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ON A CLASS OF RANDOM WALKS IN SIMPLEXES

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Abstract

We study the limit behaviour of a class of random walk models taking values in the standard d-dimensional ($d \geq 1$) simplex. From an interior point z, the process chooses one of the $d+1$ vertices of the simplex, with probabilities depending on z, and then the particle randomly jumps to a new location z' on the segment connecting z to the chosen vertex. In some special cases, using properties of the Beta distribution, we prove that the limiting distributions of the Markov chain are Dirichlet. We also consider a related history-dependent random walk model in [0, 1] based on an urn-type scheme. We show that this random walk converges in distribution to an arcsine random variable.

Keywords: Random walks in simplexes; iterated random functions; Dirichlet distribution; stick-breaking process

2010 Mathematics Subject Classification: Primary 60J05
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1. Introduction

Throughout this paper the d-dimensional standard orthogonal simplex (see e.g. [5]) is defined as

$$\mathcal{S}_d = \left\{(z_1, z_2, \ldots, z_d) \in \mathbb{R}^d : z_1 + z_2 + \cdots + z_d \leq 1,\ z_j \geq 0,\ j = 1, 2, \ldots, d\right\}.$$

We also denote the interior of $\mathcal{S}_d$, the Borel σ-algebra, and the Lebesgue measure on $\mathcal{S}_d$ by $\mathcal{S}_d^o$, $\mathcal{B}(\mathcal{S}_d)$, and λ_d respectively. Let $E_0 = (0, 0, \ldots, 0)$ be the origin, and let $E_1 = (1, 0, \ldots, 0)$, $E_2 = (0, 1, 0, \ldots, 0), \ldots, E_d = (0, \ldots, 0, 1)$ be the standard orthonormal basis vectors in $\mathbb{R}^d$, which are also the vertices of $\mathcal{S}_d$.

For some initial point $Z_0 \in \mathcal{S}_d$, we consider the following random iteration:

$$Z_{n+1} = (1 - \xi_n)Z_n + \xi_n \Theta_n, \quad n = 0, 1, 2, \ldots,$$

where

- ξ_n, $n = 0, 1, 2, \ldots$, are independent copies of some random variable ξ with support in [0, 1],
- Θ_n, $n = 0, 1, 2, \ldots$, are discrete random vectors such that

$$\mathbb{P}(\Theta_n = E_j \mid Z_0, Z_1, \ldots, Z_n; \xi_n) = p_j(Z_n), \quad j = 0, 1, 2, \ldots, d,$$

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where $p=(p_1, p_2, \ldots, p_d)$ (sometimes referred to as a *probability choice function*) is a given mapping (which is the same for all n) from $\mathcal{S}_d$ to itself such that $p_j\colon \mathcal{S}_d \to [0, 1]$, $j=1, 2, \ldots d$ are Borel-measurable functions, and $p_0(z) := 1-\sum_{j=1}^{d} p_j(z)$ for all $z \in \mathcal{S}_d$.

The aforementioned model originates from Sethuraman's construction of the Dirichlet distribution (see [10]) for the case where $p_1, p_2, \ldots, p_d$ are positive constants. Sethuraman proved that if

- $\xi \sim \text{Beta}(1, \gamma)$, where γ is some positive constant,
- Θ is a discrete random vector such that $\mathbb{P}(\Theta=E_j)=p_j$ for $j=1, 2, \ldots, d$, and $p_0 = 1-p_1-\cdots-p_d>0$,
- $Z \sim \text{Dirichlet}(p_1\gamma, p_2\gamma, \ldots, p_d\gamma, p_0\gamma)$, and
- Z, Θ, ξ are jointly independent,

then

$$Z \stackrel{d}{=} (1-\xi)Z + \xi\Theta.$$

Here Beta(a, b) denotes the usual Beta distribution with the probability density function

$$g(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0<x<1,$$

Γ is the Gamma function, and Dirichlet$(\alpha_1, \alpha_2, \ldots, \alpha_d, \alpha_{d+1})$ denotes the Dirichlet distribution with the probability density function

$$f(z_1, z_2, \ldots, z_d) = \frac{\Gamma\big(\sum_{i=1}^{d+1}\alpha_i\big)}{\prod_{i=1}^{d+1}\Gamma(\alpha_i)}\left(1-\sum_{i=1}^{d} z_i\right)^{\alpha_{d+1}-1}\prod_{i=1}^{d} z_i^{\alpha_i-1}, \quad (z_1, z_2, \ldots, z_d) \in \mathcal{S}_d^o.$$

Consequently, the stationary distribution of the Markov chain $\{Z_n\}_{n\geq 0}$ corresponding to Sethuraman's model is Dirichlet$(p_1\gamma, p_2\gamma, \ldots, p_d\gamma, p_0\gamma)$. Further extensions, where $\xi \sim \text{Beta}(k, \gamma)$ for some positive integer k, and Θ has a quasi-Bernoulli distribution, were studied by Hitczenko and Letac in [4].

In [3], Diaconis and Freedman reconsidered Sethuraman's model from the point of view of random iterated functions, and also studied the case where $p(z)$ depends on $z \in \mathcal{S}_1 = [0, 1]$. Other models in $\mathcal{S}_1$ with various special cases of $p(z)$ and ξ were studied in [7], [8], and [9]. Inspired by the work of Diaconis and Freedman, Ladjimi and Peigné in their recent work [6] studied iterated random functions with place-dependent probability choice functions, and demonstrated several applications to the one-dimensional model where $\xi \sim \text{Uniform}[0, 1]$, and $p(z)$ is a Hölder-continuous function in $[0, 1]$.

In [7], McKinlay and Borovkov gave a general condition for the ergodicity of the one-dimensional Markov chain $\{Z_n\}_{n\geq 0}$ in $\mathcal{S}_1$. By solving integral equations, they derived a closed-form expression for the stationary density function in the case where $\xi \sim \text{Beta}(1, \gamma)$, and $p(z)$ is a piecewise continuous function on $[0, 1]$. In particular, if $p(z) = (1-c)z + b(1-z)$, $b, c \in (0, 1]$, then the stationary distribution is Beta$(b\gamma, c\gamma)$.

The model, also known in the literature as a stick-breaking process, a stochastic give-and-take (see [2], [7]) or a Diaconis–Freedman chain (see [6]) has many applications in other fields such as human genetics, robot coverage algorithms, random search, etc. For further discussions, we refer the reader to [2], [9], and [7].

The rest of the paper is organized as follows. In Section 2 we give an extension of the ergodicity criterion of McKinlay and Borovkov to higher-dimensional simplexes under certain assumptions on $p(z)$ and ξ. In Section 3, in the case where ξ is Beta-distributed while the probability choice function $p(z)$ linearly depend on z, we prove that the limiting distribution of the chain is a Dirichlet distribution. Finally, in Section 4, we consider a history-dependent random walk model in [0, 1] based on urn-type schemes. Using martingales and coupling techniques, we show that the random walk converges in distribution to an arcsine random variable.

2. Existence of the limiting distribution

To prove the ergodicity of the Markov chain $\{Z_n\}_{n\geq 0}$, we will make use of the following result.

Proposition 1. (Theorems 1.3 and 2.1 in [1]) *Let Z_n, $n = 0, 1, 2, \ldots$ be a Markov chain on a measurable state space $(\mathcal{X}, \mathfrak{B})$ such that, for $n \geq 1$, $\mathbb{P}(Z_n \in A \mid Z_0 = z)$ is a measurable function of $z \in \mathcal{X}$ when $A \in \mathfrak{B}$ is fixed, while it is a probability measure of A when z is fixed.*

Then Z_n is ergodic if there exists a subset $V \in \mathfrak{B}$, $q > 0$, a probability measure φ on $(\mathcal{X}, \mathfrak{B})$, and some positive integer n_0 such that

(a) $\mathbb{P}(\tau_V < \infty \mid Z_0 = z) = 1$ *for all* $z \in \mathcal{X}$, *where* $\tau_V = \inf\{n \geq 1 \colon Z_n \in V\}$,

(b) $\sup_{z\in V} \mathbb{E}(\tau_V \mid Z_0 = z) < \infty$,

(c) $\mathbb{P}(Z_{n_0} \in B \mid Z_0 = z) \geq q\varphi(B)$ *for all* $B \in \mathfrak{B}$ *and* $z \in V$,

(d) $\gcd\{n \colon \mathbb{P}(Z_n \in B \mid Z_0 = z) \geq q\varphi(B)\} = 1$ *for* $z \in V$.

Moreover, if the above conditions are fulfilled, then there exists a unique invariant measure μ such that the distribution of Z_n converges to μ in total variation norm.

