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# DRIFT PARAMETER ESTIMATION FOR A REFLECTED FRACTIONAL BROWNIAN MOTION BASED ON ITS LOCAL TIME

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## Abstract

We consider a drift parameter estimation problem when the state process is a reflected fractional Brownian motion (RFBM) with a nonzero drift parameter and the observation is the associated local time process. The RFBM process arises as the key approximating process for queueing systems with long-range dependent and self-similar input processes, where the drift parameter carries the physical meaning of the surplus service rate and plays a central role in the heavy-traffic approximation theory for queueing systems. We study a statistical estimator based on the cumulative local time process and establish its strong consistency and asymptotic normality.

*Keywords:* Parameter estimation; fractional Brownian motion; reflected process; strong consistency; queueing model; asymptotic normality

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## 1. Introduction

The main goal of this paper is to discuss the problem of statistical parameter estimation for a one-dimensional reflected fractional Brownian motion (RFBM) with nonzero drift parameter. (See Definition 1 below for a precise definition of the RFBM process.) There have been a number of papers on the estimation of the Hurst parameter. We shall only deal with the estimation of the drift parameter. Our interest in this model stems from the fact that the RFBM model arises as the key approximating process for queueing systems with long-range dependent and self-similar input processes. We refer the reader to [13], [14], and [4] for the convergence results justifying the use of the FBM model for aggregate traffic at the connection-level time scale. The RFBM behaves like a standard FBM process in the interior of its domain $(0, \infty)$. However, when it reaches its boundary at 0, the sample path returns to the interior in a manner exercising with minimal 'pushing' force. (See (1) below for the precise description.) In the RFBM model, the drift parameter carries the physical meaning of the (suitably scaled) surplus service rate, and it also plays a central role in the heavy-traffic approximation theory for queueing systems (cf. [14]).

In practice, some important aspects of performance of a queueing system (e.g. customer waiting times, traffic intensities) may not be directly observable and, therefore, such performance measures and their related model parameters need to be statistically inferred

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from the available observed data. Previous studies of parameter estimation have mainly been concerned with the classical queueing models, such as M/M/1, M/M/$c$, GI/G/1, etc. (see, e.g. [11], [1], [3], and the references therein). In this paper we consider a drift parameter estimation problem for a queueing model with long-range dependent and self-similar input processes. We propose an estimator for the drift parameter based on the observed cumulative local time process, and establish its strong consistency and asymptotic normality. On drift parameter estimation in the fractional Brownian motion models without reflection, we refer the reader to the recent articles [9] and [8]. In the next section we introduce our one-dimensional RFBM model and the related background. The results of a strong consistency and an asymptotic normality for the proposed estimator and their proofs are presented in Section 3.

## 2. The queueing model

We begin by recalling the following definition. A real-valued process $W_H = (W_H(t) \colon t \geq 0)$ is said to be a fractional Brownian motion (FBM) with self-similarity index (or Hurst parameter) $H \in (0, 1)$ if $W_H(0) = 0$, $W_H$ has continuous sample paths, and $W_H$ is a centered Gaussian process whose covariance is given by

$$\mathbb{E}[W_H(t)W_H(s)] = \tfrac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \qquad t, s > 0.$$

In particular, when $H \in (\frac{1}{2}, 1)$, the autocorrelation sequence for $(W_H(k) - W_H(k-1) \colon k \geq 1)$ is nonsummable and FBM presents genuine long-time correlations that persist even under scaling (cf. [12]). Two fundamental properties of FBM justify its general suitability as a model in many practical problems: FBM is a self-similar process and has long-range-dependent increments, which are positively correlated if $H \in (\frac{1}{2}, 1)$. This agrees with the empirical characteristics of complex heavy-traffic networks (cf. [14]). The case $H = \frac{1}{2}$ corresponds to the standard Brownian motion.

Next, we define a one-dimensional RFBM with drift.

