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LONGEVITY, GROWTH, AND INTERGENERATIONAL EQUITY: THE DETERMINISTIC CASE

TORBEN M. ANDERSEN
Aarhus University
CEPR
CESifo
and
IZA

MARIAS H. GESTSSON
University of Iceland

Challenges raised by aging (increasing longevity) have prompted policy debates featuring policy proposals justified by reference to some notion of intergenerational equity. However, very different policies ranging from presavings to indexation of retirement ages have been justified in this way. We develop an overlapping-generations model in continuous time that encompasses different generations with different mortality rates and thus longevity. Allowing for trend increases in both longevity and productivity, we address the normative issue of intergenerational equity under a utilitarian criterion when future generations are better off in terms of both material and nonmaterial well-being. Increases in productivity and longevity are shown to have very different implications for intergenerational distribution. Further, the socially optimal retirement age, dependency ratio, and intergenerational burden sharing in the case of a trend increase in longevity are shown to depend on how individuals' utility for time/leisure is affected by age and longevity.

Keywords: Demographics, Longevity, Retirement Age, Healthy Aging

1. INTRODUCTION

A trend of increase in longevity is a major driver underlying demographic shifts in all OECD countries. According to Wilmoth (2000), longevity (life expectancy at birth) increased from 67 to 72 years for men and 75 to 79 years for women

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over the period from 1970 to 1995. Further, according to recent UN forecasts [see United Nations (2008)], the growth rate of longevity is expected to be 0.2% per year. The effects are large and a main cause of projected increases in demographic dependency ratios. The policy debate thus centers on the paradox that increases in longevity, on one hand, constitute a major welfare improvement[1] and, on the other hand, threaten the financial viability of various welfare arrangements. To reap the benefits of increased longevity, policy adjustments are needed, and in policy debates it is often stressed that the adjustments made should ensure intergenerational equity. But what is the precise meaning of intergenerational equity when different cohorts have different longevity? This issue becomes even more complicated when it is taken into account that future generations may also be richer because of productivity growth.

In policy formulations, specific proposals are often justified with reference to intergenerational equity, although this seems to lead to very different implications. To illustrate, the UK pension committee interpreted it to imply that retirement ages should be proportional to longevity:

> Over the long run, fairness between generations suggests that average pension ages should tend to rise proportionately in line with life expectancy, with each generation facing the same proportion of life contributing to and receiving state pensions. [UK Pensions Commission (2005, p. 4)]

In contrast, the Swedish fiscal policy framework has taken presaving or consolidation of public finances prior to changing demographics as being called for by intergenerational equity:[2]

> A current high level of public saving is basically motivated by the need to ensure a more equal distribution of consumption possibilities across generations. [Swedish Government (2008, p. 170)]

The aim of this paper is to clarify the notion of intergenerational equity when overlapping generations (OLG) have different mortality and thus longevity. We approach this from a utilitarian perspective (further discussed in Section 6) and consider the socially optimal allocation across generations. More precisely, the aim of the paper is to analyze the normative issue of how consumption of produced goods and leisure should be allocated to individuals belonging to different generations. This paper can thus be seen as reverting to the classical paper by Samuelson (1958) considering the social optimal allocation across generations. The novel aspect in this paper is its allowing for OLG with different mortality and thus longevity. Rather than appealing to money as a social contrivance, we assume a small open economy facing a given (and time-invariant) interest rate at which consumption possibility can be intertemporally substituted (see Section 6). This also allows us to avoid the complications arising when the capital stock is endogenized [see Diamond (1965)].

In a seminal paper, Calvo and Obstfeld (1988) consider the role of mortality in the social optimal allocation (utilitarian) in a continuous-time setting with an age-dependent survival rate.[3] Basic consumption smoothing arguments imply that the social optimum has consumption to be invariant to age. The models presented in Sheshinski (2006, 2008) analyze allocations across two individuals with different survival rates, in which case consumption smoothing entails redistribution from individuals with high mortality (low longevity) to individuals with low mortality (high longevity). However, the issue of retirement (and thus leisure on par with consumption) is not considered.

This paper makes two important extensions to these earlier papers. First, we consider different mortality rates across generations capturing the empirically observed trend increase in longevity. This implies that demographics is in transition over time/generations, precluding a steady state analysis. Overlap and redistribution across generations with different mortality (longevity) raise particular modeling issues that cannot be handled in standard models or by comparing steady state equilibrium under various assumptions concerning longevity. We present a model in continuous time in which there is overlap of cohorts with different mortality paths and thus longevity.[4] Second, allowing for changes in longevity makes the usual OLG simplification of dividing life length into periods of exogenous length denoted "young" and "old" dubious. We allow endogenous determination of these phases of life by including the retirement age as an endogenous variable under rather general specifications of the utility functions, allowing us to capture various age and health effects, including healthy aging. The endogeneity of the retirement age implies that the economic environment may display certain stationarity properties, although the underlying demographics does not necessarily do so. To focus on the issue of longevity, we assume fertility to be constant, implying that all demographic shifts are generated by changes in mortality rates.

We link changes in longevity explicitly to changes in mortality rates that are cohort-specific. Hence, we allow for a trend increase and an overlap of generations with different survival rates (and thus longevity). The specific modeling of mortality rates is inspired by the approach in Boucekkine et al. (2002) featuring age/cohort-specific mortality rates.[5] This approach can be seen as a generalization of the Yaari–Blanchard approach assuming stochastic survival with age-independent survival rates [see Blanchard (1985)]. The latter is obviously in contradiction to the empirical evidence,[6] and the formulation adopted here captures the fact that mortality rates are (almost) constant (and low) up to a certain age, after which they are increasing in age.[7] We take mortality rates as exogenous to focus on the basic issues on intergenerational equity when different cohorts have different longevity.

This paper is organized as follows: The modeling of demographics, including trend changes in mortality rates, is laid out in Section 2, together with the specification of individual utility functions. The social planner allocation problem under a utilitarian social welfare function is formulated and analyzed in Section 3, and the optimal allocation is interpreted in Section 4. Decentralization of this allocation is discussed in Section 5. Finally, a few concluding remarks are given in Section 6.

2. AN OVERLAPPING-GENERATIONS MODEL WITH COHORT-DEPENDENT LONGEVITY

Consider a setting in continuous time where a given (and constant) number of individuals are born at each instant. Individual lifetime is stochastic, but the fraction of a given cohort surviving is deterministic. The survival rates and thus longevity (life expectancy at birth) are allowed to change over time and thus to differ across cohorts. Hence, OLG alive at a given point in time differ not only in age but also in longevity. Life has two phases, "young" and "old," where *young* refers to the phase in life when individuals are working and *old* refers to the phase when they are not working (retired). The length of these two phases is endogenous, because the retirement age is a choice variable. The social planner (utilitarian) decides on consumption and work (retirement) profiles—that is, allocations across "young" and "old" at a given point in time and across time under the intertemporal resource/budget constraint. The economy is small and open in global capital markets, implying that the interest rate r is exogenous and, for simplicity, assumed to be time-invariant.[8] A deficit is therefore financed by borrowing from abroad at the interest rate r, whereas a surplus is spent on lending to foreigners at the same interest rate.

2.1. Demographics

Survival functions. The number of individuals born at each point in time is assumed to be constant and normalized to 1. Following Boucekkine et al. (2002), it is assumed that the unconditional probability for an individual born at time t of reaching age a (the survival rate) is given by the function[9]

$$\hat{m}\left(a, \beta(t)\right) = \frac{e^{-a\beta(t)} - \alpha}{1 - \alpha}, \tag{1}$$

where $\alpha > 1$, $\beta(t) < 0$, and $a \in [0, A(t)]$, where $A(t)$ is the highest age any member of the cohort born at t can reach, i.e., the age at which $\hat{m}\left(A(t), \beta(t)\right) = 0$:

$$A(t) = -\frac{\ln \alpha}{\beta(t)}. \tag{2}$$

To incorporate demographic shifts into the model, the parameter $\beta(t)$ is assumed to be time-dependent. Hence, the maximum age $A(t)$ is also time-dependent.

Using (2) in (1) gives the survival rate as

$$m\left(a, A(t)\right) = \frac{\alpha^{\frac{a}{A(t)}} - \alpha}{1 - \alpha}. \tag{3}$$

Note that $m\left(0, A(t)\right) = 1$; i.e., there is no infant mortality, and $m\left(A(t), A(t)\right) = 0$, implying that $A(t)$ is the highest age any member of the cohort born at t can reach, as discussed in the preceding.

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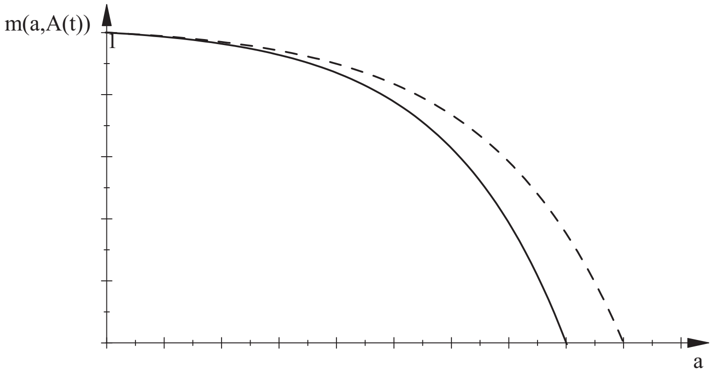

FIGURE 1. Survival rates for different cohorts with different longevity.

In the following we term this maximum age A for a given longevity for the cohort. This can be justified by noting that life expectancy at birth,

$$v(t) = \int_{a=0}^{A(t)} a\left[-\frac{\partial m\,(a, A(t))}{\partial a}\right] da = \left[\frac{\alpha \ln \alpha - (\alpha - 1)}{\ln \alpha\,(\alpha - 1)}\right] A(t), \qquad \textbf{(4)}$$

is strictly increasing in the maximum age; i.e., $\frac{\partial v(t)}{\partial A(t)} = \left[\frac{\alpha \ln \alpha - (\alpha-1)}{\ln \alpha (\alpha-1)}\right] > 0.$[10,11]

It is easily established that

$$\frac{\partial m\,(a, A(t))}{\partial a} = -\frac{\frac{\ln \alpha}{A(t)} \alpha^{\frac{a}{A(t)}}}{\alpha - 1} < 0 \quad \frac{\partial^2 m\,(a, A(t))}{\partial a^2} = -\frac{\left[\frac{\ln \alpha}{A(t)}\right]^2 \alpha^{\frac{a}{A(t)}}}{\alpha - 1} < 0,$$

$$\frac{\partial m\,(a, A(t))}{\partial A(t)} = \frac{a \frac{\ln \alpha}{A(t)^2} \alpha^{\frac{a}{A(t)}}}{\alpha - 1} > 0 \quad \frac{\partial^2 m\,(a, A(t))}{\partial A(t) \partial a} = \frac{\frac{\ln \alpha}{A(t)^2}\left[1 + a\frac{\ln \alpha}{A(t)}\right] \alpha^{\frac{a}{A(t)}}}{\alpha - 1} > 0.$$

The survival rate is strictly decreasing and concave in age. Combining this with the preceding results implies that $m\,(a, A(t)) \in [0, 1]$, i.e., the survival rates are between zero and one. Hence, the function $m\,(a, A(t))$ is well-defined for all $a \in [0, A(t)]$. Moreover, it is strictly increasing in longevity and the effect of higher longevity on the survival rate is increasing in age. Hence, an increase in the longevity of a cohort results in higher survival probabilities at every age, but the increase is greater the older an individual is. These properties are independent of the value of the parameter α (as long as $\alpha > 1$). The survival rate function is illustrated in Figure 1. The figure also shows that greater longevity $A\,(\tau) > A\,(s)$ for a generation born at τ than for a generation born at s implies an outward shift in the survival curve from the full line [corresponding to $A\,(s)$] to the dotted line [corresponding to $A\,(\tau)$], and hence the survival to any age is nondecreasing in longevity.

