Ergod. Th. & Dynam. Sys. (2017), **37**, 786–801
doi:10.1017/etds.2015.81

Flow equivalence of sofic beta-shifts

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(*Received* 7 *May* 2015 *and accepted in revised form* 17 *August* 2015)

Abstract. The Fischer, Krieger, and fiber product covers of sofic beta-shifts are constructed and used to show that every strictly sofic beta-shift is 2-sofic. Flow invariants based on the covers are computed, and shown to depend only on a single integer that can easily be determined from the β-expansion of 1. It is shown that any beta-shift is flow equivalent to a beta-shift given by some $1 < \beta < 2$, and concrete constructions lead to further reductions of the flow classification problem. For each sofic beta-shift, there is an action of $\mathbb{Z}/2\mathbb{Z}$ on the edge shift given by the fiber product, and it is shown precisely when there exists a flow equivalence respecting these $\mathbb{Z}/2\mathbb{Z}$-actions. This opens a connection to ongoing efforts to classify general irreducible 2-sofic shifts via flow equivalences of reducible shifts of finite type (SFTs) equipped with $\mathbb{Z}/2\mathbb{Z}$-actions.

1. *Introduction*

One of the simplest classes of shift spaces is the class of irreducible shifts of finite type, and Franks has given a very satisfactory classification of these up to flow equivalence in terms of a complete invariant that is both easy to compute and easy to compare [**Fra84**]. This result has been extended to general shifts of finite type by Boyle and Huang [**Boy02**, **BH03**, **Hua94**], but very little is known about the flow equivalence of the class of irreducible sofic shifts even though it constitutes a natural first generalization of the class of shifts of finite type. There is an ongoing effort to classify irreducible 2-sofic shifts up to flow equivalence by considering the reducible edge shift given by the fiber product cover and the associated $\mathbb{Z}/2\mathbb{Z}$ action [**BCEa**, **BCEb**]. The present work was motivated by a desire to apply these results to the class of sofic beta-shifts. Towards this end, canonical covers of sofic beta-shifts are constructed and used to compute a number of flow invariants. It is shown that strictly sofic beta-shifts are 2-sofic, and the flow classification of their fiber products is examined in detail. This serves to connect the present work to the machinery employed in [**BCEa**, **BCEb**], but there is still a gap between the concrete constructions carried out in this paper and the general results of the classification program. Using another angle of attack, the flow equivalence problem for sofic beta-shifts is reduced to a simpler problem through a series of concrete constructions.

In §2, notation is established and an introduction to the basic definitions and properties of beta-shifts is given. In §3, the right Fischer covers of sofic beta-shifts are determined, and it is shown that if the shift is strictly sofic, then the covering map is two-to-one on some elements and one-to-one on the rest. This result is used to show that the right Krieger cover is identical to the right Fischer cover and to construct the right fiber product cover. Section 4 concerns the flow classification of beta-shifts. It is shown that for every $\beta > 1$, there exists $1 < \beta' < 2$ such that the two beta-shifts are flow equivalent and such that the new generating sequence has a form with certain properties. Additionally, the Bowen–Franks groups of the covers considered above are computed and shown to depend only on a single integer that can easily be computed from the β-expansion of 1. There is an action of $\mathbb{Z}/2\mathbb{Z}$ on the reducible edge shift of the fiber product, and it is shown precisely when there exists a flow equivalence between the fiber product edge shifts that respect these $\mathbb{Z}/2\mathbb{Z}$-actions.

2. *Background and notation*

2.1. *Shift spaces, flow equivalence, and labeled graphs.* Here, a short introduction to the definition and properties of shift spaces is given to make the present paper self-contained; for a thorough treatment of shift spaces, see [**LM95**]. Let $\mathcal{A}$ be a finite set with the discrete topology. The *full shift* over $\mathcal{A}$ consists of the space $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology and the *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. Let $\mathcal{A}^*$ be the collection of finite words (also known as blocks) over $\mathcal{A}$. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is called a *shift space* if it is invariant under the shift map and closed. For each $\mathcal{F} \subseteq \mathcal{A}^*$, define $\mathsf{X}_{\mathcal{F}}$ to be the set of bi-infinite sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any of the *forbidden words* from $\mathcal{F}$. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a shift space if and only if there exists $\mathcal{F} \subseteq \mathcal{A}^*$ such that $X = \mathsf{X}_{\mathcal{F}}$ (cf. [**LM95**, Proposition 1.3.4]). X is said to be a *shift of finite type* (SFT) if this is possible for a finite set $\mathcal{F}$. The *language* of a shift space X is defined to be the set of all words which occur in at least one $x \in X$, and it is denoted by $\mathcal{B}(X)$. A shift space is said to be *irreducible* if it contains a point with a dense forward orbit under the shift map. For each $x \in X$, define the *left-ray* of x to be $x^- = \cdots x_{-2}x_{-1}$ and define the *right-ray* of x to be $x^+ = x_0x_1x_2\cdots$. For a shift space X, the sets of all left-rays and all right-rays are denoted by X^- and X^+, respectively. Define the *predecessor set* of a right ray $x^+ \in X^+$ to be $P^\infty(x^+) = \{x^- \in X^- \mid x^-x^+ \in X\}$. The *follower set*, $F^\infty(x^-)$, of a left-ray is defined analogously.

A bijective, continuous, and shift intertwining map between two shift spaces is called a *conjugacy*, and when such a map exists, the two shift spaces are said to be *conjugate*. A function $\pi : X_1 \to X_2$ between shift spaces X_1 and X_2 is said to be a *factor map* if it is continuous, surjective, and shift intertwining. A shift space is called *sofic* [**Wei73**] if it is the image of an SFT under a factor map.

Let X be a shift space, equip $X \times \mathbb{R}$ with the product topology, and generate an equivalence relation on $X \times \mathbb{R}$ by $(x, t) \sim (\sigma(x), t-1)$. Consider the quotient space $SX = X \times \mathbb{R}/\sim$, and let $[x, t] \in SX$ denote the equivalence class of (x, t). For each

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$[x, t] \in SX$ and $r \in \mathbb{R}$, define $[x, t] + r = [x, t + r]$. For each $z \in SX$, the set $\{z + r \in SX \mid r \in \mathbb{R}\}$ is called a *flow line*. If Y is a shift space and $\Phi : SX \to SY$ is a homeomorphism, then, for each $z \in SX$, there exists a map $\varphi_z : \mathbb{R} \to \mathbb{R}$ such that $\Phi(z + r) = \Phi(z) + \varphi_z(r)$ for all $r \in \mathbb{R}$: i.e. a homeomorphism maps flow lines to flow lines. A homeomorphism $\Phi : SX \to SY$ is said to be a *flow equivalence* if there exists, for each $z \in SX$, a monotonically increasing map $\varphi_z : \mathbb{R} \to \mathbb{R}$ such that $\Phi(z + r) = \Phi(z) + \varphi_z(r)$. In this case, X and Y are said to be *flow equivalent* and this is denoted by $X \sim_{\mathrm{FE}} Y$. Flow equivalence is generated by conjugacy and *symbol expansion* [**PS75**]. In the following, a number of lemmas from [**Joh11a**, §2.3] will be used in the construction of concrete flow equivalences between beta-shifts.