For each $z = (z_1, z_2 \ldots, z_d) \in \mathcal{S}_d$, we define $z_0 = 1 - z_1 - z_2 - \cdots - z_d$. Note that the set of all $(z_0, z_1, z_2 \ldots, z_d)$, where $(z_1, z_2 \ldots, z_d) \in \mathcal{S}_d$, constitutes the standard d-simplex in $\mathbb{R}^{d+1}$.

Assumption 1. *There exist $\delta \in (0, 1/2^d)$ and $s, t \in (\delta^{1/d}, 1 - \delta^{1/d})$, $s < t$ such that*

(i) $\mathbb{P}(\xi < 1 - \delta) := 1 - \eta < 1$,

(ii) *there is an $\varepsilon > 0$ such that, for any $1 \leq k \leq d$ and any $0 \leq j_1 < j_2 < \cdots < j_k \leq d$, we have*

$$\inf_{\substack{z\in\mathcal{S}_d,\\ z_{j_1}+\cdots+z_{j_k}\leq\delta}} \sum_{l=1}^{k} p_{j_l}(z) \geq \varepsilon,$$

(iii) *there is $c > 0$ such that, for all $B \in \mathcal{B}([0, 1])$, $B \subset [s(1-t)^{d-1} - \delta, t] \cup [(1-t)^d - \delta, 1 - s]$, we have*

$$\mathbb{P}(\xi \in B) > c\lambda(B),$$

where λ is the Lebesgue measure on [0, 1].

Remark 1. Condition (i) is quite natural in order to avoid the absorption of Z_n at the boundary of $\mathcal{S}_d$. For $d = 1$, the above conditions are very similar to the assumptions (E1–E2–E3) of

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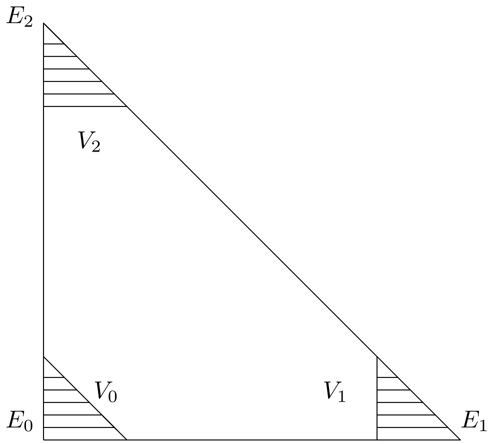

FIGURE 1. Illustration of $V_j, j = 0, 1, 2$ in case $d = 2$.

McKinlay and Borovkov in [7]. However, in contrast to our condition (iii), McKinlay and Borovkov require that ξ has a density on $[s - \delta, t]$ and $[1 - t - \delta, 1 - s]$. Also, observe that in condition (iii) the intervals are properly defined (though they may overlap).

For $j = 0, 1, \dots, d$ define

$$V_j = \{z = (z_1, \dots, z_d) \in \mathcal{S}_d \colon 1 - \delta \le z_j \le 1\}.$$

In particular,

$$V_0 = \left\{ z = (z_1, \dots, z_d) \in \mathcal{S}_d \colon \sum_{j=1}^{d} z_j \le \delta \right\}.$$

For each $x = (x_1, x_2, \dots, x_d) \in (0, 1)^d$, also define $T \colon (0, 1)^d \to \mathcal{S}_d^o$ by setting

$$T(x) = \left(x_1 \prod_{j=2}^{d} (1 - x_j),\ x_2 \prod_{j=3}^{d} (1 - x_j), \dots, x_{d-1}(1 - x_d),\ x_d \right).$$

Note that T is a homeomorphism from $(0, 1)^d$ to $\mathcal{S}_d^o$, and its inverse T^{-1} for each

$$z = (z_1, \dots, z_d) \in \mathcal{S}_d^o$$

is given by

$$T^{-1}(z) = \left(\frac{z_1}{1 - \sum_{2 \le j \le d} z_j}, \frac{z_2}{1 - \sum_{3 \le j \le d} z_j}, \dots, \frac{z_{d-1}}{1 - z_d}, z_d \right).$$

Let $z = (z_1, \dots, z_d)$, $u = (u_1, \dots, u_d) \in \mathcal{S}_d$, $z_0 = 1 - \sum_{k=1}^{d} z_k$, and $u_0 = 1 - \sum_{k=1}^{d} u_k$. For each $j = 1, 2, \dots, d$ we define the following functions:

$$R_j(u) = (u_0, u_1, \dots, u_{j-1}, u_{j+1}, u_{j+2}, \dots, u_d), \qquad j = 0, 1, \dots, d,$$
$$G_z(u) = (u_0 z_1 + u_1, u_0 z_2 + u_2, \dots, u_0 z_d + u_d).$$

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If $z_0 \neq 0$ then the map G_z is invertible; moreover, its inverse can be computed as

$$G_z^{-1}(u) = \left(u_1 - \frac{z_1 u_0}{z_0}, u_2 - \frac{z_2 u_0}{z_0}, \ldots, u_d - \frac{z_d u_0}{z_0}\right).$$

For some two real numbers s and t, such that $0 < s < t < 1$, define

$$K := \left\{(u_1, \ldots, u_d) \in \mathcal{S}_d \colon s \leq \frac{u_j}{1 - \sum_{l=j+1}^{d} u_l} \leq t, j = 1, 2, \ldots, d\right\} = T\big([s, t]^d\big).$$

The proof of the following lemma is given in the Appendix.

Lemma 1. *Assume that* $\delta \in (0, 1/2^d)$, $s, t \in (\delta^{1/d}, 1 - \delta^{1/d})$ *and* $s < t$.

(a) *If* $z \in V_0$, *then*

$$G_z^{-1}(K) \subset T([s(1-t)^{d-1} - \delta, t]^d).$$

(b) *If* $z \in V_k$ *with* $k \in \{1, 2, \ldots, d\}$, *then*

$$G_{R_k(z)}^{-1} \circ R_k(K) \subset T\big([(1-t)^d - \delta, 1 - s] \times [s(1-t)^{d-1} - \delta, t]^{d-1}\big).$$

Theorem 1. *Assume that all the conditions in Assumption 1 are fulfilled. Then the Markov chain* $\{Z_n\}_{n \geq 0}$ *converges in distribution.*

Proof. (Step 1) We define

$$V = \bigcup_{j=0}^{d} V_j.$$

From part (i) of Assumption 1 it follows that $\mathbb{P}(Z_1 \in V \mid Z_0 = z) \geq \mathbb{P}(\xi \geq 1 - \delta) = \eta > 0$ for all $z \in \mathcal{S}_d$. Therefore, for all $z \in \mathcal{S}_d$, given $Z_0 = z$, the random variable $\tau_V = \inf\{n \geq 1 \colon Z_n \in V\}$ is stochastically dominated by a geometric random variable with parameter η, thus yielding

$$\mathbb{P}(\tau_V > n \mid Z_0 = z) \leq (1 - \eta)^n.$$

Hence conditions (a) and (b) from the statement of Proposition 1 are satisfied.

(Step 2) Throughout the rest of the proof we let Const. denote some positive constant. From the definition of $\{Z_n\}_{n \geq 0}$, we observe that

$$Z_d = \zeta_0 Z_0 + \zeta_1 \Theta_0 + \zeta_2 \Theta_1 + \cdots + \zeta_d \Theta_{d-1},$$

where

$$(\zeta_1, \ldots, \zeta_d) := T(\xi_0, \ldots, \xi_{d-1}),$$
$$\zeta_0 := \prod_{j=0}^{d-1} (1 - \xi_j) = 1 - \sum_{j=1}^{d} \zeta_j.$$

For $1 \leq k \leq d$ and $0 \leq j_1 < j_2 < \cdots < j_k \leq d$, define

$$U_{j_1 j_2 \cdots j_k} := \{z = (z_1, z_2, \ldots, z_d) \in \mathcal{S}_d \colon z_{j_1} + z_{j_2} + \cdots + z_{j_k} \leq \delta\}.$$

