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EXISTENCE OF GIBBS POINT PROCESSES WITH STABLE INFINITE RANGE INTERACTION

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Abstract

We provide a new proof of the existence of Gibbs point processes with infinite range interactions, based on the compactness of entropy levels. Our main existence theorem holds under two assumptions. The first one is the standard stability assumption, which means that the energy of any finite configuration is superlinear with respect to the number of points. The second assumption is the so-called intensity regularity, which controls the long range of the interaction via the intensity of the process. This assumption is new and introduced here since it is well adapted to the entropy approach. As a corollary of our main result we improve the existence results by Ruelle (1970) for pairwise interactions by relaxing the superstabilty assumption. Note that our setting is not reduced to pairwise interaction and can contain infinite-range multi-body counterparts.

Keywords: DLR equations; entropy bounds; superstable interaction

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1. Introduction

The starting point of the theory is an energy function H defined on the space of locally finite configurations in $\mathbb{R}^d$. In the following, ω denotes such a point configuration and $H(\omega)$ its energy. Then the finite-volume Gibbs measure on a bounded set $\Lambda \subset \mathbb{R}^d$ is simply the probability measure

$$P_\Lambda = \frac{1}{Z_\Lambda} \mathrm{e}^{-H} \pi_\Lambda,$$

where π_Λ is the Poisson point process in Λ with intensity one and Z_Λ the normalization constant. The existence of P_Λ is guaranteed by the stability condition recalled below. The existence of an infinite volume measure, corresponding to the case '$\Lambda = \mathbb{R}^d$', is not obvious and cannot be achieved by the definition above. In fact, the general strategy is first to obtain a suitable thermodynamic limit for (P_Λ) when Λ tends to $\mathbb{R}^d$ and then derive a good description of the limiting point by the Dobrushin–Lanford–Ruelle (DLR) equations. The first general result in this direction is due to Ruelle in the 1970s [14, 15]. The setting was the pairwise interaction

$$H(\omega) = \sum_{x \neq y \in \omega} \phi(|x-y|),$$

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where the potential ϕ is assumed *regular*, which means roughly that ϕ is summable at infinity (i.e. $\int_R^{+\infty} r^{d-1}|\phi(r)|\,\mathrm{d}r < +\infty$ for $R > 0$ large enough) and *superstable*: there exist two constants $A > 0$ and $B \geq 0$ such that, for any bounded set $\Lambda \subset \mathbb{R}^d$ and for any finite configuration ω in Λ,

$$H(\omega) \geq -B|\omega| + \frac{A}{\lambda^d(\Lambda)}|\omega|^2,$$

where $|\omega|$ denotes the number of points in ω and λ^d the Lebesgue measure on $\mathbb{R}^d$. Under these two assumptions, Ruelle proved the existence of at least one infinite Gibbs point process. Similar results have been proved more recently in [10, 11] using functional analysis tools. In all cases the superstability assumption is required. As a corollary of our main result we improve these existence results in substituting the superstability assumption with the *stability* assumption: there exists a constant $B \geq 0$ such that, for any finite configuration ω,

$$H(\omega) \geq -B|\omega|.$$

Note that the difference between stable and superstable potential is in fact weak since any stable potential becomes superstable if a pairwise continuous non-negative and non-null-at-origin potential is added. However, there exist several pairwise stable potentials which are not superstable. For instance, any continuous, non-negative pairwise potential null at the origin is stable without being superstable. Note also that several examples of stable (and non-superstable) energy functions have been introduced recently in stochastic geometry and spatial statistics [1, 5, 13].

A multi-body interaction occurs when the energy function H is decomposed using potentials on pairs, triplets, quadruplets, k-plets of points (but not only on pairs). Our results do not use and do not depend on such decompositions. Therefore, our existence result covers several multi-body interactions, including infinite-range cases (examples are given in Section 3). We note that finite-range multi-body interactions were treated in [2] using Dobrushin's criterion.

Our main tool is the compactness of entropy level sets for the local convergence topology. This tool is particularly efficient for proving the tightness of the sequence of finite-volume Gibbs point processes. It was used for the first time in [8], and then a collection of papers followed [3, 5, 6]. Before the present paper, the entropy strategy has been used only in the setting of finite-range or random finite-range interactions. As far as we know, this is the first time that it has been applied in the setting of pure infinite-range interactions. Therefore, our main contribution here was to develop a way to control the decay of the interaction adequately with the entropy approach. This is the reason for introducing the *intensity regular* assumption (see Definition 4). It is called 'intensity regular' because the decay of the interaction is controlled via the intensity of the process. This choice is directly related to the entropy bounds, which gives uniform control of intensities for finite-volume Gibbs processes within the thermodynamic limit. In the setting of pairwise interactions, our definition is similar to the regular assumption by Ruelle.

As mentioned before, in the setting of pairwise interactions, the entropy strategy improves the existence results in relaxing the superstability assumption. However, we lose Ruelle's estimates which ensure the existence of moments of any order and some exponential and super-exponential moments. The entropy approach only provides moments of order one. Better estimates have to be obtained with different tools.

In Section 2 we introduce the definitions and notations for Gibbs point processes. Our main existence theorem is also given. Examples of energy functions are presented in Section 3. Finally, Section 4 is devoted to the proof of the theorem.

2. Notations and results

The real d-dimensional space $\mathbb{R}^d$ is equipped with the usual Euclidean distance $\|\cdot\|$ and its associated Borel σ-algebra. Any set $\Lambda \subset \mathbb{R}^d$ is assumed measurable.

2.1. Finite volume measure

The space of configurations is the set of locally finite subsets of $\mathbb{R}^d$,

$$\Omega = \{\omega \subset \mathbb{R}^d : |\omega \cap \Delta| < \infty, \text{ for all } \Delta \subset \mathbb{R}^d, \text{ bounded}\},$$

where $|\cdot|$ is the cardinal. We denote $\omega \cap \Delta$ by ω_Δ, the union $\omega \cup \omega'$ of two configurations by $\omega\omega'$, the space of finite configurations by Ω_f and the space of configurations in $\Lambda \subset \mathbb{R}^d$ by Ω_Λ.

Our space is equipped with the sigma field $\mathcal{F}$ generated by the counting functions $N_\Delta : \omega \mapsto |\omega_\Delta|$ for all $\Delta \subset \mathbb{R}^d$ bounded. In our setting, a point process is simply a probability measure on $(\Omega, \mathcal{F})$. Note that with this definition we identify a point process with its distribution. We say that the process has a finite intensity if, for all bounded subsets Δ, the expectation $\mathbb{E}_P[|\omega_\Delta|]$ is finite. If we denote this expectation by $\mu(\Delta)$ then μ is a sigma-finite measure on $\mathbb{R}^d$ called the intensity measure. When $\mu = i(P)\lambda^d$, where λ^d is the Lebesgue measure on $\mathbb{R}^d$ and $i(P) \geq 0$, we simply say that the point process has intensity $i(P)$. We also introduce

$$\xi(P) = \sup_{\substack{\Lambda \subset \mathbb{R}^d \\ 0 < \lambda^d(\Lambda) < +\infty}} \frac{\mathbb{E}^P[|\omega_\Lambda|]}{\lambda^d(\Lambda)},$$

and we say that a probability measure P has a bounded intensity if $\xi(P) < +\infty$. Obviously, if P has a finite intensity $i(P)$ then $\xi(P) = i(P)$.

A point process P is stationary if, for all $u \in \mathbb{R}^d$, $P = P \circ \tau_u^{-1}$, where τ_u is the translation of vector u. If a stationary point process has a finite intensity then its intensity measure is proportional to the Lebesgue measure and has the form $\mu = i(P)\lambda^d$.

The most popular point processes are the Poisson point processes. We consider here only the homogeneous (or stationary) case where the intensity has the form $\mu = z\lambda^d$. The process is denoted π^z, or simply π if $z = 1$. Recall briefly that π^z is the only point process in $\mathbb{R}^d$ with intensity $\mu = i(P)\lambda^d$ such that any two disjoint regions of space are independent under π^z. See the recent book [12] on the subject.

Let us now define the interaction between the points. We need to introduce an energy function.

Definition 1. An energy function is a measurable function H on the space of finite configurations Ω_f with values in $\mathbb{R} \cup \{+\infty\}$ such that:

- H is *non-degenerate*: $H(\emptyset) < +\infty$;
- H is *hereditary*: for all $\omega \in \Omega_\mathrm{f}$ and $x \in \omega$,

$$H(\omega) < +\infty \Rightarrow H(\omega \setminus \{x\}) < +\infty;$$

- H is *stationary*: for all $\omega \in \Omega_\mathrm{f}$ and $u \in \mathbb{R}^d$,

$$H(\tau_u(\omega)) = H(\omega).$$

A crucial assumption is the stability of the energy function.

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Definition 2. An energy function is said to be *stable* if there exists a constant $B \geq 0$ such that, for all $\omega \in \Omega_f$,

$$H(\omega) \geq -B|\omega|.$$

The above assumption (denoted [*Stable*]) is standard and has been treated deeply in the literature (see, for instance, [14, Section 3.2]).

Now we can define the Gibbs point processes in finite volume.

Definition 3. Let Λ be a bounded subset in $\mathbb{R}^d$. The Gibbs point process on Λ for the stable energy function H is the probability measure on Ω_Λ defined by

$$P_\Lambda(\mathrm{d}\omega) = \frac{1}{Z_\Lambda} \mathrm{e}^{-H(\omega)} \pi_\Lambda(\mathrm{d}\omega),$$

with the normalization constant $Z_\Lambda = \int \mathrm{e}^{-H(\omega)} \pi_\Lambda(\mathrm{d}\omega)$ called the partition function.

We can check with the properties of H that P_Λ is well defined (i.e. $0 < Z_\Lambda < +\infty$). In comparison with the standard formalism of Gibbs measures in statistical physics (see [14], for instance), the activity and inverse temperature parameters are included in the function H here.

2.2. Infinite volume measure

Let us turn now to the definition of Gibbs point processes in the infinite volume regime. For the following, we choose the convention $\infty - \infty = 0$. We need to introduce the local energy, which is given by $H_\Delta(\omega) = H(\omega) - H(\omega_{\Delta^c})$ for a finite configuration $\omega \in \Omega_f$. It represents the contribution of energy coming from ω_Δ in ω (the difference of energies within and without ω_Δ). We need to extend this definition to an infinite configuration. If $(\Delta_l)_{l\geq 0}$ is an increasing sequence of subsets in $\mathbb{R}^d$ with $\Delta_l \uparrow \mathbb{R}^d$, we expect that the limit $\lim_{l\to+\infty} H^l_\Delta(\omega)$ exists, where

$$H^l_\Delta(\omega) = H\big(\omega_{\Delta_l}\big) - H\big(\omega_{\Delta_l \setminus \Delta}\big) \in \mathbb{R} \cup \{+\infty\}.$$

In particular, the difference $H^{l+1}_\Delta(\omega) - H^l_\Delta(\omega)$ should go to zero as l goes to infinity. Our main assumption is a control of the expectation of this difference for point processes with bounded intensities.

Definition 4. An energy function is said to be *intensity regular* if, for all bounded subsets Δ of $\mathbb{R}^d$, we can find an increasing sequence of subsets $(\Delta_l)_{l\geq 0}$ with $\Delta_l \uparrow \mathbb{R}^d$ and $\Delta \subset \Delta_0$ such that, for all configurations ω,

$$\left| H^{l+1}_\Delta(\omega) - H^l_\Delta(\omega) \right| \leq |\omega_\Delta| G^l_\Delta(\omega_{\Delta^c}),$$

where G^l_Δ is a non-negative measurable function on Ω_{Δ^c} such that, for any probability measure P with bounded intensity, we have

$$\mathbb{E}_P\left[G^l_\Delta(\omega_{\Delta^c})\right] \leq \alpha_l\, a(\xi(P)),$$

with $a : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function and $(\alpha_l)_{l\geq 0}$ a positive summable sequence (i.e. $\sum_{l\geq 0} \alpha_l < +\infty$). Let P be a probability measure with bounded intensity. From the above assumption (denoted [*Intensity Regular*]), we have

$$\sum_{l=0}^{+\infty} \left| H^{l+1}_\Delta(\omega) - H^l_\Delta(\omega) \right| \leq |\omega_\Delta| \sum_{l=0}^{+\infty} G^l_\Delta(\omega_{\Delta^c}),$$

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and we can control the expectation of the second part of the right-hand side:

$$\mathbb{E}_P\left[\sum_{l=0}^{+\infty} G_\Delta^l(\omega_{\Delta^c})\right] \le a(\xi(P)) \sum_{l=0}^{+\infty} \alpha_l < +\infty.$$

Then, the local energy is correctly defined for P-almost every configuration ω by

$$H_\Delta(\omega) = H_\Delta^0(\omega) + \sum_{l=0}^{+\infty}\left[H_\Delta^{l+1}(\omega) - H_\Delta^l(\omega)\right].$$

In addition, if we introduce the function C_Δ^l defined on Ω_{Δ^c} by

$$C_\Delta^l(\omega_{\Delta^c}) = \sum_{j=l}^{+\infty} G_\Delta^j(\omega_{\Delta^c}),$$

then for all configurations $\omega \in \Omega$ we have the following approximation result:

$$\left|H_\Delta(\omega) - H_\Delta^l(\omega)\right| \le |\omega_\Delta| C_\Delta^l(\omega_{\Delta^c}), \tag{1}$$

with $\mathbb{E}_P[C_\Delta^l(\omega_{\Delta^c})] \le a(\xi(P)) \sum_{j=l}^{\infty} \alpha_j$.

Remark 1. Assuming that a is an increasing function in Definition 4 allows us to control the approximation (1) for a sequence $(P_n)_{n\ge 1}$ of probability measures. Indeed, if (P_n) has a uniformly bounded intensity (i.e. $\xi(P_n) \le \xi$ for all $n \ge 1$), then the expectation $\mathbb{E}_{P_n}[C_\Delta^l(\omega_{\Delta^c})]$ is uniformly bounded from above.

We are now able to give the definition of infinite-volume Gibbs point processes.

Definition 5. A probability measure P on Ω with bounded intensity is a Gibbs point process for an energy function H, satisfying assumptions [*Stable*] and [*Intensity Regular*], if, for all bounded subsets Δ and all bounded measurable functions f, we have

$$\int f(\omega)P(\mathrm{d}\omega) = \int\int f(\omega'_\Delta \omega_{\Delta^c}) \frac{1}{Z_\Delta(\omega_{\Delta^c})} \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \pi_\Delta(\mathrm{d}\omega'_\Delta) P(\mathrm{d}\omega), \tag{2}$$

with the normalization constant $Z_\Delta(\omega_{\Delta^c}) = \int \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \pi_\Delta(\mathrm{d}\omega'_\Delta)$.