**Definition 1.** (*RFBM.*) For $x \in [0, \infty)$, $\sigma \in (0, \infty)$, and $b \in (-\infty, \infty)$, a RFBM with drift $b$ is a continuous one-dimensional process $Z = \{Z(t) \colon t \geq 0\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that

(i) $Z(t) = x + bt + \sigma W_H(t) + L(t) \in [0, \infty)$ for all $t \geq 0$, $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.),

(ii) $W_H$ is a one-dimensional FBM with the Hurst parameter $H$,

(iii) $L = \{L(t) \colon t \geq 0\}$ is a one-dimensional process such that $L(0) = 0$, $\mathbb{P}$-a.s., is continuous, nondecreasing, and can increase only when $Z(\cdot)$ is on the zero boundary, i.e. $\int_0^t \mathbf{1}_{\{Z(s) \neq 0\}} \, \mathrm{d}L(s) = 0$ for all $t \geq 0$.

We recall that, in the one-dimensional case, an explicit formula for the process $L$ is well known in the literature (see, e.g. Proposition 2.3 of [7, p. 19] or Section 13.5 of [14]). That is, $L = \{L(t) \colon t \geq 0\}$ given by

$$L(t) = \max\Big\{0, \max_{s \in [0,t]} (-x - bs - \sigma W_H(s))\Big\} \quad \text{for } t \geq 0$$

is the *unique* process satisfying the defining property (iii) in Definition 1. In what follows, we provide a simple queueing model interpretation of the above-defined RFBM process.

We consider a single-server network having deterministic service rate $\mu > 0$. We assume that the initial queue is empty (i.e. $x = 0$ in Definition 1) and the cumulative amount of work input

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to the system over the time period $[0, t]$ (after being suitably rescaled) is $A(t) = \lambda t + \sigma W_H(t)$ for $t \geq 0$ and $\sigma, \lambda > 0$. (One can view the cumulative input process $A$ as a fluid inflow to the queue.) Then the workload $Z(t)$ present in the system at time $t \geq 0$ is given by

$$Z(t) = A(t) - \mu t + L(t) = (\lambda - \mu)t + \sigma W_H(t) + L(t), \tag{1}$$

where the process $L = (L(t) \colon t \geq 0)$ is given by

$$L(t) = -\min_{s\in[0,t]}[A(s) - \mu s] = \max_{s\in[0,t]}[(\mu - \lambda)s - \sigma W_H(s)]. \tag{2}$$

Recall that the process $L$ has continuous paths, increases only when $Z$ hits 0, and, hence, is often referred to as a local time process of RFBM. See [7] and [14] for additional details on this representation of the workload. Henceforth, we focus on the process $Z$ explicitly given by (1)–(2) and refer to it as the RFBM process.

The above model (1)–(2) can be viewed as the diffusion approximation under heavy-traffic conditions of a sequence of rescaled queueing systems (see, e.g. the book by Whitt [14] and the paper [4]). In this setup, the state $Z(t)$ is the workload (or the size of the queue) at time $t$, and the observation $L(t)$ is the total time the queue has spent at 0 up to time $t$, the so-called idle time.

In [10], the authors considered a stochastic filtering problem of a reflected (standard) Brownian motion with respect to its local time and derived the explicit form of the conditional law of the state process at time $t$, given the observed local time process up to time $t$. Also, we note that the main results of Zeevi and Glynn [15] (Theorems 1 and 2 therein) provide weakly consistent estimators for the drift parameter of RFBM. Those estimators are based on the maximum process of the workload and, as a result, the convergence is quite slow in general (see also related results in [2]). In fact, the results of [15] imply consistent estimation of loss probabilities in a finite buffer queue setting, but they can be easily translated into the estimation of the drift parameter for the RFBM model. To the best of our knowledge, there are no strong consistency and asymptotic normality results regarding estimators for drift parameters of RFBM processes.

## 3. Consistency and asymptotic normality

We first establish the strong consistency. