{(pseudo-complement)}$$

We shall refer to (P) as to the *property of the pseudo-complement*. On every distributive lattice L, the pseudo-complement $\neg b$ of each $b \in L$ (if it exists) is uniquely determined by the lattice structure in the following way:

$$\neg b = \bigvee \{a \in A : a \wedge b = 0\}.$$

Thus, one can consider the question whether a given distributive lattice has (an implicitly definable) pseudo-complement operation, as we will do below. The set of *dense elements* $D(\mathbf{L})$ of a p-lattice $\mathbf{L}$ is defined as for Heyting algebras:

$$D(\mathbf{L}) := \{a \vee \neg a : a \in L\} = \{a \in L : \neg a = 0\}.$$

As before, we say that a lattice filter F of $\mathbf{L}$ is *dense* if $D(\mathbf{L}) \subseteq F$. We thus have all the ingredients for adapting the twist-algebra construction of Definition 6, as follows.

Definition 13. *Let $\mathbf{L}_+ = \langle L_+; \wedge_+, \vee_+, \neg_+, 0_+, 1_+\rangle$ be a p-lattice (with order $\leq_+$), $\nabla \subseteq L_+$ a dense filter, $\mathbf{L}_- = \langle L_-; \wedge_-, \vee_-, 0_-, 1_-\rangle$ a bounded distributive lattice (with order $\leq_-$), and let $n\colon L_+ \rightarrow L_-$ and $p\colon L_- \rightarrow L_+$ be maps satisfying the following properties:*

(i) n is a bounded lattice homomorphism,
(ii) p preserves finite meets and both lattice bounds,
(iii) $n \circ p = Id_{L_-}$ and $Id_{L_+} \leq_+ p \circ n$.

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The weakly pseudo-complemented quasi-Kleene twist-algebra (WPQK twist-algebra) $Tw(\mathbf{L}_+, \mathbf{L}_-, n, p, \nabla) = \langle A; \wedge, \vee, \sim, \neg, 0, 1\rangle$ *has universe:*

$$A := \{\langle a_+, a_-\rangle \in L_+ \times L_- : a_+ \vee_+ p(a_-) \in \nabla,\ a_+ \wedge_+ p(a_-) = 0_+\}$$

and operations $\wedge, \vee, \sim, 0, 1$ *given as the corresponding ones in Definition 6. Furthermore, we let* $\neg\langle a_+, a_-\rangle := \langle \neg_+ a_+, n(a_+)\rangle$ *for all* $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in L_+ \times L_-$.

As shown in Rivieccio (2020a, Proposition 6.1), the above-defined set A is closed under the twist-algebra operations. Moreover, the conditions of Definition 13 jointly entail that the lattice $\mathbf{L}_-$ is also pseudo-complemented, the pseudo-complement operation $\neg_-$ being given by $\neg_- a_- = n(\neg_+ p(a_-))$ for all $a_- \in A_-$. If we now view $\mathbf{L}_-$ as a p-lattice, we may further observe that both maps n and p preserve the pseudo-complement (Rivieccio 2020a, Proposition 3.4). The following result motivates our interest in WPQK twist-algebras.

Theorem 2 (Rivieccio 2020a, Theorem 5.4). *Every WPQK twist-algebra is embeddable into a quasi-Nelson twist-algebra.*

The embedding of Theorem 2 is constructed as follows. Given a pair of p-lattices $\mathbf{L}_+, \mathbf{L}_-$ as per Definition 13, one considers the respective *canonical extensions* $\mathbf{L}^\sigma_+, \mathbf{L}^\sigma_-$, which are (complete) Heyting algebras into which $\mathbf{L}_+$ and $\mathbf{L}_-$ embed (see Gehrke and Harding 2001). One then shows that the maps n, p relating $\mathbf{L}_+$ and $\mathbf{L}_-$ can be extended to maps n^σ, p^σ between $\mathbf{L}^\sigma_+$ and $\mathbf{L}^\sigma_-$ that meet the requirements of Definition 6 (Rivieccio 2020a, Proposition 5.3). It is then easy to verify that $Tw(\mathbf{L}_+, \mathbf{L}_-, n, p, \nabla)$ embeds into $Tw(\mathbf{L}^\sigma_+, \mathbf{L}^\sigma_-, n^\sigma, p^\sigma, \nabla^\sigma)$, where $\nabla^\sigma = L^\sigma_+$. We note that, while the maximal choice ($\nabla^\sigma = L^\sigma_+$) certainly gives us the desired result, it is possible that $Tw(\mathbf{L}_+, \mathbf{L}_-, n, p, \nabla)$ may also embed into a smaller QNA $Tw(\mathbf{L}^\sigma_+, \mathbf{L}^\sigma_-, n^\sigma, p^\sigma, \nabla')$ with $\nabla' \subsetneq \nabla^\sigma$, which will be a proper subalgebra of $Tw(\mathbf{L}^\sigma_+, \mathbf{L}^\sigma_-, n^\sigma, p^\sigma, \nabla^\sigma)$.

Theorem 2 entails that WPQK twist-algebras are precisely the $\langle \wedge, \vee, \sim, \neg, 0, 1\rangle$-subreducts of quasi-Nelson twist-algebras (and therefore of QNAs). We proceed to show (in three steps) that this class of algebras can be axiomatised by means of identities.

Definition 14 (Sankappanavar 1987). *A* semi-De Morgan algebra *is an algebra* $\mathbf{A} = \langle A; \wedge, \vee, \sim, 0, 1\rangle$ *of type* $\langle 2, 2, 1, 0, 0\rangle$ *satisfying the following properties:*

(SD1) $\langle A; \wedge, \vee, 0, 1\rangle$ *is a bounded distributive lattice,*
(SD2) $\sim 0 \approx 1$ *and* $\sim 1 \approx 0$,
(SD3) $\sim(x \vee y) \approx \sim x \wedge \sim y$,
(SD4) $\sim\sim(x \wedge y) \approx \sim\sim x \wedge \sim\sim y$,
(SD5) $\sim x \approx \sim\sim\sim x$.

A lower quasi-De Morgan algebra *is a semi-De Morgan algebra that satisfies*

(QD) $x \ll \sim\sim x$.

A De Morgan algebra *can be defined as a semi-De Morgan algebra that satisfies the involutive identity* $\sim\sim x \approx x$.

Semi-De Morgan algebras, introduced by Sankappanavar (1987), have been subsequently studied by several authors. The following subvariety is instead quite recent and has been introduced with the aim (among others) of characterising a fragment of QNAs.

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Definition 15 (Rivieccio 2020b). *A* quasi-Kleene algebra *is a lower quasi-De Morgan algebra that additionally satisfies the following identities:*

(QK1) $x \wedge \sim x \ll y \vee \sim y$. *(the Kleene identity)*
(QK2) $\sim\sim x \wedge \sim(x \wedge y) \ll \sim x \vee \sim y$.
(QK3) $\sim\sim x \wedge \sim x \ll x$.

A Kleene algebra *can be defined as a quasi-Kleene algebra that satisfies the involutive identity:* $\sim\sim x \approx x$.

Given a semi-De Morgan algebra $\mathbf{A}$, we write

$$a \preceq b \quad \text{as a shorthand for} \quad a \leq \sim a \vee b$$

$$a \equiv b \quad \text{as a shorthand for} \quad (a \preceq b \text{ and } b \preceq a).$$

The choice of overloading the symbols $\preceq$ and $\equiv$ (already introduced in Definition 3) is due to the observation that, for every QNA $\mathbf{A}$ and all $a, b \in A$, we have $a \to b = 1$ iff $a \leq \sim a \vee b$.

Definition 16. *A* weakly pseudo-complemented quasi-Kleene algebra *(WPQK-algebra) is an algebra* $\mathbf{A} = \langle A; \wedge, \vee, \sim, \neg, 0, 1\rangle$ *of type* $\langle 2, 2, 1, 1, 0, 0\rangle$ *such that:*

- *(i)* $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ *is a quasi-Kleene algebra.*
- *(ii) For all* $a, b, c, d \in A$,
  - *(1)* $\neg 1 = 0$,
  - *(2)* $\neg(a \wedge \sim a) = 1$,
  - *(3)* $a \wedge \neg(a \wedge b) \equiv a \wedge \neg b$.

The preceding definition joins Rivieccio (2020a, Definition 4.2 and Proposition 4.12) in order to make it immediately evident that WPQK-algebras form a variety. The term 'weakly pseudo-complemented' is obviously derived from Sendlewski (1991, p. 22), who dubs *Kleene algebras with a weak pseudo-complementation* (*wp*-Kleene algebras) those Kleene algebras $\mathbf{A}$ that have an extra operation $\neg$ such that, for all $a, b \in A$,

- (i) $a \leq \neg b \quad \text{iff} \quad a \wedge b \leq \sim b$.
- (ii) $\neg(a \wedge b) = 1 \quad \text{iff} \quad \neg\neg a \leq \neg b$.

As shown in Rivieccio (2020a, Lemma 3.5), the above properties (i) and (ii) hold on every WPQK-algebra. One can easily check that, as expected, every QNA (upon defining $\neg x := x \to 0$) gives rise to a WPQK-algebra (Rivieccio, 2020a, Proposition 4.4). It is also easy to verify that every p-lattice $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ forms a WPQK-algebra if we let $\sim x := \neg x$ (cf. Sankappanavar 1987, Corollary 2.8). Sendlewski's *wp*-Kleene algebras are precisely the WPQK-algebras that satisfy the involutive identity $\sim\sim x \approx x$.

For every WPQK-algebra $\mathbf{A} = \langle A; \wedge, \vee, \sim, \neg, 0, 1\rangle$, the reduct $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a quasi-Kleene algebra. Then the above-introduced relation $\equiv$ is a congruence of the lattice reduct of $\mathbf{A}$ (Rivieccio, 2020b, Corollary 3.4) and it is, moreover, compatible with the $\neg$ operation (Rivieccio, 2020a, Corollary 4.7). Hence, we have a quotient algebra $\mathbf{A}_+ = \langle A/\!\equiv; \wedge, \vee, \neg, 0, 1\rangle$, which is a p-lattice (Rivieccio, 2020a, Proposition 4.8). The second quotient $\mathbf{A}_-$ with universe $\sim A/\!\equiv$ can be defined in the same way as for QNAs. We can view $\mathbf{A}_-$ either as a bounded distributive lattice or as a p-lattice on which the pseudo-complement is given by $\neg_-[\sim a] := [\neg \sim a]$ for all $a \in A$; since $[\neg \sim a] = [\sim\sim\neg\sim a]$ for all $a \in A$ (Rivieccio, 2020a, Proposition 4.9), we have $\neg_-[\sim a] \in A_-$

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as expected. The maps $n_{\mathbf{A}}, p_{\mathbf{A}}$ and the filter $\nabla_{\mathbf{A}}$ are defined as in the case of QNAs, giving us the following result.

Proposition 17 (Rivieccio 2020a, Theorem 10). *Every WPQK-algebra* $\mathbf{A}$ *is isomorphic to the WPQK twist-algebra* $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle$ *through the map* ι *given by* $\iota(a) = \langle [a], [\sim a]\rangle$ *for all* $a \in A$.

As for QNAs (cf. Proposition 11), the above twist representation allows us to obtain further information on the congruence lattices of WPQK-algebras. In general, for an arbitrary WPQK twist-algebra $\mathbf{A}$, it is *not* the case that $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_+)$; see Rivieccio (2020a, Section 7) for a counter-example. One can, however, establish an isomorphism between $\mathrm{Con}(\mathbf{A})$ and a sub-meet-semilattice of $\mathrm{Con}(\mathbf{A}_+)$ having the following set as universe:

$$\mathrm{Con}^*(\mathbf{A}_+) := \{\theta \in \mathrm{Con}(\mathbf{A}_+) : \langle a_+, b_+\rangle \in \theta \text{ implies } \langle pn(a_+), pn(b_+)\rangle \in \theta, \text{ for all } a_+, b_+ \in A_+\}.$$

Proposition 18. *Let* $\mathbf{A}$ *be a WPQK-algebra.*

(i) $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}^*(\mathbf{A}_+)$ *(Rivieccio 2020a, Theorem 7.8).*
(ii) If $\mathbf{A}$ *is involutive (i.e. if* $\mathbf{A}$ *is a wp-Kleene lattice), then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_+)$ *(cf. Sendlewski 1991, Theorem 5.2).*
(iii) $\mathrm{Con}(\mathbf{A}_-)$ *is isomorphic to a sub-meet-semilattice of* $\mathrm{Con}(\mathbf{A})$ *(Rivieccio 2020a, Corollary 7.14).*
(iv) If $\mathbf{A}_+$ *is a subdirectly irreducible p-lattice, then* $\mathbf{A}$ *is subdirectly irreducible (Rivieccio 2020a, Corollary 7.9).*

We note that the converse of item (iv) also need not hold in general (Rivieccio 2020a, Section 7). An inspection of the proofs leading to Propositions 5 and 17 points at one of the difficulties that may arise when working with fragments of the quasi-Nelson language: namely, that the algebraic operations needed to define the relation $\equiv$ may no longer be available. The example of WPQK-algebras shows, however, that the weak implication (employed e.g. in Definition 3) is not essential. In fact, from the definition of $a \preceq b$ as $a \leq \sim a \vee b$, one may further speculate that the weak pseudo-complement negation $\neg$ could also be dispensed with. This is indeed the case, as we now proceed to explain.

Observe that, for every quasi-Kleene algebra $\mathbf{A} = \langle A; \wedge, \vee, \sim, 0, 1\rangle$, we can define the relation $\equiv$ and quotients $\mathbf{A}_+, \mathbf{A}_-$ as for WPQK-algebras. In this case, $\mathbf{A}_+$ and $\mathbf{A}_-$ need not be pseudo-complemented but will still be bounded distributive lattices.[5] The maps $n_{\mathbf{A}} : A_+ \to A_-$, $p_{\mathbf{A}} : A_- \to A_+$ may also be defined as before and will be seen to satisfy properties (i) to (iii) of Definition 13. This allows us to define operations $\wedge, \vee, \sim, 0, 1$ on the product set $A_+ \times A_-$ as prescribed by Definition 13. The filter $\nabla_{\mathbf{A}}$ can also be defined as before, and one can show that the set:

$$Tw\langle A_+, A_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle := \{\langle [a], [b]\rangle \in A_+ \times A_- : [a] \vee_+ p([b]) \in \nabla_{\mathbf{A}},\ [a] \wedge_+ p([b]) = 0_+\}$$

is closed under the quasi-Kleene operations. Thus, we can define a quasi-Kleene twist-algebra $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle$ as per Definition 13 and check that the map ι of Proposition 17 is an embedding of $\mathbf{A}$ into the twist-algebra $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle$. However, we do not (as yet) know whether in this case the map ι needs to be surjective. This is currently an open question, and a close inspection of the proof of Rivieccio (2020a, Theorem 6.2.iv) or the corresponding result for QNAs (Rivieccio and Spinks 2020, Proposition 10.iv) suggests that the surjectivity of ι may rely essentially on the presence of the additional operations (the weak implication or the weak pseudo-complement). These observations motivate the following definition and result.

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Definition 19 (Rivieccio 2020b, Definition 4.2). *Let* $\mathbf{L}_+ = \langle L_+, \wedge_+, \vee_+, 0_+\rangle$ *and* $\mathbf{L}_- = \langle L_-, \wedge_-, \vee_-, 0_-\rangle$ *be distributive lattices having a least element. Let* $n\colon L_+ \to L_-$ *and* $p\colon L_- \to L_+$ *be maps satisfying the following properties:*

(i) n preserves finite meets, finite joins, the least element, and (if present) the greatest element;
(ii) p preserves finite meets, the least element and (if present) the greatest element;
(iii) $n \circ p = Id_{L_-}$ *and* $Id_{L_+} \leq_+ p \circ n$.

The algebra $\mathbf{L}_+ \bowtie \mathbf{L}_- = \langle L_+ \times L_-, \wedge, \vee, \sim\rangle$ *has operations defined as per Definition 6. A* quasi-Kleene twist-algebra **A** *over* $\langle \mathbf{L}_+, \mathbf{L}_-, n, p\rangle$ *is a* $\{\wedge, \vee, \sim\}$*-subalgebra of* $\mathbf{L}_+ \bowtie \mathbf{L}_-$ *with carrier set A satisfying:* $\pi_1[A] = L_+$ *and* $a_+ \wedge_+ p(a_-) = 0_+$ *for all* $\langle a_+, a_-\rangle \in A$.

In the light of the previous considerations, we have dropped the filter ∇ altogether and introduced the requirement $\pi_1[A] = L_+$ in order to ensure that each **A** embeds into a minimal algebra $\mathbf{L}_+ \bowtie \mathbf{L}_-$. We note that a quasi-Kleene twist-algebra (as a lattice) need not be bounded. This reflects the observation that the constants 0 and 1 are not definable in the $\{\wedge, \vee, \sim\}$-fragment of the language of QNAs (they become definable if one adds the weak pseudo-complement, as Definition 16.ii.2 shows). Non-necessarily bounded quasi-Kleene algebras have been dubbed *quasi-Kleene lattices* in Rivieccio (2020b).

Proposition 20 (Rivieccio 2020b, Proposition 4.7). *Every (bounded) quasi-Kleene twist-algebra is a quasi-Kleene lattice (algebra).*

Theorem 3 (Rivieccio 2020b, Theorem 4.1). *For every quasi-Kleene lattice (algebra)* **A***, the map* $\iota\colon A \to A_+ \times A_-$ *defined by* $\iota(a) := \langle [a], [\sim a]\rangle$ *is an embedding of* **A** *into* $\mathbf{A}_+ \bowtie \mathbf{A}_-$.

We do not have as yet a description of the congruence lattice of a quasi-Kleene lattice **A** in terms of those of its factors $\mathbf{A}_+, \mathbf{A}_-$, and this may indeed be one of those issues in which the missing algebraic operations (implication(s), weak pseudo-complement) turn out to be essential. It is, however, still possible to use Theorem 3 and canonical extensions to show that every quasi-Kleene lattice can be embedded into a QNA. The proof strategy is essentially the same as for WPQK-algebras.

Given a quasi-Kleene lattice **A** that embeds into $\mathbf{A}_+ \bowtie \mathbf{A}_-$, one considers the canonical extensions $\mathbf{A}_+^\sigma, \mathbf{A}_-^\sigma$, which are (complete) Heyting algebras into which $\mathbf{A}_+$ and $\mathbf{A}_-$ embed. One then shows that the maps $n_\mathbf{A}, p_\mathbf{A}$ can be extended to maps $n_\mathbf{A}^\sigma, p_\mathbf{A}^\sigma$ between $\mathbf{A}_+^\sigma$ and $\mathbf{A}_-^\sigma$ that meet the requirements of Definition 19 (Rivieccio 2020b, Proposition 6.5). Finally, one shows that **A** embeds into $Tw(\mathbf{A}_+^\sigma, \mathbf{A}_-^\sigma, n_\mathbf{A}^\sigma, p_\mathbf{A}^\sigma, \nabla)$, where $\nabla = A_+^\sigma$.

Theorem 4 (Rivieccio 2020b, Theorem 6.1). *Every quasi-Kleene algebra is embeddable into a QNA.*

4. The Algebraisable Fragment

We now turn our attention to the $\{\to, \sim\}$-fragment of the quasi-Nelson algebraic language, which has not been considered in any earlier paper. We dub it the 'algebraisable fragment' because the negation and the weak implication are the two connectives that witness the algebraisability (in the sense of Blok and Pigozzi 1989) of quasi-Nelson logic as presented in Liang and Nascimento (2019).[6] We begin with a quasi-equational presentation of the abstract class of algebras corresponding to this fragment.

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Given an algebra $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ and elements $a, b \in A$, we shall write, as before, $a \preceq b$ instead of $a \to b = 1$, and $a \equiv b$ instead of $a \to b = b \to a = 1$. We shall also employ the following new abbreviations: $a \odot b := \sim(a \to \sim b)$ and

$$q(a, b, c) := (a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to c))).$$

Definition 21. *An algebra $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ is a* quasi-Nelson implication algebra *(QNI-algebra) if the following properties are satisfied for all $a, b, c, d \in A$:*

(i) $1 \to a = a$
(ii) $a \to (b \to a) = a \to a = 0 \to a = 1$
(iii) $a \to (b \to c) = b \to (a \to c) = (a \to b) \to (a \to c)$
(iv) $\sim a \to (\sim b \to c) = (\sim a \odot \sim b) \to c$
(v) $q(a, b, a) = q(a, b, b)$
(vi) *if* $\sim a \preceq \sim b$, *then* $\sim a \preceq \sim a \odot \sim b$
(vii) $a \odot (b \odot c) \equiv (a \odot b) \odot c$
(viii) $a \odot b \equiv b \odot a$
(ix) *if* $a \equiv b$ *and* $c \equiv d$, *then* $a \to c \equiv b \to d$ *and* $a \odot c \equiv b \odot d$
(x) $\sim a = \sim\sim\sim a$
(xi) $\sim 1 = 0$ *and* $\sim 0 = 1$
(xii) $(a \to b) \to (\sim\sim a \to \sim\sim b) = 1.$
(xiii) $a \preceq \sim\sim a$
(xiv) *if* $a \preceq b$, *then* $a \odot c \preceq b \odot c$ *and* $c \odot a \preceq c \odot b$
(xv) $a \odot (a \to b) \equiv a \odot b$
(xvi) $a \odot b \equiv \sim\sim a \odot \sim\sim b$
(xvii) $\sim(a \to b) \equiv \sim(\sim\sim a \to \sim\sim b).$

We observe that the constants 0, 1 could be obviously introduced as the following abbreviations: $1 := a \to a$ and $0 := \sim 1$. This reflects the observation that, since every QNA satisfies $x \to x \approx 1$ and $\sim 1 \approx 0$, the class of $\{\to, \sim\}$-subreducts of QNAs coincides with the class of $\{\to, \sim, 0, 1\}$-subreducts.

As a sanity check, one can verify that every quasi-Nelson (twist-)algebra satisfies all the properties listed in Definition 21. Regarding the new connective $\odot$, it may be helpful to keep in mind that, on a quasi-Nelson twist-algebra $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla)$, one has, for all $\langle a_+, b_+\rangle \in H_+$ and all $\langle a_-, b_-\rangle \in H_-$,

$$\langle a_+, a_-\rangle \odot \langle b_+, b_-\rangle = \langle p(n(a_+) \wedge_- n(b_+)),\ n(a_+ \to_+ p(b_-))\rangle.$$

Our next goal is to show that every QNI-algebra embeds into a twist-algebra.

Lemma 22. *Let $\mathbf{A}$ be a QNI-algebra and $a, b, c \in A$.*

(i) $a \to 1 = 1.$
(ii) $a \equiv b$ *if and only if* $a \to c = b \to c$ *for all* $c \in A$.
(iii) $\sim a \odot \sim b \preceq \sim a$ *and* $\sim a \odot \sim b \preceq \sim b$.
(iv) *If* $a \preceq b$ *and* $b \preceq c$, *then* $a \preceq c$.
(v) $a \equiv 1$ *if and only if* $a = 1$.
(vi) $\sim a \preceq \sim b$ *if and only if* $\sim a \preceq \sim a \odot \sim b$.
(vii) $a \odot \sim a = 0.$
(viii) *If* $a \preceq b$, *then* $\sim\sim a \preceq \sim\sim b$.
(ix) $a \odot a \equiv \sim\sim a \odot \sim\sim a \equiv \sim\sim a.$

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(x) *The relation* $\leq$ *defined by* $a \leq b$ *iff* ($a \preceq b$ *and* $\sim b \preceq \sim a$) *is a partial order on* A, *with minimum* 0 *and maximum* 1.

Proof. (i). Using Definition 21.ii, we have $a \to 1 = a \to (a \to a) = 1$.

(ii). Assume $a \equiv b$, that is $a \to b = b \to a = 1$. Then, we have

$$\begin{aligned} a \to c &= 1 \to (a \to c) && \text{Definition 21.i} \\ &= (a \to b) \to (a \to c) && \text{hypothesis} \\ &= b \to (a \to c) && \text{Definition 21.iii} \\ &= (b \to a) \to (b \to c) && \text{Definition 21.iii} \\ &= 1 \to (b \to c) && \text{hypothesis} \\ &= b \to c. && \text{Definition 21.i} \end{aligned}$$

Conversely, assume $a \to c = b \to c$ for all $c \in A$. Using Definition 21.ii, we have $a \to b = b \to b = 1 = a \to a = b \to a$.

(iii). Using the identities (iii) and (i) in Definition 21, we have $(\sim a \odot \sim b) \to \sim a = \sim a \to (\sim b \to \sim a) = 1$. This gives us $\sim a \odot \sim b \preceq \sim a$, and the same proof (with an extra application of item (ii)) allows us to obtain $\sim a \odot \sim b \preceq \sim b$.

(iv). Assume $a \preceq b$ and $b \preceq c$, that is, $a \to b = b \to c = 1$. By Definition 21.i, we have $a \to c = 1 \to (a \to c)$. Using the first assumption we then have $1 \to (a \to c) = (a \to b) \to (a \to c)$. By Definition 21.iii and the second assumption, we have $(a \to b) \to (a \to c) = a \to (b \to c) = a \to 1$. Lastly, by item (i) above, we have $a \to 1 = 1$. Joining the previous equalities, $a \to c = 1 \to (a \to c) = (a \to b) \to (a \to c) = a \to (b \to c) = a \to 1 = 1$. Hence, $a \to c = 1$ and so $a \preceq c$, as required.

(v). It is clear (by Definition 21.ii) that $a = 1$ entails $a \equiv 1$. Conversely, assume $a \equiv 1$. Then $1 \to a = 1$. By Definition 21.i, we have $1 \to a = a$. Hence, $a = 1$, as required.

(vi). The rightward direction of the implication is Definition 21.vi. For the converse, assume $\sim a \preceq \sim a \odot \sim b$. By item (iii) above, we have $\sim a \odot \sim b \preceq \sim b$. Then, using the transitivity of $\preceq$ (item (v) above), we have $\sim a \preceq \sim b$, as required.

(vii). By definition, $a \odot \sim a = \sim (a \to \sim\sim a)$. By Definition 21.xiii, we have $a \to \sim\sim a = 1$, and by Definition 21.xi we have $\sim 1 = 0$. Thus, $a \odot \sim a = \sim (a \to \sim\sim a) = \sim 1 = 0$.

(viii). Assume $a \preceq b$, that is $a \to b = 1$. Then, using items (i) and (ii) in Definition 21, we have $\sim\sim a \to \sim\sim b = 1 \to (\sim\sim a \to \sim\sim b) = (a \to b) \to (\sim\sim a \to \sim\sim b) = 1$. Thus, $\sim\sim a \preceq \sim\sim b$, as required.

(ix). Observe that $a \odot a \equiv \sim\sim a \odot \sim\sim a$ is an instance of Definition 21.xvi We proceed to show $\sim\sim a \odot \sim\sim a \equiv \sim\sim a$, from which (by the transitivity of $\equiv$) we will also obtain $a \odot a \equiv \sim\sim a$. Using points (iv) and (ii) of Definition 21, we have $(\sim\sim a \odot \sim\sim a) \to \sim\sim a = \sim\sim a \to (\sim\sim a \to \sim\sim a) = \sim\sim a \to 1$, and by item (i) above we have $\sim\sim a \to 1 = 1$. Hence, $\sim\sim a \odot \sim\sim a \preceq \sim\sim a$. Conversely, using Definition 21.vi, from $\sim\sim a \preceq \sim\sim a$ we can obtain $\sim\sim a \preceq \sim\sim a \odot \sim\sim a$, as required.

(x). That $\leq$ is reflexive follows from Definition 21.ii. Transitivity follows from item (iv) above. Antisymmetry is a consequence of items (i) and (v) of Definition 21. Indeed, $a \leq b$ and $b \leq a$ give us $a \to b = b \to a = \sim a \to \sim b = \sim b \to \sim a = 1$. Then, by Definition 21.i, we have $(a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to a))) = 1 \to (1 \to (1 \to (1 \to a))) = a$ and $(a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to b))) = 1 \to (1 \to (1 \to (1 \to b))) = b$. So Definition 21.v gives us $a = b$. Recall that, by Definition 21.xi, we have $0 = \sim 1$ and $1 = \sim 0$. Then Definition 21.ii together with item (i) above imply that $0 \leq a$ and $a \leq 1$ for all $a \in A$. ☐

Let $\mathbf{A} = \langle A; \to, \sim, 0, 1 \rangle$ be a QNI algebra. Observe that the relation $\preceq$ is reflexive (by Definition 21.ii) and transitive (by Lemma 22.v). Hence, $\equiv$ is an equivalence relation on A. Letting $A_+ := A/\!\equiv$, we can thus obtain (by Definition 21.ix) a quotient algebra $\mathbf{A}_+ = \langle A_+; \to_+, 0_+, 1_+ \rangle$.

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We proceed to take a closer look at this structure; to be able to do so, we shall need some additional terminology.

Recall that *Hilbert algebras* are algebras $\langle A; \to, 1\rangle$ of type $\langle 2, 0\rangle$ that are precisely the $\{\to, 1\}$-subreducts of Heyting algebras. It is well known that $\langle A; \to, 1\rangle$ is a Hilbert algebra if and only if the following properties are satisfied: for all $a, b, c \in A$,

(H1) $a \to (b \to a) = 1$
(H2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$
(H3) if $a \to b = b \to a = 1$, then $a = b$.

Every Hilbert algebra has a natural order $\leq$ (not necessarily forming a lattice or even a semilattice) given by $a \leq b$ iff $a \to b = 1$. The top element of $\leq$ is 1. If the natural order also has a minimum element (denoted 0), then we speak of a *bounded Hilbert algebra*. Every Hilbert algebra satisfies the commutative identity $x \to (y \to z) \approx y \to (x \to z)$. Furthermore, the implication $\to$ is order-reversing in the first argument and order-preserving in the second. These observations can be used to show that every Hilbert algebra satisfies the stronger identity (H2'): $(x \to (y \to z)) \approx (x \to y) \to (x \to z))$.

Proposition 23. *For each QNI algebra* $\mathbf{A}$*, the quotient* $\mathbf{A}_+ = \langle A_+, \to_+, 0_+, 1_+\rangle$ *is a bounded Hilbert algebra.*

Proof. Definition 21.ii clearly entails (H1) and also that $\mathbf{A}_+$ is bounded. Definition 21.ii and .iii easily give us (H2). Lastly, (H3) simply follows from the definition of the quotient $A/\equiv$ in terms of the relation $\preceq$. □

Let $A_- := \{[\sim a] : a \in A\} \subseteq A_+$. We endow A_- with operations defined as follows. For all $a, b \in A$, let

$$[\sim a] \wedge_- [\sim b] := [\sim a \odot \sim b] = [\sim(\sim a \to \sim\sim b)]$$
$$0_- := [\sim 1] = [0] = 0_+$$
$$1_- := [\sim 0] = [1] = 1_+.$$

Proposition 24. *For each QNI-algebra* $\mathbf{A}$*, the quotient* $\mathbf{A}_- = \langle A_-, \wedge_-, 0_-, 1_-\rangle$ *is a bounded semilattice.*

Proof. Let us check that $\wedge_-$ is a semilattice operation. Associativity and commutativity are guaranteed by Definition 21.vii and .viii. To verify idempotency, observe that from $\sim a \preceq \sim a$ we obtain, using Definition 21.vi, $\sim a \preceq \sim a \odot \sim a$. The converse $\sim a \odot \sim a \preceq \sim a$ is a consequence of Lemma 22.iii. Thus, $\sim a \equiv \sim a \odot \sim a$, which entails $[\sim a] \wedge_- [\sim a] = [\sim a \odot \sim a]$ as required. Lastly, let us check that $\wedge_-$ behaves as expected with respect to the bounds. Using Lemma 22.i, we have $[\sim a] \wedge_- 0_- = [\sim a] \wedge_- [\sim 1] = [\sim(\sim a \to \sim\sim 1)] = [\sim(\sim a \to 1)] = [\sim 1] = 0_-$. By Lemma 22.iii, we have $\sim a \odot \sim 0 \preceq \sim a$. Also, from $\sim a \preceq \sim 0$ (Lemma 22.i), using Definition 21.vi, we obtain $\sim a \preceq \sim a \odot \sim 0$. Thus, $[\sim a] \wedge_- 1_- = [\sim a] \wedge_- [\sim 0] = [\sim a \odot \sim 0] = [\sim a]$. □

As in the previous section, we can define maps $p_{\mathbf{A}} : A_- \to A_+$ and $n_{\mathbf{A}} : A_+ \to A_-$ as follows: $p_{\mathbf{A}}$ is the identity map on A_- and $n_{\mathbf{A}}[a] := [\sim\sim a]$ for all $a \in A$.

Proposition 25. *Let* $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle$ *be a QNI-algebra, with corresponding quotient algebras* $\mathbf{A}_+ = \langle A_+; \to_+, 0_+, 1_+\rangle$, $\mathbf{A}_- = \langle A_-; \wedge_-, 0_-, 1_-\rangle$ *and maps* $n_{\mathbf{A}} : A_+ \to A_-$, $p_{\mathbf{A}} : A_- \to A_+$ *defined as before.*

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(i) $n_{\mathbf{A}}$ *and* $p_{\mathbf{A}}$ *are monotone and preserve the bounds,*
(ii) $n_{\mathbf{A}} \circ p_{\mathbf{A}} = Id_{A_-}$ *and* $Id_{A_+} \leq_+ p_{\mathbf{A}} \circ n_{\mathbf{A}}$.
(iii) $n_{\mathbf{A}}(a_+) \wedge_- n_{\mathbf{A}}(b_+) = n_{\mathbf{A}}(a_+) \wedge_- n_{\mathbf{A}}(a_+ \to_+ b_+)$, *for all* $a_+, b_+ \in A_+$.
(iv) $p_{\mathbf{A}}(a_- \wedge_- b_-) \to_+ c_+ = p_{\mathbf{A}}(a_-) \to_+ (p_{\mathbf{A}}(b_-) \to_+ c_+)$, *for all* $a_-, b_- \in A_-$ *and* $c_+ \in A_+$.

Proof. (i). It is clear that $p_{\mathbf{A}}$ preserves the bounds. To check monotonicity of $p_{\mathbf{A}}$, we need to ensure that $[\sim a] \leq_- [\sim b]$ (i.e. $[\sim a \odot \sim b] = [\sim a]$) entails $p_{\mathbf{A}}[\sim a] \leq_+ p_{\mathbf{A}}[\sim b]$, that is, $[\sim a] \to_+ [\sim b] = [\sim a \to \sim b] = [1]$. Suppose then $[\sim a \odot \sim b] = [\sim a]$, which implies $\sim a \preceq \sim a \odot \sim b$. By Lemma 22.iii, we have $\sim a \odot \sim b \preceq \sim b$. Therefore, by the transitivity of $\preceq$ (Lemma 22.v), we have $\sim a \preceq \sim b$ and so $\sim a \to \sim b = 1$, which gives us the desired result.

To check that $n_{\mathbf{A}}$ also preserves the bounds, it is sufficient to use Definition 21.xi. For monotonicity, assuming $[a] \leq_+ [b]$, that is, $[a \to b] = [1]$, we need to check that $n_{\mathbf{A}}[a] = [\sim\sim a] \leq_- [\sim\sim b] = n_{\mathbf{A}}[b]$, that is, $[\sim\sim a \odot \sim\sim b] = [\sim\sim a]$. In fact, since $\sim\sim a \odot \sim\sim b \preceq \sim\sim a$ is true in general (Lemma 22.iii), it suffices to check $\sim\sim a \preceq \sim\sim a \odot \sim\sim b$. By Lemma 22.v, the assumption $[a \to b] = [1]$ is equivalent to $a \to b = 1$, that is, $a \preceq b$. Then, using Lemma 22.viii, we obtain $\sim\sim a \preceq \sim\sim b$. From this, using Lemma 22.vi, we have $\sim\sim a \preceq \sim\sim a \odot \sim\sim b$, as required.

(ii). Let us check that $n_{\mathbf{A}} \circ p_{\mathbf{A}} = Id_{A_-}$ and $Id_{A_+} \leq_+ p_{\mathbf{A}} \circ n_{\mathbf{A}}$. As to the former, by Definition 21.x, we have $n_{\mathbf{A}} p_{\mathbf{A}}[\sim a] = n_{\mathbf{A}}[\sim a] = [\sim\sim\sim a] = [\sim a]$. As to the latter, we need to check that $[a] \leq_+ p_{\mathbf{A}} n_{\mathbf{A}}[a] = [\sim\sim a]$, that is, $[a \to \sim\sim a] = [1]$. This is Definition 21.xiii.

(iii). We need to check that $n_{\mathbf{A}}[a] \wedge_- n_{\mathbf{A}}[b] = n_{\mathbf{A}}[a] \wedge_- n_{\mathbf{A}}([a] \to_+ [b])$. On the one hand, we have $n_{\mathbf{A}}[a] \wedge_- n_{\mathbf{A}}[b] = [\sim\sim a] \wedge_- [\sim\sim b] = [\sim\sim a \odot \sim\sim b]$. On the other, $n_{\mathbf{A}}[a] \wedge_- n_{\mathbf{A}}([a] \to_+ [b]) = [\sim\sim a] \wedge_- [\sim\sim (a \to b)] = [\sim\sim a \odot \sim\sim (a \to b)]$. Observe that $b \preceq a \to b$ holds by Definition 21.ii. Then, using Lemma 22.viii, we obtain $\sim\sim b \preceq \sim\sim (a \to b)$. From this, using Definition 21.xiv, we have $\sim\sim a \odot \sim\sim b \preceq \sim\sim a \odot \sim\sim (a \to b)$. To show $\sim\sim a \odot \sim\sim (a \to b) \preceq \sim\sim a \odot \sim\sim b$, observe that $a \odot (a \to b) \preceq a \odot b$ holds by Definition 21.xv. From this, using Definition 21.xvi, we obtain $\sim\sim a \odot \sim\sim (a \to b) \equiv a \odot (a \to b) \preceq a \odot b \equiv \sim\sim a \odot \sim\sim b$, as required.

(iv). Lastly, we need to check that $p_{\mathbf{A}}([\sim a] \wedge_- [\sim b_-]) \to_+ [c] = p_{\mathbf{A}}[\sim a] \to_+ (p_{\mathbf{A}}[\sim b] \to_+ [c])$. We have, on the one hand, $p_{\mathbf{A}}([\sim a] \wedge_- [\sim b_-]) \to_+ [c] = p_{\mathbf{A}}[\sim a \odot \sim b] \to_+ [c] = [(\sim a \odot \sim b) \to c]$. On the other, $p_{\mathbf{A}}[\sim a] \to_+ (p_{\mathbf{A}}[\sim b] \to_+ p[c]) = p_{\mathbf{A}}[\sim a] \to_+ [\sim b \to c] = [\sim a \to (\sim b \to c)]$. Then Definition 21.iv gives us the desired result. □

Propositions 23–26 motivate the following definition.