The equations (2) for all Δ and f are the DLR equations. To be correctly defined we need to check that $0 < Z_\Delta(\omega_{\Delta^c}) < +\infty$ for P-almost all configurations ω. A lower bound can easily be obtained with

$$Z_\Delta(\omega_{\Delta^c}) \ge \mathrm{e}^{-H_\Delta(\omega_{\Delta^c})} \pi_\Delta(\emptyset) = \mathrm{e}^{-\lambda^d(\Delta)} > 0.$$

For an upper bound, we use assumptions [*Stable*] and [*Intensity Regular*]. From inequality (1), and assuming that $H(\omega_{\Delta_0 \setminus \Delta}) < +\infty$, we have

$$\begin{aligned} H_\Delta(\omega) &\ge H_\Delta^0(\omega) - |\omega_\Delta| C_\Delta^0(\omega_{\Delta^c}) \\ &= H(\omega_{\Delta_0}) - H(\omega_{\Delta_0 \setminus \Delta}) - |\omega_\Delta| C_\Delta^0(\omega_{\Delta^c}) \\ &\ge -(B + C_\Delta^0(\omega_{\Delta^c}))|\omega_\Delta| - B|\omega_{\Delta_0 \setminus \Delta}| - H(\omega_{\Delta_0 \setminus \Delta}). \end{aligned}$$

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Then, we obtain

$$\begin{aligned}
Z_\Delta(\omega_{\Delta^c}) &\leq \exp\left(B|\omega_{\Delta_0\backslash\Delta}| + H(\omega_{\Delta_0\backslash\Delta})\right) \int \mathrm{e}^{\left(C_\Delta^0(\omega_{\Delta^c})+B\right)|\omega'_\Delta|}\pi_\Delta(\mathrm{d}\omega'_\Delta) \\
&= \exp\left(B|\omega_{\Delta_0\backslash\Delta}| + H(\omega_{\Delta_0\backslash\Delta})\right) \sum_{n=0}^{+\infty} \frac{\left(\lambda^d(\Delta)\mathrm{e}^{\left(C_\Delta^0(\omega_{\Delta^c})+B\right)}\right)^n}{n!} \mathrm{e}^{-\lambda^d(\Delta)} \\
&= \exp\left(B|\omega_{\Delta_0\backslash\Delta}| + H(\omega_{\Delta_0\backslash\Delta}) + \lambda^d(\Delta)\left(\mathrm{e}^{\left(C_\Delta^0(\omega_{\Delta^c})+B\right)} - 1\right)\right) \\
&< +\infty.
\end{aligned}$$

If $H(\omega_{\Delta_0\backslash\Delta}) = +\infty$ then, by the hereditary property, for all $l \geq 0$ we have $H(\omega_{\Delta_l\backslash\Delta}) = +\infty$ and $H_\Delta^l(\omega) = 0$ (according to our convention $\infty - \infty = 0$ given above), which gives $H_\Delta(\omega) = 0$ and $Z_\Delta(\omega_{\Delta^c}) < +\infty$.

Our main result is the following theorem, which is proved in Section 4.

Theorem 1. *For any energy function H satisfying assumptions [*Stable*] and [*Intensity Regular*], there exists at least one stationary Gibbs point process P with finite intensity and locally finite energy (i.e. $P(H(\omega_\Delta) < \infty,$ for all $\Delta \subset \mathbb{R}^d,$ bounded$) = 1$).*

Remark 2. (Temperedness) Our formalism of Gibbs point processes does not really need the introduction of tempered configurations as it is usually done [11, 15]. But, the definition of the DLR equations requires a space of good configurations. In our case, this space is the space of configurations produced by any point processes with bounded intensity. It is not always obvious how to compare this with the different definitions of temperedness mentioned above.

3. Examples

Let us give some examples of energy functions satisfying assumptions [*Stable*] and [*Intensity Regular*] of Theorem 1. Since the assumption [*Stable*] is studied deeply in the literature, we focus mainly on interesting examples satisfying assumption [*Intensity Regular*]. In the following, $B(z, \rho)$ is the ball of center z and radius ρ, $\oplus$ is the Minkowski sum of two sets, and $c_d = \lambda^d(B(0, 1))$.

3.1. Finite-range interaction

An energy function H has a finite range if there exists $R > 0$ such that, for all finite configurations $\omega \in \Omega_\mathrm{f}$ and bounded subset Δ, $H_\Delta(\omega) = H_\Delta\left(\omega_{\Delta\oplus B(0,R)}\right)$. It is easy to see that a finite-range energy function verifies assumption [*Intensity Regular*], with any increasing sequence of subsets $(\Delta_l)_{l\geq 0}$ such that $\Delta_0 = \Delta \oplus B(0, R)$. Therefore, in this setting of finite-range interaction, only the assumption [*Stable*] is required to ensure the existence of Gibbs point processes.

Corollary 1. *For a finite-range and stable energy there exists at least one stationary Gibbs point process with finite intensity and locally finite energy.*

This result was proved previously in [5] (see also [4] for a simpler and pedagogical proof).

3.2. Pairwise interaction

An energy function H is pairwise if there exists a symmetric function $\phi : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$, called a potential, such that

$$H(\omega) = \sum_{\{x,y\}\subset\omega} \phi(x - y).$$

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We do not assume the finite-range property and so the support of ϕ can be unbounded. However, we still need an assumption on the long-range behavior of ϕ. The potential ϕ is *regular* if there exists a decreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_A^{+\infty}\psi(r)r^{d-1}\,\mathrm{d}r<+\infty$ for some $A>0$ and $|\phi(x)|\le\psi(\|x\|)$ for all $x\in\mathbb{R}^d$. Let us show that this energy function satisfies the assumption [*Intensity Regular*].

Let Δ be a bounded subset of $\mathbb{R}^d$. We choose $z\in\mathbb{R}^d$ and $\rho>0$ such that $\Delta\subset B(z,\rho)$, and we introduce $\Delta_l=B(z,\rho+A+l)$. We have

$$H_\Delta^l(\omega)=\sum_{\substack{\{x,y\}\subset\omega\cap\Delta_l\\ \{x,y\}\cap\Delta\neq\emptyset}}\phi(x-y),$$

which gives

$$H_\Delta^{l+1}(\omega)-H_\Delta^l(\omega)=\sum_{x\in\Delta}\sum_{y\in\Delta_{l+1}\setminus\Delta_l}\phi(x-y).$$

By definition, if $x\in\Delta$ and $y\in\Delta_{l+1}\setminus\Delta_l$ then $\|x-y\|\ge A+l$ and $|\phi(x-y)|\le\psi(A+l)$. Introducing the function $G_\Delta^l(\omega_{\Delta^c})=|\omega_{\Delta_{l+1}\setminus\Delta_l}|\psi(A+l)$, we have $|H_\Delta^{l+1}(\omega)-H_\Delta^l(\omega)|\le|\omega_\Delta|G_\Delta^l(\omega_{\Delta^c})$ and, for all P with bounded intensity,

$$\begin{aligned}\mathbb{E}_P\left[G_\Delta^l(\omega_{\Delta^c})\right]&\le\xi(P)\lambda^d(\Delta_{l+1}\setminus\Delta_l)\psi(A+l)\\&\le\xi(P)c_d\left((\rho+A+l+1)^d-(\rho+A+l)^d\right)\psi(A+l)\\&\le\xi(P)c_d d(\rho+A+l+1)^{d-1}\psi(A+l).\end{aligned}$$

Since $\int_A^{+\infty}\psi(r)r^{d-1}\,\mathrm{d}r<+\infty$ and ψ is decreasing, we have that

$$\sum_{l\ge0}(\rho+A+l+1)^{d-1}\psi(A+l)<+\infty.$$

Hence, the energy H satisfies the assumption [*Intensity Regular*] with $a=id$ and $\alpha_l=c_d d(\rho+A+l+1)^{d-1}\psi(A+l)$.