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Let $B \in \mathcal{B}(\mathcal{S}_d)$. Then, if $Z_0 = z \in V_0$ and $\Theta_0 = E_1, \Theta_1 = E_2, \ldots, \Theta_{d-1} = E_d$, then $Z_j \in U_{j+1,j+2,\ldots,d}$ for $j = 0, 1, \ldots, d$. Therefore, from part (ii) of Assumption 1 it follows that

$$\begin{aligned}
&\mathbb{P}(Z_d \in B \mid Z_0 = z) \\
&\quad \geq \mathbb{P}(Z_d \in B \cap K, (\Theta_0, \Theta_1, \ldots, \Theta_{d-1}) = (E_1, E_2, \ldots, E_d) \mid Z_0 = z) \\
&\quad \geq \left[\prod_{l=1}^{d} \inf_{z \in U_{l,l+1,\ldots,d}} \left(\sum_{j=l}^{d} p_j(z) \right) \right] \times \mathbb{P}(\zeta_0 z + \zeta_1 E_1 + \cdots + \zeta_d E_d \in B \cap K) \\
&\quad \geq \varepsilon^d \mathbb{P}((\zeta_0 z_1 + \zeta_1, \zeta_0 z_2 + \zeta_2, \ldots, \zeta_0 z_d + \zeta_d) \in B \cap K) \\
&\quad = \varepsilon^d \mathbb{P}((\zeta_1, \ldots, \zeta_d) \in G_z^{-1}(B \cap K)) = \varepsilon^d \mathbb{P}((\xi_0, \ldots, \xi_{d-1}) \in T^{-1} \circ G_z^{-1}(B \cap K)). \quad (1)
\end{aligned}$$

(Step 3) For $B \in \mathcal{B}(\mathcal{S}_d)$ and $z \in V_0$, from part (iii) of Assumption 1 and Lemma 1, we have

$$\mathbb{P}((\xi_0, \xi_1, \ldots, \xi_{d-1}) \in T^{-1} \circ G_z^{-1}(B \cap K)) \geq c^d \lambda_d(T^{-1} \circ G_z^{-1}(B \cap K)). \quad (2)$$

We shall demonstrate below that

$$\lambda_d(T^{-1} \circ G_z^{-1}(B \cap K)) \geq \lambda_d(B \cap K). \quad (3)$$

Indeed, for any injective continuously differentiable map $Q \colon K \to [0, 1]^d$ and any measurable subset $A \subset K$,

$$\lambda_d(Q(A)) \geq \inf_{u \in A} \left| \det \left(\frac{\partial}{\partial u} Q(u) \right) \right| \cdot \lambda_d(A). \quad (4)$$

We also observe that

$$\det \left(\frac{\partial}{\partial u} G_z^{-1}(u) \right) = \det \begin{pmatrix}
1 + \frac{z_1}{z_0} & \frac{z_1}{z_0} & \frac{z_1}{z_0} & \cdots & \frac{z_1}{z_0} \\
\frac{z_2}{z_0} & 1 + \frac{z_2}{z_0} & \frac{z_2}{z_0} & \cdots & \frac{z_2}{z_0} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\frac{z_{d-1}}{z_0} & \cdots & \frac{z_{d-1}}{z_0} & 1 + \frac{z_{d-1}}{z_0} & \frac{z_{d-1}}{z_0} \\
\frac{z_d}{z_0} & \cdots & \frac{z_d}{z_0} & \frac{z_d}{z_0} & 1 + \frac{z_d}{z_0}
\end{pmatrix}$$

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$$
\begin{aligned}
&= \det \begin{pmatrix}
1+\dfrac{z_1}{z_0} & -1 & -1 & -1 & \cdots & -1 \\
\dfrac{z_2}{z_0} & 1 & 0 & 0 & \cdots & 0 \\
\dfrac{z_3}{z_0} & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
\dfrac{z_{d-1}}{z_0} & 0 & \cdots & 0 & 1 & 0 \\
\dfrac{z_d}{z_0} & 0 & \cdots & 0 & 0 & 1
\end{pmatrix} \\
&= 1+\frac{z_1}{z_0}+\frac{z_2}{z_0}+\cdots+\frac{z_d}{z_0} \\
&= \frac{1}{z_0} \\
&\geq 1,
\end{aligned}
$$

where the second matrix is obtained from the first one by subtracting the first column from each of the remaining columns, and then we use the *Schur determinant identity*, i.e.

$$
\det \begin{pmatrix} C & D \\ E & F \end{pmatrix} = \det(F)\cdot\det(C-DF^{-1}E)
$$

when F is invertible; here F is the $d-1$ square identity matrix, $C=[1+z_1/z_0]$, etc. Furthermore,

$$
\begin{aligned}
&\det\left(\frac{\partial}{\partial v}T^{-1}(v)\right) \\
&= \det \begin{pmatrix}
\dfrac{1}{1-\sum_{j=2}^{d} v_j} & \dfrac{v_1}{\left(1-\sum_{j=2}^{d} v_j\right)^2} & \dfrac{v_1}{\left(1-\sum_{j=2}^{d} v_j\right)^2} & \cdots & \dfrac{v_1}{\left(1-\sum_{j=2}^{d} v_j\right)^2} \\
0 & \dfrac{1}{1-\sum_{j=3}^{d} v_j} & \dfrac{v_2}{\left(1-\sum_{j=3}^{d} v_j\right)^2} & \cdots & \dfrac{v_2}{\left(1-\sum_{j=3}^{d} v_j\right)^2} \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & \dfrac{1}{1-v_d} & \dfrac{v_{d-1}}{(1-v_d)^2} \\
0 & \cdots & 0 & 0 & 1
\end{pmatrix} \\
&= \left[\prod_{j=1}^{d}\left(1-\sum_{l=j+1}^{d} v_l\right)\right]^{-1} \\
&\geq 1.
\end{aligned}
$$

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Therefore inequality (3) is obtained by applying (4) to the map $Q = T^{-1} \circ G_z^{-1}$.

Combining (1), (2), and (3), we conclude that for each $B \in \mathcal{B}(\mathcal{S}_d)$ and $z \in V_0$

$$\mathbb{P}(Z_d \in B \mid Z_0 = z) \geq \text{Const. } \lambda_d(B \cap K). \tag{5}$$

(Step 4) For each $k \in \{1, 2, \ldots, d\}$, $B \in \mathcal{B}(\mathcal{S}_d)$, and $z \in V_k$, we have

$$\begin{aligned}
&\mathbb{P}(Z_d \in B \mid Z_0 = z) \\
&\quad \geq \mathbb{P}(Z_d \in B \cap (K), (\Theta_0, \ldots, \Theta_{d-1}) = (E_0, E_1, \ldots, E_{k-1}, E_{k+1}, \ldots, E_d) \mid Z_0 = z) \\
&\quad \geq \text{Const. } \mathbb{P}(R_k^{-1} \circ G_{R_k(z)}(\zeta_1, \zeta_2, \ldots, \zeta_d) \in B \cap K) \\
&\quad = \text{Const. } \mathbb{P}((\zeta_1, \ldots, \zeta_d) \in G_{R_k(z)}^{-1} \circ R_k(B \cap K)) \\
&\quad = \text{Const. } \mathbb{P}((\xi_0, \xi_1, \ldots, \xi_{d-1}) \in T^{-1} \circ G_{R_k(z)}^{-1} \circ R_k(B \cap K)),
\end{aligned}$$

where we use the fact that, for $u \in \mathcal{S}_d$ and $z \in V_k$,

$$R_k^{-1}(G_{R_k(z)}(u)) = (u_0 z_1 + u_2, \ldots, u_0 z_{k-1} + u_k, u_0 z_k, u_0 z_{k+1} + u_{k+1}, \ldots, u_0 z_d + u_d).$$

Similarly to the inequalities (2) and (3), we have

$$\mathbb{P}((\xi_0, \xi_1, \ldots, \xi_{d-1}) \in T^{-1} \circ G_{R_k(z)}^{-1} \circ R_k(B \cap K)) \geq \text{Const. } \lambda_d(B \cap K).$$

It follows that for each $B \in \mathcal{B}(\mathcal{S}_d)$, $k = 1, 2, \ldots, d$ and $z \in V_k$,

$$\mathbb{P}(Z_d \in B | Z_0 = z) \geq \text{Const. } \lambda_d(B \cap K). \tag{6}$$

Next, we define the probability measure φ as

$$\varphi(B) = \frac{\lambda_d(B \cap K)}{\lambda_d(K)}$$

for each $B \in \mathcal{B}(\mathcal{S}_d)$. From (5) and (6), we can conclude that condition (c) in Proposition 1 is verified.