Definition 26. *Let* $\mathbf{H}_+ = \langle H_+; \to_+, 0_+, 1_+ \rangle$ *be a bounded Hilbert algebra, let* $\mathbf{M}_- = \langle M_-; \wedge, 0_-, 1_- \rangle$ *be a bounded semilattice, and let* $n \colon H_+ \to M_-$ *and* $p \colon M_- \to H_+$ *be maps satisfying the following properties:*

(i) n *and* p *are monotone and preserve the bounds,*
(ii) $n \circ p = Id_{A_-}$ *and* $Id_{H_+} \leq_+ p \circ n$.
(iii) $n(a_+) \wedge_- n(b_+) = n(a_+) \wedge_- n(a_+ \to_+ b_+)$, *for all* $a_+, b_+ \in H_+$.
(iv) $p(a_- \wedge_- b_-) \to_+ c_+ = p(a_-) \to_+ (p(b_-) \to_+ c_+)$, *for all* $a_-, b_- \in A_-$ *and* $c_+ \in H_+$.

The algebra $\mathbf{H}_+ \bowtie \mathbf{M}_- = \langle H_+ \times M_-;\ \to, \sim, 0, 1 \rangle$ *is defined as follows. The operations* $\to, \sim$ *are given as the corresponding ones in Definition 6. A* quasi-Nelson implicative twist-algebra *(QNI twist-algebra)* $\mathbf{A}$ *over* $\langle \mathbf{H}_+, \mathbf{M}_-, n, p \rangle$ *is a* $\{\to, \sim, 0, 1\}$*-subalgebra of* $\mathbf{H}_+ \bowtie \mathbf{M}_-$ *with carrier set* A *satisfying:* $\pi_1[A] = H_+$ *and* $n(a_+) \wedge_- a_- = 0_-$ *for all* $\langle a_+, a_- \rangle \in A$.

To ensure that the notion of twist-algebra is well defined, let us check that the set:

$$A := \{\langle a_+, a_- \rangle \in H_+ \times M_- : n(a_+) \wedge_- a_- = 0_-\}$$

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is closed under the algebraic operations and is therefore the universe of the largest twist-algebra over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$. The case of the constants and of the unary operation $\sim$ are quite straightforward. Regarding the binary operation $\to$, we need to check that

$$n(a_+ \to_+ b_+) \wedge_- n(a_+) \wedge_- b_- = 0_-$$

whenever $n(a_+) \wedge_- a_- = n(b_+) \wedge_- b_- = 0_-$. Applying Definition 26.iii, we have $n(a_+ \to_+ b_+) \wedge_- n(a_+) \wedge_- b_- = n(a_+) \wedge_- n(b_+) \wedge_- b_- = n(a_+) \wedge_- 0_- = 0_-$, as required.

On every QNI twist-algebra $\mathbf{A}$ over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$, we introduce the derived connective $\odot$ as before. Therefore, we have $\langle a_+, a_-\rangle \odot \langle b_+, b_-\rangle = \langle p(n(a_+) \wedge_- n(b_+)),\ n(a_+ \to_+ p(b_-))\rangle$. We also define the relation $\preceq$ by:

$$\langle a_+, a_-\rangle \preceq \langle b_+, b_-\rangle \quad \text{iff} \quad \langle a_+, a_-\rangle \to \langle b_+, b_-\rangle = \langle 1_+, 0_-\rangle.$$

It is easy to check that $\langle a_+, a_-\rangle \preceq \langle b_+, b_-\rangle$ iff $a_+ \leq_+ b_+$. The symmetric relation $\equiv$ is defined by:

$$\langle a_+, a_-\rangle \equiv \langle b_+, b_-\rangle \quad \text{iff} \quad (\langle a_+, a_-\rangle \preceq \langle b_+, b_-\rangle \text{ and } \langle b_+, b_-\rangle \preceq \langle a_+, a_-\rangle).$$

We then have $\langle a_+, a_-\rangle \equiv \langle b_+, b_-\rangle$ iff $a_+ = b_+$. Also observe that $\sim\langle a_+, a_-\rangle \preceq \sim\langle b_+, b_-\rangle$ iff $a_- \leq_- b_-$, and therefore $\sim\langle a_+, a_-\rangle \equiv \sim\langle b_+, b_-\rangle$ iff $a_- = b_-$. Indeed, $\sim\langle a_+, a_-\rangle \preceq \sim\langle b_+, b_-\rangle$ means $p(a_-) \leq_+ p(b_-)$. Then using the monoticity of n and the identity $n \circ p = Id_{A_-}$, we have $np(a_-) = a_- \leq_- b_- = np(b_-)$.

Remark 27. Since n and p are monotone maps and $n \circ p \leq_- Id_{M_-}$ and $Id_{H_+} \leq_+ p \circ n$, we have that n and p form an adjoint pair from the poset $\langle H_+, \leq_+\rangle$ to the poset $\langle M_-, \leq_-\rangle$. This entails that n preserves arbitrary existing joins and p arbitrary existing meets (cf. Definition 6 above). Moreover, in our case, for all $a_-, b_- \in M_-$, the meet of $\{p(a_-), p(b_-)\}$ does exist in $\mathbf{H}_+$ and is $p(a_- \wedge_- b_-)$. To see this, observe that $a_- \wedge_- b_- \leq_- a_-, b_-$ gives us, by monotonicity of p, that $p(a_- \wedge_- b_-) \leq_+ p(a_-), p(b_-)$. Thus, $p(a_- \wedge_- b_-)$ is a lower bound of $\{p(a_-), p(b_-)\}$. Suppose $c_+ \in H_+$ is also a lower bound of $\{p(a_-), p(b_-)\}$, that is, $c_+ \leq_+ p(a_-), p(b_-)$. Then, using monotonicity of n and the identity $n \circ p = Id_{M_-}$, we have $n(c_+) \leq_- np(a_-) = a_-$ and likewise $n(c_+) \leq_- b_-$. Hence, $n(c_+) \leq_- a_- \wedge_- b_-$. Applying p to both sides of the inequality, we have $pn(c_+) \leq_+ p(a_- \wedge_- b_-)$. By $Id_{H_+} \leq_+ p \circ n$ we have $c_+ \leq_+ pn(c_+) \leq_+ p(a_- \wedge_- b_-)$ and so $c_+ \leq_+ p(a_- \wedge_- b_-)$. Thus, $p(a_- \wedge_- b_-)$ is the greatest lower bound of $\{p(a_-), p(b_-)\}$.

Lemma 28. *Let $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$ be as per Definition 26. Let $a_+, b_+, c_+ \in H_+$ and $a_-, b_- \in M_-$.*

(i) $pn(a_+ \to_+ b_+) \to_+ (pn(a_+) \to_+ pn(b_+)) = 1_+$.
(ii) $n(a_+) \leq_- n(b_+ \to_+ c_+)$ *implies* $n(a_+) \wedge_- n(b_+) \leq_- n(c_+)$.
(iii) $p(a_-) \to_+ p(b_-) = pn(p(a_-) \to_+ p(b_-))$.

Proof. (i). From $n(a_+) \wedge_- n(b_+) \leq_- n(b_+)$, using the monotonicity of p we have $p(n(a_+) \wedge_- n(b_+)) \leq_+ pn(b_+)$, so $p(n(a_+) \wedge_- n(b_+)) \to_+ pn(b_+) = 1_+$. Then:

$$\begin{aligned} & pn(a_+ \to_+ b_+) \to_+ (pn(a_+) \to_+ pn(b_+)) \\ & = p(n(a_+ \to_+ b_+) \wedge_- n(a_+)) \to_+ pn(b_+) && \text{by Definition 26.iv} \\ & = p(n(a_+) \wedge_- n(b_+)) \to_+ pn(b_+) && \text{by Definition 26.ii} \\ & = 1_+. \end{aligned}$$

(ii). Assume $n(a_+) \leq_- n(b_+ \to_+ c_+)$. Using the monotonicity of p, this gives us $pn(a_+) \leq_+ pn(b_+ \to_+ c_+)$. By item (i) above, we have $pn(b_+ \to_+ c_+) \leq_+ pn(b_+) \to_+ pn(c_+)$. Hence, $pn(a_+) \leq_+ pn(b_+) \to_+ pn(c_+)$, which is equivalent to $pn(a_+) \to_+ (pn(b_+) \to_+ pn(c_+)) = 1_+$. By Definition 26.iv, this gives us $1_+ = pn(a_+) \to_+ (pn(b_+) \to_+ pn(c_+)) = p(n(a_+) \wedge_- n(b_+)) \to_+$

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$pn(c_+)$. Thus, $p(n(a_+) \wedge_- n(b_+)) \leq_+ pn(c_+)$. Using the monotonicity of n and Definition 26.ii, we then have $np(n(a_+) \wedge_- n(b_+)) = n(a_+) \wedge_- n(b_+) \leq_- npn(c_+) = n(c_+)$, as required.

(iii). The inequality $p(a_-) \to_+ p(b_-) \leq_+ pn(p(a_-) \to_+ p(b_-))$ follows from Definition 26.ii. As to the converse inequality, using item (i) above and Definition 26.ii, we have $pn(p(a_-) \to_+ p(b_-)) \leq_+ pnp(a_-) \to_+ pnp(b_-) = p(a_-) \to_+ p(b_-)$. □

Proposition 29. *Let $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$ be as per Definition 26. For all $a_-, b_- \in M_-$, define the operation $\to_-$ as $a_- \to_- b_- := n(p(a_-) \to_+ p(b_-))$. Then $\mathbf{M}_- = \langle M_-; \wedge_-, \to_-, 0_-, 1_-\rangle$ is a bounded implicative semilattice[7]. Moreover, the map p preserves the implication, that is, $p(a_- \to_- b_-) = p(a_-) \to_+ p(b_-)$ for all $a_-, b_- \in M_-$.*

Proof. Since $p(a_- \to_- b_-) = pn(p(a_-) \to_+ p(b_-))$, the last claim, that is, $p(a_- \to_- b_-) = p(a_-) \to_+ p(b_-)$ for all $a_-, b_- \in M_-$, has been shown in Lemma 28.iii. We proceed to show that, for all $a_-, b_-, c_- \in M_-$, one has $a_- \wedge_- b_- \leq_- c_-$ iff $a_- \leq_- b_- \to_- c_-$. Assuming $a_- \wedge_- b_- \leq_- c_-$, by Definition 26.i we have $p(a_- \wedge_- b_-) \leq_+ p(c_-)$. Then, using Definition 26.iv, we have $1_+ = p(a_- \wedge_- b_-) \to_+ p(c_-) = p(a_-) \to_+ (p(b_-) \to_+ p(c_-))$. So $p(a_-) \leq_+ p(b_-) \to_+ p(c_-)$. Then (using items (i) and (ii) of Definition 26), we obtain $np(a_-) = a_- \leq_- n(p(b_-) \to_+ p(c_-)) = b_- \to_- c_-$, as required. Conversely, assume $a_- \leq_- b_- \to_- c_-$. Using Definition 26.i, we have $p(a_-) \leq_+ p(b_- \to_- c_-) = p(b_-) \to_+ p(c_-)$, the last equality holding because p preserves the implication. Hence, $p(a_-) \to_+ (p(b_-) \to_+ p(c_-)) = 1_+$. By Definition 26.iv, we then have $1_+ = p(a_-) \to_+ (p(b_-) \to_+ p(c_-)) = p(a_- \wedge_- b_-) \to_+ p(c_-)$. Thus $p(a_- \wedge_- b_-) \leq_+ p(c_-)$ and, by items (i) and (ii) of Definition 26, we obtain $np(a_- \wedge_- b_-) = a_- \wedge_- b_- \leq_- c_- = np(c_-)$. □

Proposition 29 suggests that in Definition 26, we could equivalently have required $\mathbf{M}_-$ to be an implicative semilattice. An interesting consequence of Proposition 29 is that $\mathbf{M}_-$ is a *distributive* semilattice in the sense of Grätzer (1978, Section II.5), that is, the lattice of filters of $\mathbf{M}_-$ is distributive. As is well known, the lattice of (implicative) filters of a Hilbert algebra such as $\mathbf{H}_+$ is also distributive (Celani et al. 2009, p. 477).

We proceed to check that every QNI twist-algebra satisfies the conditions of Definition 21 and is therefore a QNI-algebra.