In accordance with empirical evidence (see the Introduction), we assume that there is a trend increase in longevity.[12] Hence, we make the following assumption:

Assumption 1. Longevity of the generation born at time t, i.e., $A(t)$, follows the process

$$dA(t) = \mu_A(t)dt; \quad \mu_A(t) \geq 0, \; \frac{\partial \mu_A(t)}{\partial t} \leq 0. \tag{5}$$

Hence, longevity of the generation born at some time t relates to longevity for the generation born at $s < t$ as

$$A(t) = A(s) + \int_{j=s}^{t} \mu_A(j)\, dj \quad \text{for } t \geq s.$$

Equation (5) implies that each new generation has a longevity that is no less than that of the previous generation. Further, it is assumed that growth in longevity is nonincreasing over time.[13] Note that because we can have $\mu_A(t) = 0$ for some t and $\frac{\partial \mu_A(t)}{\partial t} \leq 0$ for all t, it follows that a special case of this setup is one where there is an upper bound to longevity.

As discussed earlier, the longevity of an individual born at time t is denoted by $A(t)$. Hence, the longevity of an individual aged a at time t is denoted by $A(t-a)$, because he is born at time $t-a$. At any point in time the last person from some generation passes away, i.e., the generation becomes extinct. At time t this happens for the generation born at time $t - \tilde{A}(t)$ with longevity $\tilde{A}(t)$; i.e., $\tilde{A}(t)$ denotes the longevity of the generation that becomes extinct at time t. Using this and (5), the longevity of the generation aged a at time t relates to the longevity of the generation that becomes extinct at time t as

$$A(t-a) = \tilde{A}(t) + \int_{j=t-\tilde{A}(t)}^{t-a} \mu_A(j)\, dj \quad \text{for } \tilde{A}(t) \geq a \geq 0. \tag{6}$$

This relation allows us to restate the survival probability function (3) conveniently in terms of the longevity of the generation that becomes extinct today (at time t); i.e.,

$$\tilde{m}(a, \tilde{A}(t)) \equiv m(a, A(t-a)), \tag{7}$$

where $A(t-a)$ is given in (6). From (3), (6), and (7), it is obvious that $\tilde{m}\left(0, \tilde{A}(t)\right) = 1$ and $\tilde{m}\left(\tilde{A}(t), \tilde{A}(t)\right) = 0$. Moreover,[14]

$$\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial a} < 0 \quad \frac{\partial^2 \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial a^2} < 0$$
$$\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} > 0 \quad \frac{\partial^2 \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)\partial a} > 0.$$

Hence, the $\tilde{m}$-function has properties similar to those of the m-function, $0 \leq \tilde{m}\left(a, \tilde{A}(t)\right) \leq 1$, and it is bounded on $a \in \left[0, \tilde{A}(t)\right]$ and hence its integral exists. These properties are important because they ensure that the population size, i.e., the number of young and the number of old individuals, is well-defined in the model.

Population composition. Although the birth rate is constant, the population size is not, because survival rates and longevity change. Because, by assumption, *one* individual is born at each point in time, the number of individuals aged a in time t is $\tilde{m}\left(a, \tilde{A}(t)\right)$. Hence, the total population at time t is given as

$$N(t) = \int_{a=0}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da. \quad \textbf{(8)}$$

The retirement age of an individual born at time t is denoted by $R(t)$. It follows that we may classify individuals born at time t as *young* when their age is between 0 and $R(t)$, i.e., $a \in [0, R(t)]$, and *old* when their age is between $R(t)$ and the maximum age $A(t)$, i.e., $a \in (R(t), A(t)]$. Further, the retirement age of an individual aged a at time t is denoted by $R(t-a)$ because he is born at time $t-a$. At any point in time, individuals from some generation retire. At time t, this happens for the generation born at time $t - \tilde{R}(t)$ with retirement age $\tilde{R}(t)$, i.e., $\tilde{R}(t)$ denotes the retirement age of the generation that retires at time t. It follows that individuals aged between 0 and $\tilde{R}(t)$, i.e., $a \in \left[0, \tilde{R}(t)\right]$, are young at time t, and individuals aged between $\tilde{R}(t)$ and $\tilde{A}(t)$, i.e., $a \in \left(\tilde{R}(t), \tilde{A}(t)\right]$, are old at time t. Note that the retirement age is allowed to depend on time and thus to be cohort-specific.

The numbers of young (working) and old (retired) individuals, respectively, at time t are therefore

$$N_w(t) = \int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da = N_w\left(\tilde{R}(t), \tilde{A}(t)\right) > 0, \quad \textbf{(9)}$$

$$N_o(t) = \int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da = N_o\left(\tilde{R}(t), \tilde{A}(t)\right) > 0. \quad \textbf{(10)}$$

Hence, both the number of young and the number of old individuals at time t can be expressed as functions of the longevity of the generation that becomes extinct at time t, i.e., $\tilde{A}(t)$, and the retirement age of the generation that retires at time t, i.e., $\tilde{R}(t)$. Obviously, total population size is given as $N(t) = N_w(t) + N_o(t)$, as can be verified from (8), (9), and (10).

The dependency ratio[15] is defined as the ratio between the number of old and young individuals:

$$K(t) = \frac{N_o(t)}{N_w(t)} = \frac{N_o\left(\tilde{R}(t), \tilde{A}(t)\right)}{N_w\left(\tilde{R}(t), \tilde{A}(t)\right)} \equiv K\left(\tilde{R}(t), \tilde{A}(t)\right). \tag{11}$$

Note that even though demographic stock variables need not be stationary, because of a trend increase in longevity, it follows that compositional variables such as the dependency ratio may be stationary, because they depend on how the retirement age responds to demographic changes. This turns out to be crucial in the following.

We can now work out how the upward trend in longevity (5) affects the key demographic variables $N(t)$, $N_w(t)$, $N_o(t)$, and $K(t)$. We have (see Appendix B) that both the number of young and the number of old increase [for a given retirement age $\tilde{R}(t)$]. The latter is straightforward and the former arises because an increase in longevity implies a decrease in mortality rates at all ages (see Figure 1). However, the number of old individuals increases by more than the number of young individuals, and hence we have that the dependency ratio unambiguously increasing. In summary, we have

$$\begin{aligned} dN(t) &> 0, \\ dN_w(t) &> 0, \\ dN_o(t) &> 0, \\ dK(t) &> 0. \end{aligned}$$

Note that $\tilde{R}(t)$ is endogenous in the social planner's maximization problem and, hence, so are $N_w(t)$, $N_o(t)$, and $K(t)$. This prevents the dependency ratio $K(t)$ becoming infinitely large over time.

2.2. Individual Utility

Individuals consume when young and old, but work when young only. Utility is assumed in the conventional way to depend (separably) on utility from consumption and leisure/time. Working hours are exogenous and hence the only dimension of labor supply is along the extensive margin related to retirement. Upon retirement, the agent gains more leisure, which under standard assumptions is a utility gain. One simple approach is to assume age-independent and constant disutility from work or utility from leisure, in which case retirement at any age releases a constant utility gain along the leisure/time dimension for the remaining life, which then has to be traded off against the implications for consumption [see, e.g., Sheshinski (1978), Crawford and Lilien (1981), and Kalemli-Ozcan and Weil (2002)]. This approach cannot readily be applied in a context such as here where mortality rates and thus implicitly health status differ across generations. Age and health effects will in general affect both the disutility from work and the utility of time/leisure. The following thus assumes a relationship between age, as well as longevity, and health; i.e., health is decreasing with age, and health is better at a given age, the higher the longevity. We propose a general formulation capturing these effects and that embeds various approaches taken in the literature.[16,17]

The utility function for a young (working) individual aged a at time t is given as a separable function specified over consumption[18] $c_w(t)$ and time, i.e.,

$$W_w(t,a) = U(c_w(t)) + L\left(a, A(t-a)\right) \;\; \forall \;\; a \in \left[0, \tilde{R}(t)\right], \tag{12}$$

where U is the utility function specified over consumption, and satisfies standard conditions. The utility from leisure/time is given by the function $L(a, A(t-a))$ depending on age and longevity (recall that working hours are exogenous and thus implicit in the formulation).[19] We assume that disutility from work is nondecreasing with age, and hence L is nonincreasing in a. The disutility from work is assumed to be nonincreasing in health, captured by longevity $A(t-a)$. We have that

$$\frac{\partial L\left(a, A(t-a)\right)}{\partial a} \leq 0, \tag{13}$$

$$\frac{\partial L\left(a, A(t-a)\right)}{\partial A\left(t-a\right)} \geq 0. \tag{14}$$

Hence, the utility from time for a young individual aged a is greater (disutility from work lower) when longevity of his generation is A_1 years than when it is $A_2 < A_1$ years.[20]

The utility of an old (retired) individual aged a at time t is similarly given as

$$W_o(t,a) = Q(c_o(t)) + H\left(a, A(t-a)\right) \;\; \forall \;\; a \in \left(\tilde{R}(t), \tilde{A}(t)\right], \tag{15}$$

where Q is the utility function specified over consumption, and satisfies standard conditions. The value of leisure is assumed to depend on age and health/longevity, capturing the fact that the ability to use time is affected by the two. Specifically, the value of leisure time is nonincreasing in age and nondecreasing in longevity; i.e.,

$$\frac{\partial H\left(a, A(t-a)\right)}{\partial a} \leq 0, \tag{16}$$

$$\frac{\partial H\left(a, A(t-a)\right)}{\partial A\left(t-a\right)} \geq 0. \tag{17}$$

Observe that the different specifications of utility from consumption [$U(\cdot)$ and $Q(\cdot)$] when working and retired capture different needs in the two situations, but all qualitative results that follow hold if the utility functions are the same.

The utility generated from time uses (leisure) takes into account various possible time-consuming activities such as work, leisure activities, and rest,[21] and the formulation is thus rather general. Note that we need to specify utility of time both when working and when retired, and hence the separate functions[22] $L\left(\cdot\right)$ and $H\left(\cdot\right)$. Importantly, at retirement there is a discrete change in utility from leisure/time from $L(\cdot)$ to $H(\cdot)$; cf. the following. To ensure that there is an interior retirement age, we assume that

$$\frac{\partial\left[L\left(a, A(t-a)\right)-H\left(a, A(t-a)\right)\right]}{\partial a} < 0, \tag{18}$$

$$\frac{\partial\left[L\left(a, A(t-a)\right)-H\left(a, A(t-a)\right)\right]}{\partial A(t-a)} > 0, \tag{19}$$

$$\lim_{a \to A(t-a)}\left[L\left(a, A(t-a)\right)-H\left(a, A(t-a)\right)\right] \to -\infty. \tag{20}$$

The first assumption implies that the urge to retire increases with age, whereas the second says that it is reduced by higher longevity. This captures the fact that health and the ability to enjoy time (working or spent on leisure activities, etc.) worsen with age but increase with longevity (a healthy aging effect; see also Section 4.1). This captures the fact that health (morbidity) is related to age. The age effect (18) says that the utility from time when working decreases more than the utility from time when retired; that is, the gain in the utility from time when retired relative to when working increases with age. This can, for example, be because working becomes more physically challenging when individuals become older and their health worsens. The longevity effect (19) says that increased longevity tends to increase the utility from time more when working than when retired (lower disutility from work); that is, the gain in utility from time of retirement at a given age decreases with longevity. This is consistent with empirical evidence found by Haliday and Podor (2009) showing that improvements in health status have large and positive effects on time allocated to home and market production and large negative effects on time spent on watching TV, sleeping, and consumption of other types of leisure activities. Finally, (20) is needed to ensure an interior solution for the retirement age.

When the functions L and H are homogeneous of degree zero, any increase in longevity makes health increase proportionally for all ages; that is, the utility from time (disutility from work) for a person at age a and with longevity A is the same as for a person at age λa and longevity λA (for all $\lambda > 0$). However, when the degree of homogeneity is positive, health at different ages increases more than proportionally to longevity. Bloom et al. (2007) assume that disutility from work is increasing in age and decreasing in life expectancy as well as homogeneous of degree zero in the two. We make a more general assumption by allowing the value of time for retired persons also to depend on age and longevity (life expectancy) and by not restricting the utility functions to be homogeneous of degree zero in the two arguments; cf. the following.

In the following it turns out to be more convenient to analyze the model when the value of leisure is expressed in terms of the longevity of the generation becoming

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extinct at time t, i.e., $\tilde{A}(t)$. By use of (5), we have that

$$\tilde{L}\left(a, \tilde{A}(t)\right) \equiv L\left(a, A(t-a)\right),$$
$$\tilde{H}\left(a, \tilde{A}(t)\right) \equiv H\left(a, A(t-a)\right),$$

where $A(t-a)$ is given in (6), and hence utilities for young and old are given as

$$\tilde{W}_w\left(t, a\right) = U(c_w(t)) + \tilde{L}\left(a, \tilde{A}(t)\right),$$
$$\tilde{W}_o\left(t, a\right) = Q(c_o(t)) + \tilde{H}\left(a, \tilde{A}(t)\right),$$

where we, by the use of (6) and the properties of the L and H functions, have

$$\frac{\partial \tilde{L}}{\partial a} = \frac{\partial L}{\partial a} - \mu_A(t-a)\frac{\partial L}{\partial A(t-a)} < 0,$$
$$\frac{\partial \tilde{L}}{\partial \tilde{A}(t)} = \left[1 + \mu_A\left(t - \tilde{A}\right)\right]\frac{\partial L}{\partial A(t-a)} > 0,$$
$$\frac{\partial \tilde{H}}{\partial a} = \frac{\partial H}{\partial a} - \mu_A(t-a)\frac{\partial H}{\partial A(t-a)} \leq 0,$$
$$\frac{\partial \tilde{H}}{\partial \tilde{A}(t)} = \left[1 + \mu_A\left(t - \tilde{A}\right)\right]\frac{\partial H}{\partial A(t-a)} \geq 0.$$

It is easily verified that this changes none of the qualitative insights from the preceding. Note that (18), (19), and (20) are readily shown to hold for the modified values of time for young and old ($\widetilde{L}$, $\widetilde{H}$).

2.3. Productivity

The earnings capability is assumed to be exogenous but to be driven by productivity growth. Hence, we set forth the following assumption:

Assumption 2. Output of a young individual at time t, i.e., $y(t)$, follows the process

$$dy(t) = \mu_y y(t)dt; \quad \mu_y \geq 0. \tag{21}$$

Total output in the economy $N_w(t)y(t)$ is endogenous because $N_w(t)$ is endogenous. Note that productivity is only included for comparative purposes, allowing a comparison of intergenerational distributions arising from different longevity or economic possibilities across cohorts.

2.4. Planner Allocation

The aim of the paper is to analyze how consumption and leisure (retirement) should be allocated to individuals belonging to different generations. Hence, the social planner decides on consumption and work (retirement age). It is easiest to characterize and interpret the social planner allocation if it is cast in terms of taxes and transfers relative to the reference outcome where individuals consume their labour income. Therefore, consumption of a young and an old individual, respectively, at time t is

$$c_w(t) = y(t) - T_w(t), \tag{22}$$

$$c_o(t) = T_o(t), \tag{23}$$

where $T_w(t)$ is net taxes paid by a young individual and $T_o(t)$ is net transfers to an old individual at time t. We cast the model in this way both because it is analytically more tractable and because it gives a simple relation to the analysis of social security schemes in standard two-period OLG models. Further, because $y(t)$ is exogenous, choosing $T_w(t)$ and $T_o(t)$ is tantamount to choosing $c_w(t)$ and $c_o(t)$ in the model.

We have made the informational restriction that the social planner cannot make taxes and transfers age-dependent, but only dependent on labor market status (working or not working). This can be motivated in terms of information and transactions costs, which apparently are high in reality, because actual schemes in general satisfy the condition imposed here. Hence, at a given point in time, the social planner has to collect the same amount of tax from each young (working) individual and give the same amount of transfer to each old (retired) individual. Because the output of each young individual is only time-dependent, it follows from (22) and (23) that, at a given point in time, consumption of each young and each old individual is independent of age. This fits the specification of the utility functions from before, i.e., that the marginal utility from consumption is independent of age [see $U(\cdot)$ and $Q(\cdot)$ in (12) and (15), respectively].

The policy package thus includes three elements, namely, the tax levied on the young (working), the transfer to the old (retired). and the retirement age. The policy problem is thus to choose $\{T_w(i), T_o(i), \tilde{R}(i)\}_{i=t}^{\infty}$ at each point in time t.

The primary budget balance of this scheme in any period t reads

$$B(t) = N_w(t)T_w(t) - N_o(t)T_o(t). \tag{24}$$

whereas the intertemporal budget constraint at time t can be written as[23]

$$D(t) = \int_{i=t}^{\infty} e^{-r(i-t)}B(i)di, \tag{25}$$

where r is the interest rate. From (25), debt dynamics can be written

$$\begin{aligned} dD(t) &\equiv D(t+dt) - D(t) \\ &= r dt D(t) - B(t)dt \\ &= [rD(t) - B(t)]\, dt, \end{aligned} \tag{26}$$

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where it is used that $e^r \approx 1 + r$. Note that $D(t)$ stands for foreign debt and there is no other storage technology present, such as capital.

3. SOCIAL OPTIMUM

We consider the social optimum under a utilitarian criterion.[24] Although this criterion is neither unproblematic nor uncontroversial, it is useful to illustrate some basic trade-offs that arise, and we consider it a useful benchmark case for studying intergenerational distribution; cf. the discussion in Section 6.

The objective of the social planner at time t is to maximize the sum of the present values of lifetime utilities of generations born at time t or later and the present values of utilities for the remaining lives of generations alive at time t. We assume that the social planner discounts at the same subjective rate as individuals (θ), which may be considered the "pure" utilitarian case where utility achieved at any point in time gets the same weight irrespective of who obtains the utility.

Under the utilitarian criterion, the objective of the social planner at time t can be written as the present value of instantaneous utility generated to all living individuals, as is shown in Appendix C. The social welfare function can therefore be written as

$$W(t) = \int_{i=t}^{\infty} e^{-\theta(i-t)} Z(i) di, \qquad (27)$$

where

$$Z(i) = \int_{a=0}^{\tilde{R}(i)} \tilde{m}\left(a, \tilde{A}(i)\right) \tilde{W}_w(i, a) da + \int_{a=\tilde{R}(i)}^{\tilde{A}(i)} \tilde{m}\left(a, \tilde{A}(i)\right) \tilde{W}_o(t, a) da,$$

where (6) and (7) are used.

The problem facing the policy maker is

$$\underset{\left\{T_w(i), T_o(i), \tilde{R}(i)\right\}_{i=t}^{\infty}}{\text{Max}} W(t), \qquad (28)$$

subject to the budget constraint from (25) [and (24)] and given (5) and (21).

The problem is solved by setting up the Hamilton–Jacobi–Bellman (HJB) equation, which determines the value function

$$V(D(t), y(t), \tilde{A}(t)) = \underset{\left\{T_w(i), T_o(i), \tilde{R}(i)\right\}_{i=t}^{\infty}}{\text{Max}} W(t),$$

where $\tilde{A}(t)$ is defined in the following. The HJB equation is in shorthand (suppressing time indices)

$$\theta V\left(D, y, \tilde{A}\right) = \underset{T_w, T_o, \tilde{R}}{\text{Max}} \left\{ \begin{array}{c} \int_{a=0}^{\tilde{R}} \tilde{m}\left(a, \tilde{A}\right)\left[U\left(y - T_w\right) + \tilde{L}\left(a, \tilde{A}\right)\right] da \\ + \int_{a=\tilde{R}}^{\tilde{A}} \tilde{m}\left(a, \tilde{A}\right)\left[Q\left(T_o\right) + \tilde{H}\left(a, \tilde{A}\right)\right] da \\ + \frac{1}{dt} dV\left(D, y, \tilde{A}\right) \end{array} \right\}$$

$$\text{s.t.}$$

$$\begin{aligned} dD &= \left\{rD - \left[N_w\left(\tilde{R}, \tilde{A}\right) T_w - N_o\left(\tilde{R}, \tilde{A}\right) T_o\right]\right\} dt \\ d\tilde{A} &= \hat{\mu}_A dt \\ dy &= \mu_y y dt, \end{aligned} \quad \mathbf{(29)}$$

with N_w and N_o given from (9) and (10), respectively, and $\hat{\mu}_A \equiv \frac{\mu_A}{1+\mu_A}$.[25] This gives the following first-order conditions for the optimal T_w, T_o, and $\tilde{R}$, respectively:[26]

$$V_D(\cdot) = -U'\left(y - T_w\right), \quad \mathbf{(30)}$$

$$V_D(\cdot) = -Q'\left(T_o\right), \quad \mathbf{(31)}$$

$$\left[T_w + T_o\right] V_D(\cdot) = U\left(y - T_w\right) + \tilde{L}\left(\tilde{R}, \tilde{A}\right) - Q\left(T_o\right) - \tilde{H}\left(\tilde{R}, \tilde{A}\right). \quad \mathbf{(32)}$$

Further, this gives the following law of motion for the marginal value function:

$$dV_D(\cdot) = (\theta - r) V_D dt. \quad \mathbf{(33)}$$

To give an intuition for the meaning of the marginal value function $V_D(\cdot)$, consider the first two equations [(30) and (31)]. These state that consumption of young and old should be such that the marginal utilities from increased consumption (U' and Q') are equal to the decrease in the value function ($-V_D$) due to lower future consumption possibilities following an increase in consumption today (the budget constraint has to hold).

3.1. Neutral Generational Weighting

The relation between the subjective (θ) and the objective (r) discount rates determines whether the marginal value function is decreasing or increasing over time; cf. (33). Like models of intertemporal consumption choices, a subjective discount rate higher (lower) than the objective discount rate implies a profile for consumption that is decreasing (increasing) over time; see, e.g., Blanchard and Fischer (1989). This source of reallocation across time and thus generations is standard. In the following we therefore consider only the case where the subjective and objective (world market) discount rates are identical, $\theta = r$. The assumption is standard in a small open economy setting because it ensures that there is no systematic accumulation or decumulation of assets vis-à-vis the world [see, e.g., Blanchard and Fischer (1989)]. This may also be interpreted as a case of neutral generational weighting in the sense that the subjective and objective discount rates are equal. Hence, discounting per se does not imply any profile benefitting current or future generations in a particular way. In short, we refer to this case as neutral generational weighting. Under this assumption we have (relaxing the shorthand)

$$dV_D\left(D(i), y(i), \tilde{A}(i)\right) = 0 \tag{34}$$

for all $i \geq t$; i.e., the optimal policy package is such that the marginal value function $V_D(\cdot)$ is the same for all $i \geq t$.

Applying this to (30)–(32) gives

$$V_D(t) = -U'\left(y(i) - T_w(i)\right), \tag{35}$$

$$V_D(t) = -Q'\left(T_o(i)\right), \tag{36}$$

and

$$[T_w(i) + T_o(i)]\, V_D(t) = U\left(y(i) - T_w(i)\right) + \tilde{L}\left(\tilde{R}(i), \tilde{A}(i)\right) - Q\left(T_o(i)\right) - \tilde{H}(\tilde{R}(i), \tilde{A}(i)), \tag{37}$$

for all $i \geq t$, where $V_D(t)$ is written as a function of t to indicate that it is the same for all $i \geq t$.

Combining (35)–(37) gives

$$U'\left(y(i) - T_w(i)\right) = Q'\left(T_o(i)\right) \tag{38}$$

and

$$[T_w(i) + T_o(i)]\, Q'\left(T_o(i)\right) = Q\left(T_o(i)\right) + \tilde{H}(\tilde{R}(i), \tilde{A}(i)) - U\left(y(i) - T_w(i)\right) - \tilde{L}\left(\tilde{R}(i), \tilde{A}(i)\right). \tag{39}$$

The RHS of (39) gives the discrete change in utility upon retirement, that is, the difference between utility when working and not working at the retirement age (and given the longevity of the retiring cohort). The LHS is the cost given as the product of the total tax wedge on the retirement decision (sum of taxes paid as working and transfers received as retired) times the marginal utility of consumption. Clearly the socially optimal allocation balances the benefits and costs of retirement. It is an implication of (39) that the agent at the moment of retirement experiences a utility gain, because

$$\tilde{H}(\tilde{R}(i), \tilde{A}(i)) + Q\left(T_o(i)\right) > U\left(y(i) - T_w(i)\right) + \tilde{L}(\tilde{R}(i), \tilde{A}(i))$$

follows, because $Q'\left(T_o(i)\right) > 0$, $T_w > 0$, and $T_o > 0$. Because there is a cost of retirement captured by the tax wedge, it follows that retirement is not taking place at the age at which the utility when retired equals the utility when working.