For countable sets E^0 and E^1 and maps $r, s : E^1 \to E^0$, the quadruple $E = (E^0, E^1, r, s)$ is called a *directed graph*. The elements of E^0 and E^1 are the vertices and the edges of the graph, respectively. For each edge $e \in E^1$, $s(e)$ is the source of e and $r(e)$ is the range of e. A *path* $\lambda = e_1 \cdots e_n$ is a sequence of edges such that $r(e_i) = s(e_{i+1})$ for all $i \in \{1, \ldots n - 1\}$. For each $n \in \mathbb{N}_0$, the set of paths of length n is denoted by E^n, and the set of all finite paths is denoted by E^*. Extend the maps r and s to E^* in the natural way. E is said to be *essential* if every vertex emits and receives an edge. For a finite directed graph E, the *edge shift* X_E is defined by

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbb{Z}} \mid r(x_i) = s(x_{i+1}) \forall\, i \in \mathbb{Z}\}.$$

For a finite graph E with $n \times n$ integer adjacency matrix A, the *Bowen–Franks* group of X_E is the cokernel $\mathrm{BF}(\mathsf{X}_E) = \mathbb{Z}^n / \mathbb{Z}^n (\mathrm{Id} - A)$ [**LM95**, Definition 7.4.15]. If $\mathsf{X}_A, \mathsf{X}_B$ are irreducible edge shifts not flow equivalent to the trivial shift, then $\mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$ if and only if $\mathrm{BF}(\mathsf{X}_A) = \mathrm{BF}(\mathsf{X}_B)$ and $\operatorname{sgn} \det(\mathrm{Id} - A) = \operatorname{sgn} \det(\mathrm{Id} - B)$ [**Fra84**]. The pair consisting of the Bowen–Franks group and the sign of the determinant will be denoted by BF_+.

A *labeled graph* $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E and a surjective labeling map $\mathcal{L} : E^1 \to \mathcal{A}$. Extend the labeling map to $\mathcal{L} : E^* \to \mathcal{A}^*$ by defining $\mathcal{L}(e_1 \cdots e_n) = \mathcal{L}(e_1) \cdots \mathcal{L}(e_n) \in \mathcal{A}^*$. For a finite labeled graph $(E, \mathcal{L})$, define the shift space $\mathsf{X}_{(E,\mathcal{L})}$ by

$$\mathsf{X}_{(E,\mathcal{L})} = \{(\mathcal{L}(x_i))_i \in \mathcal{A}^{\mathbb{Z}} \mid x \in \mathsf{X}_E\}.$$

The labeled graph $(E, \mathcal{L})$ is said to be a *presentation* of the shift space $\mathsf{X}_{(E,\mathcal{L})}$, and a *representative* of a word $w \in \mathcal{B}(\mathsf{X}_{(E,\mathcal{L})})$ is a path $\lambda \in E^*$ such that $\mathcal{L}(\lambda) = w$. Representatives of rays are defined analogously. For each $v \in E^0$, the *follower set* $F^\infty \subseteq \mathsf{X}^+_{(E,\mathcal{L})}$, will denote the set of right-rays which have a presentation starting at v. The predecessor set, $P^\infty(v)$, is defined analogously.

Every SFT is sofic, and a sofic shift which is not an SFT is called *strictly sofic*. A shift space is sofic if and only if it can be presented by a finite labeled graph [**Fis75**]. An irreducible sofic shift X has a unique (up to graph isomorphism) minimal follower-separated presentation called the right *Fischer cover* [**Fis75**]. The right *Krieger cover* gives a canonical presentation of an arbitrary sofic shift [**Kri84**]. A map $f : X \to Y$ is said to be *k-to-one* if $|f^{-1}(\{y\})| = k$ for all $y \in Y$. Similarly, the map is said to be *at most k-to-one* if the inverse of every singleton set has at most k elements. The map is said to be

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(one and two)-to-one if it is at most two-to-one, but neither two-to-one nor one-to-one. A strictly sofic shift is said to be *2-sofic* if the factor map induced by the labeling of either the left or the right Fischer cover is at most two-to-one, but not one-to-one.

2.2. *Beta-shifts.* Here, a short introduction to the basic definitions and properties of beta-shifts is given. For a more detailed treatment of beta-shifts, see [**Bla89**]. Let $\beta \in \mathbb{R}$ with $\beta > 1$. For each $t \in [0, 1]$ define sequences $(r_n(t))_{n\in\mathbb{N}}$ and $(x_n(t))_{n\in\mathbb{N}}$ by

$$r_1(t) = \langle \beta t \rangle, \quad r_n(t) = \langle \beta r_{n-1}(t) \rangle,$$
$$x_1(t) = \lfloor \beta t \rfloor, \quad x_n(t) = \lfloor \beta r_{n-1}(t) \rfloor,$$

where $\lfloor y \rfloor$ and $\langle y \rangle$ are the integer part and the fractional part of $y \in \mathbb{R}$, respectively. The sequence $x_1(t)x_2(t)\cdots$ is said to be the *β-expansion* of t, and by [**Rén57**],

$$t = \sum_{n=1}^{\infty} \frac{x_n(t)}{\beta^n}.$$

The β-expansion of 1 is denoted by $e(\beta)$. As an example, consider $\beta = (1 + \sqrt{5})/2$ where the β-expansion of 1 is $e(\beta) = 11000\cdots$. The β-expansion of t is said to be *finite* if there exists $N \in \mathbb{N}$ such that $x_n(t) = 0$ for all $n \geq N$. It is said to be *eventually periodic* if there exist $N, p \in \mathbb{N}$ such that $x_{n+p}(t) = x_n(t)$ for all $n \geq N$. Define the *generating sequence of β* to be

$$g(\beta) = \begin{cases} e(\beta), & e(\beta) \text{ is infinite,} \\ (a_1 a_2 \cdots a_{k-1}(a_k - 1))^{\infty}, & e(\beta) = a_1 \cdots a_k 00 \cdots \text{ and } a_k \neq 0. \end{cases}$$

Define $\mathcal{B}_\beta = \{x_n(t)\cdots x_m(t) \mid 1 \leq n \leq m, t \in [0, 1]\}$. It is easy to check that $\mathcal{B}_\beta$ is the language of a shift space X_β. Such a shift space is called a *beta-shift*. The alphabet $\mathcal{A}_\beta$ of X_β is either $\{0, \ldots, \beta - 1\}$ or $\{0, \ldots, \lfloor \beta \rfloor\}$ depending on whether β is an integer. If $\beta \in \mathbb{N}$, then X_β is the full $\{0, \ldots, \beta - 1\}$-shift. For $\beta = (1 + \sqrt{5})/2$ as considered above, X_β is conjugate to the golden mean shift.

The words in $\mathcal{B}(\mathsf{X}_\beta)$ and right-rays in X_β^+ are ordered by lexicographical order $\preceq$, and the following two theorems use this order to give fundamental descriptions of beta-shifts.

THEOREM 2.1. [**Rén57**] *Let $\beta > 1$, let $g(\beta)$ be the generating sequence, let $\mathcal{A}_\beta$ be the alphabet of X_β, and let $x^+ = x_1 x_2 \ldots \in \mathcal{A}_\beta^{\mathbb{N}}$. Then $x^+ \in \mathsf{X}_\beta^+$ if and only if $x_k x_{k+1} \cdots \preceq g(\beta)$ for all $k \in \mathbb{N}$.*

THEOREM 2.2. [**Par60**] *A sequence $a_1 a_2 \cdots$ is the β-expansion of* 1 *for some $\beta > 1$ if and only if $a_k a_{k+1} \cdots \prec a_1 a_2 \cdots$ for all $k \in \mathbb{N}$. Such a β is uniquely given by the expansion of* 1.

For detailed treatments of the connections that beta-shifts provide between symbolic dynamics and number theory, see, for example, [**Sch80**, **BM86**, **Par60**, **DGS76**, **Lin84**].

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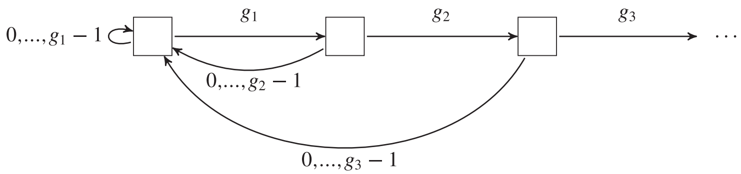

FIGURE 1. The standard loop graph of a beta-shift with generating sequence $g(\beta) = (g_n)_{n\in\mathbb{N}}$. For the sake of clarity, an edge labeled $0, \ldots, g_i - 1$, terminating at the leftmost vertex in the figure, represents g_i individual edges labeled 0 through $g_i - 1$.

3. *Covers of sofic beta-shifts*

Let $\beta > 1$, let $g(\beta) = g_1 g_2 \cdots$ be the generating sequence of X_β, and define an infinite labeled graph $\mathcal{G}_\beta = (G_\beta, \mathcal{L}_\beta)$ with vertices $G^0_\beta = \{v_i \mid i \in \mathbb{N}\}$ and edges $G^1 = \{e^k_i \mid i \in \mathbb{N}, 0 \leq k \leq g_i\}$ such that

$$s(e^k_i) = v_i, \quad r(e^k_i) = \begin{cases} v_{i+1}, & k = g_i, \\ v_1, & k < g_i, \end{cases} \quad \text{and} \quad \mathcal{L}_\beta(e^k_i) = k$$

for all $i \in \mathbb{N}$ and $0 \leq k \leq g_i$. It is easy to check that the language of X_β is $\mathcal{L}_\beta(G^*_\beta)$: i.e. one can think of $(G_\beta, \mathcal{L}_\beta)$ as an infinite presentation of X_β. The labeled graph $(G_\beta, \mathcal{L}_\beta)$ is called the *standard loop graph presentation* of X_β, and it is sketched in Figure 1. Note that the structure of the standard loop graph shows that every beta-shift is irreducible.

3.1. *Fischer cover.* Parry proved that X_β is an SFT if and only if the generating sequence is periodic [**Par60**], and Bertrand-Mathis proved that X_β is sofic if and only if the generating sequence is eventually periodic [**BM**]. The latter result is apparently only available in a preprint, so an argument for this fact is given as part of the proof of the following proposition.

PROPOSITION 3.1. *Given $\beta > 1$, the beta-shift X_β is sofic if and only if the generating sequence $g(\beta)$ is eventually periodic. For minimal $n, p \in \mathbb{N}$ with $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$, the right Fischer cover of X_β is the labeled graph $(F_\beta, \mathcal{L}_\beta)$ shown in Figure 2 which has $F^0_\beta = \{v_1, \ldots, v_{n+p}\}$, $F^1_\beta = \{e^k_i \mid 1 \leq i \leq n+p, 0 \leq k \leq g_i\}$, and*

$$s(e^k_i) = v_i, \quad r(e^k_i) = \begin{cases} v_1, & k < g_i, \\ v_{i+1}, & k = g_i, i < n+p, \\ v_{n+1}, & k = g_{n+p}, i = n+p, \end{cases} \quad \textit{and} \quad \mathcal{L}_\beta(e^k_i) = k.$$

Note that the right Fischer cover $(F_\beta, \mathcal{L}_\beta)$ of a sofic beta-shift with generating sequence $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$ is the labeled graph obtained from the subgraph of the standard loop graph $\mathcal{G}_\beta$ induced by the first $n+p$ vertices $v_1, \ldots, v_{n+p}$ of G^0_β by adding an additional edge labeled g_{n+p} from v_{n+p} to v_{n+1}.

Proof of Proposition 3.1. Assume that $g(\beta)$ is not eventually periodic and let v, w be finite prefixes of $g(\beta)$ with $v \neq w$. Choose $x^+_v, x^+_w \in \mathsf{X}^+_\beta$ such that $v x^+_v = g(\beta) = w x^+_w$. Since $g(\beta)$ is not eventually periodic, $x^+_v \neq x^+_w$. Assume, without loss of generality, that

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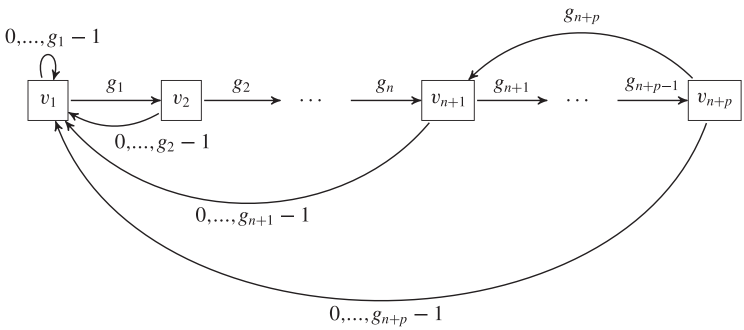

FIGURE 2. Right Fischer cover of a sofic beta-shift with generating sequence $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$ for minimal n, p. An edge labeled $0, \ldots, g_i - 1$ represents g_i individual edges labeled 0 through $g_i - 1$.

$x_v^+ > x_w^+$. Then $wx_v^+ > wx_w^+ = g(\beta)$, so by Theorem 2.1, $wx_v^+ \notin \mathsf{X}_\beta^+$. Hence, $P_\infty(x_v^+) \neq P_\infty(x_w^+)$. Since $g(\beta)$ is not eventually periodic, this proves that X_β has an infinite number of different predecessor sets, so it is not a sofic shift [**Kri84**, §2]. The standard loop graph is right-resolving, so it follows from the observation preceding this proof that $(F_\beta, \mathcal{L}_\beta)$ is a right-resolving presentation of X_β when $g(\beta)$ is eventually periodic. It is easy to check that $(F_\beta, \mathcal{L}_\beta)$ is also follower-separated, so it is the right Fischer cover of X_β. □

Remark 3.2. A direction was chosen in the definition of X_β which makes right-rays behave qualitatively differently from left-rays. While it is generally straightforward to write down the left Fischer cover of a sofic beta-shift, the author knows of no closed form description of the left Fischer cover corresponding to the one given above for the right Fischer cover.

LEMMA 3.3. [**Joh99**, Proposition 2.5.1] *Let $\beta > 1$, let X_β be sofic, and let $(F_\beta, \mathcal{L}_\beta)$ be the right Fischer cover of X_β. If there is a path $\lambda \in F_\beta^*$ with $s(\lambda) = v_1$ such that $\mathcal{L}_\beta(\lambda) = w$ is not a factor of $g(\beta)$, then every path in F_β^* labeled w terminates at $r(\lambda)$.*

Some cases were left unchecked in the proof in [**Joh99**], but a complete proof is given in [**Joh11a**, Lemma 4.6].

PROPOSITION 3.4. *Let $\beta > 1$ with eventually periodic $g(\beta)$, let $(F_\beta, \mathcal{L}_\beta)$ be the right Fischer cover of X_β, and let $\pi : \mathsf{X}_{F_\beta} \to \mathsf{X}_\beta$ be the covering map. If $g(\beta)$ is periodic, then π is one-to-one, and if not, then it is (one and two)-to-one.*

Proof. Let $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$ for minimal $n, p \in \mathbb{N}$, and let $\mathfrak{p} = g_{n+1} \cdots g_{n+p}$. The first goal is to prove that if $x \in \mathsf{X}_\beta$ has more than one presentation in $(F_\beta, \mathcal{L}_\beta)$, then there exists $k \in \mathbb{Z}$ such that $\cdots x_{k-1} x_k = \mathfrak{p}^\infty$.

Let $x \in \mathsf{X}_\beta$, and let λ be a bi-infinite path in F_β with $\mathcal{L}_\beta(\lambda) = x$. If there is a lower bound l on the set

$$A = \{ j \in \mathbb{Z} \mid \exists i < j : \mathcal{L}_\beta(\lambda_{[i,j]}) \text{ is not a factor of } g(\beta) \},$$

then there exists $k \leq l$ such that $\cdots x_{k-1} x_k = \mathfrak{p}^\infty$, since x is bi-infinite while $g(\beta)$ is not. By Proposition 3.1, the only circuit in $(F_\beta, \mathcal{L}_\beta)$ that does not pass through v_1 is labeled $\mathfrak{p}$,

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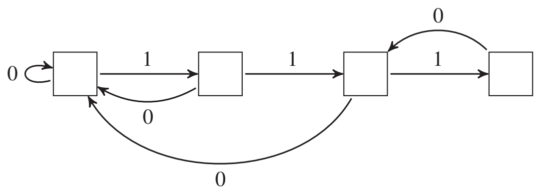

FIGURE 3. Right Fischer cover of the sofic beta-shift from Example 3.5.

so if there is a lower bound l on the set $B = \{i \in \mathbb{Z} \mid r(\lambda_i) = v_1\}$, then there exists $k < l$ such that $\cdots x_{k-1}x_k = \mathfrak{p}^\infty$. Assume that both A and B are unbounded below, let $\mu \in \mathsf{X}_{F_\beta}$ with $\mathcal{L}(\mu) = x$, and let $k \in \mathbb{Z}$. Then there exist $i < j < k$ with $s(\lambda_i) = v_1$ such that $w = \mathcal{L}_\beta(\lambda_{[i,j]})$ is not a factor of $g(\beta)$. By Lemma 3.3, $r(\mu_j) = r(\lambda_j)$, and $(F_\beta, \mathcal{L}_\beta)$ is right-resolving, so this implies that $\mu_k = \lambda_k$. Since k was arbitrary, $\lambda = \mu$, and this proves the claim.

If $g(\beta)$ is periodic, then every circuit in F_β passes through v_1, and only one of these is labeled $\mathfrak{p}$, since p is the minimal period. Hence, there is precisely one vertex in F^0 where a presentation of $\mathfrak{p}^\infty$ can end, so π_β is one-to-one and X_β is an SFT.

Assume that $g(\beta)$ is eventually periodic without being periodic. Then $\mathfrak{p} < g(\beta)_{[1,p]}$, so there is a circuit μ in F_β passing through v_1 with $\mathcal{L}_\beta(\mu) = \mathfrak{p}$. Choose $0 \leq i \leq p$ such that $\mathfrak{p}_i \cdots \mathfrak{p}_p\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}$ is maximal among the cyclic permutations of the letters of $\mathfrak{p}$. The number i is unique because p is the minimal period. Now $s(\mu_i) = v_i$, so μ is unique because $(F_\beta, \mathcal{L}_\beta)$ is right-resolving. The only circuit that does not pass through v_1 is also labeled $\mathfrak{p}$, so π_β is (one and two)-to-one. □

Example 3.5. Use Theorem 2.2 to choose $\beta > 1$ such that $g(\beta) = 11(10)^\infty$. The right Fischer cover of X_β is shown in Figure 3. Note that there are two presentations of, for example, the bi-infinite sequence $(10)^\infty$.

The following result was proved by Parry [**Par60**], but it is repeated here since it follows immediately from Propositions 3.1 and 3.4.

COROLLARY 3.6. *For $\beta > 1$, the beta-shift X_β is an SFT if and only if the generating sequence $g(\beta)$ is periodic.*

3.2. *Krieger cover.* Let X be an irreducible sofic shift with right Fischer cover $(G, \mathcal{L})$. A left-ray $x^- \in X^-$ is said to be $1/k$-*synchronizing* if k is the smallest integer such that there exist $v_1, \ldots, v_k \in G^0$ with $F^\infty(x^-) = \bigcup_{i=1}^k F^\infty(v_i)$. See [**Joh11b**] for a discussion of the relation between this concept and the structure of the Krieger cover. In particular, it is shown that the right Krieger cover is identical to the right Fischer cover when every right-ray is 1-synchronizing.

PROPOSITION 3.7. *If $\beta > 1$ and the beta-shift X_β is sofic, then the right Krieger cover of X_β is identical to the right Fischer cover $(F_\beta, \mathcal{L}_\beta)$ of X_β.*

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Proof. Let n, p be minimal such that $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$ and let $F_\beta^0 = \{v_1, \ldots, v_{n+p}\}$ as in Proposition 3.1. Let $x^- \in \mathsf{X}_\beta^-$, and let $r(x^-) \subseteq F_\beta^0$ be the set of vertices where a presentation of x^- can end. The goal is to prove that x^- is 1-synchronizing.

Assume first that $x^- = (g_{n+1} \cdots g_{n+p})^\infty g_{n+1} \cdots g_{n+k}$ for some $k \le p$. By the proof of Proposition 3.4, there is a unique i such that there is a circuit labeled $\mathfrak{p}' = g_{n+k+1} \cdots g_{n+p} g_{n+1} \cdots g_{n+k}$ passing through v_1 and terminating at v_i. Similarly, there is a unique $j \neq i$ such that there is a circuit labeled $\mathfrak{p}'$ which terminates at v_j without passing through v_1. Now, $r(x^-) = \{v_i, v_j\}$ and $F_\infty(v_j) \subseteq F_\infty(v_i)$, so x^- is 1-synchronizing.

If x^- is not of the form considered above, then the proof of Proposition 3.4 shows that $|r(x^-)| = 1$, so x^- is 1-synchronizing. Hence, the right Krieger cover is equal to the right Fischer cover by [**Joh11b**, §4]. □

This result also follows from [**KMW98**], where it is shown that the Matsumoto algebra associated with X_β is simple.

3.3. *Fiber product.* By Proposition 3.4, the covering map of the Fischer cover of a sofic beta-shift is (one and two)-to-one, so every strictly sofic beta-shift is 2-sofic. There is an ongoing effort to classify irreducible 2-sofic shifts up to flow equivalence via flow equivalence of certain derived reducible shift spaces equipped with an action of $\mathbb{Z}/2\mathbb{Z}$ [**BCEa**, **BCEb**]. These tools are not yet general enough to be applied to beta-shifts, but the present work was motivated by a desire to pave the way for such an investigation. This section contains an introduction to fiber products and a construction of the right fiber product covers of sofic beta-shifts.

Definition 3.8. Let X be an irreducible sofic shift and let $(F, \mathcal{L}_F)$ be the right Fischer cover of X. The *(right) fiber product graph* of X is defined to be the labeled graph with vertex set $\{(u, v) \mid u, v \in F^0\}$ where there is an edge labeled a from (u_1, v_1) to (u_2, v_2) if and only if there are edges labeled a from u_1 to u_2 and from v_1 to v_2 in F.

The fiber product graph of X is a presentation of X, and the right Fischer cover is isomorphic to the subgraph induced by the diagonal vertices $\{(v, v) \mid v \in F^0\}$. The fiber product graph is generally not essential, so it is often useful to pass to the maximal essential subgraph. This subgraph will be called the *fiber product cover* in the following.

Let X be a sofic shift with right Fischer cover $(F, \mathcal{L}_F)$ such that the covering map $\pi : \mathsf{X}_F \to X$ is (one and two)-to-one, and let $(P, \mathcal{L}_P)$ be the fiber product cover of X. Let $\lambda \in \mathsf{X}_P$ be a bi-infinite sequence, let $i \in \mathbb{Z}$, let $s(\lambda_i) = (u_i, v_i) \in P^0$, and let $r(\lambda_i) = (u_{i+1}, v_{i+1}) \in P^0$. Then (v_i, u_i), $(v_{i+1}, u_{i+1}) \in P^0$ as well, and there is a unique edge $\bar{\lambda}_i \in P^1$ labeled $\mathcal{L}_P(\lambda_i)$ from (v_i, u_i) to (v_{i+1}, u_{i+1}). Define a map $\varphi : \mathsf{X}_P \to \mathsf{X}_P$ by $\varphi(\lambda) = (\bar{\lambda}_i)_{i \in \mathbb{Z}}$, and note that $\varphi^2 = \mathrm{id}$. In this way, the labeling induces a continuous and shift commuting $\mathbb{Z}/2\mathbb{Z}$ action on the edge shift X_P called the *fiber product involution*. This also induces a corresponding continuous $\mathbb{Z}/2\mathbb{Z}$ action on the suspension $S\mathsf{X}_P$. See [**BS05**] for an investigation of equivariant flow equivalence of shift spaces equipped with continuous free shift commuting actions of a finite group, but note that the fiber product involution defined from the Fischer cover will never be free.

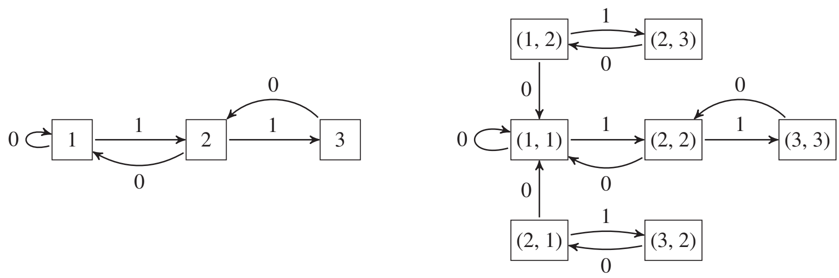

FIGURE 4. Fischer cover and Fiber product cover of the beta-shift considered in Example 3.9.

Example 3.9. Use Theorem 2.2 to find $1 < \beta < 2$ with $g(\beta) = 1(10)^\infty$. It is straightforward to check that the labeled graphs in Figure 4 are the Fischer cover and the fiber product cover of this beta-shift. Note that the Fischer cover is isomorphic to the subgraph induced by the vertices (1, 1), (2, 2), and (3, 3).

The following proposition shows that the fiber product cover always has the structure seen in the previous example.

PROPOSITION 3.10. *Let $\beta > 1$ such that $g(\beta)$ is eventually periodic but not periodic, then the fiber product cover of X_β is the graph shown in Figure 5.*

Proof. Let $(F_\beta, \mathcal{L}_\beta)$ be the right Fischer cover of X_β, and let n, p be minimal such that $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$. By the proof of Proposition 3.4, a left-ray will have a unique presentation unless it is equal to w^∞ for some cyclic permutation w of the period $g_{n+1} \cdots g_{n+p}$. To find the fiber product cover, it is therefore sufficient to consider such periodic left-rays.

By the proof of Proposition 3.4, there exist $u_0, \ldots, u_{p-1}, u'_0, \ldots, u'_{p-1} \in F^0_\beta$ with $u_i \neq u'_i$ such that there are edges labeled g_{n+i+1} from u_i to $u_{i+1 (\text{mod } p)}$ and from u'_i to $u'_{i+1 (\text{mod } p)}$ for each $0 \leq i \leq p-1$. Now $(u_0, u'_0), \ldots, (u_{p-1}, u'_{p-1})$ are the only off-diagonal vertices in the fiber product cover. For each $0 \leq i \leq p-1$, the fiber product cover has an edge labeled g_{n+i+1} from (u_i, u'_i) to $(u_{i+1 (\text{mod } p)}, u'_{i+1 (\text{mod } p)})$ and edges labeled $0, \ldots, g_{n+i+1} - 1$ from (u_i, u'_i) to (v_1, v_1), where v_1 is the first vertex of the right Fischer cover. This gives the labeled graph shown in Figure 5. □

4. *Invariants and equivalences*

In this section, the acquired knowledge about the structure of covers of sofic beta-shifts will be used to compute flow invariants and to reduce the flow classification problem through a series of concrete constructions.

4.1. *Flow classification of beta-shifts of finite type.*

The characterization of beta-shifts of finite type given in Corollary 3.6 makes it possible to give a complete flow classification of such shifts.

PROPOSITION 4.1. *Given that $\beta > 1$ such that X_β is an SFT, and choose minimal $p \in \mathbb{N}$ such that the generating sequence is $g(\beta) = (g_1 \cdots g_p)^\infty$. Then $\mathrm{BF}_+(\mathsf{X}_\beta) = -\mathbb{Z}/S\mathbb{Z}$ with $S = \sum_{j=1}^{p} g_j$. In particular, every SFT beta-shift is flow equivalent to a full shift.*

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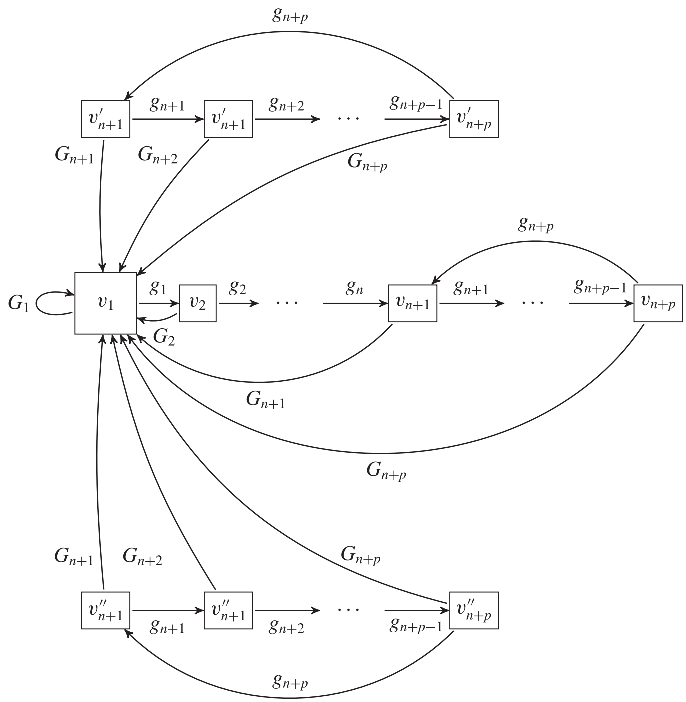

FIGURE 5. Fiber product cover of a sofic beta-shift with $g(\beta) = g_1 \cdots g_n(g_{n+1} \cdots g_{n+p})^\infty$ for minimal n, p. Here, each edge labeled G_i represents g_i individual edges labeled 0 through $g_i - 1$.

Note that $\sum_{j=1}^{p} g_j = -1 + \sum_{i=1}^{p} a_i$ when $e(\beta) = a_1 \cdots a_p 00 \cdots$ and $a_p \neq 0$.