Lemma 30. *Let $\mathbf{A}$ be a QNI twist-algebra over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$, and let $a_+, b_+ \in H_+$, $a_-, b_- \in M_-$.*

(i) If $\langle 1_+, 0_-\rangle \preceq \langle a_+, a_-\rangle$, then $\langle 1_+, 0_-\rangle = \langle a_+, a_-\rangle$.
(ii) $\langle 1_+, 0_-\rangle \to \langle a_+, a_-\rangle = \langle a_+, a_-\rangle$.
(iii) $\langle a_+, a_-\rangle \to (\langle b_+, b_-\rangle \to \langle a_+, a_-\rangle) = \langle a_+, a_-\rangle \to \langle a_+, a_-\rangle = \langle 0_+, 1_-\rangle \to \langle a_+, a_-\rangle = 1$.
(iv) $\langle a_+, a_-\rangle \to (\langle b_+, b_-\rangle \to \langle c_+, c_-\rangle) = \langle b_+, b_-\rangle \to (\langle a_+, a_-\rangle \to \langle c_+, c_-\rangle) = (\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle) \to (\langle a_+, a_-\rangle \to \langle c_+, c_-\rangle)$.
(v) $\sim\langle a_+, a_-\rangle \to (\sim\langle b_+, b_-\rangle \to \langle c_+, c_-\rangle) = (\sim\langle a_+, a_-\rangle \odot \sim\langle b_+, b_-\rangle) \to \langle c_+, c_-\rangle$.
(vi) $(\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle) \to ((\langle b_+, b_-\rangle \to \langle a_+, a_-\rangle) \to ((\sim\langle a_+, a_-\rangle \to \sim\langle b_+, b_-\rangle) \to ((\sim\langle b_+, b_-\rangle \to \sim\langle a_+, a_-\rangle) \to \langle a_+, a_-\rangle))) =$
$(\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle) \to ((\langle b_+, b_-\rangle \to \langle a_+, a_-\rangle) \to ((\sim\langle a_+, a_-\rangle \to \sim\langle b_+, b_-\rangle) \to ((\sim\langle b_+, b_-\rangle \to \sim\langle a_+, a_-\rangle) \to \langle b_+, b_-\rangle)))$.
(vii) If $\sim\langle a_+, a_-\rangle \preceq \sim\langle b_+, b_-\rangle$, then $\sim\langle a_+, a_-\rangle \preceq \sim\langle a_+, a_-\rangle \odot \sim\langle b_+, b_-\rangle$.
(viii) $\langle a_+, a_-\rangle \odot (\langle b_+, b_-\rangle \odot \langle c_+, c_-\rangle) \equiv (\langle a_+, a_-\rangle \odot \langle b_+, b_-\rangle) \odot \langle c_+, c_-\rangle$.
(ix) $\langle a_+, a_-\rangle \odot \langle b_+, b_-\rangle \equiv \langle b_+, b_-\rangle \odot \langle a_+, a_-\rangle$.
(x) If $\langle a_+, a_-\rangle \equiv \langle b_+, b_-\rangle$ and $\langle c_+, c_-\rangle \equiv \langle d_+, d_-\rangle$, then $\langle a_+, a_-\rangle \to \langle c_+, c_-\rangle \equiv \langle b_+, b_-\rangle \to \langle d_+, d_-\rangle$ and $\langle a_+, a_-\rangle \odot \langle c_+, c_-\rangle \equiv \langle b_+, b_-\rangle \odot \langle d_+, d_-\rangle$.
(xi) $\sim\langle a_+, a_-\rangle = \sim\sim\sim\langle a_+, a_-\rangle$.
(xii) $\sim 1 = 0$ and $\sim 1 = 0$.
(xiii) $(a_+ \to_+ b_+) \to_+ (pn(a_+) \to_+ pn(b_+)) = 1_+$.

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(xiv) $\langle a_+, a_- \rangle \preceq \sim\sim\langle a_+, a_- \rangle$.
(xv) *If* $\langle a_+, a_- \rangle \preceq \langle b_+, b_- \rangle)$, *then* $\langle a_+, a_- \rangle \odot \langle c_+, c_- \rangle \preceq \langle b_+, b_- \rangle \odot \langle c_+, c_- \rangle$ *and* $\langle c_+, c_- \rangle \odot \langle a_+, a_- \rangle \preceq \langle c_+, c_- \rangle \odot \langle b_+, b_- \rangle$.
(xvi) $\langle a_+, a_- \rangle \odot (\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle) \equiv \langle a_+, a_- \rangle \odot \langle b_+, b_- \rangle$.
(xvii) $\langle a_+, a_- \rangle \odot \langle b_+, b_- \rangle \equiv \sim\sim\langle a_+, a_- \rangle \odot \sim\sim\langle b_+, b_- \rangle$.
(xviii) $\sim(\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle) \equiv \sim(\sim\sim\langle a_+, a_- \rangle \to \sim\sim\langle b_+, b_- \rangle)$.

Proof.

(i) This is a useful lemma for subsequent proofs. Assume $\langle 1_+, 0_- \rangle \preceq \langle a_+, a_- \rangle$. Then $1_+ = a_+$. Recalling the requirement $a_+ \wedge_+ p(a_-) = 0_+$, we then have $a_+ \wedge_+ p(a_-) = 1_+ \wedge_+ p(a_-) = p(a_-) = 0_+$. From $p(a_-) = 0_+$, applying n to both sides of the equation, and recalling that n preserve the bounds and that $n \circ p = Id_{A_-}$, we obtain $np(a_-) = a_- = 0_- = n(0_+)$. Thus $\langle a_+, a_- \rangle = \langle 1_+, 0_- \rangle$, as claimed.

(ii) This follows from the equation $1 \to x = x$ (which is valid on all Hilbert algebras) and the fact that n preserves the top element. We have $\langle 1_+, 0_- \rangle \to \langle a_+, a_- \rangle = \langle 1_+ \to_+ a_+, n(1_+) \wedge_- a_- \rangle = \langle a_+, 1_- \wedge_- a_- \rangle = \langle a_+, a_- \rangle$.

(iii) Recalling item (i) above, we see that the claim easily follows from the corresponding identities $x \to (y \to x) \approx x \to x \approx 0 \to x \approx 1$, which are valid on all bounded Hilbert algebras.

(iv) We will use the identities $x \to (y \to z) \approx y \to (x \to z) \approx (x \to y) \to (x \to z)$, which hold on every Hilbert algebra (and commutativity of the $\wedge_-$ operation). We have

$$
\begin{aligned}
&\langle a_+, a_- \rangle \to (\langle b_+, b_- \rangle \to \langle c_+, c_- \rangle) = \\
&= \langle a_+ \to (b_+ \to_+ c_+), n(a_+) \wedge_- n(b_+) \wedge_- c_- \rangle = \\
&= \langle b_+ \to (a_+ \to_+ c_+), n(b_+) \wedge_- n(a_+) \wedge_- c_- \rangle = \\
&= \langle b_+, b_- \rangle \to (\langle a_+, a_- \rangle \to \langle c_+, c_- \rangle) = \\
&= \langle b_+ \to (a_+ \to_+ c_+), n(b_+) \wedge_- n(a_+) \wedge_- c_- \rangle = \\
&= \langle b_+ \to (a_+ \to_+ c_+), n(a_+ \to b_+) \wedge_- n(a_+) \wedge_- c_- \rangle = && \text{Definition 26.iii} \\
&= \langle (a_+ \to_+ b_+) \to_+ (a_+ \to_+ c_+), n(a_+ \to b_+) \wedge_- n(a_+) \wedge_- c_- \rangle = \\
&= (\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle) \to (\langle a_+, a_- \rangle \to \langle c_+, c_- \rangle).
\end{aligned}
$$

(v) Recall that $x \odot y := \sim(x \to \sim y)$. We then have

$$
\begin{aligned}
&\sim\langle a_+, a_- \rangle \to (\sim\langle b_+, b_- \rangle \to \langle c_+, c_- \rangle) = \\
&= \langle p(a_-) \to_+ (p(b_-)_+ \to_+ c_+), np(a_-) \wedge_- np(b_-) \wedge_- c_- \rangle = \\
&= \langle p(a_-) \to_+ (p(b_-)_+ \to_+ c_+), a_- \wedge_- b_- \wedge_- c_- \rangle = && \text{Definition 26.ii} \\
&= \langle p(a_- \wedge_- b_-) \to_+ c_+, a_- \wedge_- b_- \wedge_- c_- \rangle = && \text{Definition 26.iv} \\
&= \langle p(np(a_-) \wedge_- b_-) \to_+ c_+, np(np(a_-) \wedge_- b_-) \wedge_- c_- \rangle = && \text{Definition 26.ii} \\
&= \sim(\sim\langle a_+, a_- \rangle \to \sim\sim\langle b_+, b_- \rangle) \to \langle c_+, c_- \rangle = \\
&= (\sim\langle a_+, a_- \rangle \odot \sim\langle b_+, b_- \rangle) \to \langle c_+, c_- \rangle.
\end{aligned}
$$

(vi) We have $(\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle) \to ((\langle b_+, b_- \rangle \to \langle a_+, a_- \rangle) \to ((\sim\langle a_+, a_- \rangle \to \sim\langle b_+, b_- \rangle) \to ((\sim\langle b_+, b_- \rangle \to \sim\langle a_+, a_- \rangle) \to \langle a_+, a_- \rangle))) = \langle (a_+ \to_+ b_+) \to_+ ((b_+ \to_+ a_+) \to_+ ((p(a_-) \to_+ p(b_-)) \to_+ ((p(b_-) \to_+ p(a_-)) \to_+ a_+))), n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- n(p(a_-) \to_+ p(b_-)) \wedge_- n(p(b_-) \to_+ p(a_-)) \wedge_- a_- \rangle$. Using (ii) and (iii) in Definition 26, the second component of the latter pair can be simplified as follows: $n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- n(p(a_-) \to_+ p(b_-)) \wedge_- n((p(b_-) \to_+ p(a_-))$

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$\wedge_- a_- = n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- n(p(a_-) \to_+ p(b_-)) \wedge_- n(p(b_-) \to_+ p(a_-)) \wedge_- np(a_-) = n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- np(b_-) \wedge_- n(p(b_-) \to_+ p(a_-)) \wedge_- np(a_-) = n(a_+ \to_+ b_+) \wedge_- n((b_+ \to_+ a_+)) \wedge_- np(b_-) \wedge_- np(a_-) \wedge_- np(a_-) = n(a_+ \to_+ b_+) \wedge_- n((b_+ \to_+ a_+)) \wedge_- a_- \wedge_- b_-$. Since $(\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle) \to ((\langle b_+, b_- \rangle \to \langle a_+, a_- \rangle) \to ((\sim\langle a_+, a_- \rangle \to \sim\langle b_+, b_- \rangle) \to ((\sim\langle b_+, b_- \rangle \to \sim\langle a_+, a_- \rangle) \to \langle b_+, b_- \rangle))) = \langle (a_+ \to_+ b_+) \to_+ ((b_+ \to_+ a_+) \to_+ ((p(a_-) \to_+ p(b_-)) \to_+ ((p(b_-) \to_+ p(a_-)) \to_+ b_+))), n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- n(p(a_-) \to_+ p(b_-)) \wedge_- n(p(b_-) \to_+ p(a_-)) \wedge_- b_- \rangle$, we can similarly simplify the second component as follows: $n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- n(p(a_-) \to_+ p(b_-)) \wedge_- n(p(b_-) \to_+ p(a_-)) \wedge_- b_- = n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- np(a_-) \wedge_- np(b_-) \wedge_- np(b_-) \wedge_- np(a_-) = n(a_+ \to_+ b_+) \wedge_- n(b_+ \to_+ a_+) \wedge_- a_- \wedge_- b_-$. The equality of the second components is thus clear. Concerning the equality of the first, observe that $(x \to y) \to ((y \to x) \to x) \approx (x \to y) \to ((y \to x) \to y)$ is valid on every Hilbert algebra (see e.g. Celani et al. 2009, Lemma 1.1.8), which easily implies the required result.

(vii) We compute only the first components, which are those that matter. The assumption $\sim\langle a_+, a_- \rangle \preceq \sim\langle b_+, b_- \rangle$ is equivalent to $p(a_-) \leq_+ p(b_-)$. Applying n to both sides of the inequality (Definition 26.i) and recalling $n \circ p = Id_{A_-}$ (Definition 26.ii), we have $np(a_-) = a_- \leq_- b_- = np(b_-)$. Then $a_- = a_- \wedge_- b_-$ and so $p(a_-) = p(a_- \wedge_- b_-)$. Since $\sim\langle a_+, a_- \rangle \preceq \sim\langle a_+, a_- \rangle \odot \sim\langle b_+, b_- \rangle$ reduces to $p(a_-) \leq_+ p(np(a_-) \wedge_- np(b_-)) = p(a_- \wedge_- b_-)$, the result follows.

(viii) As before, only the first components matter. Then $\langle a_+, a_- \rangle \odot (\langle b_+, b_- \rangle \odot \langle c_+, c_- \rangle)$ gives us $p(n(a_+) \wedge_- np(n(b_+) \wedge_- n(c_+))) = p(n(a_+) \wedge_- n(b_+) \wedge_- n(c_+))$ and $(\langle a_+, a_- \rangle \odot \langle b_+, b_- \rangle) \odot \langle c_+, c_- \rangle$ gives us $p(np(n(a_+) \wedge_- n(b_+)) \wedge_- n(c_+)) = p(n(a_+) \wedge_- n(b_+) \wedge_- n(c_+))$.

(ix) Once more only the first components matter, and it is clear that the term $p(n(x) \wedge_- n(y))$ is commutative.