(Step 5) For each $B \in \mathcal{B}(\mathcal{S}_d)$ and $z \in V$,

$$\begin{aligned}
\mathbb{P}(Z_{d+1} \in B \mid Z_0 = z) &\geq \mathbb{P}(Z_{d+1} \in B, Z_1 \in V | Z_0 = z) \\
&= \mathbb{P}(Z_{d+1} \in B \mid Z_1 \in V, Z_0 = z) \cdot \mathbb{P}(Z_1 \in V \mid Z_0 = z) \\
&\geq \eta \mathbb{P}(Z_{d+1} \in B \mid Z_1 \in V) \\
&\geq \text{Const. } \lambda_d(B \cap K).
\end{aligned}$$

Since $\gcd\{d, d+1\} = 1$, condition (d) in Proposition 1 is also fulfilled. □

3. Beta walks with linearly place-dependent probabilities

Suppose that the conditions of Assumption 1 are fulfilled. Since convergence in total variation implies convergence in distribution, as $n \to \infty$, Z_n converges in distribution to a random vector Z having the invariant measure. By the definition of the invariant measure

$$Z \stackrel{d}{=} (1 - \xi) Z + \xi \Theta, \tag{7}$$

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where Θ is a discrete random vector satisfying

$$\mathbb{P}(\Theta = E_j \mid Z = z) = p_j(z), \quad j = 0, 1, \ldots, d, \tag{8}$$

and ξ is independent of Z and Θ.

Lemma 2. *Assume that*

- *$p = (p_1, p_2, \ldots, p_d)$ is a Borel-measurable probability choice function and Θ satisfies (8),*
- *ξ is independent of Z and Θ,*
- *Z and ξ have the probability density functions f and g respectively (with respect to Lebesgue measures λ_d and λ_1).*

Then (7) holds if and only if f and g satisfy the following equation:

$$f(z) = \sum_{j=0}^{d} T_j(z) \quad \lambda_d\text{-a.e. on } \mathcal{S}_d, \tag{9}$$

where

$$T_0(z_1, z_2, \ldots, z_d) = \int_{z_1+z_2+\cdots+z_d}^{1} \frac{1}{u^d} f\left(\frac{z_1}{u}, \frac{z_2}{u}, \ldots, \frac{z_d}{u}\right) p_0\left(\frac{z_1}{u}, \frac{z_2}{u}, \ldots, \frac{z_d}{u}\right) g(1-u)\,\mathrm{d}u,$$

$$T_j(z_1, z_2, \ldots, z_d) = \int_{1-z_j}^{1} \frac{1}{u^d} f\left(\frac{z_1}{u}, \ldots, \frac{z_{j-1}}{u}, \frac{z_j - 1 + u}{u}, \frac{z_{j+1}}{u}, \ldots, \frac{z_d}{u}\right)$$
$$\times\, p_j\left(\frac{z_1}{u}, \ldots, \frac{z_{j-1}}{u}, \frac{z_j - 1 + u}{u}, \frac{z_{j+1}}{u}, \ldots, \frac{z_d}{u}\right) g(1-u)\,\mathrm{d}u$$

for $j = 1, 2, \ldots, d$. (The integrals above are understood in the Lebesgue sense.)

Proof. Denote $\tilde{Z} = (1-\xi)Z + \xi\Theta$. For each $z \in \mathcal{S}_d$, we have

$$\mathbb{P}(\tilde{Z} \le y) = \int_0^1 \sum_{j=0}^{d} \mathbb{P}(uZ + (1-u)\Theta \le y,\ \Theta = E_j) g(1-u)\,\mathrm{d}u$$
$$= \sum_{j=0}^{d} \int_{\mathcal{S}_d \times [0,1]} \mathbf{1}_{\{z \le u^{-1}(y-(1-u)E_j)\}} f(z) p_j(z) g(1-u)\,\mathrm{d}z\,\mathrm{d}u, \tag{10}$$

where for $y = (y_1, y_2, \ldots, y_d)$, $z = (z_1, z_2, \ldots, z_d) \in \mathcal{S}_d$, we write $z \le y$ if $z_j \le y_j$ for all $j = 1, 2, \ldots, d$.

For each $j \in \{0, 1, \ldots, d\}$, $u \in (0, 1)$, and $y \in \mathcal{S}_d^o$, changing the variable $x = \varphi(z) := uz + (1-u)E_j$ we have

$$\int_{\mathcal{S}_d} \mathbf{1}_{\{z \le u^{-1}(y-(1-u)E_j)\}} f(z) p_j(z)\,\mathrm{d}z$$
$$= \int_{\varphi(\mathcal{S}_d)} \mathbf{1}_{\{x \le y\}} f(\varphi^{-1}(x)) p_j(\varphi^{-1}(x)) \mathsf{D}\varphi^{-1}(x)\,\mathrm{d}x$$
$$= \int_{\{x \in \mathcal{S}_d\colon 0 \le x \le y,\, 1-u \le x_j \le 1\}} \frac{1}{u^d} f\left(\frac{1}{u}(x - (1-u)E_j)\right) p_j\left(\frac{1}{u}(x - (1-u)E_j)\right) \mathrm{d}x, \tag{11}$$

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where $\mathsf{D}\varphi^{-1}$ denotes the Jacobian of φ^{-1}. Combining (10) and (11), and applying Fubini's theorem, we obtain

$$\mathbb{P}(\tilde{Z} \le y)$$
$$= \int_{\{x\in\mathcal{S}_d\ :\ 0\le x\le y\}} \left[\sum_{j=0}^{d} \int_{1-x_j}^{1} \frac{1}{u^d} f\left(\frac{1}{u}(x-(1-u)E_j)\right) p_j\left(\frac{1}{u}(x-(1-u)E_j)\right) g(1-u)\,\mathrm{d}u \right] \mathrm{d}x.$$

Therefore

$$\tilde{f}(z) := \sum_{j=0}^{d} \int_{1-z_j}^{1} \frac{1}{u^d} f\left(\frac{1}{u}(z-(1-u)E_j)\right) p\left(\frac{1}{u}(z-(1-u)E_j)\right) g(1-u)\,\mathrm{d}u = \sum_{j=0}^{d} T_j(z)$$

is a probability density function of $\tilde{Z}$, which is unique up to a set of measure zero. Hence, $f(z) = \tilde{f}(z)$ for almost all $z \in \mathcal{S}_d$, and the lemma is thus proved. □

Theorem 2. *Assume that*

(a) $\xi \sim \mathrm{Beta}(1, \gamma)$, *where* $\gamma > 0$ *is some constant,*

(b) $p = (p_1, p_2, \ldots, p_d)\colon \mathcal{S}_d \to \mathcal{S}_d$ *is defined by*

$$p_k(z_1, z_2, \ldots, z_d) = \beta_k(1-z_k) + \left(1 - \sum_{j=1}^{d+1} \beta_j + \beta_k\right) z_k, \quad k = 1, 2, \ldots, d,$$

where $\beta_k > 0$ *and* $\sum_{j=1}^{d+1} \beta_j - \beta_k < 1$ *for* $k = 1, 2, \ldots, d+1$,

(c) $Z \sim \mathrm{Dirichlet}(\beta_1\gamma, \beta_2\gamma, \ldots, \beta_d\gamma, \beta_{d+1}\gamma)$.

Then $Z \overset{d}{=} (1-\xi)Z + \xi\Theta$, *and thus* Z_n *converges to a Dirichlet distribution by Theorem 1 and Lemma 2.*

Proof. Let f and g, respectively, be the probability density functions of

$$\mathrm{Dirichlet}(\beta_1\gamma, \beta_2\gamma, \ldots, \beta_d\gamma, \beta_{d+1}\gamma) \quad \text{and} \quad \mathrm{Beta}(1, \gamma).$$

It suffices to check that f and g satisfy the integral equation (9).

We have

$$T_0(z_1, \ldots, z_d) = \frac{\Gamma\big(\gamma \sum_{j=1}^{d+1} \beta_j\big)}{\prod_{j=1}^{d+1} \Gamma(\beta_j\gamma)} \prod_{j=1}^{d} z_j^{\beta_j\gamma-1} \int_{\sum_{j=1}^{d} z_j}^{1} \left(u - \sum_{j=1}^{d} z_j\right)^{\gamma\beta_{d+1}-1} u^{-\gamma\left(\sum_{j=1}^{d+1}\beta_j - 1\right)-1}$$
$$\times \left[\gamma\beta_{d+1}u - \gamma\left(\sum_{j=1}^{d+1}\beta_j - 1\right)\left(u - \sum_{j=1}^{d} z_j\right)\right] \mathrm{d}u$$
$$= \frac{\Gamma\big(\gamma \sum_{j=1}^{d+1} \beta_j\big)}{\prod_{j=1}^{d+1} \Gamma(\beta_j\gamma)} \left(1 - \sum_{j=1}^{d} z_j\right)^{\beta_{d+1}\gamma} \prod_{j=1}^{d} z_j^{\beta_j\gamma-1},$$

where we use the fact that

$$\int_z^1 u^{-b-1}(u-z)^{a-1}[au - b(u-z)]\,\mathrm{d}u = (1-z)^a.$$

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Similarly, for $k = 1, 2, \ldots, d$, we also obtain that

$$T_k(z_1, \ldots, z_d) = \frac{\Gamma\left(\gamma \sum_{j=1}^{d+1} \beta_j\right)}{\prod_{j=1}^{d+1} \Gamma(\beta_j \gamma)} \left(1 - \sum_{j=1}^{d} z_j\right)^{\beta_{d+1}\gamma - 1} z_k \prod_{j=1}^{d} z_j^{\beta_j \gamma - 1}.$$

Therefore

$$\sum_{k=0}^{d} T_k(z_1, \ldots, z_d) = \frac{\Gamma\left(\gamma \sum_{j=1}^{d+1} \beta_j\right)}{\prod_{j=1}^{d+1} \Gamma(\beta_j \gamma)} \left(1 - \sum_{j=1}^{d} z_j\right)^{\beta_{d+1}\gamma - 1} \prod_{j=1}^{d} z_j^{\beta_j \gamma - 1} = f(z_1, z_2, \ldots, z_d).$$

□

We want to conclude this section with the following observations.