Proof. By Proposition 3.1, the adjacency matrix of the underlying graph of the right Fischer cover of X_β is

$$A = \begin{pmatrix} g_1 & 1 & 0 & \cdots & 0 & 0 \\ g_2 & 0 & 1 & & 0 & 0 \\ g_3 & 0 & 0 & & 0 & 0 \\ \vdots & & & \ddots & & \\ g_{p-1} & 0 & 0 & & 0 & 1 \\ g_p + 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Now it is straightforward to compute the complete invariant by finding the Smith normal form and determinant of $\mathrm{Id} - A$. □

It is also not hard to construct a concrete flow equivalence between the beta-shift considered in Proposition 4.1 and the full $(S+1)$-shift.

Example 4.2. If $\beta > 1$, then the entropy of X_β is $\log \beta$ [**Par60**, **Rén57**]. In particular, beta-shifts X_{β_1} and X_{β_2} with $\beta_1 \neq \beta_2$ are never conjugate. However, by Theorem 2.2, there exist $1 < \beta_1 < 2$ and $2 < \beta_2 < 3$ such that $(110)^\infty$ is the generating sequence of X_{β_1} and $(20)^\infty$ is the generating sequence of X_{β_2}, and the beta-shifts X_{β_1} and X_{β_2} are flow equivalent by Proposition 4.1.

4.2. *Bowen–Franks groups.* The Bowen–Franks groups of the underlying graphs of the covers from §3 will be computed in this section.

PROPOSITION 4.3. *Let $\beta > 1$ with sofic X_β, and let n, p be minimal such that $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$. Let A_F and A_P be the adjacency matrices of the underlying graphs of the Fischer cover and the fiber product cover, respectively. Then $\mathrm{BF}_+(A_F) = -\mathbb{Z}/S\mathbb{Z}$ and $\mathrm{BF}(A_P) = (\mathbb{Z}/S\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$, where $S = \sum_{i=1}^{p} g_{n+i}$.*

Proof. By Proposition 3.1,

$$A_F = \left(\begin{array}{cccccc|cccccc}
g_1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
g_2 & 0 & 1 & & 0 & 0 & 0 & 0 & 0 & & 0 & 0\\
g_3 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & & 0 & 0\\
\vdots & & & \ddots & & \vdots & \vdots & & & \ddots & & \vdots\\
g_{n-1} & 0 & 0 & & 0 & 1 & 0 & 0 & 0 & & 0 & 0\\
g_n & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ \hline
g_{n+1} & & & & & & 0 & 1 & 0 & \cdots & 0 & 0\\
g_{n+2} & & & & & & 0 & 0 & 1 & & 0 & 0\\
g_{n+3} & & & & & & 0 & 0 & 0 & & 0 & 0\\
\vdots & & & 0 & & & \vdots & & & \ddots & & \vdots\\
g_{n+p-1} & & & & & & 0 & 0 & 0 & & 0 & 1\\
g_{n+p} & & & & & & 1 & 0 & 0 & \cdots & 0 & 0
\end{array}\right).$$

It is straightforward to find the invariant by computing the Smith normal form and determinant of $\mathrm{Id} - A_F$. The other Bowen–Franks group is computed in the same manner. □

Note that these Bowen–Franks groups only contain information about the sum of the numbers in the periodic part of the generating sequence. This is partially explained by the results of the following section.

4.3. *Concrete constructions.* This section contains recipes for concrete constructions of flow equivalences reducing the complexity of beta-shifts. Let $n \in \mathbb{N}$ and $n - 1 < \beta < n$ be given, and let $X = \mathsf{X}_\beta$. Define $\varphi : \mathcal{A}_\beta \to \{0, 1\}^*$ by $\varphi(0) = 0$ and

$$\varphi(j) = 1^j 0 \quad j \in \mathbb{N}.$$

Extend this to a map $\varphi : \mathcal{A}_\beta^* \to \{0, 1\}^*$ by defining $\varphi(a_1 \cdots a_k) = \varphi(a_1) \cdots \varphi(a_k)$. It is straightforward to check that the shift closure of $\varphi(\mathcal{B}(X))$ is the language of a shift space X' flow equivalent to X (see [**Joh11a**, §2.3] for a collection of similar constructions). Define the induced map $\varphi : X^+ \to X'^+$ by $\varphi(x_1 x_2 \cdots) = \varphi(x_1)\varphi(x_2) \cdots$.

LEMMA 4.4. *The map $\varphi : X^+ \to X'^+$ constructed above is surjective and order preserving.*

Proof. The map preserves the lexicographic order by construction. Let $x'^+ \in X'^+$ be given. By construction, there exists $x^+ = x_1 x_2 \cdots \in X^+$ such that $x'^+ =$

$w\varphi(x_2)\varphi(x_3)\cdots$, where w is a suffix of $\varphi(x_1)$. Now there exists $1 \le a \le x_1$ such that $w = \varphi(a)$. Since $ax_2x_3\cdots \le x_1x_2\cdots \le g(\beta)$, it follows from Theorem 2.1 that $ax_2x_3\cdots \in X^+$, so $x'^+ \in \varphi(X^+)$. □

THEOREM 4.5. *For every $\beta > 1$ there exists $1 < \beta' < 2$ such that $\mathsf{X}_\beta \sim_{FE} \mathsf{X}_{\beta'}$.*

Proof. Let $X = \mathsf{X}_\beta$, and construct $X' \sim_{\mathrm{FE}} X$ and $\varphi : X^+ \to X'^+$ as above. Use Theorem 2.2 to choose $1 < \beta' < 2$ such that $g(\beta') = \varphi(g(\beta))$. The aim is to prove that $X' = \mathsf{X}_{\beta'}$.

Given $x'^+ = x'_1x'_2\cdots \in X'^+$ and $k \in \mathbb{N}$, let $x'_{k+} = x'_kx'_{k+1}\cdots$. Use Lemma 4.4 to choose $x^+ \in X^+$ such that $\varphi(x^+) = x'_{k+}$. Now $x^+ \le g(\beta)$ and φ is order preserving, so $x'_{k+} \le g(\beta')$. By Theorem 2.1, this means that $x'^+ \in \mathsf{X}^+_{\beta'}$.

Let $x'^+ = x'_1x'_2\cdots \in \mathsf{X}^+_{\beta'}$ and let $n = \max \mathcal{A}_\beta$. Consider the extension $\varphi : \{0, \ldots, n\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ and note that x'^+ does not contain 1^{n+1} as a factor, so there exists $x^+ = x_1x_2\cdots \in \{0, \ldots, n\}^{\mathbb{N}}$ such that $\varphi(x^+) = x'^+$. Let $k \in \mathbb{N}$ be given and let $x_{k+} = x_kx_{k+1}\cdots$. Then there exists $l \ge k$ such that $\varphi(x_{k+}) = x'_lx'_{l+1}\cdots \le g(\beta') = \varphi(g(\beta))$, but φ is order preserving, so this means that $x_{k+} \le g(\beta)$. Hence, $x^+ \in \mathsf{X}^+_\beta$ and $x'^+ \in X'^+$ as desired. □

This shows that it is sufficient to consider $1 < \beta < 2$ when trying to classify sofic beta-shifts up to flow equivalence. The next goal is to find a standard form that any sofic beta-shift can be reduced to.