3.2. Optimal Policy Package

An optimal policy package $\{T_w(i), T_o(i), \tilde{R}(i)\}_{i=t}^{\infty}$ must satisfy (35)–(37) as well as the intertemporal budget constraint

$$D(t) = \int_{i=t}^{\infty} e^{-r(i-t)} \begin{bmatrix} N_w(\tilde{R}(i), \tilde{A}(i))T_w(i) \\ -N_o(\tilde{R}(i), \tilde{A}(i))T_o(i) \end{bmatrix} di, \tag{40}$$

where $N_w\left(\tilde{R}(i), \tilde{A}(i)\right)$ and $N_o(\tilde{R}(i), \tilde{A}(i))$ are given in (9) and (10). Hence, the optimal policy package satisfies conditions for fiscal sustainability [see, e.g., European Commission (2006)].

PROPOSITION 1. *The social optimum implies that the following holds for any growth rates of longevity $\mu_A(i)$ and output μ_y from Assumptions 1 and 2, respectively:*

(i) Taxes and transfers:

$$\begin{aligned} T_w(i) &= T_w(t) + y(t)\left(e^{\mu_y[i-t]} - 1\right), \\ T_o(i) &= T_o(t), \end{aligned} \tag{41}$$

for all $i \geq t$.
(ii) Consumption:

$$\begin{aligned} c_w(i) &= c_w(t), \\ c_o(i) &= c_o(t), \end{aligned} \tag{42}$$

for all $i \geq t$.
(iii) Retirement:

$$d\tilde{R}(i) = \eta_y(i)dy(i) + \eta_A(i)d\tilde{A}(i),$$

where

$$\begin{aligned} \eta_y(i) &= \frac{V_D(t)}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} > 0, \\ \eta_A(i) &= -\frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} > 0, \end{aligned} \tag{43}$$

for all $i \geq t$.

Proof. See Appendix D. ■

According to Proposition 1, the tax payment of young individuals is given as some time-invariant component plus the income growth since time t; i.e., all income growth is fully taxed. Old individuals receive a time-invariant transfer. This implies that the consumption level of both the young and the old is constant over time. Income growth is thus smoothed across generations; that is, it affects the overall level of consumption but not the profile (recall the assumption $\theta = r$). To put it differently, current and future generations all share the gains from future productivity increases in terms of a higher consumption level.

According to (35) and (36), the optimal policy implies that the marginal utility of consumption is equal for young and old; i.e.,

$$U'\left(y(i) - T_w(i)\right) = Q'\left(T_o(i)\right)$$

or

$$U'\left(c_w(i)\right) = Q'\left(c_o(i)\right),$$

from (22) and (23), capturing the well-known finding that a utilitarian policy maker redistributes to ensure equal marginal utilities of consumption (income). If utility functions over consumption are the same for young and old, $U(\cdot) = Q(\cdot)$, it is an immediate implication that they will have the same consumption level. This generalizes the Calvo and Obstfeld (1988) result to a setting with time-varying mortality rates and thus different longevity across generations (and income growth). If the marginal utility of consumption is higher when working (for young individuals), the solution implies that the consumption level for young individuals is higher than that for old ones.

The preceding results on consumption are neither surprising nor new, because they follow straightforwardly from standard consumption-smoothing arguments under separable utility. More interesting and novel are the implications for the retirement age. Two issues are important here, namely, the level of the retirement age and its profile over time (due to growth in productivity and longevity). The former is discussed in Section 5, where the allocation under the social optimum is compared with the allocation under individual decision making, whereas results for the latter are stated in Proposition 1 and discussed next.

Consider the two drivers, productivity growth and increasing longevity. First, we have that productivity growth unambiguously implies that the retirement age should be increasing over time, because $\partial \tilde{R}(i)/\partial y(i) > 0$ from (43) and because output is increasing over time from (21). The intuition is that the generations with higher productivity should work more than the generations with lower productivity. Hence, although productivity growth implies front loading of consumption in the sense that both current and future generations benefit from productivity growth in terms of a higher consumption level, it implies a rising profile for the retirement age and thus a shift of the work load onto future generations.

Second, the effect of increasing longevity on the retirement age implies that the retirement age should increase over time, because $\partial \tilde{R}(i)/\partial \tilde{A}(i) > 0$ from (43). Using (6), this implies that

$$\frac{\partial \tilde{R}(i)}{\partial A(i - \tilde{R})} > 0, \qquad \textbf{(44)}$$

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that is, generations with greater longevity should retire later. This reflects that greater longevity (better health) decreases the direct utility gain from retirement [from (19)], and therefore retirement is delayed. The retirement age increases basically because of better health, measured in terms of the value of time. Hence, although longevity does not influence the consumption profiles for working and retired persons, it does influence retirement ages.[27]

To conclude, longevity and productivity do not affect the socially optimal consumption profiles (development over time), but they do affect the retirement age profile.

4. IMPLICATIONS

As discussed in the Introduction, policies are often motivated by some notion of intergenerational equity, although this has been interpreted in quite different ways. This section interprets the properties of the social optimum in relation to some of these policy views.[28]

4.1. Retirement Age and Healthy Aging

Consider first the perception that intergenerational equity implies that the retirement age should evolve proportionally to longevity. We consider under which assumptions this is implied by the socially optimal policy derived in the preceding and then argue that these assumptions are closely related to the notion of healthy aging. This argument involves only longevity, and we therefore disregard productivity growth; i.e., $dy(i) = 0$.

The first result is given in the following lemma (remember that $\tilde{R}(i)$ is the retirement age of the generation that retires at time i, and $\tilde{A}(i)$ is the longevity of the generation that passes away at time i):

LEMMA 1. *The social optimum implies that*

$$\tilde{R}(i) = \kappa \tilde{A}(i) \tag{45}$$

for all $i \geq t$, where κ is some constant (> 0), iff the $\tilde{L}$ and $\tilde{H}$ functions are homogeneous of degree zero.

Proof. See Appendix E. ■

Lemma 1 gives the conditions that need to be fulfilled for the social optimum to imply that the retirement age should be proportional to the longevity of the generation that becomes extinct today. Intuitively the retirement age should be related to the longevity of the generation retiring rather than the longevity of the generation becoming extinct. The following lemma shows an equivalence:

LEMMA 2. *Given constant growth in longevity, there exist κ and ψ ($< \kappa$) such that*

$$\tilde{R}(i) = \kappa \tilde{A}(i) \quad \textit{iff} \quad \tilde{R}(i) = \psi A\left(i - \tilde{R}\right)$$

for all $i \geq t$.

Proof. See Appendix E. ■

Hence, given constant growth in longevity, a retirement age being proportional to the longevity of the generation that becomes extinct today [$\tilde{A}(i)$] is equivalent to a retirement age being proportional to the longevity of the generation that retires today [$A\left(i-\tilde{R}\right)$]. Therefore Lemma 1 also gives the necessary conditions for the social optimum to imply the retirement age being proportional to longevity of the generation that retires today.

Empirical evidence strongly supports healthy aging or the so-called compressed morbidity hypothesis; that is, life extension is associated with a postponement of the age at which morbidity appears [see Payne et al. (2007) and Fries et al. (2011)]. Empirical research on the relationship between age and health also confirms this and finds support for the time-to-death approach; see also OECD (2006). This approach counts backward from a given fixed reference point (date of death) rather than using the forward measure given by age; see, e.g., Batljan and Lagergren (2004) and Werblow et al. (2007). This suggests an approach[29] where time until death replaces age as the demographic indicator of health status corresponding to the homogeneity property mentioned previously. To see this, note that healthy aging implies that the health conditions at a given age improve proportionally with longevity. That is, the direct utility consequences of retirement at age a_0 for a person belonging to a generation with longevity $A(0)$ are the same as the consequences of retiring at an age $a_1 = a_0 \frac{A(1)}{A(0)}$ for a person from a generation with longevity $A(1)$ ($\neq A(0)$).[30] This is implied in the present setup if the L and H functions are homogeneous of degree zero,[31] which is implied by zero homogeneity of the $\tilde{L}$ and $\tilde{H}$ functions when there is constant growth in longevity, as the following lemma shows:

LEMMA 3. *Given constant growth in longevity, the $\tilde{L}$ and $\tilde{H}$ functions are homogeneous of degree λ iff the L and H functions are homogeneous of degree λ.*

Proof. See Appendix E. ■

This leads us to the following proposition:

PROPOSITION 2. *Given constant growth in longevity and homogeneity of the L and H functions, social optimum implies*

$$\tilde{R}(i) = \kappa \tilde{A}(i),$$
$$\tilde{R}(i) = \psi A(i-\tilde{R}),$$

for all $i \geq t$ where κ and ψ are positive constants ($\psi < \kappa$), iff the L and H functions are homogeneous of degree zero (healthy aging).

Proof. See Appendix E. ■

In the presence of healthy aging the social optimum thus implies that the retirement age should be proportional to longevity. It is interesting to observe that pension reforms in many countries in recent years have moved in the direction of linking statutory retirement ages to longevity; see OECD (2011). This implies that the tendency for the fraction of life spent in retirement to increase has been reversed.

4.2. Dependency Ratio

Policy debates center on the dependency ratio for the obvious reason that it gives the balance between contributing and receiving persons in a pay-as-you-go type of scheme. The basic problem is that aging under unchanged policies implies an increasing dependency ratio and worsening budget balance; cf. Section 2 and (24). When the retirement age is a policy instrument, the choice of the retirement age may ensure that a balance between the shares of the population working and nonworking is maintained. Hence, even if demographic stock variables may display nonstationarity, it does not necessarily follow that the key economic variable—the dependency ratio—does so. We therefore consider the implications of the socially optimal policy for the dependency ratio.

The change in the dependency ratio is given as [from (11)]

$$dK(i) = \frac{\partial K}{\partial \tilde{R}(i)} d\tilde{R}(i) + \frac{\partial K}{\partial \tilde{A}(i)} d\tilde{A}(i) \tag{46}$$

for all $i \geq t$. We have already shown in Section 2 that $dK(i) > 0$ for a fixed retirement age, i.e., $\partial K / \partial \tilde{A}(i) > 0$, and obviously a higher retirement age reduces the dependency ratio, i.e.,

$$\begin{aligned} \frac{\partial K(i)}{\partial \tilde{R}(i)} &= \frac{\frac{\partial N_o}{\partial \tilde{R}(i)} N_w(i) - N_o(i) \frac{\partial N_w}{\partial \tilde{R}(i)}}{N_w(i)^2} \\ &= \frac{-\frac{\partial N_w}{\partial \tilde{R}(i)} \left[N_w(i) + N_o(i) \right]}{N_w(i)^2} < 0, \end{aligned} \tag{47}$$

because $\frac{\partial N_w}{\partial \tilde{R}(i)} = -\frac{\partial N_o}{\partial \tilde{R}(i)} > 0$.

The case of productivity growth only is trivial ($d\tilde{A}(i) = 0$). In this case $\tilde{A}(i)$ is constant and the retirement age is increasing, cf. from (43). Hence, the dependency ratio is declining. The following proceeds under the assumption that productivity growth is zero, i.e., $dy(i) = 0$, to focus on the effects of longevity. It follows from (46) that

$$\frac{\partial K(i)}{\partial \tilde{A}(i)} \gtreqless 0 \quad \text{iff} \quad \frac{\partial \tilde{R}(i)}{\partial \tilde{A}(i)} \lesseqgtr -\frac{\frac{\partial K}{\partial \tilde{A}(i)}}{\frac{\partial K}{\partial \tilde{R}(i)}} > 0 \tag{48}$$

for all $i \geq t$ when the retirement age is allowed to respond to changes in longevity. This emphasizes that the dependency ratio increases unless the optimal policy implies a sufficiently large increase in the retirement age. The question is thus

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whether this is implied by the socially optimal policy and what conditions need to be met for it to hold.