- If $d = 1$, then $p_1(z) = \beta_1(1 - z) + (1 - \beta_2)z$. This is, in fact, the one-dimensional case considered by McKinlay and Borovkov in [7].
- If $d \geq 1$ and $\sum_{j=1}^{d+1} \beta_j = 1$, then we obtain the model considered by Sethuraman in [10].

4. Random walks in [0, 1] based on urn-type schemes

Let $h\colon [0, 1] \to \mathbb{R}_+$ be some non-random measurable function. In this section we will study a random walk on the unit interval $\mathcal{S}_1 = [0, 1]$ with the following properties.

(1) At time $n = 0, 1, 2, \ldots,$ the system is characterized by $Z_n \in [0, 1]$ (location of the particle) and two positive numbers L_n and R_n. We assume that $R_0 = L_0 = 1$.

(2) At time $n + 1$, with probability $L_n/(L_n + R_n)$ the quantity L_n increases by $h(Z_n)$, i.e. a function of the distance from 0 to the current position of the particle, and then the particle jumps to a new location Z_{n+1}, which is uniformly distributed on the interval $(0, Z_n)$. With the complementary probability $R_n/(L_n + R_n)$, the quantity R_n increases by $h(1 - Z_n)$, i.e. a function of the distance from 1 to the current position of the particle, and then the particle jumps to a new location Z_{n+1}, uniformly distributed on the interval $(Z_n, 1)$.

One can think of L_n and R_n as numbers of two different kinds of balls in an urn, and the direction of the walk is governed by the kind of ball that is drawn randomly from the urn at time n. The number of balls of the chosen type then increases by yet another random quantity, depending on the position of the walk. The number of balls in our model can, in general, be non-integer; however, this is allowed for the generalized Pólya urn models.

Formally, we can write the model as the following recursion: let $Z_0 \in (0, 1)$ be some non-random quantity, and for $n = 1, 2, \ldots$ let

$$\begin{aligned} Z_n &= Z_{n-1}\xi_n \cdot \mathbf{1}_{\{U_n < L_{n-1}/(L_{n-1}+R_{n-1})\}} + [Z_{n-1} + (1 - Z_{n-1})\,\xi_n] \cdot \mathbf{1}_{\{U_n \geq L_{n-1}/(L_{n-1}+R_{n-1})\}}, \\ L_n &= L_{n-1} + h(Z_{n-1}) \cdot \mathbf{1}_{\{U_n < L_{n-1}/(L_{n-1}+R_{n-1})\}}, \\ R_n &= R_{n-1} + h(1 - Z_{n-1}) \cdot \mathbf{1}_{\{U_n \geq L_{n-1}/(L_{n-1}+R_{n-1})\}}, \end{aligned} \tag{12}$$

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where $\{\xi_n, U_n\}_{n=1}^{\infty}$ is a set of i.i.d. uniform $U[0, 1]$ random variables. Since the probabilities of jumps to the left (and right, respectively) depend on (L_n, R_n), the distribution of Z_{n+1} is generally dependent on the whole history of the random walk up to time n. Also, let

$$\mathcal{F}_n = \sigma(U_1, \ldots, U_n, \xi_1, \ldots, \xi_n),$$
$$\mathcal{G}_n = \sigma(U_1, \ldots, U_n, \xi_1, \ldots, \xi_{n-1}),$$

and note that Z_n is $\mathcal{F}_n$-measurable, while L_n and R_n are $\mathcal{G}_n$-measurable.

If $h(x) \equiv 0$, then L_n and R_n do not change with time, and Z_n is a Markov chain satisfying Theorem 2 with $\gamma = 1$ and

$$p \equiv p_1 \equiv \frac{R_0}{L_0 + R_0},$$

so Z_n converges in distribution to

$$\mathrm{Beta}\left(\frac{R_0}{L_0 + R_0}, \frac{L_0}{L_0 + R_0}\right).$$

If $h(x) \equiv \beta$ for some constant $\beta > 0$, then the process (L_n, R_n) is the classical Pólya urn. We conjecture that under some regularity conditions on the function h, the random walk Z_n converges either almost surely to a Bernoulli random variable, or weakly to some non-trivial distribution with full support on $[0, 1]$ (compare with Section 2.1 in [3]). Even though we were not able to deal with the general case, there is one non-trivial situation where we have explicit results, as follows.

In the remaining part of this section we consider only the case where

$$h(x) = x, \quad x \in [0, 1].$$

It turns out that even in this seemingly 'simple' case, there are challenges to rigorously obtaining the limiting distribution (see Theorem 3 below).

Lemma 3. *We have*

(a) $\limsup_{n\to\infty} L_n/R_n \le 4$ *and* $\limsup_{n\to\infty} R_n/L_n \le 4$ *almost surely,*

(b) $L_n \to \infty$ *and* $R_n \to \infty$ *almost surely as* $n \to \infty$.

Proof. First of all, observe that the probability that the sequence Z_n eventually becomes monotone is zero, namely

$$\mathbb{P}(\exists N\colon Z_{n+1} \le Z_n \text{ for all } n \ge N) = 0 \quad \text{and} \quad \mathbb{P}(\exists N\colon Z_{n+1} \ge Z_n \text{ for all } n \ge N) = 0.$$

Since R_n is non-decreasing in n and $L_n \le L_0 + n$, we have

$$\mathbb{P}(Z_{n+1} \in (Z_n, 1) \mid \mathcal{F}_n) = \frac{R_n}{L_n + R_n} \ge \frac{R_0}{L_n + R_0} \ge \frac{R_0}{R_0 + L_0 + n}.$$

Since

$$\sum_{n=1}^{\infty} \frac{R_0}{R_0 + L_0 + n} = \infty,$$

by Lévy's extension to the Borel–Cantelli lemma the event in the above display happens infinitely often with probability 1, hence there are infinitely many ns for which Z_n decreases. By the identical argument, Z_n cannot eventually become increasing.

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Let us prove part (a) now. We know that Z_n makes a.s. infinitely many steps to the left as well as to the right. Hence there exists a sequence of finite stopping times with respect to the filtration $\mathcal{G}$:

$$\begin{aligned}\tau_1 &= 0,\\ \eta_i &= \inf\{n > \tau_i \colon Z_n \geq Z_{n-1}\},\\ \tau_{i+1} &= \inf\{n > \eta_i \colon Z_n < Z_{n-1}\},\end{aligned}$$

for $i = 1, 2, \ldots$. Moreover,

$$\tau_1 < \eta_1 < \tau_2 < \eta_2 < \cdots,$$

$\tau_n, \eta_n \to \infty$ as $n \to \infty$, and

$$\begin{aligned} Z_n &< Z_{n-1} \quad \text{if } n \in [\tau_i, \eta_i - 1],\\ Z_n &\geq Z_{n-1} \quad \text{if } n \in [\eta_i, \tau_{i+1} - 1],\end{aligned}$$

for each $i = 2, 3, \ldots$ (note that, in fact, the probability of the event $Z_n = Z_{n-1}$ is zero).