LEMMA 4.6. *Let $1 < \beta < 2$ such that $g(\beta)$ is aperiodic and let n be the largest number such that 1^n is a prefix of $g(\beta)$. Then $\mathsf{X}_\beta \sim_{FE} \mathsf{X}_{\beta'}$, where $g(\beta')$ is obtained from $g(\beta)$ by deleting a 0 immediately after each occurrence of 1^n.*

Proof. Note that $1^{n+1} \notin \mathcal{B}(\mathsf{X}_\beta)$, so each occurrence of 1^n in X_β is followed by 0. Define a map $\phi_{1^n0\mapsto 1^n} : \mathcal{B}(\mathsf{X}_\beta) \to \{0, 1\}^*$ by mapping each word w to the word obtained by deleting the 0 in each occurrence of 1^n0. It is straightforward to prove that $\phi_{1^n0\mapsto 1^n}(\mathcal{B}(\mathsf{X}_\beta))$ is the language of a shift space $\mathsf{X}_\beta^{1^n0\mapsto 1^n}$ (see [**Joh11a**, §2.3]). There is an upper bound on $\{k \mid (1^n0)^k \in \mathcal{B}(\mathsf{X}_\beta)\}$ since 1^n0 is a prefix of $g(\beta)$, which is aperiodic, and therefore $\mathsf{X}_\beta \sim_{\mathrm{FE}} \mathsf{X}_\beta^{1^n0\mapsto 1^n}$, by arguments analogous to the ones used in [**Joh11a**, §2.4].

Define $\varphi : \mathsf{X}^+_\beta \to \{0, 1\}^{\mathbb{N}}$ such that $\varphi(x^+)$ is the sequence obtained from x^+ by deleting one 0 immediately after each occurrence of 1^n. This is an order preserving map. Use Theorem 2.2 to choose β' such that $g(\beta') = \varphi(g(\beta))$. Given $y \in \mathsf{X}_\beta^{1^n0\mapsto 1^n}$ and $k \in \mathbb{N}$, define x^+ to be the sequence obtained from $y_ky_{k+1}\cdots$ by inserting 0 after y_j if there exists $l \in \mathbb{N}$ such that $y_{[j-ln,j]} = 01^{ln}$ or such that $j = k + ln - 1$ and $y_{[k,j]} = 1^{ln}$. Now, $x^+ \in \mathsf{X}^+_\beta$, so $y_ky_{k+1}\cdots = \varphi(x^+) \le \varphi(g(\beta)) = g(\beta')$, and $y \in \mathsf{X}_{\beta'}$ by Theorem 2.1. A similar argument proves the other inclusion. □

PROPOSITION 4.7. *Let $1 < \beta < 2$ such that $g(\beta)$ is aperiodic and let n be the largest number such that 1^n is a prefix of $g(\beta)$. For each $k > n/2$, $\mathsf{X}_\beta \sim_{FE} \mathsf{X}_{\beta'}$, where $g(\beta')$ is obtained from $g(\beta)$ by inserting a 0 immediately after the initial 1^k and after each subsequent occurrence of 01^k.*

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Proof. Apply Lemma 4.6 to $\mathsf{X}_{\beta'}$. □

Example 4.8. Consider $\beta > 1$ such that $g(\beta) = 1101101(0101100)^\infty$. Use Lemma 4.6 to see that $\mathsf{X}_\beta \sim_{\mathrm{FE}} \mathsf{X}_{\beta_1}$ when $g(\beta_1) = 11111(010110)^\infty$. By Proposition 4.7, $\mathsf{X}_{\beta_1} \sim_{\mathrm{FE}} \mathsf{X}_{\beta_2}$ when

$$g(\beta_2) = 111110(010110)^\infty = 111(110010)^\infty.$$

Note how this operation permutes the period of the generating sequence. Use Proposition 4.7 again to show that $\mathsf{X}_{\beta_2} \sim_{\mathrm{FE}} \mathsf{X}_{\beta_3}$ when

$$g(\beta_3) = 1110(110010)^\infty = 11(101100)^\infty.$$

An additional application of Proposition 4.7 will not reduce the aperiodic beginning of the generating sequence further, since it will also add an extra 0 inside the period. Note, in particular, that the length of neither the period nor the beginning of the generating sequence is a flow invariant. The sum of entries in the period of the generating sequence is a flow invariant by Proposition 4.3, but the same is apparently not true for the sum of entries in the aperiodic beginning. Indeed, it is straightforward to use Lemma 4.6 and Proposition 4.7 to show that $\mathsf{X}_\beta \sim_{\mathrm{FE}} \mathsf{X}_{\beta'}$ if

$$g(\beta') = 1^{3n}(110010)^\infty \quad \text{or}$$
$$g(\beta') = 1^{3n+2}(101100)^\infty$$

for some $n \in \mathbb{N}$. However, at this stage, it is still unclear whether $\mathsf{X}_\beta \sim_{\mathrm{FE}} \mathsf{X}_{\beta''}$ when $g(\beta'') = 1^{3n}(101100)^\infty$.

The following proposition shows how the tools applied in Example 4.8 can be used to modify the generating sequence of a general sofic beta-shift to produce a beta-shift flow equivalent to the original.

PROPOSITION 4.9. *Let $1 < \beta < 2$ with $g(\beta) = \mathfrak{b}\mathfrak{p}^\infty$, where $\mathfrak{b} = g_1 \cdots g_n$ and $\mathfrak{p} = g_{n+1} \cdots g_{n+p} = 1^{p_1}0^{q_1} \cdots 1^{p_m}0^{q_m}$. Assume that n and p are minimal, and let $S_\mathfrak{b} = \sum_{i=1}^n g_i$, and $S_\mathfrak{p} = \sum_{i=1}^p g_{n+i} = \sum_{j=1}^m p_j$. Given $1 \le k \le m$, $\mathsf{X}_\beta \sim_{FE} \mathsf{X}_{\beta'}$ when*

$$g(\beta') = 1^{S_\mathfrak{b} + lS_\mathfrak{p} + p_1 + \cdots + p_k}(1^{p_{k+1}}0^{q_{k+1}} \cdots 1^{p_m}0^{q_m}1^{p_1}0^{q_1} \cdots 1^{p_k}0^{q_k})^\infty$$

for $l \in \mathbb{Z}$ with $S_\mathfrak{b} + lS_\mathfrak{p} + p_1 + \cdots + p_k > 0$.

Proof. By Lemma 4.6, X_β is flow equivalent to the beta-shift with generating sequence $1^{S_\mathfrak{b}}\mathfrak{p}$. The rest of the statement follows by applying Lemma 4.6 and Proposition 4.7 as in Example 4.8. □

Remark 4.10. In particular, Proposition 4.9 can be used to show that a beta-shift X_β with $g(\beta) = \mathfrak{b}\mathfrak{p}^\infty$ is flow equivalent to some beta-shift $\mathsf{X}_{\beta'}$, where $g(\beta') = 1^n(\mathfrak{p}')^\infty$ with $n \le |\mathfrak{p}'|$.

Remark 4.11. By Proposition 4.3, two flow equivalent sofic beta-shifts must have the same sum of elements in the period, but it is currently unknown whether this number is a complete invariant.