To address this issue, we consider the case where growth in productivity is constant. Under the assumption of zero growth in productivity we have from (41) that $T_o(i) = T_o(t)$ and $T_w(i) = T_w(t)$ for all $i \geq t$. Using this in (37) implies that the development of the retirement age over time under the social optimal policy can be written as a function of longevity only,

$$\tilde{R}(i) = R\left(\tilde{A}(i)\right), \tag{49}$$

for all $i \geq t$ and where we, by use of (43), have $R' > 0$. Under the social optimal policy the relative retirement age is related to longevity as

$$\phi(i) = \frac{\tilde{R}(i)}{\tilde{A}(i)} \equiv \phi\left(\tilde{A}(i)\right) \tag{50}$$

for all $i \geq t$. Note that we have $\phi' = 0$ and $\tilde{R}' = \phi(i) = \kappa$ (a constant from before) in the case of a retirement age proportional to longevity. In general, we have the following relationship between the relative retirement age and the dependency ratio:

LEMMA 4. *Given constant growth in longevity,*

$$\frac{\partial K(i)}{\partial \tilde{A}(i)} \gtreqqless 0 \quad \textit{iff} \quad \phi' \lesseqqgtr 0$$

for all $i \geq t$.

Proof. See Appendix F. ■

Hence, if the social optimal policy is such that the relative retirement age increases following an increase in longevity, then the dependency ratio decreases. It follows directly that if the retirement age is proportional to longevity, $\phi' = 0$, we have $\partial K(i)/\partial \tilde{A}(i) = 0$. In this case the social optimal policy implies that the dependency ratio is constant over time. Combining this with the results from last section gives

COROLLARY 1. *Given constant growth in longevity and healthy aging, the social optimum implies that the dependency ratio is constant over time.*

The so-called retirement–consumption puzzle states that consumption when young is greater than when old. This is ensured if $U(c_w(i)) \geq Q(c_o(i))$ holds in the socially optimal solution. Assuming this gives the following:

LEMMA 5. *Given (i) constant growth in longevity, (ii) that utility from consumption when young is no less than when old, and (iii) that the L and H functions are homogeneous of degree λ, where λ is sufficiently close to zero, it holds that*

$$\phi' \lesseqqgtr 0 \quad \textit{iff} \quad \lambda \gtreqqless 0$$

for all $i \geq t$.

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Proof. See Appendix F. ■

Hence, if homogeneity is less (greater) than zero, the socially optimal policy is such that the relative retirement age increases (decreases) over time. This leads to the following proposition, which has Corollary 1 as a special case:[32]

PROPOSITION 3. *Given (i) constant growth in longevity, (ii) utility from consumption when young no less than when old, and (iii) L and H functions homogeneous of degree λ, where λ is sufficiently close to zero, the social optimum implies that*

$$\frac{\partial K(i)}{\partial \tilde{A}(i)} \gtreqless 0 \quad \textit{iff} \quad \lambda \gtreqless 0$$

for all $i \geq t$.

Proof. See Appendix F. ■

Hence, the dependency ratio is increasing (decreasing, constant) over time under socially optimal policy if the homogeneity of the L and H functions is greater than (less than, equal to) zero. Hence, only in the case of healthy aging does the socially optimal policy imply a constant dependency ratio.

4.3. Presavings

The idea of presavings is very predominant in debates on how to cope with the financial problems arising from increasing dependency ratios; see, e.g., IMF(2004) and European Commission (2006, 2009). The argument is that consolidation of the government's budget is needed in advance of expenditure increases driven by the demographic transitions. This is sometimes phrased "unpaid bills should not be left in the nursing room." Although such consolidation may seem common sense, it is less obvious from a normative perspective, taking into account that future generations may enjoy both higher productivity and greater longevity. The present framework makes it possible to address this issue.

Having formulated the problem in terms of taxes and transfers makes it possible to assess the extent of intergenerational redistribution by considering the budget profile. From (9), (10), and (24), we have that the primary budget balance at time i reads

$$B(i) = N_w(\tilde{R}(i), \tilde{A}(i))T_w(i) - N_o\left(\tilde{R}(i), \tilde{A}(i)\right)T_o(i), \qquad \textbf{(51)}$$

for all $i \geq t$. Taking the total difference and using (41), (9), and (10) gives

$$\begin{aligned} dB(i) &= \left\{[T_w(i) + T_o(t)]\,\tilde{m}(\tilde{R}(i), \tilde{A}(i))\right\} d\tilde{R}(i) \\ &\quad + \begin{bmatrix} T_w(i)\frac{\partial N_w\left(\tilde{R}(i),\tilde{A}(i)\right)}{\partial \tilde{A}(i)} \\ -T_o(t)\frac{\partial N_o\left(\tilde{R}(i),\tilde{A}(i)\right)}{\partial \tilde{A}(i)} \end{bmatrix} d\tilde{A}(i) \\ &\quad + \left[N_w\left(\tilde{R}(i), \tilde{A}(i)\right)\right] dy(i) \end{aligned} \qquad \textbf{(52)}$$

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for all $i \geq t$, where $T_w(i)$ is given by (41) and it has been used that $dT_o(i) = 0$ and $\tilde{m}\left(\tilde{A}(i), \tilde{A}(i)\right) = 0$.

Considering (52), we have three channels affecting the evolution of the budget balance, namely (i) a higher retirement age improves the budget balance because more individuals work and pay taxes and fewer individuals are retired and receive transfers, (ii) the direct effect of aging (increasing longevity) is in general ambiguous because it implies both more young (working) individuals and thus higher tax revenue and more old (retired) individuals and thus expenditures on transfers, and (iii) productivity growth improves the budget via improved tax revenue.

Consider first the case with productivity growth only, i.e., $d\tilde{A}(i) = 0$. Point (iii) implies positive effects of productivity growth on the budget balance. In addition to this, productivity growth has positive effects on the retirement age and, hence, on the budget balance under the optimal policy; cf. (43). It can thus be concluded that productivity growth unambiguously implies an "upward" profile for the primary budget balance; that is, the budget balance improves over time. Hence, productivity growth tends to imply current borrowing to be matched by future savings (i.e., no presavings). The intuition for this derives directly from the consumption smoothing implied by the optimal policy, because if all generations are to share the fruits of future productivity growth, then intertemporal substitution calls for current borrowing.

We now turn to the role of increasing longevity in the absence of productivity growth, i.e., $dy(i) = 0$. To simplify, assume moreover that the initial debt level is zero, i.e., $D(t) = 0$. This enables us to determine whether presaving is an optimal policy by looking at the primary budget balance at time t, i.e., $B(t)$.

Consider the expression for the budget balance (51) rewritten in terms of the dependency ratio from (11),

$$B(i) = \left[T_w(i) - K\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(i)\right] N_w\left(\tilde{R}(i), \tilde{A}(i),\right)$$

for all $i \geq t$. Note that in the absence of productivity growth we have $T_w(i) = T_w(t)$ and $T_o(i) = T_o(t)\ \forall i \geq t$ from (41). Hence,

$$\begin{aligned} dB(i) = &-T_o(t) N_w\left(\tilde{R}(i), \tilde{A}(i)\right) dK(\cdot) \\ &+ \left[T_w(t) - K\left(\tilde{R}(i), \tilde{A}(i)\right) T_o(t)\right] dN_w(\cdot) \end{aligned}$$

for all $i \geq t$ and where [from (9)]

$$dN_w(\cdot) = \frac{\partial N_w}{\partial \tilde{R}(i)} d\tilde{R}(i) + \frac{\partial N_w}{\partial \tilde{A}(i)} d\tilde{A}(i) > 0,$$

because $\partial N_w / \partial \tilde{A}(i) > 0$, $\partial N_w / \partial \tilde{R}(i) > 0$, and $d\tilde{R}(i) > 0$ from (43).

Observe first that if the optimal policy implies that the dependency ratio is constant, i.e., $\partial K / \partial \tilde{A}(i) = 0$, as is the case when there are constant growth in longevity and healthy aging (from Corollary 1), then it follows that the primary budget must balance in all periods to satisfy the intertemporal budget constraint in (40), i.e.,

$$T_w(t) - K(\tilde{R}(i), \tilde{A}(i))T_o(t) = 0.$$

Hence, the initial primary budget balance is zero, $B(t) = 0$, and remains so, and presaving is not implied by socially optimal policy.

More generally, if the optimal policy implies that $\partial K/\partial \tilde{A}(i) > 0$, we must have that the primary budget balance at t is positive, i.e.,

$$T_w(t) - K\left(\tilde{R}(t), \tilde{A}(t)\right) T_o(t) > 0,$$

because $dN_w(\cdot) > 0$, and (40) must hold. Hence, presaving is an optimal policy if the dependency ratio is increasing over time. However, if the socially optimal policy is such that $\partial K/\partial \tilde{A}(i) < 0$, we must have the converse,

$$T_w(t) - K\left(\tilde{R}(t), \tilde{A}(t)\right) T_o(t) < 0,$$

because $dN_w(\cdot) > 0$ and (40) must hold.

Hence, for presaving to be implied by socially optimal policy, the dependency ratio must increase with longevity (after adjustment of the retirement age) over time. Connecting this to the results obtained earlier indicates that it is necessary that the value of time functions for the young and old be homogeneous of degree greater than zero (nonhealthy aging) for presaving to be an optimal policy. These results are summarized in the following:

Summary 1. Given (i) constant growth in longevity, (ii) that utility from consumption when young is no less than when old, and (iii) that the L and H functions are homogeneous of degree λ, where λ is sufficiently close to zero, the social optimum implies that

Homogeneity	Dependency ratio	Presaving
$\lambda < 0$	Decreases	No
$\lambda = 0$	Constant	No
$\lambda > 0$	Increases	Yes

As this discussion indicates, it is not generally the case that presaving is implied by the socially optimal policy. In the benchmark case of healthy aging, the dependency ratio is constant, and given an initial budget balance, there are no intergenerational transfers.[33] Further, relaxing the assumption of zero growth in productivity, it becomes less likely that presaving can be a socially optimal policy, because productivity growth implies current borrowing (from abroad) financed by future savings to smooth consumption.

5. DECENTRALIZED EQUILIBRIUM

Finally, consider the question of whether the socially optimal allocation can be decentralized. In an OLG setting with age heterogeneity w.r.t. mortality risk and perfect annuities markets to insure individuals against uncertain lifetimes, Calvo and Obstfeld (1988) show that the social optimal allocation can be decentralized provided there is a sufficiently rich set of age- and time-dependent instruments. In that setting, longevity and output are constant over time and the population is stationary. Similarly, Sheshinski (2008) shows that a first-best solution is obtained in decentralized equilibrium when annuities markets are available to insure individuals against uncertain lifetimes. In that setup, there are no intergenerational differences; i.e., output and longevity are constant over time. With a sufficiently rich set of instruments this result should carry over to the present setting.

In the following we briefly consider whether the social optimum allocation derived in Section 3 in our setting with changing mortality and thus longevity can be decentralized when policy instruments (taxes and transfers) cannot be made dependent on age but only on labor market status. This case is interesting because it matches current policy settings.

To solve for the decentralized equilibrium, we follow Sheshinski (2008) in how the individual's problem is set up. An individual chooses consumption and retirement conditional on taxes and transfers levied on individuals. The government then chooses taxes and transfers now and in the future.[34] Note the difference from previous sections. Before the social planner was choosing $\{T_w(i), T_o(i), \tilde{R}(i)\}_{i=t}^{\infty}$, but now the government's policy package is $\{T_w(i), T_o(i)\}_{i=t}^{\infty}$, and individuals from different cohorts choose $\{\tilde{R}(i)\}_{i=t}^{\infty}$ (as well as consumption). The question is whether the individual's chosen retirement age can be made to coincide with the socially optimal retirement age.