Corollary 2. *For a* stable *pairwise energy associated with a* regular *potential, there exists at least one stationary Gibbs point process with finite intensity and locally finite energy.*

The stability assumption for a pairwise potential is delicate and was investigated a long time ago. We refer to [14] for several results. We mention that a non-negative potential which is equal to zero in the neighborhood of zero and positive outside this neighborhood is a simple example of a stable but not a superstable potential.

Note that the sum of two energy functions satisfying assumption [*Intensity Regular*] also satisfies the assumption [*Intensity Regular*]. The following corollary follows.

Corollary 3. *For a* stable *energy function that is the sum of a finite-range energy function and a pairwise energy function associated with a regular potential, there exists at least one stationary Gibbs point process with finite intensity and locally finite energy.*

3.3. Cloud interaction

In this last example, we provide an energy function which is infinite range and not reducible, at any scale, to a pairwise interaction. It is a multibody interaction between a germ–grain interaction (see, for instance, the Quermass model [3] or the Widom–Rowlinson interaction [16])

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and a pairwise interaction. We call it a cloud interaction because each point of the configuration is diluted in a cloud around itself and the pairwise interaction is integrated on this cloud. Precisely, for any finite configuration ω,

$$H(\omega)=\sum_{x\in\omega}\int_{L^R(\omega)}\phi(x-y)\,\mathrm{d}y,$$

where $L^R(\omega)=\bigcup_{x\in\omega}B(x,R)$ is the cloud produced by the configuration ω ($R>0$ is a fixed parameter) and $\phi:\mathbb{R}^d\mapsto\mathbb{R}\cup\{\infty\}$ is a symmetric function. This energy function can be viewed as an approximation of the pairwise interaction introduced above. Indeed, $H(\omega)/R^d$ tends to the pairwise interaction function (times a multiplicative constant) when R goes to zero.

We assume that there exists a decreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that, for all $x\in\mathbb{R}^d$, $|\phi(x)|\le\psi(\|x\|)$ and $\int\psi(r)r^{d-1}\,\mathrm{d}r<+\infty$. This ensures that the energy function is well defined and satisfies the assumption [*Stable*] since

$$|H(\omega)|\le|\omega|\int|\psi(r)|r^{d-1}\,\mathrm{d}r.$$

Note that H is not superstable. Indeed, this would require that for all $\omega\in\Omega_\Lambda$, $H(\omega)\ge-B|\omega|+A|\omega|^2/\lambda^d(\Lambda)$ with $B\ge0$ and $A>0$, which is impossible with the previous upper bound of H.

Concerning assumption [*Intensity Regular*], for Δ a bounded subset of $\mathbb{R}^d$, the local energy is given by

$$\begin{aligned}H^l_\Delta(\omega)&=H(\omega_{\Delta_l})-H(\omega_{\Delta_l\setminus\Delta})\\&=\sum_{x\in\omega_{\Delta_l}}\int_{L^R(\omega_{\Delta_l})}\phi(\|x-y\|)\,\mathrm{d}y-\sum_{x\in\omega_{\Delta_l\setminus\Delta}}\int_{L^R(\omega_{\Delta_l\setminus\Delta})}\phi(\|x-y\|)\,\mathrm{d}y\\&=\sum_{x\in\omega_{\Delta_l\setminus\Delta}}\int_{L^R(\omega_{\Delta_l})\setminus L^R(\omega_{\Delta_l\setminus\Delta})}\phi(\|x-y\|)\,\mathrm{d}y+\sum_{x\in\omega_\Delta}\int_{L^R(\omega_{\Delta_l})}\phi(\|x-y\|)\,\mathrm{d}y.\end{aligned}$$

We can find $z\in\mathbb{R}^d$ and $\rho>0$ such that $\Delta\subset B(z,\rho)$, and, for the following, we choose $\Delta_0=B(z,\rho+2R)$, which implies that

$$L^R(\omega_{\Delta_l})\setminus L^R(\omega_{\Delta_l\setminus\Delta})=L^R(\omega_{\Delta_0})\setminus L^R(\omega_{\Delta_0\setminus\Delta})\stackrel{\text{def}}{=}L^R_\Delta(\omega).$$

Using this notation we have

$$H^l_\Delta(\omega)=\sum_{x\in\omega_{\Delta_l\setminus\Delta}}\int_{L^R_\Delta(\omega)}\phi(\|x-y\|)\,\mathrm{d}y+\sum_{x\in\omega_\Delta}\int_{L^R(\omega_{\Delta_l})}\phi(\|x-y\|)\,\mathrm{d}y.$$

The first term corresponds to the interaction of the points outside of Δ with the cloud created by the points in Δ, and the second corresponds to the interaction of the points in Δ with the full cloud. We can compute the cost of adding a shell:

$$\begin{aligned}H^{l+1}_\Delta(\omega)-H^l_\Delta(\omega)=&\sum_{x\in\omega_{\Delta_{l+1}\setminus\Delta_l}}\int_{L^R_\Delta(\omega)}\phi(\|x-y\|)\,\mathrm{d}y\\&+\sum_{x\in\omega_\Delta}\int_{L^R(\omega_{\Delta_{l+1}})\setminus L^R(\omega_{\Delta_l})}\phi(\|x-y\|)\,\mathrm{d}y.\end{aligned}$$

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If we choose $\Delta_l = B(z, \rho + 2R + l)$, for $x \in \omega_{\Delta_{l+1}\setminus\Delta_l}$ and $y \in L^R_\Delta(\omega)$ or for $x \in \omega_\Delta$ and $y \in L^R(\omega_{\Delta_{l+1}}) \setminus L^R(\omega_{\Delta_l})$, then we have $\|x - y\| > A + l$. Then, we have

$$|H^{l+1}_\Delta(\omega) - H^l_\Delta(\omega)| \le \Big(|\omega_{\Delta_{l+1}\setminus\Delta_l}|\lambda^d\big(L^R_\Delta(\omega)\big) + |\omega_\Delta|\lambda^d\big(L^R(\omega_{\Delta_{l+1}\setminus\Delta_l})\big)\Big)\,\psi(A + l).$$

With the union bound we have

$$\lambda^d\big(L^R_\Delta(\omega)\big) \le |\omega_\Delta| c_d R^d \text{ and } \lambda^d\big(L^R(\omega_{\Delta_{l+1}\setminus\Delta_l})\big) \le |\omega_{\Delta_{l+1}\setminus\Delta_l}| c_d R^d.$$

If we introduce the function $G^l_\Delta(\omega_{\Delta^c}) = 2|\omega_{\Delta_{l+1}\setminus\Delta_l}| c_d R^d \psi(A + l)$, we obtain $|H^{l+1}_\Delta(\omega) - H^l_\Delta(\omega)| \le |\omega_\Delta| G^l_\Delta(\omega_{\Delta^c})$ and, for all P with bounded intensity,

$$\begin{aligned}\mathbb{E}_P\Big[G^l_\Delta(\omega_{\Delta^c})\Big] &\le \xi(P)\lambda^d(\Delta_{l+1} \setminus \Delta_l)\psi(A + l)\\ &\le \xi(P) d c_d(\rho + A + l + 1)^{d-1}\psi(A + l).\end{aligned}$$

Since $\int_A^{+\infty} \psi(r) r^{d-1}\,\mathrm{d}r < +\infty$ and ψ is decreasing, this implies that

$$\sum_{l\ge 0} (\rho + A + l + 1)^{d-1}\psi(A + l) < +\infty.$$

Hence, the energy H satisfies assumption [*Intensity Regular*].