Observe that, for each $k \geq 1$ and $1 \leq \ell \leq k-1$,

$$\begin{aligned}&\{\eta_i = k,\ \tau_i = k-\ell\}\\ &\quad = \{Z_k \geq Z_{k-1},\ Z_{k-1} < Z_{k-2} < \cdots < Z_{k-\ell}, \tau_i = k-\ell\}\\ &\quad = \left\{U_k \geq \frac{L_{k-1}}{L_{k-1}+R_{k-1}},\ U_{k-1} < \frac{L_{k-2}}{L_{k-2}+R_{k-2}}, \ldots, U_{k-\ell+1} < \frac{L_{k-\ell}}{L_{k-\ell}+R_{k-\ell}},\ \tau_i = k-\ell\right\}.\end{aligned}$$

However, L_{k-j} and R_{k-j} depend only on L_{k-j-1}, R_{k-j-1}, Z_{k-j-1}, and U_{k-j}, $j = 1, 2, \ldots, \ell$ (see (12)). Hence the event above is $\sigma(U_1, \ldots, U_k, \xi_1, \ldots, \xi_{k-2})$-measurable, and it is thus independent of ξ_{k-1}; as a result $V_{i+1} := \xi_{\eta_i - 1}$ is independent of

$$\mathcal{H}_i := \sigma(U_1, U_2, \ldots, U_{\eta_i}, \xi_1, \xi_2, \ldots, \xi_{\eta_i - 2}).$$

On the other hand, since $\eta_{i-1} - 1 \leq \eta_i - 2$, V_i is $\mathcal{H}_i$-measurable. So $\{V_i\}_{i=1}^{\infty}$ is an i.i.d. sequence of Uniform[0, 1] random variables.

We have

$$R_{\eta_i} - R_{\eta_i - 1} = 1 - Z_{\eta_i - 1} = 1 - Z_{\eta_i - 2}\xi_{\eta_i - 1} \geq 1 - \xi_{\eta_i - 1} = 1 - V_{i+1},$$

hence, due to the monotonicity of R_n,

$$R_{\eta_i} \geq \sum_{k=1}^{i} [1 - V_{k+1}].$$

By the strong law of large numbers

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} [1 - V_{i+1}] = \frac{1}{2} \quad \text{a.s.},$$

hence

$$\liminf_{k\to\infty} \frac{R_{\eta_k}}{\eta_k} \geq \frac{1}{2} \quad \text{a.s.}$$

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Next, for $i \in \mathbb{N}$, we have

$$\begin{aligned} L_{\tau_{i+1}-1} - L_{\eta_i-1} &= 0, \\ L_{\eta_i-1} - L_{\tau_i-1} &= Z_{\tau_i-1}(1+\xi_{\tau_i}+\xi_{\tau_i}\xi_{\tau_i+1}+\cdots+\xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_{\eta_i-2}) \\ &\le 1+\xi_{\tau_i}+\xi_{\tau_i}\xi_{\tau_i+1}+\cdots+\xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_{\eta_i-2} \le \tilde{V}_i, \end{aligned}$$

where

$$\begin{aligned} \tilde{V}_i &= [1+\xi_{\tau_i}+\xi_{\tau_i}\xi_{\tau_i+1}+\cdots+\xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_{\eta_i-2}] \\ &\quad + \xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_{\eta_i-2}\tilde{\xi}^{(i)}_{\eta_i-1} + \xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_{\eta_i-2}\tilde{\xi}^{(i)}_{\eta_i-1}\tilde{\xi}^{(i)}_{\eta_i} \\ &\quad + \xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_{\eta_i-2}\tilde{\xi}^{(i)}_{\eta_i-1}\tilde{\xi}^{(i)}_{\eta_i}\tilde{\xi}^{(i)}_{\eta_i+1} + \cdots \\ &= 1 + \sum_{k=\tau_i}^{\eta_i-2} \xi_{\tau_i}\xi_{\tau_i+1}\cdots\xi_k + \sum_{k=\eta_i-1}^{\infty} \xi_{\tau_i}\cdots\xi_{\eta_i-2}\cdot\tilde{\xi}^{(i)}_{\eta_i-1}\tilde{\xi}^{(i)}_{\eta_i}\cdots\tilde{\xi}^{(i)}_k \end{aligned}$$

and $\tilde{\xi}^{(i)}_k$, $i, k \in \mathbb{N}$ are i.i.d. copies of ξ, independent of everything else. By construction, $\tilde{V}_i$ is a $\tilde{\mathcal{H}}_i$-measurable random variable, where

$$\tilde{\mathcal{H}}_i := \sigma(U_1, U_2, \ldots, U_{\eta_i}; \xi_1, \xi_2, \ldots, \xi_{\eta_i-2}; \tilde{\xi}^{(\ell)}_k, \ \ell = 1, 2, \ldots, i, \ k \in \mathbb{N}).$$

On the other hand, one can easily show that the variables $\xi_{\tau_{i+1}+j}$, $\tilde{\xi}^{(i+1)}_{j+1}$, $j = 0, 1, 2, \ldots,$ are independent of $\tilde{\mathcal{H}}_i$, and therefore $\tilde{V}_{i+1}$ is independent of $\tilde{\mathcal{H}}_i$. Consequently, $\tilde{V}_i$, $i = 1, 2, \ldots,$ are independent random variables with expectation

$$\mathbb{E}\left[1 + \sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\xi_i\right)\right] = 2,$$

and hence by the strong law we have

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\tilde{V}_i = 2 \quad \text{a.s.},$$

yielding

$$\limsup_{k\to\infty}\frac{L_{\eta_k}}{\eta_k} \le 2 \quad \text{a.s.} \tag{13}$$

Combining this with (13), and taking into account that $\eta_i \to \infty$, we get

$$\limsup_{n\to\infty}\frac{L_n}{R_n} \le \frac{2}{1/2} = 4 \quad \text{a.s.}$$

Due to the symmetry, the complementary inequality can be proved identically.

Let us now prove part (b). From part (a) we obtain that a.s. either both L_n and R_n increase to ∞, or both stay bounded, i.e. $\sup_{n\ge 0} L_n < \infty$ and $\sup_{n\ge 0} R_n < \infty$ for all n. Let us show that

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the latter case a.s. cannot happen. Again, from (a) we get that a.s. there exists a (random) N such that, for $n \geq N$,

$$\mathbb{P}(Z_{n+1} < Z_n | \mathcal{F}_n) = \frac{L_n}{L_n + R_n} = \frac{1}{1 + R_n/L_n} > \frac{1}{5}, \quad \mathbb{P}(Z_{n+1} > Z_n | \mathcal{F}_n) = \frac{R_n}{L_n + R_n} > \frac{1}{5}.$$

As a result, for $n \geq N$, we have for $S_n = L_n + R_n$

$$\begin{aligned}
&\mathbb{P}\left(S_{n+1} - S_n > \frac{1}{4} \mid \mathcal{F}_n\right) \\
&\quad = \mathbb{P}\left(S_{n+1} - S_n > \frac{1}{4} \mid \mathcal{F}_n, Z_n \leq \frac{1}{2}\right) + \mathbb{P}\left(S_{n+1} - S_n > \frac{1}{4} \mid \mathcal{F}_n, Z_n > \frac{1}{2}\right) \\
&\quad \geq \mathbb{P}\left(R_{n+1} - R_n > \frac{1}{4} \mid \mathcal{F}_n, Z_n \leq \frac{1}{2}\right) + \mathbb{P}\left(L_{n+1} - L_n > \frac{1}{4} \mid \mathcal{F}_n, Z_n > \frac{1}{2}\right) \\
&\quad = \mathbb{P}\left(R_{n+1} - R_n > \frac{1}{4},\ Z_{n+1} > Z_n \mid \mathcal{F}_n, Z_n \leq \frac{1}{2}\right) \\
&\qquad + \mathbb{P}\left(L_{n+1} - L_n > \frac{1}{4},\ Z_{n+1} < Z_n \mid \mathcal{F}_n, Z_n > \frac{1}{2}\right) \\
&\quad \geq \frac{1}{2} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{5} \\
&\quad = \frac{1}{5} > 0.
\end{aligned}$$

Since S_n is non-decreasing for any n, this implies that $S_n \to \infty$ a.s., contradicting the assumption that both L_n and R_n remain bounded. □

Lemma 4.

$$\zeta_n := \frac{L_n}{L_n + R_n}$$

converges almost surely to $\zeta_\infty \in (0, 1)$ *as* $n \to \infty$.