(x) Follows easily from the component-wise definitions of $\to$ and $\odot$, together with the observation made earlier that $\langle a_+, a_- \rangle \equiv \langle b_+, b_- \rangle$ holds iff $a_+ = b_+$.

(xi) Follows from $n \circ p = Id_{A_-}$ (Definition 26.ii).

(xii) Follows from the requirement that n and p preserve the bounds (Definition 26.i).

(xiii) Since $Id_{A_+} \leq_+ p \circ n$, we have $a_+ \to_+ b_+ \leq_+ pn(a_+ \to_+ b_+)$. Thus, showing $pn(a_+ \to_+ b_+) \to_+ (pn(a_+) \to_+ pn(b_+)) = 1_+$ is sufficient to establish the claim. Recalling that $x \to x = 1$ holds on every Hilbert algebra, we have

$$
\begin{aligned}
& pn(a_+ \to_+ b_+) \to_+ (pn(a_+) \to_+ pn(b_+)) \\
&\quad = p(n(a_+ \to_+ b_+) \wedge_- n(a_+)) \to_+ pn(b_+) && \text{Definition 26.iv} \\
&\quad = p(n(a_+) \wedge_- n(b_+)) \to_+ pn(b_+) && \text{Definition 26.iii} \\
&\quad = pn(a_+) \to_+ (pn(b_+) \to_+ pn(b_+)) && \text{Definition 26.iv} \\
&\quad = pn(a_+) \to_+ 1_+ \\
&\quad = 1_+.
\end{aligned}
$$

(xiv) Follows easily from the two properties postulated in Definiton 26.ii.

(xv) As noted earlier, the assumption $\langle a_+, a_- \rangle \preceq \langle b_+, b_- \rangle)$ simply means $a_+ \leq_+ b_+$. By monotonicity of n, from this we obtain $n(a_+) \leq_- n(b_+)$. Thus, we have $n(a_+) \wedge_- n(c_+) \leq_- n(b_+) \wedge_- n(c_+)$ for all $c_+ \in A_+$. Using the monotonicity of p, we thus obtain $p(n(a_+) \wedge_- n(c_+)) \leq_- p(n(b_+) \wedge_- n(c_+))$, which means $\langle a_+, a_- \rangle \odot \langle c_+, c_- \rangle \preceq \langle b_+, b_- \rangle \odot \langle c_+, c_- \rangle$. A similar reasoning shows that $\langle c_+, c_- \rangle \odot \langle a_+, a_- \rangle \preceq \langle c_+, c_- \rangle \odot \langle b_+, b_- \rangle$.

(xvi) We compute the first component, which is the only one that matters. Using Definiton 26.iii, $\langle a_+, a_- \rangle \odot (\langle a_+, a_- \rangle \to \langle b_+, b_- \rangle)$ gives us $p(n(a_+) \wedge_- n(a_+ \to_+ b_+)) = p(n(a_+) \wedge_- n(b_+))$, as desired.

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(xvii) We compute the first component, which is the only one that matters. It is then sufficient to check that, by Definition 26.ii, $\sim\sim\langle a_+, a_-\rangle \odot \sim\sim\langle b_+, b_-\rangle$ gives us $p(npn(a_+) \wedge_- npn(a_+)) = p(n(a_+) \wedge_- n(a_+))$.

(xviii) As before, we compute the first component. Then, $\sim(\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle)$ gives us $p(n(a_+) \wedge_- b_-)$ and, recalling Definiton 26.ii, $\sim(\sim\sim\langle a_+, a_-\rangle \to \sim\sim\langle b_+, b_-\rangle)$ gives us $p(npn(a_+) \wedge_- b_-) = p(n(a_+) \wedge_- b_-)$.

□

Lemma 30 immediately entails the following.

Proposition 31. *Every QNI twist-algebra is a QNI-algebra.*

Theorem 5. *Every QNI algebra* $\mathbf{A}$ *is isomorphic to a QNI twist-algebra over* $\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}\rangle$ *through the map* $\iota\colon A \to A_+ \times A_-$ *given by* $\iota(a) := \langle [a], [\sim a]\rangle$ *for all* $a \in A$.

Proof. It follows from our previous observations that the tuple $\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}\rangle$ satisfies the conditions postulated in Definition 26 for constructing the algebra $\mathbf{A}_+ \bowtie \mathbf{A}_-$. Let us check that ι is an embedding of $\mathbf{A}$ into $\mathbf{A}_+ \bowtie \mathbf{A}_-$. Definition 21.i guarantees that ι is injective. It is also easy to check that ι preserves $\sim$ and the bounds. Regarding the implication, we have

$$
\begin{aligned}
\iota(a \to b) &= \langle [a \to b], [\sim(a \to b)]\rangle \\
&= \langle [a \to b], [\sim(\sim\sim a \to \sim\sim b)]\rangle && \text{by Definition 21.xvii} \\
&= \langle [a \to b], [\sim\sim a \odot \sim b]\rangle \\
&= \langle [a \to b], [\sim\sim a] \wedge_- [\sim b]\rangle \\
&= \langle [a] \to_+ [b], n[a] \wedge_- [\sim b]\rangle \\
&= \langle [a], [\sim a]\rangle \to \langle [b], [\sim b]\rangle \\
&= \iota(a) \to \iota(b).
\end{aligned}
$$

It is clear that $\pi_1[A] = A_+$. Lastly, recalling Lemma 22.vii, we have, for all $\langle [a], [\sim a]\rangle \in \iota(A)$, $n[a] \wedge_- [\sim a] = [\sim\sim a] \wedge_- [\sim a] = [\sim\sim a \odot \sim a] = [0] = 0_-$, as required by Definition 26. □

The result of Theorem 5 (like that of Theorem 4) is, for the time being, only an embedding and not an isomorphism. However, the expressive power of the weak implication allows one to obtain certain results on QNI-algebras analogous to, for example, Proposition 11. We are going to show how in the next subsection.

4.1 Congruences of QNI-algebras

In this subsection, we take a look at congruences of QNI-algebras. In particular, we are going to show that this variety is congruence-distributive and has both the strong congruence extension property and equationally definable principal congruences (see e.g. Blok and Pigozzi 1994 for the relevant definitions).

Lemma 32. *Let* $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$ *be as per Definition 26. Then* $a_+ \to_+ p(n(a_+) \wedge_- a_-) = a_+ \to_+ p(a_-)$ *for all* $a_+ \in H_+$ *and all* $a_- \in M_-$.

Proof. Since $n(a_+) \wedge_- a_- \leq_- a_-$, we have $p(n(a_+) \wedge_- a_-) \leq_+ p(a_-)$. Since the Hilbert implication is order-preserving in the second argument, this gives us $a_+ \to_+ p(n(a_+) \wedge_- a_-) \leq_+ a_+ \to_+ p(a_-)$. To show the converse inequality, observe that $(a_+ \to_+ p(a_-)) \to_+ (a_+ \to_+ p(n(a_+) \wedge_- a_-)) = a_+ \to_+ (p(a_-) \to_+ p(n(a_+) \wedge_- a_-))$ because $x \to (y \to z) \approx (x \to y) \to (x \to z)$ holds

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on every Hilbert algebra (Celani et al. 2009, Lemma 1.1.4). Thus, we may show that $a_+ \to_+ (p(a_-) \to_+ p(n(a_+) \wedge_- a_-)) = 1_+$. Using Proposition 29 and the properties of the implicative semilattice implication, we have

$$
\begin{aligned}
& a_+ \to_+ (p(a_-) \to_+ p(n(a_+) \wedge_- a_-)) = \\
& = a_+ \to_+ p(a_- \to_- (n(a_+) \wedge_- a_-)) && \text{(Proposition 29)} \\
& = a_+ \to_+ p((a_- \to_- n(a_+)) \wedge_- (a_- \to_- a_-)) \\
& = a_+ \to_+ p((a_- \to_- n(a_+)) \wedge_- 1_-) \\
& = a_+ \to_+ p(a_- \to_- n(a_+)) \\
& = a_+ \to_+ (p(a_-) \to_+ pn(a_+)) && \text{(Proposition 29)} \\
& = p(a_-) \to_+ (a_+ \to_+ pn(a_+)) && \text{(Celani et al. 2009, Lemma 1.1.3)} \\
& = p(a_-) \to_+ 1_+ && \text{(Definition 26.ii)} \\
& = 1_+.
\end{aligned}
$$
□

As before, for every QNI-algebra **A** and for all $a, b, c \in A$, we abbreviate

$$q(a, b, c) := (a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to c))).$$

Lemma 33. *Let* **A** *be a QNI-algebra and* $a, b, c, d \in A$.

(i) $q(a, a, b) = b$.
(ii) $q(a, b, a) = q(a, b, b)$.
(iii) $q(a, b, \sim c) = q(a, b, \sim q(a, b, c))$.
(iv) $q(a, b, c \to d) = q(a, b, q(a, b, c) \to q(a, b, d))$.

Proof.

(i) Using items (i) and (ii) of Definition 21, we have $q(a, a, b) = (a \to a) \to ((a \to a) \to ((\sim a \to \sim a) \to ((\sim a \to \sim a) \to b))) = 1 \to (1 \to (1 \to (1 \to b))) = b$.

(ii) This is precisely Definition 21.v.

(iii) Assuming **A** is a QNI twist-algebra, we let $a = \langle a_+, a_- \rangle$, $b = \langle b_+, b_- \rangle$ etc. To improve readability, let us abbreviate $\alpha_1 := a_+ \to_+ b_+$, $\alpha_2 := b_+ \to_+ a_+$, $\alpha_3 := p(a_-) \to_+ p(b_-)$, $\alpha_4 := p(b_-) \to_+ p(a_-)$. Then the first component of $q(a, b, \sim c)$ is $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ p(c_-))))$ and the second one is $n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(c_+)$. Similarly, we compute the first component of $q(a, b, \sim q(a, b, c))$, which is

$$\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- c_-)))))$$

and the second, which is

$$n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))).$$

By Lemma 32, we have $\alpha_4 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- c_-) = \alpha_4 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- c_-)$, so the first component of $q(a, b, \sim q(a, b, c))$ simplifies to

$$\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- c_-))))).$$

Recalling that $x \to (y \to z) \approx y \to (x \to z)$ is valid on Hilbert algebras, we see that the latter is equal to

$$\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_4 \to_+ (\alpha_3 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- c_-)))))$$

which similarly simplifies to $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_4 \to_+ (\alpha_3 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- c_-)))))$. By a similar reasoning, the latter is equal to

$$\alpha_1 \to_+ (\alpha_4 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- c_-)))))$$

which is equal to $\alpha_1 \to_+ (\alpha_4 \to_+ (\alpha_3 \to_+ (\alpha_2 \to_+ p((n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- c_-)))))$, which in turn simplifies to $\alpha_1 \to_+ (\alpha_4 \to_+ (\alpha_3 \to_+ (\alpha_2 \to_+ p((n(\alpha_1) \wedge_- c_-)))))$. By a similar reasoning, from the latter, we can obtain $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ p(c_-))))$ as required. Let us now turn to the second components. Applying repeatedly Definition 26.iii, we have

$$n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) =$$
$$= n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))$$
$$= n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3 \to_+ (\alpha_4 \to_+ c_+)) =$$
$$= n(\alpha_4) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4 \to_+ c_+) =$$
$$= n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(c_+)$$

as required.

(iv) Maintaining the preceding abbreviations, the first component of $q(a, b, c \to d)$ is $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ (c_+ \to_+ d_+))))$ and the second is $n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(c_+) \wedge_- d_-$. On the other hand, the first component of $q(a, b, q(a, b, c) \to q(a, b, d))$ is $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ ((\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) \to_+ (\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ d_+))))))))$ and the second one is $n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- d_-$. Using the lattice properties and (repeatedly) Definition 26.iii, this last expression can be simplified as follows: $n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- d_- = n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) \wedge_- d_- = n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+))) \wedge_- d_- = n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n((\alpha_3 \to_+ (\alpha_4 \to_+ c_+)) \wedge_- d_- = n(\alpha_4) \wedge_- n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4 \to_+ c_+) \wedge_- d_- = n(\alpha_1) \wedge_- n(\alpha_2) \wedge_- n(\alpha_3) \wedge_- n(\alpha_4) \wedge_- n(c_+) \wedge_- d_-$, as required.

Regarding the first component, observe that, using (H2) from the definition of Hilbert algebras, we have

$$((\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) \to_+ (\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ d_+)))) =$$
$$= \alpha_1 \to_+ ((\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+))) \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ d_+)))) =$$
$$= \alpha_1 \to_+ (\alpha_2 \to_+ ((\alpha_3 \to_+ (\alpha_4 \to_+ c_+)) \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ d_+)))) =$$
$$= \alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ ((\alpha_4 \to_+ c_+) \to_+ (\alpha_4 \to_+ d_+)))) =$$
$$= \alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ (c_+ \to_+ d_+).$$

Thus, $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ ((\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ c_+)))) \to_+ (\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ d_+))))))))$ simplifies to

$$\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ (\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ (c_+ \to_+ d_+))))))))$$

which, using again (H2) and the commutative identity $x \to (y \to z) \approx y \to (x \to z)$, simplifies to $\alpha_1 \to_+ (\alpha_2 \to_+ (\alpha_3 \to_+ (\alpha_4 \to_+ (c_+ \to_+ d_+))))$, as required.