Compare first the socially optimal retirement decisions and retirement in the decentralized equilibrium. Consider an arbitrary generation born at time $i - R_d$. According to the decentralized equilibrium, this generation retires at the age R_d at time i. If we assume that the same consumption levels are arising in the social planner's and decentralized equilibrium allocations (partial equilibrium, direct effects), we have that the retirement age differs because

$$R_s(i) \gtreqless R_d(i) \quad \text{for} \quad T_w(i) + T_o(i) \gtreqless 0. \tag{53}$$

The reason is that there is a common pool problem not taken into account in the individual decision making. Specifically, a marginal increase in the retirement age affects public finances by $T_w(i) + T_o(i)$. If this expression is positive (negative), retirement imposes a burden (gain) on other cohorts, which the individual does not take into account. Therefore the individually chosen retirement age differs in general from the socially optimal retirement age. In the empirically relevant case the individual tends to retire too early from a social point of view.[35]

These results are a strong indication that the social planner's allocation cannot be attained in the decentralized equilibrium. More formally, we have the following proposition, whose proof is available from the authors upon request:[36]

PROPOSITION 4. *The socially optimal allocation cannot in general be decentralized.*

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Note that the failure to decentralize the socially optimal allocation holds despite the presence of annuities markets handling the problems caused by uncertain individual lifetimes.

6. CONCLUDING REMARKS

The notion of intergenerational equity has been considered under a utilitarian criterion in an OLG model allowing for both productivity growth and increasing longevity. The latter is the more interesting aspect, both because of its relevance to current debates on demographics and because it requires a modelling approach allowing for OLG with different mortality rates and thus longevity. As in Calvo and Obstfeld (1988), the socially optimal allocation is found under generational neutral weighting to imply consumption smoothing across generations and time; i.e., young and old have the same consumption flows at all times. This is a straightforward implication of intertemporal substitution under the utilitarian criterion and with separable utility functions. However, in addition, the retirement age differs across generations, and both productivity growth and increasing longevity tend to call for increasing retirement ages over time. The former holds generally and derives from a higher return to work when its marginal productivity is high, whereas the latter follows under mild conditions amounting to increasing longevity being reflected in better health at given ages.

In policy debates strong assertions are often made about the implications of intergenerational equity, such as retirement ages following longevity proportionally or presavings. We show that the former holds under so-called healthy aging, whereas the latter arises if the retirement age does not increase sufficiently to avoid an increasing (economic) dependency ratio. Note that even if the socially optimal policy implies some presavings, the needed savings are affected by the fact that retirement ages increase across generations because of increasing longevity.

Intergenerational distribution and equity have been considered from a utilitarian perspective. Utilitarianism is the dominant approach in welfare analyses, and for comparative purposes it is thus a natural starting point for addressing the issue of changing mortality rates (and productivity). Working out the implications of utilitarianism makes it easier to discuss its pros and cons. It should be noted that the analysis assumes the same underlying utility function across generations, weighted in a way that does not imply any generational preference or bias. The criticism of utilitarianism [see, e.g., Konow (2003) for a survey and references] includes aspects such as interpersonal (generational) comparability of utility (cardinal measurement) and consequentialism assessing outcomes only in terms of utility, disregarding the underlying processes or elements affecting well-being. Utilitarianism implies redistribution based on the ability to generate utility at the margin (marginal utilities) rather than the level of utility per se. This is reflected in the present analysis, and it may be questioned whether future generations should work more (retire later) because they have higher longevity and are more productive.

This paper has taken mortality rates to be exogenous. Although this is a natural starting point to clarify the basic issues involved, it is also clear that changes in mortality rates and thus longevity are driven by both individual (lifestyle, eating habits, housing, etc.) and public decisions (health care). A small but growing literature is exploring the consequences of endogenizing longevity via these channels [see Philipson and Becker (1998), Leroux et al. (2008)]. It is an obvious agenda for future research to endogenize mortality rates.

Finally, a small open economy assumption has been adopted with respect to financial markets. This is a reasonable assumption to make for most countries, given financial globalization. Consequences of changes in the interest rate profile can readily be analyzed within the model. Because aging is a global phenomenon, there is an issue with respect to how aging itself and the induced policy changes will affect global interest rates. Although countries acting non-cooperatively take interest rates as exogenous, there are likely to be important interdependencies. It is an interesting topic for future research to analyze these issues.

NOTES

1. The human development index (HDI) published by the UNDP has longevity to weight by 1/3 [see United Nations Development Programme (2008)].

2. Balassone et al. (2009) also argue that intergenerational equity calls for presavings. The now common metric of fiscal sustainability, S2 [see, e.g., European Commission (2006, 2009)] giving the needed permanent change in the primary budget balance implies presavings if the underlying demographic changes cause a trend deterioration in the primary budget balance.

3. They also allow the social planner to weight the utility of future generations differently than implied by the subjective discount rate of the individuals. In the egalitarian or pure utilitarian case where the two are the same, the flat consumption profile follows.

4. These issues have also been addressed in standard two-period OLG models where longevity changes are interpreted as extending the length of the second period; see Auerbach and Hassett (2007) and Andersen (2008).

5. Heijdra and Romp (2008) adopt a similar approach. Both modeling approaches give a fairly good approximation to observed mortality rates. The main difference is that here there is a given maximum age, whereas in the Heijdra and Romp (2008) model survival approaches zero in the limit for a high age.

6. This holds when mortality rates apply to individuals, as is the case in this paper. When these apply to families the assumption of constant mortality rates is more acceptable, as is discussed in Blanchard (1985).

7. This is in accordance with empirical evidence; see, e.g., Wilmoth (2000).

8. This is a simplifying assumption, because increasing longevity is a global phenomenon and it is likely that increased longevity results in changes in world interest rates. Whether it results in higher or lower rates is, however, generally uncertain. Some empirical studies do show evidence of a positive relationship between longevity and aggregate savings [see discussion in Sheshinski (2008)]. This implies that interest rates decrease as longevity increases.

9. This function captures the "rectangular" shape of the data-based survival curve shown in Wilmoth (2000).

10. This holds because $\lim_{\alpha \to 1^+} [\alpha \ln \alpha - [\alpha - 1]] = 0$, $\frac{\partial [\alpha \ln \alpha - [\alpha - 1]]}{\partial \alpha} = \ln \alpha > 0$, and hence $\alpha \ln \alpha - [\alpha - 1] > 0 \; \forall \, \alpha > 1$.

11. Note that $-\frac{\partial m(a,A(t))}{\partial a}$ is the unconditional probability of passing away at age a for an individual born at time t.

12. Historically there has been a trend to increased longevity [see Oeppen and Vaupel (2002)], and demographic evidence does not show signs that human lifespans are approaching fixed limits imposed by biology or other factors [see also Wilmoth (2000) and Christensen et al. (2009)].

13. The idea is that the growth in longevity is nonincreasing in longevity. Because there is a one-to-one relationship between longevity and time in the model, this is identical to having the growth in longevity nonincreasing in time. The results of the paper can be shown to generalize to the case where there is a constant upward trend in longevity to some upper bound:

$$dA(t) = \mu_A dt \quad \text{for } A(t) < \overline{A}$$

$$dA(t) = 0 \quad \text{for } A(t) = \overline{A}.$$

All qualitative results in the paper hold under this assumption. The calculations showing this are available upon request from the authors.

14. The derivatives are shown in Appendix A.

15. This may be termed the economic dependency ratio because it depends on the retirement age, which is endogenous, and not some exogenously given age.

16. Bloom et al. (2007) use a similar specification to capture the disutility from work: $v(z, t)$, where z is life expectancy, or longevity, and t is age. They propose that the v function is homogeneous of degree zero, which can be interpreted as reflecting healthy aging (see Section 4).

17. Further, Diamond (2003) assumes that disutility of labor is a nonincreasing function of the survival probability.

18. Assuming that the marginal utility of consumption is dependent on age would create a question of optimal distribution of taxes (consumption) among young individuals, on one hand, and among old individuals, on the other. Hence, we assume that it is independent of age, as is implied by the separable formulation adopted here.

19. Alternatively, it could be formulated as $L(a, A(t-A)) = -D(a, A(t-A))$, where D denotes disutility from work, and D is increasing in a and decreasing in A.

20. In the case where the function is homogeneous of degree zero, as is proposed in Bloom et al. (2007), the value of time for a 40-year-old individual with longevity 80 years for his generation is the same as the value of time for a 50-year-old with longevity 100 years for his generation.

21. Think of time as being spent on three main types of activities: (i) work (h_w), (ii) leisure activities such as travel, sports, hobbies, and home production (h_t), and (iii) rest (h_r), where $h_w + h_t + h_r = 1$; i.e., the available time is normalized to one and the various time uses are exogenous. Using the time constraint, the utility from time use measured relative to time spent on the fallback option "rest" can be written

$$L(a, A) = \kappa_w(a, A)\,[h_w - h_r] + \kappa_t(a, A)\,[h_t - h_r]\,,$$

where κ_w and κ_t give the respective marginal values of time spent on work and leisure activities. Assuming that $\partial\kappa_w/\partial a < 0$ and $\partial\kappa_t/\partial a < 0$ corresponds to less utility from work (more disutility from work) and leisure activities with age, whereas assuming that $\partial\kappa_w/\partial A > 0$ and $\partial\kappa_t/\partial A > 0$ implies that the utility from work and utility from leisure activities increase at any age a (disutility decreases) when longevity increases. Note that the formulation allows for $\kappa_w(a, A)$ being positive for some values of (a, A), implying that people up to some limit like to work, e.g., to use skills and qualifications acquired in education.

22. Alternatively, one could specify a general function as $G(1-h, a, A(t-a))$, where 1 is the time endowment and h the exogenous hours of work. Hence $L(a, A(t-a)) \equiv G(1-h, a, A(t-a))$ and $H(a, A(t-a)) \equiv G(1, a, A(t-a))$.

23. This constraint makes the economywide resource constraint hold.

24. Note that it is more accurate to call this a constrained social optimum because the social planner is constrained by having to collect the same amount of tax from each young individual and give the same amount of transfer to each old one.

25. Setting $a = 0$ in (6) gives the relationship between the longevity of the generation born today and the generation that passes away today. Taking the total difference of this, using (5), and rearranging gives $d\tilde{A}(i) = \frac{\mu_A(i-\tilde{A})}{1+\mu_A(i-\tilde{A})}di$.

26. The derivation of the first-order conditions and the law of motion for the marginal value function are available upon request from the authors. Also, proofs of (i) the first-order conditions being necessary and sufficient for solving the maximization problem in (29) and (ii) the HJB equation and the budget constraint in (25) [and (24)] being necessary for solving (28) are available from the authors upon request.

27. It can be shown that generations with sufficiently high longevity are net contributors in the sense that their present values of consumption are lower than their present values of production, whereas those with sufficiently low longevity are net receivers. The calculations showing this are available from the authors upon request.

28. The main results in this chapter hold under nonconstant consumption ($\theta \neq r$) as well as under the assumption of autarky (balanced budget $D(t) = 0\ \forall t$). The calculations showing this are available from the authors upon request.

29. The compression in morbidity can be in either an absolute or a relative form. In the absolute form the number of years with morbidity problems is unchanged irrespective of longevity, whereas in the relative form the morbidity years are a constant share of longevity. The absolute version is a stronger form of healthy aging than the relative version, and the present paper adopts the relative version.

30. In, e.g., OECD (2006), healthy aging is interpreted in the sense that the need for health care and thus age-dependent health expenditures shift proportionally with longevity to higher ages.

31. The homogeneity assumption is made in, e.g., Bloom et al. (2007) and Andersen (2008).

32. Note that Corollary 1 does not require an assumption concerning the utilities from consumption.

33. The case of healthy aging implies the optimality of a balanced budget and, hence, the optimality of a pay-as-you-go type of scheme.

34. A description of the model and a derivation of the decentralized equilibrium are available from the authors upon request.

35. It can be shown that the retirement age tends to decline when $T_w(i) + T_o(i)$ is positive. The demonstration is available from the authors upon request.

36. Necessary conditions for allocation in the decentralized equilibrium being the same as the social planners's only hold when the number of young individuals is very close to the number of old individuals, which gives the proposition.

37. The time separability and exponential discounting following Yaari (1965) imply that problems of time inconsistency do not arise, and that agents display risk neutrality with respect to the length of life [see Bommier (2006) and Bommier et al. (2009)].

38. This is available from the authors upon request.

REFERENCES

Andersen, Torben M. (2008) Longevity and social security reform—A legislative procedure approach. *Journal of Public Economics* 92, 633–646.

Auerbach, Alan J. and Kevin Hassett (2007) Optimal long-run fiscal policy: Constraints, preferences and the resolution of uncertainty. *Journal of Economic Dynamics and Control* 31, 1451–1472.