Remark 2. Lemma 3 implies only that

$$\frac{1}{5} \leq \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n \leq \frac{4}{5} \quad \text{a.s.}$$

Proof of Lemma 4. We introduce the quantity

$$W_n = \left(\frac{1}{2} - \frac{L_n + Z_n}{L_n + R_n}\right)^2 + \frac{1}{L_n + R_n}$$

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which will be shown to be a supermartingale. Indeed,

$$\begin{aligned}
&\mathbb{E}(W_{n+1}|\mathcal{F}_n)\\
&= \frac{L_n}{L_n+R_n}\mathbb{E}\left(W_{n+1} \mid U_{n+1} < \frac{L_n}{L_n+R_n}, \mathcal{F}_n\right)\\
&\quad + \frac{R_n}{L_n+R_n}\mathbb{E}\left(W_{n+1} \mid U_{n+1} > \frac{L_n}{L_n+R_n}, \mathcal{F}_n\right)\\
&= \frac{L_n}{L_n+R_n}\int_0^{Z_n} \frac{1}{Z_n}\left[\left(\frac{1}{2}-\frac{L_n+Z_n+u}{L_n+R_n+Z_n}\right)^2+\frac{1}{L_n+R_n+Z_n}\right]\mathrm{d}u\\
&\quad + \frac{R_n}{L_n+R_n}\int_{Z_n}^{1} \frac{1}{1-Z_n}\left[\left(\frac{1}{2}-\frac{L_n+u}{L_n+R_n+1-Z_n}\right)^2+\frac{1}{L_n+R_n+1-Z_n}\right]\mathrm{d}u\\
&= \frac{L_n}{L_n+R_n}\,\frac{3(L_n-R_n)^2+12(L_n-R_n)Z_n+12(L_n+R_n+Z_n)+13Z_n^2}{12(L_n+R_n+Z_n)^2}\\
&\quad + \frac{R_n}{L_n+R_n}\,\frac{3(L_n-R_n)^2+12(L_n-R_n)Z_n+12(L_n+R_n+Z_n)+13(1-Z_n)^2}{12(L_n+R_n+1-Z_n)^2}.
\end{aligned}$$

Substituting

$$\zeta_n = \frac{L_n}{L_n+R_n} \quad \text{and} \quad \varepsilon_n = \frac{1}{L_n+R_n},$$

or equivalently

$$L_n = \frac{\zeta_n}{\varepsilon_n},\ R_n = \frac{1-\zeta_n}{\varepsilon_n},$$

we obtain that

$$\mathbb{E}(W_{n+1}-W_n|\mathcal{F}_n, Z_n=z) = \frac{\varepsilon_n\left[r_0(\zeta_n,z)+r_1(\zeta_n,z)\varepsilon_n+\cdots+r_5(\zeta_n,z)\varepsilon_n^5\right]}{6(\varepsilon_n z+1)^2(1+\varepsilon_n(1-z))^2}, \tag{14}$$

where

$$\begin{aligned}
r_0(\zeta,z) &= -24z\zeta^3+36z\zeta^2+12\,\zeta^3-18z\zeta-24\zeta^2+3z+15\zeta-3\\
&= -3(2\zeta-1)^2(\zeta z+(1-z)(1-\zeta)),\\
r_1(\zeta,z) &= -30\,z^2\zeta^2-12\,z\zeta^3+30\,z^2\zeta+24\,z\zeta^2+6\,\zeta^3\\
&\qquad -7\,z^2-38\,z\zeta-12\,\zeta^2+14\,z+13\,\zeta-7,\\
r_2(\zeta,z) &= 10\,z^3\zeta-12\,z^2\zeta^2-5\,z^3-9\,z^2\zeta+7\,z^2-19\,z\zeta+4\,z+6\,\zeta-6,\\
r_3(\zeta,z) &= -z(6\,z^3\zeta^2-6\,z^3\zeta-12\,z^2\zeta^2-17\,z^3-12\,z^2\zeta\\
&\qquad +6\,z\zeta^2+28\,z^2+30\,z\zeta-23\,z-6\,\zeta+12),\\
r_4(\zeta,z) &= -6\,z^2(1-z)(z^2+2z\zeta(1-z)+1),\\
r_5(\zeta,z) &= -6z^4(1-z)^2.
\end{aligned}$$

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One can show that $\max_{x,y\in[0,1]} r_i(x, y) \leq 0$ for $i = 0, 2, 3, 4, 5$. From the assertion of Lemma 3, $\varepsilon_n \to 0$ almost surely as $n \to \infty$, and one can show that $\max_{x,y\in[0,1]} r_0(x, y) + r_1(x, y)\varepsilon \leq 0$ for $0 \leq \varepsilon < 0.5$. Hence W_n is a supermartingale. Therefore, by Doob's martingale convergence theorem, a.s. there exists $W_\infty := \lim_{n\to\infty} W_n$. Observe that

$$\zeta_n \in \left\{\frac{1}{2} - \sqrt{W_n - \frac{1}{L_n + R_n} - \frac{Z_n}{L_n + R_n}}, \frac{1}{2} + \sqrt{W_n - \frac{1}{L_n + R_n} - \frac{Z_n}{L_n + R_n}}\right\}.$$

On the other hand, note that

$$|\zeta_{n+1} - \zeta_n| \leq \max\left\{\frac{L_n}{L_n + R_n} \cdot \frac{1 - Z_n}{L_{n+1} + R_{n+1}}, \frac{R_n}{L_n + R_n} \cdot \frac{Z_n}{L_{n+1} + R_{n+1}}\right\} \to 0$$

as $n \to \infty$. As a result, $\limsup_{n\to\infty} \zeta_n = \liminf_{n\to\infty} \zeta_n =: \zeta_\infty$ almost surely. □

Lemma 5. $\zeta_\infty = \frac{1}{2}$ *almost surely.*

Proof. Suppose $\mathbb{P}(\zeta_\infty = 1/2) < 1$. Then there exists $\varepsilon > 0$ such that $\mathbb{P}(|\zeta_\infty - 1/2| > \varepsilon) > 0$. Let us denote the stopping time

$$\tau_m = \inf\left\{n \geq m \colon \left|\zeta_n - \frac{1}{2}\right| < \frac{\varepsilon}{2}\right\}.$$

Since $\mathbb{P}(|\zeta_\infty - 1/2| > \varepsilon) > 0$, there exists m such that $\mathbb{P}(\tau_m = \infty) > 0$. Let us consider $Y_n = W_{n\wedge\tau_m}$. Y_n is also a supermartingale, hence there exists $Y_\infty = \lim_{n\to\infty} Y_n$ as well. From (14) it follows that

$$\mathbb{E}(W_{n+1} - W_n \mid \mathcal{F}_n) = \frac{1}{6}\varepsilon_n(r_0(\zeta_n, Z_n) + \varepsilon_n \rho_n),$$

where

$$\rho_n = \frac{1}{(1 + \varepsilon_n Z_n)^2(1 + \varepsilon_n(1 - Z_n))^2}(r_1(\zeta_n, Z_n) + r_2(\zeta_n, Z_n)\varepsilon_n + \cdots + r_5(\zeta_n, Z_n)\varepsilon_n^4$$
$$- (1 + (1 - Z_n)Z_n\varepsilon_n)(2 + \varepsilon_n + (1 - Z_n)Z_n\varepsilon_n^2)r_0(\zeta_n, Z_n)).$$

Furthermore, $|\rho_n|$ is bounded by a non-random constant. This fact implies that, for $N > 0$,

$$\mathbb{E}(W_{m+N}) - \mathbb{E}(W_m) = \frac{1}{6}\mathbb{E}\left(\sum_{n=m}^{m+N} \varepsilon_n(r_0(\zeta_n, Z_n) + O(\varepsilon_n))\right).$$

Therefore

$$\mathbb{E}(Y_\infty) - \mathbb{E}(Y_m) = \frac{1}{6}\mathbb{E}\left(\sum_{n=m}^{\tau_m} \varepsilon_n(r_0(\zeta_n, Z_n) + O(\varepsilon_n))\right). \tag{15}$$

Note that

$$\left|\zeta_n - \frac{1}{2}\right| \geq \frac{\varepsilon}{2} \quad \text{for all } n \in [m, \tau_m).$$

Hence, combining with Remark 2, it follows that on the event $\{\tau_m = \infty\}$

$$r_0(\zeta_n, Z_n) = -3(2\zeta_n - 1)^2(Z_n\zeta_n + (1 - Z_n)(1 - \zeta_n)) \leq -3\varepsilon^2 \min\{\zeta_n, 1 - \zeta_n\} \leq -\frac{3\varepsilon^2}{13}$$

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for large enough n. Since $\mathbb{P}(\tau_m = \infty) > 0$ and $\varepsilon_n \geq 1/(2+n)$, the left-hand side of (15) is finite while the right-hand side is divergent. This contradiction proves the lemma. $\square$

Theorem 3. *As $n \to \infty$, Z_n converges in distribution to a* $\mathrm{Beta}(\frac{1}{2}, \frac{1}{2})$ *random variable.*

Proof. Let us fix a small $\varepsilon > 0$. By Lemma 5 there exists a (random) N such that

$$\frac{1}{2} - \varepsilon \leq \frac{L_n}{L_n + R_n} \leq \frac{1}{2} + \varepsilon$$

for all $n \geq N$. Fix a large non-random N_0. For this fixed N_0 we couple $\{Z_n\}_{n\geq 0}$ with two random walks $\{\tilde{Z}_n\}_{n\geq 0}$ and $\{\hat{Z}_n\}_{n\geq 0}$ defined as follows.