□

Lemma 33 allows us to apply (Blok and Pigozzi 1994, Theorem 2.3, Corollary 2.4) to obtain the following result.

Corollary 34. *$q(x, y, z)$ is a (commutative, non-regular) ternary deduction term in the sense of Blok and Pigozzi (1994). Therefore, the variety of QNI-algebras has equationally definable principal congruences and the strong congruence extension property (Blok and Pigozzi, 1994, Theorem 2.12).*

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Corollary 34 applies to the variety of QNAs as well, and more generally to all the subreducts that include at least the operations $\sim$ and $\to$.

Lemma 35. *Let* $\mathbf{A}$ *be a QNI-algebra,* $a, b \in A$ *and* $\theta \in \mathrm{Con}(\mathbf{A})$*. The following conditions are equivalent:*

(i) $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$.
(ii) $\langle a \to c, b \to c\rangle \in \theta$ *for all* $c \in A$.

Proof. The proof resembles and generalises that of Lemma 22.ii. Assume $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$. Let $c \in A$. From the first assumption (using also Definition 21.i), we have $\langle (a \to b) \to (a \to c), 1 \to (a \to c)\rangle = \langle (a \to b) \to (a \to c), a \to c\rangle \in \theta$. Using Definition 21.iii, we have $(a \to b) \to (a \to c) = a \to (b \to c) = b \to (a \to c) = (b \to a) \to (b \to c)$. Thus, $\langle (b \to a) \to (b \to c), a \to c\rangle \in \theta$. From the assumption $\langle b \to a, 1\rangle \in \theta$, reasoning as before, we get $\langle (b \to a) \to (b \to c), 1 \to (b \to c)\rangle = \langle (b \to a) \to (b \to c), b \to c\rangle \in \theta$. Hence, by symmetry and transitivity of θ, we obtain $\langle a \to c, b \to c\rangle \in \theta$, as claimed.

Conversely, assume $\langle a \to c, b \to c\rangle \in \theta$ for all $c \in A$. Then, using using Definition 21.ii, we have $\langle a \to b, b \to b\rangle = \langle a \to b, 1\rangle \in \theta$ and likewise $\langle b \to a, a \to a\rangle = \langle b \to a, 1\rangle \in \theta$, as required. □

The following observation is folklore on Hilbert algebras, but we provide a short self-contained proof.

Lemma 36. *Let* $\mathbf{H} = \langle H; \to, 1\rangle$ *be a Hilbert algebra,* $a, b \in A$ *and* $\eta \in \mathrm{Con}(\mathbf{H})$*. The following conditions are equivalent:*

(i) $\langle a, b\rangle \in \eta$.
(ii) $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \eta$.

Proof. Let $\langle a, b\rangle \in \eta$. Using the identity $x \to x \approx x$, we have $\langle a \to b, b \to b\rangle = \langle a \to b, 1\rangle \in \eta$ and $\langle b \to a, b \to b\rangle = \langle b \to a, 1\rangle \in \eta$, as required. Conversely, consider the quotient $\mathbf{H}/\eta$. As observed in Celani et al. (2009, p. 464), the class of Hilbert algebras is a variety (hence closed under homomorphic images). Thus, $\mathbf{H}/\eta$ is a Hilbert algebra. Now, assuming $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \eta$, in $\mathbf{H}/\eta$ we have $[a]_\eta \to [b]_\eta = [b]_\eta \to [a]_\eta = [1]_\eta$. Thus, by (H3), we have $[a]_\eta = [b]_\eta$, that is, $\langle a, b\rangle \in \eta$. □

Proposition 37. *Let* $\mathbf{A}$ *be a QNI-algebra, and let* $\mathbf{A}_+$ *be the corresponding bounded Hilbert algebra (as per Theorem 5). Let* $\theta \in \mathrm{Con}(\mathbf{A})$ *and* $\eta \in \mathrm{Con}(\mathbf{A}_+)$*.*

(i) $\theta_+ \in \mathrm{Con}(\mathbf{A}_+)$, *where* $\theta_+ := \{\langle [a], [b]\rangle \in A_+ \times A_+ : \langle a \to e, b \to e\rangle \in \theta$ *for all* $e \in A\}$.
(ii) $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$, *where* $\eta^{\bowtie} := \{\langle a, b\rangle \in A \times A : \langle [a], [b]\rangle, \langle [\sim a], [\sim b]\rangle \in \eta\}$.
(iii) $(\theta_+)^{\bowtie} = \theta$.
(iv) $(\eta^{\bowtie})_+ = \eta$.

Proof. (i) To check that θ_+ is well defined, let $\langle [a], [b]\rangle \in \theta_+$. Assume $[a] = [a']$ and $[b] = [b']$ for some $a', b' \in A$. Then, by Lemma 22.ii, we have $\langle a' \to e, b' \to e\rangle = \langle a \to e, b \to e\rangle \in \theta$, as required. Let us now check that $\theta_+ \in \mathrm{Con}(\mathbf{A}_+)$. To this end, assume $\langle [a], [b]\rangle, \langle [c], [d]\rangle \in \theta_+$, that is, $\langle a \to e, b \to e\rangle, \langle c \to e, d \to e\rangle \in \theta$ for all $e \in A$. We need to prove that $\langle (a \to c) \to e, (b \to d) \to e\rangle \in \theta$ for all $e \in A$. Observe that the assumptions imply $\langle (a \to c) \to (b \to d), 1\rangle, \langle (b \to d) \to (a \to c), 1\rangle \in \theta$. We prove only the former. Indeed, taking $e = c$ in the first assumption, $\langle a \to c, b \to c\rangle \in \theta$, and using Definition 21.iii, we have $\langle (a \to c) \to (b \to d), (b \to c) \to (b \to d)\rangle = \langle (a \to c) \to (b \to d), b \to (c \to d)\rangle \in \theta$. Now, taking $e =$

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d, from the assumption $\langle c \to d, d \to d\rangle \in \theta$, recalling Definition 21.i and Lemma 22.i, we have $\langle b \to (c \to d), b \to (d \to d)\rangle = \langle b \to (c \to d), b \to 1\rangle = \langle b \to (c \to d), 1\rangle \in \theta$. Then, by transitivity of θ, we obtain $\langle (a \to c) \to (b \to d), 1\rangle \in \theta$, as claimed. A similar reasoning allows us to get $\langle (b \to d) \to (a \to c), 1\rangle \in \theta$. Recalling Definition 21.i, from $\langle (a \to c) \to (b \to d), 1\rangle \in \theta$ we have $\langle ((a \to c) \to (b \to d)) \to ((a \to c) \to e), 1 \to ((a \to c) \to e)\rangle = \langle ((a \to c) \to (b \to d)) \to ((a \to c) \to e), (a \to c) \to e\rangle \in \theta$. By Definition 21.iii, we have $((a \to c) \to (b \to d)) \to ((a \to c) \to e) = (a \to c) \to ((b \to d) \to e) = (b \to d) \to ((a \to c) \to e) = ((b \to d) \to (a \to c)) \to ((b \to d) \to e)$. Thus, $\langle ((b \to d) \to (a \to c)) \to ((b \to d) \to e), (a \to c) \to e\rangle \in \theta$. Having established earlier that $\langle (b \to d) \to (a \to c), 1\rangle \in \theta$, using Definition 21.i we obtain $\langle ((b \to d) \to (a \to c)) \to ((b \to d) \to e), 1 \to ((b \to d) \to e)\rangle = \langle ((b \to d) \to (a \to c)) \to ((b \to d) \to e), (b \to d) \to e\rangle \in \theta$. Hence, by symmetry and transitivity of θ, we get $\langle (a \to c) \to e, (b \to d) \to e\rangle \in \theta$, as required.

(ii) Assuming $\langle a, b\rangle \in \eta^{\bowtie}$, we first need to show $\langle \sim a, \sim b\rangle \in \eta^{\bowtie}$, that is, $\langle [\sim a], [\sim b]\rangle, \langle [\sim\sim a], [\sim\sim b]\rangle \in \eta$, or indeed just $\langle [\sim\sim a], [\sim\sim b]\rangle \in \eta$. From the assumption $\langle [a], [b]\rangle \in \eta$ (and using the equation $x \to x = 1$, which holds on any Hilbert algebra), we get $\langle [a] \to_+ [b], [b] \to_+ [b]\rangle = \langle [a \to b], [1]\rangle \in \eta$ and $\langle [b] \to_+ [a], [b] \to_+ [b]\rangle = \langle [b \to a], [1]\rangle \in \eta$. From $\langle [a \to b], [1]\rangle \in \eta$, recalling Definition 21.i and. xi, we have $\langle [a \to b] \to_+ [\sim\sim a \to \sim\sim b], [1] \to_+ [\sim\sim a \to \sim\sim b]\rangle = \langle [(a \to b) \to (\sim\sim a \to \sim\sim b)], [1 \to (\sim\sim a \to \sim\sim b)]\rangle = \langle [1], [\sim\sim a \to \sim\sim b]\rangle = \langle 1_+, [\sim\sim a] \to_+ [\sim\sim b]\rangle \in \eta$. Likewise, from $\langle [b \to a], [1]\rangle \in \eta$ we can obtain $\langle 1_+, [\sim\sim b] \to_+ [\sim\sim a]\rangle \in \eta$. Then, applying Lemma 36, we have $\langle [\sim\sim a], [\sim\sim b]\rangle \in \eta$, as required. Thus $\eta^{\bowtie}$ is compatible with $\sim$. Let us check the compatibility with $\to$.

Assume $\langle a, b\rangle, \langle c, d\rangle \in \eta^{\bowtie}$, that is, $\langle [a], [b]\rangle, \langle [\sim a], [\sim b]\rangle, \langle [c], [d]\rangle, \langle [\sim c], [\sim d]\rangle \in \eta$. We need to check that $\langle [a \to c], [b \to d]\rangle, \langle [\sim (a \to c)], [\sim (b \to d)]\rangle \in \eta$. From $\langle [a], [b]\rangle, \langle [c], [d]\rangle \in \eta$, we immediately have $\langle [a] \to_+ [c], [b] \to_+ [d]\rangle = \langle [a \to c], [b \to d]\rangle \in \eta$. We proceed to show $\langle [\sim (a \to c)], [\sim (b \to d)]\rangle \in \eta$. Let $e \in A$ be an arbitrary element. From the assumption $\langle [\sim c], [\sim d]\rangle \in \eta$, we have $\langle [\sim c] \to_+ [e], [\sim d] \to_+ [e]\rangle = \langle [\sim c \to e], [\sim d \to e]\rangle \in \eta$. Also, as proved earlier, the assumption $\langle [a], [b]\rangle \in \eta$ implies $\langle [\sim\sim a], [\sim\sim b]\rangle \in \eta$. From this and $\langle [\sim c \to e], [\sim d \to e]\rangle \in \eta$, we have $\langle [\sim\sim a] \to_+ [\sim c \to e], [\sim\sim b] \to_+ [\sim d \to e]\rangle = \langle [\sim\sim a \to (\sim c \to e)], [\sim\sim b \to (\sim d \to e)]\rangle \in \eta$. Observe that, by Definition 21.iv, we have $\sim\sim a \to (\sim c \to e) = (\sim\sim a \odot \sim c) \to e = \sim (\sim\sim a \to \sim\sim c) \to e$, and likewise $\sim\sim b \to (\sim d \to e) = \sim (\sim\sim b \to \sim\sim d) \to e$. We then have $\langle [\sim\sim a \to (\sim c \to e)], [\sim\sim b \to (\sim d \to e)]\rangle = \langle [\sim (\sim\sim a \to \sim\sim c) \to e], [\sim (\sim\sim b \to \sim\sim d) \to e]\rangle \in \eta$. Using Definition 21.xvi, we have $[\sim(\sim\sim a \to \sim\sim c) \to e] = [\sim(\sim\sim a \to \sim\sim c)] \to_+ [e] = [\sim (a \to c)] \to_+ [e]$, and likewise $[\sim (\sim\sim b \to \sim\sim d) \to e] = [\sim (b \to d)] \to_+ [e]$. Hence, $\langle [\sim (a \to c)] \to_+ [e], [\sim (b \to d)] \to_+ [e]\rangle \in \eta$. Recalling that the element e is arbitrary, we can let $e = \sim (b \to d)$ and we have $\langle [\sim(a \to c)] \to_+ [\sim(b \to d)], [\sim(b \to d)] \to_+ [\sim (b \to d)]\rangle = \langle [\sim (a \to c)] \to_+ [\sim (b \to d)], [1]\rangle \in \eta$. Similarly, letting $e = \sim (a \to c)$, we obtain $\langle [\sim (a \to c)] \to_+ [\sim(a \to c)], [\sim(b \to d)] \to_+ [\sim (a \to c)]\rangle = \langle [1], [\sim (b \to d)] \to_+ [\sim (a \to c)]\rangle$. Thus, we can invoke Lemma 36 to obtain $\langle [\sim (a \to c)], [\sim (b \to d)]\rangle \in \eta$, as was required. Hence, $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$.