Corollary 4. *For the cloud interaction, there exists at least one stationary Gibbs point process with finite intensity and locally finite energy.*

4. Proof of the theorem

4.1. Construction of an infinite volume measure

The first step of the proof is to build an accumulation point of a sequence of finite-volume Gibbs measures. Using entropy bounds and the stability of the energy, we prove the existence of such an accumulation point for the local convergence topology. This strategy and its tools have been used several times in the literature [3, 4, 5, 6], and we recall here only the main ideas.

For n a positive integer, we denote by Λ_n the set $]-n, n]^d$. We consider the sequence of Gibbs measures in finite volume given by

$$P_n(\mathrm{d}\omega) = P_{\Lambda_n}(\mathrm{d}\omega) = \frac{1}{Z_n}\mathrm{e}^{-H(\omega)}\pi_{\Lambda_n}(\mathrm{d}\omega),$$

with the normalization constant $Z_n = \int \mathrm{e}^{-H(\omega)}\pi_{\Lambda_n}(\mathrm{d}\omega)$. Since our tension tool will be defined for stationary measures, we need to modify $(P_n)_{n\ge1}$. We defined the periodized version P_n^{per} by the probability measure $\bigotimes_{u\in\mathbb{Z}^d} P_n \circ \tau^{-1}_{2nu}$, and the stationarized version by

$$P_n^{\text{sta}} = \frac{1}{\lambda^d(\Lambda_n)}\int_{\Lambda_n} P_n^{\text{per}} \circ \tau_u^{-1}\,\mathrm{d}u.$$

Definition 6. A function f is said to be local if there exists a bounded set Δ such that f is $\mathcal{F}_\Delta$ measurable (i.e. for all configurations ω in Ω, we have $f(\omega) = f(\omega_\Delta)$). A sequence of measures (μ_n) converges to μ for the local convergence topology if, for all bounded local functions f,

$$\int f\,\mathrm{d}\mu_n \underset{n\to+\infty}{\to} \int f\,\mathrm{d}\mu.$$

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Given two probabilities measures μ and ν on Ω, we recall that the relative entropy of μ with respect to ν on Λ is defined as

$$I_\Lambda(\mu \mid \nu) = \begin{cases} \int \log f \, \mathrm{d}\mu_\Lambda & \text{if } \mu_\Lambda \ll \nu_\Lambda \text{ and } f = \frac{\mathrm{d}\mu_\Lambda}{\mathrm{d}\nu_\Lambda}, \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 7. Let μ be a stationary probability measure with finite intensity on Ω. For $z > 0$ the specific entropy of μ with respect to π^z is defined by

$$I_z(\mu \mid \pi^z) = \lim_{n \to +\infty} \frac{I_{\Lambda_n}(\mu \mid \pi^z)}{\lambda^d(\Lambda_n)} = \sup_{\substack{\Lambda \subset \mathbb{R}^d \\ 0 < \lambda^d(\Lambda) < +\infty}} \frac{I_\Lambda(\mu \mid \pi^z)}{\lambda^d(\Lambda)}. \tag{3}$$

For details we refer to [7, Chapter 15]. The next result, stated in [9], is our tension tool.

Proposition 1. *For any $z > 0$ and $c > 0$, the set of probability measures*

$$\{\mu \text{ stationary with finite intensity}, I_z(\mu) \le c\}$$

is compact and sequentially compact for the local convergence topology.

In order to apply this proposition in our case, we need to compute the specific entropy of the probability measure P_n^{sta}. Using the affine property of the specific entropy, it is well known that

$$I_z(P_n^{\text{sta}}) = \frac{1}{\lambda^d(\Lambda_n)} I_{\Lambda_n}(P_n \mid \pi^z).$$

What remains is to compute the relative entropy of the Gibbs measure P_n with respect to the Poisson point process π^z on Λ_n:

$$\begin{aligned} I_{\Lambda_n}(P_n \mid \pi^z) &= \int \log\left(\frac{\mathrm{d}P_n}{\mathrm{d}\pi^z}\right) \mathrm{d}P_n \\ &= \int \left[\log\left(\frac{\mathrm{d}P_n}{\mathrm{d}\pi}\right) + \log\left(\frac{\mathrm{d}\pi}{\mathrm{d}\pi^z}\right)\right] \mathrm{d}P_n \\ &= \int \left[-\log(Z_n) - H(\omega) + \log\left(\mathrm{e}^{(z-1)\lambda^d(\Lambda_n)} \left(\frac{1}{z}\right)^{|\omega|}\right)\right] P_n(\mathrm{d}\omega) \\ &= (z-1)\lambda^d(\Lambda_n) - \log(Z_n) + \int \left[-H(\omega) - \log(z)|\omega|\right] P_n(\mathrm{d}\omega). \end{aligned}$$

The normalization constant can easily be bounded from below:

$$Z_n = \int \mathrm{e}^{-H} \, \mathrm{d}\pi \ge \mathrm{e}^{-H(\emptyset)} \mathrm{e}^{-\lambda^d(\Lambda_n)}.$$

Then, using the stability of H, we have

$$I_{\Lambda_n}(P_n \mid \pi^z) \le z\lambda^d(\Lambda_n) + H(\emptyset) + \int (B - \log(z))|\omega| P_n(\mathrm{d}\omega).$$

If we choose $z > 0$ such that $B - \log(z) \le 0$, we have $I_z(P_n^{\text{sta}}) \le z + H(\emptyset)$ for all $n \ge 1$. According to Proposition 1 we can exhibit a subsequence of $(P_n^{\text{sta}})_{n \ge 1}$ which converges to a

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stationary measure P with finite intensity. To simplify the notation we can suppose that we have changed the indexation of the sequence $(\Lambda_n)_{n\geq 1}$ such that $(P_n^{\text{sta}})_{n\geq 1}$ converges locally to P.

We can prove that P is also an accumulation point of the sequence

$$\bar{P}_n = \frac{1}{\lambda^d(\Lambda_n)} \int_{\Lambda_n} P_n \circ \tau_u^{-1} \, \mathrm{d}u.$$

See [3, Lemma 3.5] for details.

We need to verify that P has locally finite energy. If Δ is a bounded subset of $\mathbb{R}^d$, the event $\{H(\omega_\Delta) < +\infty\}$ is a local event and

$$P(H(\omega_\Delta) < +\infty) = \lim_{n\to+\infty} \bar{P}_n(H(\omega_\Delta) < +\infty) = 1,$$

because by heredity we have $\bar{P}_n(H(\omega_\Delta) = +\infty) = 0$. We deduce, again using heredity, that

$$P\Big(H(\omega_\Delta) < +\infty, \text{ for all } \Delta \subset \mathbb{R}^d, \text{ bounded}\Big) = P\big(H(\omega_{\Lambda_n})) < +\infty, \text{ for all } n \geq 1\big) = 1.$$

We finish this section by giving the crucial property of uniform control of intensities for the sequence $(\bar{P}_n)$.