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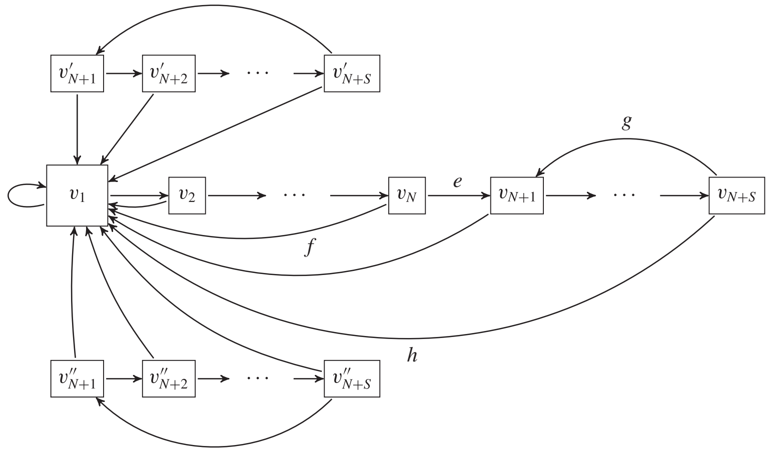

FIGURE 6. An unlabeled graph defining an edge shift flow equivalent to the edge shift of the underlying graph of the fiber product cover of a sofic beta-shift with minimal $n,\ p$, such that $g(\beta) = g_1 \cdots g_n(g_{n+1} \cdots g_{n+p})^\infty$. Here, $N = \sum_{i=1}^{n} g_i$ and $S = \sum_{i=1}^{p} g_{n+i}$. Every vertex emits precisely two edges, one of which terminates at v_1.

Assume that $\mathsf{X}_\beta \sim_{\text{FE}} \mathsf{X}_{\beta'}$ whenever $g(\beta) = \mathfrak{b}\mathfrak{p}^\infty$ and $g(\beta') = \mathfrak{b}(\mathfrak{p}')^\infty$ with $\mathfrak{p}'$ obtained from $\mathfrak{p}$ by inserting a 0 after an arbitrary 1 in $\mathfrak{p}$, and $\mathfrak{p}'$ is minimal in the sense that no shorter period can give the same generating sequence. Then it follows from Proposition 4.9 that two arbitrary beta-shifts are flow equivalent if and only if they have the same sum of entries in the period. In this way, the flow classification of sofic beta-shifts can be reduced to the question of whether it is possible to add zeroes in the period as specified above using conjugacies, symbol expansions and symbol reductions. However, it is still unknown whether it is possible to carry out this final reduction of the problem to show that the sum of entries in the period of the generating sequence is a complete invariant of flow equivalence of sofic beta-shifts.

4.4. *Flow equivalence of fiber products.* Let $1 < \beta < 2$ with sofic X_β, let n, p be minimal such that $g(\beta) = g_1 \cdots g_n(g_{n+1} \cdots g_{n+p})^\infty$, and let $(P_\beta, \mathcal{L}_\beta)$ be the fiber product cover of X_β. The goal of this section is to study the flow class of the reducible edge shift X_{P_β} defined by the underlying graph. Let $N = \sum_{i=1}^{n} g_i$ and $S = \sum_{i=1}^{p} g_{n+i}$. By Remark 4.10, it can be assumed that $N \leq S$, without loss of generality.

Let $P_\beta^0 = \{v_1, \ldots, v_{n+p}, v'_{n+1}, \ldots, v'_{n+p}, v''_{n+1}, \ldots, v'_{n+p}\}$ as in Figure 5. Let $e \in P_\beta^1$ with $\mathcal{L}_\beta(e) = 0$ and $r(e) \neq v_1$. Then there exists $f \in P_\beta^1$ such that for $\lambda \in \mathsf{X}_{P_\beta}$, $\lambda_0 = f$ if and only if $\lambda_1 = e$. Hence, all these edges can be removed using symbol contraction, and this leaves the edge shift of the graph shown in Figure 6. In this graph, the vertices v_N and v_{N+S} each emit one edge to v_{N+1} and one edge to v_1. Use in-amalgamation to merge these two vertices. This identifies the edges e and g and the edges f and h marked in Figure 6. The result is a graph of the same form, where the size of N is reduced by one. Repeat this process N times to show that X_{P_β} is flow equivalent to the graph in Figure 7. This leads to the following proposition.

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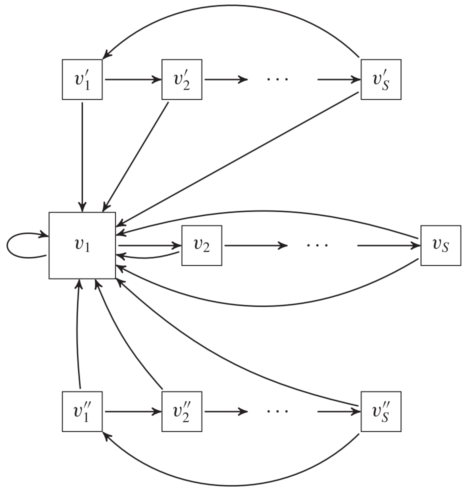

FIGURE 7. An unlabeled graph defining an edge shift flow equivalent to the edge shift of the underlying graph of the fiber product cover of a sofic beta-shift with minimal n, p, such that $g(\beta) = g_1 \cdots g_n (g_{n+1} \cdots g_{n+p})^\infty$. Here, $S = \sum_{i=1}^{p} g_{n+i}$.

PROPOSITION 4.12. *For $i \in \{1, 2\}$, let $\beta_i > 1$ with minimal n_i, p_i such that $g(\beta) = g_1^i \cdots g_{n_i}^i (g_{n_i+1}^i \cdots g_{n_i+p_i}^i)^\infty$, let $S_i = \sum_{i=1}^{p_i} g_{n_i+i}$, and let $(P_{\beta_i}, \mathcal{L}_{P_{\beta_i}})$ be the fiber product cover of X_{β_i}. Then $S_1 = S_2$ if and only if there exists a flow equivalence $\Phi : S\mathsf{X}_{P_{\beta_1}} \to S\mathsf{X}_{P_{\beta_2}}$, which commutes with the fiber product involutions.*

Proof. If there is such a flow equivalence, then the Bowen–Franks groups of the edge shifts $\mathsf{X}_{P_{\beta_1}}$ and $\mathsf{X}_{P_{\beta_2}}$ must be equal [**Fra84**], so $S_1 = S_2$ by Proposition 4.3. Reversely, if $S_1 = S_2$, then the preceding arguments prove that there is a flow equivalence $\Phi : S\mathsf{X}_{P_{\beta_1}} \to S\mathsf{X}_{P_{\beta_2}}$. Furthermore, the fiber product involutions on $\mathsf{X}_{P_{\beta_i}}$ are respected by the conjugacies and symbol reductions used in the construction, so Φ commutes with the actions. □

5. *Perspectives*

The concrete constructions of flow equivalences culminating in Remark 4.11 give a drastic reduction of the flow classification problem for sofic beta-shifts by showing that the general problem can be reduced to a question of whether it is possible to add extra zeroes inside the period of the generating sequence. However, it is still unknown whether this final step can be carried out in general. For example, there is no known way to determine whether the beta-shifts X_{β_1} and X_{β_2} are flow equivalent when β_1, β_2 have generating sequences

$$g(\beta_1) = 1(110)^\infty \quad \text{and} \quad g(\beta_2) = 11(110)^\infty.$$

This is arguably the simplest question about the flow equivalence of sofic beta-shifts that cannot be answered directly by the tools presented above.

Sofic beta-shifts do not belong to the class of irreducible 2-sofic shifts currently covered by the classification results obtained in [**BCEa**, **BCEb**]. However, if it is possible to extend these results to cover sofic beta-shifts then Proposition 4.12 will be sufficient to prove that the sum of elements in the period of the generating sequence is a complete invariant of flow equivalence of sofic beta-shifts.

Acknowledgements. Supported by VILLUM FONDEN through the network for Experimental Mathematics in Number Theory, Operator Algebras, and Topology. Supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

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