Balassone, Fabrizio, Jorge Cunha, Geert Langenus, Bernhard Manzke, Jaenne Pavot, Doris Prammer, and Pietro Tommasino (2009) Fiscal Sustainability and Implications for the Euro Area. ECB working paper 994.

Batljan, Ilija and Marten Lagergren (2004) Inpatient/outpatient health care costs and remaining years of life—Effect of decreasing mortality on future acute health care demand. *Social Science and Medicine* 59, 2459–2466.

Blanchard, Olivier J. (1985) Debts, deficits, and finite horizons. *Journal of Political Economy* 93, 223–247.

Blanchard, Olivier J. and Stanley Fischer (1989) *Lectures on Macroeconomics*. Cambridge, MA: MIT Press.

Bloom, David E., David Canning, Richard K. Mansfield, and Michael Moore (2007) Demographic change, social security systems, and savings. *Journal of Monetary Economics* 54, 92–114.

Bommier, Antoine (2006) Uncertain lifetime and intertemporal choice: Risk aversion as a rationale for time discounting. *International Economic Review* 47, 1223–1246.

Bommier, Antoine, Marie-L. Leroux, and Jean-M. Lozachmeur (2009) On the Public Economics of Annuities with Differential Mortality. ECORE discussion paper 2009/56.

Boucekkine, Raouf, David de la Croix, and Omar Licandro (2002) Vintage human capital, demographic trends, and endogenous growth. *Journal of Economic Theory* 104, 340–375.

Calvo, Guillermo A. and Maurice Obstfeld (1988) Optimal time-consistent fiscal policy with finite lifetimes. *Econometrica* 56, 411–432.

Christensen, Kaare, Gabriele Doblhammer, Roland Rau, and James W. Vaupel (2009) Ageing populations: The challenges ahead. *Lancet* 375, 1196–1208.

Crawford, Vincent P. and David M. Lilien (1981) Social security and the retirement decision. *Quarterly Journal of Economics* 95, 505–529.

Diamond, Peter A. (1965) National debt in a neoclassical growth model. *American Economic Review* 55, 1126–1150.

Diamond, Peter A. (2003) *Taxation, Incomplete Markets, and Social Security: The 2000 Munich Lectures*. Cambridge, MA: MIT Press.

European Commission (2006) The Long-Term Sustainability of Public Finances in the European Union. A report by the Community Services.

European Commission (2009) Sustainability Report 2009. Brussels.

Fries, James F., Bonnie Bruce, and Eliza Chakravarty (2011) Compression of morbidity 1980–2011: A focused review of paradigms and progress. *Journal of Aging Research* 2011, 261702.

Halliday, Timothy J. and Melinda Podor (2009) Health Status and the Allocation of Time. IZA working paper 5368.

Heijdra, Ben J. and Ward E. Romp (2008) A life-cycle overlapping generations model of the small open economy. *Oxford Economic Papers* 60, 88–121.

IMF (2004) World Economic Outlook, September.

Kalemli-Ozcan, Sebnem and David N. Weil (2002) Mortality Change, the Uncertainty Effect, and Retirement. NBER working paper 8742.

Konow, James (2003) Which is the fairest one of all? A positive analysis of justice theories. *Journal of Economic Literature* 41, 1188–1239.

Leroux, Marie-L., Pierre Pestieau, and Gregory Ponthiere (2008) Optimal Linear Taxation under Endogenous Longevity. CORE working paper 2008/51.

OECD (2006) Projecting OECD Health and Long-Term Care Expenditures: What Are the Main Drivers? OECD Economics Department working paper 477.

OECD (2011) *Pensions at a Glance*. Paris: OECD.

Oeppen, Jim and James W. Vaupel (2002) Broken limits to life expectancy. *Science* 296, 1029–1031.

Payne, Greg, Audrey Laporte, Raisa Deber, and Peter C. Coyte (2007) Counting backward to health care's future: Using time-to-death modeling to identify changes in end-of-life morbidity and the impact of aging on health care expenditures. *Milbank Quarterly* 85, 213–257.

Philipson, Tomas J. and Gary S. Becker (1998) Old-age longevity and mortality-contingent claims. *Journal of Political Economy* 106, 551–573.

Samuelson, Paul A. (1958) An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy* 66, 467–482.

Sheshinski, Eytan (1978) A model of social secutiry and retirement decisions. *Journal of Public Economics* 10, 337–360.

Sheshinski, Eytan (2006) Note on longevity and aggregate savings. *Scandinavian Journal of Economics* 108, 353–356.

Sheshinski, Eytan (2008) *The Economic Theory of Annuities*. Princeton, NJ: Princeton University Press.

Swedish Government (2008) *Vårpropositionen*. Stockholm: Swedish Government.

UK Pensions Commission (2005) A New Pension Settlement for the Twenty-First Century. The second peport of the Pensions Commission. London.

United Nations (2008) World Population Prospects—The 2008 Revision. Population database http://esa.un.org/unpp/.

United Nations Development Programme (2008) Human Development Reports. http://hdr.undp.org/en/statistics/indices/hdi/.

Werblow, Andreas, Stefan Felder, and Peter Zweifel (2007) Population ageing and health care expenditure: A school of "red herrings"? *Health Economics* 16, 1109–1126.

Wilmoth, J. R. (2000) Demography of longevity: Past, present and future trends. *Experimental Gerontology* 35, 1111–1129.

Yaari, Menahem E. (1965) Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies* 32, 137–150.

APPENDIX A: DERIVATIVES OF THE $\tilde{m}$ FUNCTION

The derivatives are as follows:

$$\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a} = -\frac{1}{A(t-a)}\left[1+\frac{a}{A\left(t-a\right)}\mu_A(t-a)\right]\frac{\ln\alpha\,\alpha^{\frac{a}{A(t-a)}}}{\alpha-1} < 0,$$

$$\frac{\partial^2 \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial a^2} = -\frac{1}{A(t-a)^2}\left\{\begin{array}{c}\left[1+\frac{a}{A(t-a)}\mu_A(t-a)\right]\\ \times\left[\begin{array}{c}\left[1+\frac{a}{A(t-a)}\mu_A(t-a)\right]\ln\alpha\\ +2\mu_A(t-a)\end{array}\right]\\ -a\mu'_A\mu_A(t-a)\end{array}\right\}\frac{\ln\alpha\,\alpha^{\frac{a}{A(t-a)}}}{\alpha-1} < 0,$$

$$\frac{\partial \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)} = \frac{\left[1+\mu_A\left(t-\tilde{A}\right)\right]a}{A(t-a)^2}\frac{\ln\alpha\,\alpha^{\frac{a}{A(t-a)}}}{\alpha-1} > 0,$$

$$\frac{\partial^2 \tilde{m}\left(a,\tilde{A}(t)\right)}{\partial \tilde{A}(t)\partial a} = \frac{\left[1+\mu_A\left(t-\tilde{A}\right)\right]}{A(t-a)^2}\left\{\begin{array}{c}1+2\frac{a\mu_A(t-a)}{A(t-a)}\\ +\frac{a\ln\alpha}{A(t-a)}\left[1+\frac{a}{A(t-a)}\mu_A(t-a)\right]\end{array}\right\}\frac{\ln\alpha\,\alpha^{\frac{a}{A(t-a)}}}{\alpha-1} > 0,$$

where $A(t-a)$ is given in (6).

APPENDIX B: CALCULATIONS FOR POPULATION COMPOSITION $dN(t) > 0$, $dN_w(t) > 0$, AND $dN_w(t) > 0$

Applying Taylor approximation to (8), (9), and (10) and erasing terms that contain "dt" raised to a higher power than 1 [because time is continuous ($dt \to 0$)] gives the following:

$$dN(t) = \frac{\partial N}{\partial \tilde{A}(t)} d\tilde{A}(t) = \left[\int_{a=0}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da \right] d\tilde{A}(t) > 0,$$

$$dN_w(t) = \frac{\partial N_w}{\partial \tilde{A}(t)} d\tilde{A}(t) = \left[\int_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da \right] d\tilde{A}(t) > 0,$$

$$dN_o(t) = \frac{\partial N_o}{\partial \tilde{A}(t)} d\tilde{A}(t) = \left[\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da \right] d\tilde{A}(t) > 0,$$

where (5) and properties of the $\tilde{m}$ function (from Appendix A) are used. Hence, $dN(t) > 0$, $dN_w(t) > 0$, and $dN_o(t) > 0$ all hold.

Proof of $dK(t) > 0$**.** Applying Taylor approximation to (11) and erasing terms that contain "dt" raised to a higher power than 1 [because time is continuous ($dt \to 0$)] gives the following:

$$dK(t) = \frac{\frac{\partial N_o}{\partial \tilde{A}(t)} N_w(t) - N_o(t) \frac{\partial N_w}{\partial \tilde{A}(t)}}{N_w(t)^2} d\tilde{A}(t) = \frac{N_o(t)}{N_w(t)} \left[\frac{\frac{\partial N_o}{\partial \tilde{A}(t)}}{N_o(t)} - \frac{\frac{\partial N_w}{\partial \tilde{A}(t)}}{N_w(t)} \right] d\tilde{A}(t).$$

Hence, using (5), $dK(t) > 0$ iff

$$\frac{\frac{\partial N_o}{\partial \tilde{A}(t)}}{N_o(t)} > \frac{\frac{\partial N_w}{\partial \tilde{A}(t)}}{N_w(t)},$$

or, by using (9) and (10),

$$\frac{\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da} > \frac{\int_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da}. \quad \textbf{(B.1)}$$

Because $\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial a} < 0$ and $\frac{\partial^2 \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t) \partial a} > 0$, we have that

$$\int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da > \tilde{R}(t) \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right),$$

$$\int_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da < \tilde{R}(t) \left. \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} \right|_{a=\tilde{R}(t)},$$

$$\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da < \left[\tilde{A}(t) - \tilde{R}(t)\right] \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right), \quad \text{and}$$

$$\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da > \left[\tilde{A}(t) - \tilde{R}(t)\right] \left.\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}.$$

Hence,

$$\frac{\int_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da} < \frac{\tilde{R}(t) \left.\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\tilde{R}(t) \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)} \quad \text{and}$$

$$\times \frac{\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da} > \frac{\left[\tilde{A}(t) - \tilde{R}(t)\right] \left.\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\left[\tilde{A}(t) - \tilde{R}(t)\right] \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)}.$$

This gives

$$\frac{\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int_{a=\tilde{R}(t)}^{\tilde{A}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da} > \frac{\left[\tilde{A}(t) - \tilde{R}(t)\right] \left.\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\left[\tilde{A}(t) - \tilde{R}(t)\right] \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)} = \frac{\left.\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)}$$

$$= \frac{\tilde{R}(t) \left.\frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)}\right|_{a=\tilde{R}(t)}}{\tilde{R}(t) \tilde{m}\left(\tilde{R}(t), \tilde{A}(t)\right)} > \frac{\int_{a=0}^{\tilde{R}(t)} \frac{\partial \tilde{m}\left(a, \tilde{A}(t)\right)}{\partial \tilde{A}(t)} da}{\int_{a=0}^{\tilde{R}(t)} \tilde{m}\left(a, \tilde{A}(t)\right) da}.$$

Hence, (B.1) holds as well as $dK(t) > 0$. ■

APPENDIX C: THE WELFARE OBJECTIVE DERIVED

The objective of the social planner at time t is to maximize the sum of the present values of lifetime utilities of generations born at time t or later and the present value of utilities for the remaining lives of generations alive at time t. Hence, the social welfare function is[37]

$$W(t) = \int_{i=t}^{\infty} e^{-\theta(i-t)} W(i, t)\, di + \int_{i=t-\tilde{A}(t)}^{t} e^{-\theta(i-t)} W(i, t)\, di, \qquad \textbf{(C.1)}$$

where

$$W(i, t) = \int_{a=0}^{R(i)} e^{-\theta a} m(a, A(i))\, W_w(i+a, a) da + \int_{a=R(i)}^{A(i)} e^{-\theta a} m(a, A(i))\, W_o(i+a, a) da$$

for all $i \geq t$,

$$W(i, t) = \int_{a=t-i}^{R(i)} e^{-\theta a} m(a, A(i))\, W_w(i+a, a) da + \int_{a=R(i)}^{A(i)} e^{-\theta a} m(a, A(i))\, W_o(i+a, a) da$$

for all $t > i \geq t - \tilde{R}(t)$, and

$$W(i, t) = \int_{a=t-i}^{A(i)} e^{-\theta a} m(a, A(i))\, W_o(i+a, a) da$$

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for all $t - \tilde{R}(t) > i \geq t - \tilde{A}(t)$, where $W_w(i+a, a)$ and $W_o(i+a, a)$ are given in (12) and (15). It can be shown that (C.1) is well-defined.[38]

We assume that the social planner discounts at the same subjective rate as individuals (θ), which may be considered the "pure" utilitarian case where utility achieved at any point in time gets the same weight irrespective of who obtains the utility. Under the utilitarian criterion, (C.1) gives the same welfare measure as is obtained by calculating the present value of instantaneous utility generated to all living individuals [see Calvo and Obstfeld (1988)]. The social welfare function can therefore be written as in (27).