Lemma 1. *We can find $\xi \geq i(P)$ such that, for all integers $n \geq 1$, $\xi(\bar{P}_n) \leq \xi$.*

Proof. We use the entropic inequality $\int g \, \mathrm{d}\mu \leq I(\mu \mid \nu) + \log\big(\int \mathrm{e}^g \, \mathrm{d}\nu\big)$ to obtain

$$\mathbb{E}_{P_n^{\text{sta}}}[|\omega_{\Lambda_n}|] \leq I_{\Lambda_n}(P_n^{\text{sta}} \mid \pi^z) + \log\Big(E_{\pi^z}\big[\mathrm{e}^{|\omega_{\Lambda_n}|}\big]\Big).$$

Using the expression of the specific entropy as a supremum (3), we have

$$I_{\Lambda_n}\big(P_n^{\text{sta}} \mid \pi^z\big) \leq I_z\big(P_n^{\text{sta}}\big)\lambda^d(\Lambda_n) \leq (z + H(\emptyset))\lambda^d(\Lambda_n).$$

Under π^z, the random variable $|\omega_{\Lambda_n}|$ follows a Poisson law of parameter $z\lambda^d(\Lambda_n)$, so

$$\mathbb{E}_{\pi^z}\big[\mathrm{e}^{|\omega_{\Lambda_n}|}\big] = \mathrm{e}^{-z\lambda^d(\Lambda_n)} \sum_{p=0}^{\infty} \frac{\big(z\lambda^d(\Lambda_n)\big)^p}{p!} \mathrm{e}^p = \exp\,(z\lambda^d(\Lambda_n)(\mathrm{e}-1)).$$

Then, we obtain

$$i\big(P_n^{\text{sta}}\big)\lambda^d(\Lambda_n) = \mathbb{E}_{P_n^{\text{sta}}}[|\omega_{\Lambda_n}|] \leq (z\mathrm{e} + H(\emptyset))\lambda^d(\Lambda_n),$$

and since $\xi(\bar{P}_n) \leq i(P_n^{\text{sta}})$, we deduce the lemma.

4.2. Some remarks on the local energy

Lemma 2. *If P is a point process with bounded intensity, then for P-almost all ω the following equivalence holds for all $l \geq 0$:*

$$H_\Delta(\omega) = +\infty \iff H_\Delta^l(\omega) = +\infty.$$

Proof. Let ω be a configuration. With our convention $\infty - \infty = 0$ and the intensity regular assumption, for $l \geq 1$, having $H_\Delta^l(\omega) = +\infty$ implies that $H_\Delta^{l-1}(\omega) = +\infty$ and $H_\Delta^{l+1}(\omega) = +\infty$.

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Then, by induction we deduce that, for all $l \geq 1$, we have $H^l_\Delta(\omega) = +\infty$ if and only if $H^0_\Delta(\omega) = +\infty$.

The local energy is defined by

$$H_\Delta(\omega) = H^0_\Delta(\omega) + \sum_{l=0}^{+\infty} \left[H^{l+1}_\Delta(\omega) - H^l_\Delta(\omega) \right],$$

and since for a point process P with bounded intensity the second part of the right-hand side is finite, we deduce that, for P-almost all ω, $H_\Delta(\omega) = +\infty$ if and only if $H^0_\Delta(\omega) = +\infty$.

Lemma 2 ensures that the event $\{H_\Delta(\omega) = +\infty\}$ is local. In particular, the hard-core part of the energy has to be finite range.

For a finite configuration $\omega \in \Omega_\mathrm{f}$, the sum in the definition of the local energy $H_\Delta(\omega)$ is a finite sum, and then Lemma 2 also holds. Hence, we can write

$$H_\Delta(\omega) = H^l_\Delta(\omega) = H(\omega_{\Delta_l}) - H(\omega_{\Delta_l^c}) = H(\omega) - H(\omega_{\Delta^c})$$

for some $l \geq 0$ (depending on ω). Moreover, thanks to the heredity of the energy and our convention $\infty - \infty = 0$, we have, for all $\omega \in \Omega_\mathrm{f}$,

$$H(\omega) = H_\Delta(\omega) + H(\omega_{\Delta^c}).$$

This equality is useful for proving the DLR equations for finite-volume Gibbs measures.

4.3. The DLR equation

We prove in this section that the accumulation point P satisfies the DLR equations stated in Definition 5. By a standard monotone class argument we can replace the class of bounded measurable functions by the class of bounded local functions. Let f be a bounded local function and Δ be a bounded measurable subset of $\mathbb{R}^d$; we have to show that $\int f \,\mathrm{d}P = \int f_\Delta \,\mathrm{d}P$ with

$$f_\Delta(\omega) = \int f(\omega'_\Delta \omega_{\Delta^c}) \frac{1}{Z_\Delta(\omega_{\Delta^c})} \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \pi_\Delta(\mathrm{d}\omega'_\Delta).$$

We fix $\varepsilon > 0$.

Step 1 There exists $K' > 0$ such that, for all $k \geq K'$, we have $P(|\omega_\Delta| > k) \leq \varepsilon$; this implies that, for all $k \geq K'$,

$$\left| \int f_\Delta(\omega) P(\mathrm{d}\omega) - \int f_\Delta(\omega) \mathbf{1}_{|\omega_\Delta| \leq k} P(\mathrm{d}\omega) \right| \leq \|f\|_\infty \varepsilon. \tag{4}$$

The introduction of this indicator function will be useful in Step 4.

Step 2 We approximate f_Δ by $f^l_{\Delta,k}$, which corresponds to the approximation of the local energy H_Δ by H^l_Δ and a restriction to configurations having less than k points in Δ, which means

$$f^l_{\Delta,k}(\omega) = \frac{1}{Z^l_{\Delta,k}(\omega_{\Delta^c})} \int f(\omega'_\Delta \omega_{\Delta^c}) \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| \leq k} \pi_\Delta(\mathrm{d}\omega'_\Delta),$$

with the normalization constant $Z^l_{\Delta,k}(\omega_{\Delta^c}) = \int \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| \leq k} \pi_\Delta(\mathrm{d}\omega'_\Delta)$. We prove that we can find $K \geq K'$ and l (depending on K) such that

$$\left| \int f_\Delta(\omega) \mathbf{1}_{|\omega_\Delta| \leq K} P(\mathrm{d}\omega) - \int f^l_{\Delta,K}(\omega) \mathbf{1}_{|\omega_\Delta| \leq K} P(\mathrm{d}\omega) \right| \leq 6\|f\|_\infty \varepsilon. \tag{5}$$

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We introduce an event $\mathcal{A}_\varepsilon$ such that $P(\mathcal{A}_\varepsilon) \le \varepsilon$. Its precise definition is given later. We have

$$\left| \int f_\Delta(\omega) \mathbf{1}_{|\omega_\Delta| \le k} P(\mathrm{d}\omega) - \int f^l_{\Delta,k}(\omega) \mathbf{1}_{|\omega_\Delta| \le K} P(\mathrm{d}\omega) \right| \\ \le \int_{\mathcal{A}^c_\varepsilon} |f_\Delta(\omega) - f^l_{\Delta,k}(\omega)| \mathbf{1}_{|\omega_\Delta| \le k} P(\mathrm{d}\omega) + 2\|f\|_\infty \varepsilon. \tag{6}$$

We must estimate the approximation error

$$f_\Delta(\omega) - f^l_{\Delta,k}(\omega) \\ = \frac{1}{Z_\Delta(\omega_{\Delta^c})} \int f(\omega'_\Delta \omega_{\Delta^c}) \left(\mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} - \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| \le k} \right) \pi_\Delta(\mathrm{d}\omega'_\Delta) \\ + \left(\frac{1}{Z_\Delta(\omega_{\Delta^c})} - \frac{1}{Z^l_{\Delta,k}(\omega_{\Delta^c})} \right) \int f(\omega'_\Delta \omega_{\Delta^c}) \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| \le k} \pi_\Delta(\mathrm{d}\omega'_\Delta).$$