- For $0 \leq n \leq N_0$, set $\tilde{Z}_n = \hat{Z}_n = Z_n$.
- For $n \geq N_0$, set

$$\tilde{Z}_{n+1} = \begin{cases} \xi_{n+1}\tilde{Z}_n & \text{if } U_{n+1} \leq \frac{1}{2} - \varepsilon, \\ \tilde{Z}_n + \xi_{n+1}(1 - \tilde{Z}_n) & \text{if } U_{n+1} > \frac{1}{2} - \varepsilon, \end{cases}$$

  and

$$\hat{Z}_{n+1} = \begin{cases} \xi_{n+1}\hat{Z}_n & \text{if } U_{n+1} \leq \frac{1}{2} + \varepsilon, \\ \hat{Z}_n + \xi_{n+1}(1 - \hat{Z}_n) & \text{if } U_{n+1} > \frac{1}{2} + \varepsilon. \end{cases}$$

Let $A_{N_0} = \{N \leq N_0\}$.

Assume that for some $n \geq N_0$, $\hat{Z}_n \leq Z_n \leq \tilde{Z}_n$ (this is definitely true for $n = N_0$). We observe that on A_{N_0}:

- When $\tilde{Z}_n$ chooses left, Z_n also chooses left since

$$U_{n+1} \leq \frac{1}{2} - \varepsilon < \frac{L_n}{L_n + R_n}.$$

  In this case, $Z_{n+1} = \xi_{n+1} Z_n \leq \xi_{n+1}\tilde{Z}_n = \tilde{Z}_{n+1}$. When $\tilde{Z}_n$ chooses right, Z_n might choose left or right, but we still have

$$Z_{n+1} \leq Z_n + \xi_{n+1}(1 - Z_n) \leq \tilde{Z}_n + \xi_{n+1}(1 - \tilde{Z}_n) = \tilde{Z}_{n+1}.$$

- When Z_n chooses left, $\hat{Z}_n$ also chooses left since

$$U_{n+1} \leq \frac{L_n}{L_n + R_n} < \frac{1}{2} + \varepsilon.$$

  In this case, $Z_{n+1} = \xi_{n+1} Z_n \geq \xi_{n+1}\hat{Z}_n = \hat{Z}_{n+1}$. When Z_n chooses right, $\hat{Z}_n$ might choose left or right, but we still have

$$\hat{Z}_{n+1} \leq \hat{Z}_n + \xi_{n+1}(1 - \hat{Z}_n) \leq Z_n + \xi_{n+1}(1 - Z_n) = Z_{n+1}.$$

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By induction, we obtain that on A_{N_0} for all $n \geq 0$, $\hat{Z}_n \leq Z_n \leq \tilde{Z}_n$. Therefore we have

$$\mathbb{P}(\tilde{Z}_n \leq x,\ A_{N_0}) \leq \mathbb{P}(Z_n \leq x,\ A_{N_0}) \leq \mathbb{P}(\hat{Z}_n \leq x,\ A_{N_0})$$

for all $n \geq 0$ and $x \in [0, 1]$. On the other hand, by Theorem 2, $\tilde{Z}_n$ and $\hat{Z}_n$ converge weakly to $\mathrm{Beta}(\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon)$ and $\mathrm{Beta}(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ respectively, as $n \to \infty$. Since ε is arbitrarily small and $\mathbb{P}(A_{N_0}) \to 1$ as $N_0 \to \infty$, the theorem is proved. □

Appendix A

Proof of Lemma 1.

(a) For $u = (u_1, \ldots, u_d) \in K$ and $z = (z_1, \ldots, z_d) \in V_0$, define

$$v = (v_1, v_2, \ldots, v_d) := G_z^{-1}(u).$$

Note that $u_0 \leq 1 - u_d \leq 1 - \delta \leq z_0$, $z_j \leq 1 - z_0 \leq \delta$, and thus

$$s(1-t)^{d-1} - \delta \leq u_j - z_j \leq v_j = u_j - \frac{u_0}{z_0} z_j \leq u_j \leq t$$

for $j = 1, 2, \ldots, d$. Therefore, for $j = 1, 2, \ldots, d$, we have

$$\begin{aligned} s(1-t)^{d-1} - \delta \leq v_j \leq \frac{v_j}{1 - \sum_{l=j+1}^d v_l} &= \frac{u_j - z_j u_0/z_0}{1 - \sum_{l=j+1}^d u_l + \sum_{l=j+1}^d z_l u_0/z_0} \\ &\leq \frac{u_j}{1 - \sum_{l=j+1}^d u_l} \leq t. \end{aligned}$$

This implies that $v = G_z^{-1}(u) \in K_0$ for each $u \in K$ and $z \in V_0$, where we denote

$$K_0 = \left\{ (v_1, \ldots, v_d) \in \mathcal{S}_d : s(1-t)^{d-1} - \delta \leq \frac{v_j}{1 - \sum_{l=j+1}^d v_l} \leq t, j = 1, 2, \ldots, d \right\}.$$

Observe that $T^{-1}(K_0) = [s(1-t)^{d-1} - \delta, t]^d$. Thus $T^{-1} \circ G_z^{-1}(K) \subset [s(1-t)^{d-1} - \delta, t]^d$.

(b) For $u \in K$, $z \in V_k$, let

$$\begin{aligned} v &= (v_1, v_2, \ldots\ldots, v_k) \\ &:= G^{-1}_{R_k(z)}(R_k(u)) \\ &= \left(u_0 - z_0 \frac{u_k}{z_k}, u_1 - z_1 \frac{u_k}{z_k}, \ldots, u_{k-1} - z_{k-1} \frac{u_k}{z_k}, u_{k+1} - z_{k+1} \frac{u_k}{z_k}, \ldots, u_d - z_d \frac{u_k}{z_k} \right). \end{aligned}$$

Note that $z_l \leq 1 - z_k \leq \delta$ for $l \in \{0, 2, \ldots, d\} \setminus \{k\}$ and $u_k \leq \max\{u_d, 1 - u_d\} \leq 1 - \delta \leq z_k$. Therefore we observe the following.

(i) For $k + 1 \leq j \leq d$,

$$\frac{v_j}{1 - \sum_{l=j+1}^d v_l} = \frac{u_j - z_j u_k/z_k}{1 - \sum_{l=j+1}^d u_l + \sum_{l=j+1}^d z_l u_k/z_k} \leq \frac{u_j}{1 - \sum_{l=j+1}^d u_l} \leq t$$

and

$$\frac{v_j}{1 - \sum_{l=j+1}^d v_l} \geq v_j = u_j - z_j \frac{u_k}{z_k} \geq u_j - z_j \geq s(1-t)^{d-1} - \delta.$$

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(ii) For $j=1$, we have

$$\begin{aligned}\frac{v_1}{1-\sum_{l=2}^d v_l} &= \frac{u_0 - z_0 u_k/z_k}{u_0+\sum_{l=1}^d z_l u_k/z_k}\\ &= 1-\frac{u_k}{\big(1-\sum_{l=1}^d u_l\big)z_k + u_k\sum_{l=1}^d z_l}\\ &\le 1-\frac{u_k}{\big(1-\sum_{l=k}^d u_l\big)z_k+u_k}\\ &\le 1-\frac{u_k}{1-\sum_{l=k+1}^d u_l}\\ &\le 1-s\end{aligned}$$

and

$$\frac{v_1}{1-\sum_{l=2}^d v_l} \ge v_1 = u_0 - z_0\frac{u_k}{z_k} \ge (1-t)^d-\delta.$$

(iii) For $2\le j\le k$,

$$s(1-t)^{d-1}-\delta \le v_j \le \frac{v_j}{1-\sum_{l=j+1}^d v_l} = \frac{u_{j-1}-z_{j-1}\,u_k/z_k}{1-\sum_{l=j}^d u_l + \sum_{l=j}^d z_l\,u_k/z_k} \le \frac{u_{j-1}}{1-\sum_{l=j}^d u_l} \le t.$$

(c) Therefore

$$v\in T\big([(1-t)^d-\delta,\, 1-s]\times[s(1-t)^{d-1}-\delta,\, t]^{d-1}\big). \qquad \square$$

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