APPENDIX D: A PROOF FOR THE OPTIMAL POLICY PACKAGE

Proof of Proposition 1. (35) and (36) imply that

$$dT_w(i) = dy(i),$$
$$dT_o(i) = 0,$$

for all $i \geq t$. Using (21) in these gives (41). Using (22) and (23) along with (21) in (41) gives (42).

(37) and (41) imply that

$$V_D(t)dT_w(i) = \left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)} d\tilde{R}(i) + \left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)} d\tilde{A}(i),$$

for all $i \geq t$, implying that the retirement age evolves according to

$$d\tilde{R}(i) = \frac{V_D(t)}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} dy(i) - \frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} d\tilde{A}(i),$$

for all $i \geq t$, where $V_D(t) < 0$ from (35), $\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)} < 0$ from (18), and $\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)} > 0$ from (19). Hence,

$$\frac{V_D(t)}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} > 0,$$

$$-\frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} > 0,$$

for all $i \geq t$. This gives (43). ■

APPENDIX E: PROOFS FOR RETIREMENT AGE AND HEALTHY AGING

Proof of Lemma 1. "Only if": (45) holds only if

$$\frac{d\tilde{R}(i)}{\tilde{R}(i)} = \frac{d\tilde{A}(i)}{\tilde{A}(i)}, \qquad \textbf{(E.1)}$$

for all $i \geq t$. The expression for the evolution of the optimal retirement age from (43) implies that this holds only if

$$-\frac{\left[\frac{\partial \tilde{L}}{\partial \tilde{A}(i)} - \frac{\partial \tilde{H}}{\partial \tilde{A}(i)}\right]_{a=\tilde{R}(i)}}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\tilde{R}(i)}} = \frac{\tilde{R}(i)}{\tilde{A}(i)}, \qquad \textbf{(E.2)}$$

for all $i \geq t$. Assume that the $\tilde{L}\ (a, \tilde{A}(i))$ and $\tilde{H}(a, \tilde{A}(i))$ functions are homogeneous of degree λ, i.e., $\tilde{L}\left(a, \tilde{A}(i)\right) = \tilde{A}(i)^{\lambda} \tilde{L}\left(\frac{a}{\tilde{A}(i)}, 1\right)$ and $\tilde{H}\left(a, \tilde{A}(i)\right) = \tilde{A}(i)^{\lambda} \tilde{H}\left(\frac{a}{\tilde{A}(i)}, 1\right)$. When this is used in (E.2), (E.1) holds only if

$$-\lambda\left[\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) - \tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right)\right] = 0,$$

for all $i \geq t$. Hence, as long as $\tilde{L}(\tilde{R}(i)/\tilde{A}(i), 1) \neq \tilde{H}(\tilde{R}(i)/\tilde{A}(i), 1)$ and therefore $\tilde{L}(\tilde{R}(i), \tilde{A}(i)) \neq \tilde{H}\left(\tilde{R}(i), \tilde{A}(i)\right)$ because of the homogeneity assumption, which is likely to hold from (35)–(37), we have that (E.2) holds only if $\lambda = 0$. That is, (E.2) holds only if the $\tilde{L}$ and $\tilde{H}$ functions are both homogeneous of degree 0. This completes the first half of the proof.

"If": From (37), (41), and (21), assuming constant output per young individual ($dy(i) = 0$) gives

$$[T_w(t) + T_o(t)]\, V_D(t) = U\,(y(t) - T_w(t)) + \tilde{L}(\tilde{R}(i), \tilde{A}(i)) - Q\,(T_o(t)) - \tilde{H}(\tilde{R}(i), \tilde{A}(i)), \qquad \textbf{(E.3)}$$

for all $i \geq t$. Assuming that $\tilde{L}\ (a, \tilde{A}(i))$ and $\tilde{H}(a, \tilde{A}(i))$ are homogeneous of degree 0 implies from (E.3) that

$$\tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) - \tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1\right) = \text{constant}.$$

This can only hold if $\tilde{R}(i)/\tilde{A}(i)$ is constant, because $\partial\tilde{L}/\partial a < \partial\tilde{H}/\partial a$ from (18) and (19). Hence, we have that (45) must hold. ■

Proof of Lemma 2. Assuming constant growth in longevity in (6) gives the following relationship between the longevity of the generation that retires today and the longevity of the generation that becomes extinct today:

$$A\left(i - \tilde{R}\right) = \tilde{A}(i) - \mu_A\left[\tilde{R}(i) - \tilde{A}(i)\right]. \qquad \textbf{(E.4)}$$

"Only if": Assuming that

$$\tilde{R}(i) = \kappa \tilde{A}(i)$$

holds and plugging into (E.4) gives

$$\tilde{R}(i) = \frac{\kappa}{1 + \mu_A [1 - \kappa]} A\left(i - \tilde{R}\right).$$

Choosing $\psi = \frac{\kappa}{1+\mu_A[1-\kappa]}$ completes the first half of the proof.

"If": Assuming that

$$\tilde{R}(i) = \psi A\left(i - \tilde{R}\right)$$

holds and plugging into (E.4) gives

$$\tilde{R}(i) = \frac{\psi [1 + \mu_A]}{1 + \psi \mu_A} \tilde{A}(i).$$

Choosing $\kappa = \frac{\psi[1+\mu_A]}{1+\psi\mu_A}$ completes the second half of the proof. ■

Proof of Lemma 3. "Only if": Using the definition of the $\tilde{L}$ function, (6), under constant growth in longevity and assuming that the $\tilde{L}$ function is homogeneous of degree λ gives

$$\begin{aligned} & k^{\lambda} \tilde{L}\left(a, \tilde{A}(i)\right) \\ & = \tilde{L}(ka, k\tilde{A}(i)) \\ & = L\left(ka, k\tilde{A}(i) - \mu_A\left[ka - k\tilde{A}(i)\right]\right) \\ & = L\left(ka, k\left[\tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right]\right) \\ & = L\left(ka, kA\left(i - a\right)\right) \\ & = k^{\lambda} L\left(a, A\left(i - a\right)\right), \end{aligned}$$

where k is some constant (> 0) and the sixth line uses that $\tilde{L}\left(a, \tilde{A}(i)\right) = L\left(a, A\left(i - a\right)\right)$ by definition. Hence, the $\tilde{L}$ function is homogeneous of degree λ only if the L function is also homogeneous of degree λ. Obviously, a similar proof goes for the H function. This completes the first half of the proof.

"If": Using the definition of the $\tilde{L}$ function, (6), under constant growth in longevity and assuming that the L function is homogeneous of degree λ gives

$$\begin{aligned} & k^{\lambda} L\left(a, A\left(i - a\right)\right) \\ & = L\left(ka, kA\left(i - a\right)\right) \\ & = L\left(ka, k\left[\tilde{A}(i) - \mu_A\left[a - \tilde{A}(i)\right]\right]\right) \\ & = L\left(ka, \left[k\tilde{A}(i) - \mu_A\left[ka - k\tilde{A}(i)\right]\right]\right) \\ & = \tilde{L}(ka, k\tilde{A}(i)) \\ & = k^{\lambda} \tilde{L}\left(a, \tilde{A}(i)\right), \end{aligned}$$

where k is some constant (> 0) and the sixth line uses that $\tilde{L}\left(a, \tilde{A}(i)\right) = L\left(a, A\left(i - a\right)\right)$ by definition. Hence, the $\tilde{L}$ function is homogeneous of degree λ if the L function is also homogeneous of degree λ. Obviously, a similar proof goes for the H function. ■

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Proof of Proposition 2. This follows directly from Lemmas 1–3. ■

APPENDIX F: PROOFS FOR THE DEPENDENCY RATIO

Proof of Lemma 4. Plugging (50) into (9), (10), and (6) assuming constant growth in longevity, and the resulting equations into (11), gives

$$K(i) = \frac{\tilde{A}(i)\int_{\tilde{a}=\phi(i)}^{1} \frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}}{\tilde{A}(i)\int_{\tilde{a}=0}^{\phi(i)} \frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}} = \frac{\int_{\tilde{a}=\phi(i)}^{1} \frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}}{\int_{\tilde{a}=0}^{\phi(i)} \frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}},$$

where $\tilde{a} \equiv a/\tilde{A}(i)$ and

$$\frac{\partial K}{\partial \tilde{A}(i)} = -\phi' \frac{\left[\frac{\alpha^{\frac{\phi(i)}{1+\mu_A[1-\phi(i)]}}-\alpha}{1-\alpha}\right]}{\left[\int_{\tilde{a}=0}^{\phi(i)} \frac{\alpha^{\frac{\tilde{a}}{1+\mu_A[1-\tilde{a}]}}-\alpha}{1-\alpha}d\tilde{a}\right]}[1+K(i)].$$

It is clear from this that $\partial K/\partial \tilde{A}(i) \gtreqless 0$ iff $\phi' \lesseqgtr 0$. ■

Proof of Lemma 5. Assuming that the L and H functions, and hence the $\tilde{L}$ and $\tilde{H}$ functions from Lemma 3, are homogeneous of degree λ, we have from (43) that

$$R' = -\left\{\lambda \frac{\left[\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right) - \tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right)\right]}{\left[\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right]_{a=\frac{\tilde{R}(i)}{\tilde{A}(i)},\tilde{A}(i)=1}} - \frac{\tilde{R}(i)}{\tilde{A}(i)}\right\}.$$

From (50) we have that $\phi' \gtreqless 0$ iff $R' \gtreqless \phi(i) = \tilde{R}(i)/\tilde{A}(i)$. This implies that $\phi' \gtreqless 0$ iff

$$\underbrace{-\lambda \frac{\left[\tilde{L}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right) - \tilde{H}\left(\frac{\tilde{R}(i)}{\tilde{A}(i)},1\right)\right]}{\left(\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a}\right)_{a=\frac{\tilde{R}(i)}{\tilde{A}(i)},\tilde{A}(i)=1}}}_{\text{LHS}} \gtreqless 0.$$

Assume that the social optimal solution is such that consumption and utility when young is no less than when old [$c_w(i) \geq c_o(i)$ and $U(c_w(i)) \geq Q(c_o(i))$], which implies from (37) that $\tilde{H}(\tilde{R}(i), \tilde{A}(i)) > \tilde{L}(\tilde{R}(i), \tilde{A}(i))$ and hence that $\tilde{H}(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1) > \tilde{L}(\frac{\tilde{R}(i)}{\tilde{A}(i)}, 1)$ because of the homogeneity assumption. We therefore have that $\phi' = 0$ iff $\lambda = 0$, because $(\frac{\partial \tilde{L}}{\partial a} - \frac{\partial \tilde{H}}{\partial a})_{a=\frac{\tilde{R}(i)}{\tilde{A}(i)},\tilde{A}(i)=1} < 0$ from (18) and (19). Further, we have that $\frac{\partial \text{LHS}}{\partial \lambda}_{\lambda=0} < 0$ and, hence, $\phi' > 0$ (< 0) when (and only when) λ is slightly less (greater) than zero. The general result is $\phi' \lesseqgtr 0$ iff $\lambda \gtreqless 0$. ■

Proof of Proposition 3. This follows directly from Lemmas 4 and 5. ■

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