Since the difference between the normalization constants is

$$Z^l_{\Delta,k}(\omega_{\Delta^c}) - Z_\Delta(\omega_{\Delta^c}) = \int \left(\mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} - \mathbf{1}_{|\omega'_\Delta| \le k} \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \right) \pi_\Delta(\mathrm{d}\omega'_\Delta),$$

we obtain the upper bound

$$|f_\Delta(\omega) - f^l_{\Delta,k}(\omega)| \le \frac{2\|f\|_\infty}{Z_\Delta(\omega_{\Delta^c})} \int \left| \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} - \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| \le k} \right| \pi_\Delta(\mathrm{d}\omega'_\Delta) \\ = \frac{2\|f\|_\infty}{Z_\Delta(\omega_{\Delta^c})} \int \left| \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} - \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} \right| \mathbf{1}_{|\omega'_\Delta| \le k} \pi_\Delta(\mathrm{d}\omega'_\Delta) \\ + \frac{2\|f\|_\infty}{Z_\Delta(\omega_{\Delta^c})} \int \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| > k} \pi_\Delta(\mathrm{d}\omega'_\Delta). \tag{7}$$

Using Lemma 2, the inequality $|\mathrm{e}^b - \mathrm{e}^a| \le |b-a|\mathrm{e}^{|b-a|+a}$ and the approximation (1), we obtain the upper bound

$$\left| \mathrm{e}^{-H^l_\Delta(\omega'_\Delta \omega_{\Delta^c})} - \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \right| \mathbf{1}_{|\omega'_\Delta| \le K} \le K C^l_\Delta(\omega_{\Delta^c}) \mathrm{e}^{K C^l_\Delta(\omega_{\Delta^c}) - H_\Delta(\omega'_\Delta \omega_{\Delta^c})}. \tag{8}$$

By combining (7) and (8), we have

$$|f_\Delta(\omega) - f^l_{\Delta,k}(\omega)| \le 2\|f\|_\infty K C^l_\Delta(\omega_{\Delta^c}) \mathrm{e}^{K C^l_\Delta(\omega_{\Delta^c})} \\ + \frac{2\|f\|_\infty}{Z_\Delta(\omega_{\Delta^c})} \int \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| > k} \pi_\Delta(\mathrm{d}\omega'_\Delta). \tag{9}$$

Now we need to control the two terms of the right-hand side of (9). For the second one, by the dominated convergence theorem, we can find $K \ge K'$ such that

$$\int \frac{1}{Z_\Delta(\omega_{\Delta^c})} \int \mathrm{e}^{-H_\Delta(\omega'_\Delta \omega_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta| > K} \pi_\Delta(\mathrm{d}\omega'_\Delta) P(\mathrm{d}\omega) \le \varepsilon.$$

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Once K is chosen, we can control the first term. If $W:\mathbb{R}_+\to\mathbb{R}_+$ is the inverse function of $x\mapsto xe^x$ (W is the so-called Lambert function), using Markov's inequality we have

$$P\Big(KC^l_\Delta(\omega_{\Delta^c})\mathrm{e}^{KC^l_\Delta(\omega_{\Delta^c})}>\varepsilon\Big)=P(KC^l_\Delta(\omega_{\Delta^c})>W(\varepsilon))\le\frac{K\mathbb{E}_P\big[C^l_\Delta(\omega_{\Delta^c})\big]}{W(\varepsilon)}.$$

As $\mathbb{E}^P[C^l_\Delta(\omega_{\Delta^c})]$ goes to zero when l goes to infinity, we can choose l (depending on K) such that

$$P\Big(KC^l_\Delta(\omega_{\Delta^c})\mathrm{e}^{KC^l_\Delta(\omega_{\Delta^c})}>\varepsilon\Big)\le\varepsilon.$$

Introducing the event $\mathcal{A}_\varepsilon=\{KC^l_\Delta(\omega_{\Delta^c})\mathrm{e}^{KC^l_\Delta(\omega_{\Delta^c})}>\varepsilon\}$ in (6), and with our choice of K and l, we finally have the approximation (5).

According to Lemma 1, the point processes $(\bar{P}_n)_{n\ge1}$ have uniformly bounded intensities and, as mentionned in Remark 1, the expectation $\mathbb{E}_{\bar{P}_n}[C^l_\Delta(\omega_{\Delta^c})]$ is bounded from above uniformly in $n\ge1$. Hence, l could be chosen such that, for all $n\ge1$,

$$\bar{P}_n\left(KC^l_\Delta(\omega_{\Delta^c})\mathrm{e}^{KC^l_\Delta(\omega_{\Delta^c})}>\varepsilon\right)\le\varepsilon. \tag{10}$$

This will be useful later, in Step 4.

Step 3 For n large enough (depending on K and l), we have

$$\left|\int f^l_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\le K}P(\mathrm{d}\omega)-\int f^l_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega)\right|\le\|f\|_\infty\varepsilon. \tag{11}$$

This is simply a consequence of the local convergence of the sequence $(\bar{P}_n)_{n\ge1}$ to P.

Step 4 For all $n\ge1$ we show the approximation

$$\left|\int f^l_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega)-\int f_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega)\right|\le4\|f\|_\infty\varepsilon, \tag{12}$$

where

$$f_{\Delta,K}(\omega)=\frac{1}{Z_{\Delta,K}(\omega_{\Delta^c})}\int f(\omega'_\Delta\omega_{\Delta^c})\mathrm{e}^{-H_\Delta(\omega'_\Delta\omega_{\Delta^c})}\mathbf{1}_{|\omega'_\Delta|\le K}\pi_\Delta(\mathrm{d}\omega'_\Delta),$$

with the normalization constant $Z_{\Delta,K}(\omega_{\Delta^c})=\int\mathrm{e}^{-H_\Delta(\omega'_\Delta\omega_{\Delta^c})}\mathbf{1}_{|\omega'_\Delta|\le K}\pi_\Delta(\mathrm{d}\omega'_\Delta)$.

Similarly to the upper bounds (7) and (8), we obtain

$$\begin{aligned}|f^l_{\Delta,K}(\omega)-f_{\Delta,K}(\omega)|&\le\frac{2\|f\|_\infty}{Z_{\Delta,K}(\omega_{\Delta^c})}\int\left|\mathrm{e}^{-H_\Delta(\omega'_\Delta\omega_{\Delta^c})}-\mathrm{e}^{-H^l_\Delta(\omega'_\Delta\omega_{\Delta^c})}\right|\mathbf{1}_{|\omega'_\Delta|\le K}\pi_\Delta(\mathrm{d}\omega'_\Delta)\\&\le2\|f\|_\infty\mathrm{e}^{KC^l_\Delta(\omega_{\Delta^c})}KC^l_\Delta(\omega_{\Delta^c}).\end{aligned}$$

From our previous choice of K and l in estimate (10), and using a restriction to a specific event as in (6), we obtain the approximation (12).