(iii) By definition, we have $\langle a, b\rangle \in (\theta_+)^{\bowtie}$ iff $\langle [a], [b]\rangle, \langle [\sim a], [\sim b]\rangle \in \theta_+$ iff $\langle a \to e, b \to e\rangle, \langle \sim a \to e, \sim b \to e\rangle \in \theta$ for all $e \in A$. It is thus easy to see that $\theta \subseteq (\theta_+)^{\bowtie}$. To show $(\theta_+)^{\bowtie} \subseteq \theta$, assume $\langle a \to e, b \to e\rangle, \langle \sim a \to e, \sim b \to e\rangle \in \theta$ for all $e \in A$. By Lemma 35, we then have $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle, \langle \sim a \to \sim b, 1\rangle, \langle \sim b \to \sim a, 1\rangle \in \theta$. From $\langle \sim b \to \sim a, 1\rangle \in \theta$, using Definition 21.i, we get $\langle (\sim b \to \sim a) \to a, 1 \to a\rangle = \langle (\sim b \to \sim a) \to a, a\rangle \in \theta$. Likewise, from $\langle (\sim b \to \sim a) \to a, a\rangle \in \theta$ and $\langle \sim a \to \sim b, 1\rangle \in \theta$, we have $\langle (\sim a \to \sim b) \to ((\sim b \to \sim a) \to a), 1 \to a\rangle = \langle (\sim a \to \sim b) \to ((\sim b \to \sim a) \to a), a\rangle \in \theta$. Going on in this fashion, we use the two remaining assumptions $\langle a \to b, 1\rangle, \langle b \to$

$a, 1\rangle \in \theta$ to successively obtain $\langle(b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to a)), a\rangle \in \theta$ and $\langle(a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to a))), a\rangle \in \theta$. A similar reasoning gives us $\langle(a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to b))), b\rangle \in \theta$. By Definition 21.v, $(a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to a))) = (a \to b) \to ((b \to a) \to ((\sim a \to \sim b) \to ((\sim b \to \sim a) \to b)))$. Then, by symmetry and transitivity of θ, we obtain $\langle a, b\rangle \in \theta$, as required.

(iv) By definition, we have $\langle [a], [b]\rangle \in (\eta^{\bowtie})_+$ iff $\langle a \to e, b \to e\rangle \in \eta^{\bowtie}$ for all $e \in A$ iff $\langle [a \to e], [b \to e]\rangle, \langle [\sim (a \to e)], [\sim (b \to e)]\rangle \in \eta$ for all $e \in A$. To check that $(\eta^{\bowtie})_+ \subseteq \eta$, assume $\langle [a \to e], [b \to e]\rangle, \langle [\sim (a \to e)], [\sim (b \to e)]\rangle \in \eta$. Then $\langle [a] \to_+ [e], [b] \to_+ [e]\rangle \in \eta$. Taking first $e = a$ and then $e = b$, we obtain $\langle [a] \to_+ [a], [b] \to_+ [a]\rangle = \langle [1], [b] \to_+ [a]\rangle \in \eta$ and $\langle [a] \to_+ [b], [b] \to_+ [b]\rangle = \langle [a] \to_+ [b], [1]\rangle \in \eta$. Thus, we can invoke Lemma 36 to obtain $\langle [a], [b]\rangle \in \eta$, as desired.

For the converse inclusion $\eta \subseteq (\eta^{\bowtie})_+$, assume $\langle [a], [b]\rangle \in \eta$ and $c \in A$. Then, we immediately have $\langle [a] \to_+ [c], [b] \to_+ [c]\rangle = \langle [a \to c], [b \to c]\rangle \in \eta$. To check that $\langle [\sim (a \to c)], [\sim (b \to c)]\rangle \in \eta$, we reason in a similar way to item (ii) above. We have shown there that $\langle [a], [b]\rangle \in \eta$ implies $\langle [\sim\sim a], [\sim\sim b]\rangle \in \eta$. Let $d \in A$ be an arbitrary element. From $\langle [\sim\sim a], [\sim\sim b]\rangle \in \eta$ and $\langle [\sim c \to d], [\sim c \to d]\rangle \in \eta$ (which holds just by reflexivity of η) we have $\langle [\sim\sim a] \to_+ [\sim c \to d], [\sim\sim b] \to_+ [\sim c \to d]\rangle = \langle [\sim\sim a \to (\sim c \to d)], [\sim\sim b \to (\sim c \to d)]\rangle \in \eta$. Observe that, by Definition 21.iv, we have $\sim\sim a \to (\sim c \to d) = (\sim\sim a \odot \sim c) \to d = \sim(\sim\sim a \to \sim\sim c) \to d$, and likewise $\sim\sim b \to (\sim c \to d) = \sim(\sim\sim b \to \sim\sim c) \to d$. We then have $\langle [\sim\sim a \to (\sim c \to d)], [\sim\sim b \to (\sim c \to d)]\rangle = \langle [\sim(\sim\sim a \to \sim\sim c) \to d], [\sim(\sim\sim b \to \sim\sim c) \to d]\rangle \in \eta$. Using Definition 21.xvi, we have $[\sim(\sim\sim a \to \sim\sim c) \to d] = [\sim(\sim\sim a \to \sim\sim c)] \to_+ [d] = [\sim (a \to c)] \to_+ [d]$, and likewise $[\sim(\sim\sim b \to \sim\sim c) \to d] = [\sim (b \to c)] \to_+ [d]$. Hence, $\langle [\sim (a \to c)] \to_+ [d], [\sim (b \to c)] \to_+ [d]\rangle \in \eta$. Since the element d was arbitrary, we can let $d = \sim (b \to c)$ and we have $\langle [\sim(a \to c)] \to_+ [\sim (b \to c)], [\sim(b \to c)] \to_+ [\sim(b \to c)]\rangle = \langle [\sim(a \to c)] \to_+ [\sim(b \to c)], [1]\rangle \in \eta$. Similarly, letting $d = \sim (a \to c)$, we obtain $\langle [\sim(a \to c)] \to_+ [\sim(a \to c)], [\sim(b \to c)] \to_+ [\sim(a \to c)]\rangle = \langle [1], [\sim (b \to c)] \to_+ [\sim (a \to c)]\rangle$. Thus, we can invoke Lemma 36 to obtain $\langle [\sim (a \to c)], [\sim (b \to c)]\rangle \in \eta$, as was required. □

Theorem 6. *For every QNI-algebra* $\mathbf{A}$, *one has* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_+)$.

Proof. It suffices to join together the results of Proposition 37. Observe that the map $(.)_+$ given by $\theta \mapsto \theta_+$ is order-preserving, as is the map $(.)^{\bowtie}$ given by $\eta \mapsto \eta^{\bowtie}$. Thus, by item (iii) in Proposition 37, the map $(.)_+$ is also order-reflecting. Hence, $(.)_+$ is an order-embedding of the lattice $\mathrm{Con}(\mathbf{A})$ into $\mathrm{Con}(\mathbf{A}_+)$. By item (iv) we also have that $(.)_+$ is surjective. Thus, $(.)_+$ is an order-isomorphism (and therefore a lattice isomorphism). □

It is well known that the lattice of congruences of every Hilbert algebra $\mathbf{A}_+$ is distributive (see e.g. Celani et al. 2009, p. 477). Thus Theorem 6 entails the following result.

Corollary 38. *QNI-algebras are congruence-distributive.*

5. Future Work

We would like to conclude the paper by mentioning a few directions for research to be pursued in future publications.

1. As observed earlier, the twist constructions for quasi-Kleene and QNI-algebras only yield an embedding result. A natural question to be addressed next is whether these constructions can be improved to obtain a full representation. Previous experience (see e.g. Rivieccio 2014) suggests

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that this issue may be a hard one to tackle due to the reduced expressive power or the language of quasi-Kleene and QNI-algebras. Regarding the latter, we also note that we have not yet established a result entailing that every QNI-algebra is embeddable into a QNA. This problem appears to be feasible but will require a deeper study of the notion of canonical extension in the context of Hilbert algebras.

2. Rivieccio (2020a) introduced a variation of the twist construction that, due to space limitations, we have altogether passed over in the present paper. The idea is quite simple, and not entirely new in the context of algebras of non-classical logics. Thanks to the properties of the maps n and p, it is possible to replace (say) a structure of type $\langle \mathbf{A}_+, \mathbf{A}_-, n, p, \nabla \rangle$ by a structure of type $\langle \mathbf{A}_+, f, \nabla \rangle$ where $f = p \circ n$ is a map from A_+ to itself satisfying suitable properties. The algebra $\mathbf{A}_-$ can then be recovered by defining algebraic operations on the direct image $f[A_+]$, and maps n, p between A_+ and A_- can be defined straightforwardly. The properties that characterise f are reminiscent of certain modal operators, more precisely the operators known in the literature as *nuclei* (for further details, see Rivieccio 2020a and the references cited therein). This 'nuclear twist construction' approach can be pursued so as to obtain alternative representations for all the classes of algebras considered in the present paper (quasi-Nelson, quasi-Kleene, WPQK and QNI-algebras). Furthermore, as observed by Rivieccio (2020a), this perspective can indeed provide new insight into the above-mentioned classes of algebras (in particular, concerning congruences and duality). We plan to work out the details of these representations in a forthcoming publication.

3. The approach outlined in the present paper, and in particular the twist representation, can be applied to the study of other fragments of the quasi-Nelson language; indeed, as observed in Rivieccio (2020a, Section 10), the landscape of fragments appears to be much more complex and interesting in a non-involutive setting than in that of Nelson algebras. The insight gained thanks to the twist representation can in turn be applied to the study of these classes of algebras from different standpoints, for example, from the point of view of topological duality. This line of research is current work in progress (see Rivieccio et al. 2020b) and will be further pursued in subsequent publications.

4. We conclude with a general question that, we believe, may deserve further investigation. As illustrated in Subsection 3.1, the twist representation provides a bridge between subvarieties of QNAs and subvarieties of Heyting algebras, and similar results can be obtained on other classes of subreducts (e.g. WPQK-algebras). These correspondences can be formulated as categorical equivalences (as done e.g. in Rivieccio et al. 2020b) between algebraic categories corresponding to (say) subvarieties of QNAs and categories having objects of type $\langle \mathbf{H}_+, \mathbf{H}_-, n, p, \nabla \rangle$ with $\mathbf{H}_+, \mathbf{H}_-$ Heyting algebras (or, as explained above, objects of type $\langle \mathbf{H}, f, \nabla \rangle$ with $\mathbf{H}$ a Heyting algebra and f a nucleus). In this setting, however, it appears that non-equational conditions imposed on n, p or ∇ (cf. Proposition 8 in Subsection 3.1) correspond to equations in the language of QNAs. The question then arises of whether it may be possible to define translations between, for example, the language of (possibly enriched) Heyting algebras and that of, for example, QNAs that would allow one to place the correspondences mentioned in Subsection 3.1 in a unified and systematic framework. This project will likely entail addressing the issue of how the class operators of universal algebra (H, S, P, etc.) interact with the twist construction in the non-involutive setting.

Notes

1 We use $\approx$ for the formal equality symbol in the formal languages used to express equations and quasi-equations.

2 In fact, since n and p are monotone maps and $n \circ p \leq_- Id_{H_-}$ and $Id_{H_+} \leq_+ p \circ n$, we have that n and p form an adjoint pair from the poset $\langle H_+, \leq_+ \rangle$ to the poset $\langle H_-, \leq_- \rangle$. This entails that n preserves arbitrary existing joins and p preserves arbitrary existing meets (cf. Remark 27 below).

3 To be more precise, the classes (i) and (iv) coincide, (i) and (iii) are term equivalent, and (iii) is the closure of (ii) under isomorphic copies.

4 In fact, (the logical counterpart of) this result can also be obtained as an application of Galatos et al. (2007, Lemma 5.25)

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5 Indeed, one may even consider unbounded Kleene algebras, in which case $\mathbf{A}_+, \mathbf{A}_-$ will still have a least but not necessarily the greatest element. See below and Rivieccio (2020b) for further details.

6 There are obviously other fragments of quasi-Nelson logic that are also algebraisable, and in which the connectives $\to$ and $\sim$ may not be definable. For example, the $\{\Rightarrow\}$-fragment, in which 1 and $\to$ are definable but $\sim$ is not. This fragment may turn out to be beyond the scope of applicability of twist constructions.

7 Bounded implicative semilattices are the $\langle \wedge, \to, 0, 1\rangle$-subreducts of Heyting algebras and correspond to the conjunction-implication-negation fragment of intuitionistic logic. Abstractly, an algebra $\langle M; \wedge, \to, 0, 1\rangle$ is a bounded implicative semilattice if and only if (i) $\langle M; \wedge, 0, 1\rangle$ is a bounded semilattice and (ii) $\to$ is the residuum of $\wedge$, that is, $a \wedge b \leq c$ iff $a \leq b \to c$ for all $a, b, c \in M$.

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