Step 5 We use the DLR equations for finite-volume Gibbs processes to prove that

$$\left|\int f_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega)-\int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega)\right|\le2\|f\|_\infty\varepsilon. \tag{13}$$

Let us introduce $\Lambda_n^* = \{u \in \Lambda_n : \tau_u^{-1}(\Delta) \subset \Lambda_n\}$. Note that if $\Delta \subset \Lambda_k$ and $n \geq k$ then $\Lambda_{n-k} \subset \Lambda_n^*$ and $(n-k)^d/n^d \leq \lambda^d(\Lambda_n^*)/\lambda^d(\Lambda_n) \leq 1$. We choose n large enough such that $\lambda^d(\Lambda_n^*)/\lambda^d(\Lambda_n) \geq 1-\varepsilon$, and if we denote

$$\bar{P}_n^* = \frac{1}{\lambda^d(\Lambda_n)} \int_{\Lambda_n^*} P_n \circ \tau_u^{-1},$$

we have the approximation

$$\left| \int f_{\Delta,K}(\omega) \mathbf{1}_{|\omega_\Delta| \leq K} \bar{P}_n(\mathrm{d}\omega) - \int f_{\Delta,K}(\omega) \mathbf{1}_{|\omega_\Delta| \leq K} \bar{P}_n^*(\mathrm{d}\omega) \right| \leq \|f\|_\infty \varepsilon.$$

Let us detail the term

$$\begin{aligned}
&\int f_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\leq K}\bar{P}_n^*(\mathrm{d}\omega)\\
&= \frac{1}{\lambda^d(\Lambda_n)} \int_{\Lambda_n^*} \int f_{\Delta,K}(\tau_u(\omega))\mathbf{1}_{|\tau_u(\omega)_\Delta|\leq K} P_n(\mathrm{d}\omega)\,\mathrm{d}u\\
&= \frac{1}{\lambda^d(\Lambda_n)} \int_{\Lambda_n^*} \iint f(\omega'_\Delta \tau_u(\omega)_{\Delta^c}) \frac{1}{Z_{\Delta,K}(\tau_u(\omega)_{\Delta^c})} \mathrm{e}^{-H_\Delta(\omega'_\Delta \tau_u(\omega)_{\Delta^c})} \mathbf{1}_{|\omega'_\Delta|\leq K} \pi_\Delta(\mathrm{d}\omega'_\Delta)\\
&\quad \mathbf{1}_{|\tau_u(\omega)_\Delta|\leq K} P_n(\mathrm{d}\omega)\,\mathrm{d}u.
\end{aligned}$$

For $u \in \Lambda_n^*$, using the fact that $\tau_u(\omega)_{\Delta^c} = \tau_u\big(\omega_{\tau_u^{-1}(\Delta)^c}\big)$, that π_Δ has the same law as $\pi_{\tau_u^{-1}(\Delta)} \circ \tau_u^{-1}$, and that $H_\Delta(\tau_u(\omega)) = H_{\tau_u^{-1}(\Delta)}(\omega)$, we have

$$\begin{aligned}
Z_{\Delta,K}(\tau_u(\omega)_{\Delta^c}) &= \int \mathrm{e}^{-H_\Delta\left(\omega'_\Delta \tau_u\left(\omega_{\tau_u^{-1}(\Delta)^c}\right)\right)} \mathbf{1}_{|\omega'_\Delta|\leq K} \pi_\Delta(\mathrm{d}\omega'_\Delta)\\
&= \int \mathrm{e}^{-H_\Delta\left(\tau_u\left(\omega'_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\right)\right)} \mathbf{1}_{\left|\omega'_{\tau_u^{-1}(\Delta)}\right|\leq K} \pi_{\tau_u^{-1}(\Delta)}\left(\mathrm{d}\omega'_{\tau_u^{-1}(\Delta)}\right)\\
&= \int \mathrm{e}^{-H_{\tau_u^{-1}(\Delta)}\left(\omega'_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\right)} \mathbf{1}_{\left|\omega'_{\tau_u^{-1}(\Delta)}\right|\leq K} \pi_{\tau_u^{-1}(\Delta)}\left(\mathrm{d}\omega'_{\tau_u^{-1}(\Delta)}\right)\\
&= Z_{\tau_u^{-1}(\Delta),K}\left(\omega_{\tau_u^{-1}(\Delta)^c}\right).
\end{aligned}$$

Then, by a similar calculation, we find

$$\begin{aligned}
&\int f_{\Delta,K}(\tau_u(\omega))\mathbf{1}_{|\tau_u(\omega)_\Delta|\leq K} P_n(\mathrm{d}\omega)\\
&\quad = \iint f\left(\tau_u\left(\omega'_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\right)\right) \frac{1}{Z_{\tau_u^{-1}(\Delta),K}\left(\omega_{\tau_u^{-1}(\Delta)^c}\right)} \mathrm{e}^{-H_{\tau_u^{-1}(\Delta)}\left(\omega'_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\right)}\\
&\qquad \mathbf{1}_{|\omega'_{\tau_u^{-1}(\Delta)}|\leq K} \pi_{\tau_u^{-1}(\Delta)}(\mathrm{d}\omega'_{\tau_u^{-1}(\Delta)}) \mathbf{1}_{|\omega_{\tau_u^{-1}(\Delta)}|\leq K} P_n(\mathrm{d}\omega).
\end{aligned}$$

But we can write the measure in finite volume as

$$\begin{aligned}
P_n(d\omega) = \frac{1}{Z_n} &\mathrm{e}^{-H_{\tau_u^{-1}(\Delta)}\left(\omega_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\right)} \mathrm{e}^{-H\left(\omega_{\tau_u^{-1}(\Delta)^c}\right)}\\
&\pi^z_{\Lambda_n \setminus \tau_u^{-1}(\Delta)}\left(\mathrm{d}\omega_{\tau_u^{-1}(\Delta)^c}\right) \pi_{\tau_u^{-1}(\Delta)}\left(\mathrm{d}\omega_{\tau_u^{-1}(\Delta)}\right),
\end{aligned}$$

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and integration with respect to the measure $\pi_{\tau_u^{-1}(\Delta)}$ will give the normalization constant (thanks to the indicator function introduced in Step 1). After simplification we have for the translated $P_n \circ \tau_u^{-1}$ with $u \in \Lambda_n^*$ a finite-volume DLR equation:

$$\int f_{\Delta,K}(\tau_u(\omega))\mathbf{1}_{|\tau_u(\omega)_\Delta|\le K} P_n(\mathrm{d}\omega)$$
$$= \frac{1}{Z_n}\iint f\big(\tau_u\big(\omega'_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\big)\big)\mathrm{e}^{-H\left(\omega'_{\tau_u^{-1}(\Delta)}\omega_{\tau_u^{-1}(\Delta)^c}\right)}\mathbf{1}_{|\tau_u(\omega)_\Delta|\le K}$$
$$\pi^z_{\Lambda_n\setminus\tau_u^{-1}(\Delta)}\big(\mathrm{d}\omega_{\tau_u^{-1}(\Delta)^c}\big)\pi_{\tau_u^{-1}(\Delta)}\big(\mathrm{d}\omega'_{\tau_u^{-1}(\Delta)}\big)$$
$$= \int f(\tau_u(\omega))\mathbf{1}_{|\tau_u(\omega)_\Delta|\le K} P_n(\mathrm{d}\omega).$$

This DLR-type equation is then verified for the mixture $\bar{P}_n^*$:

$$\int f_{\Delta,K}(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n^*(\mathrm{d}\omega) = \int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n^*(\mathrm{d}\omega).$$

Since we have the approximation

$$\left|\int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n^*(\mathrm{d}\omega) - \int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega)\right| \le \|f\|_\infty\varepsilon,$$

we finally obtain (13).

Step 6 We show the last approximation,

$$\left|\int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega) - \int f(\omega)P(\mathrm{d}\omega)\right| \le 2\|f\|_\infty\varepsilon. \tag{14}$$

Using the local convergence of $(P_n)_{n\ge1}$ to P again, we have, for n large enough,

$$\left|\int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}\bar{P}_n(\mathrm{d}\omega) - \int f(\omega)\mathbf{1}_{|\omega_\Delta|\le K}P(\mathrm{d}\omega)\right| \le \|f\|_\infty\varepsilon.$$

With our choice of K we have $P(|\omega_\Delta| > K) \le \varepsilon$ and we obtain (14).

Conclusion

Gathering the approximations (4), (5), (11), (12), (13) and (14), we finally have

$$\left|\int f_\Delta\,\mathrm{d}P - \int f\,\mathrm{d}P\right| \le 16\|f\|_\infty\varepsilon.$$

The inequality is true for every $\varepsilon > 0$, which ends the proof of Theorem 1.

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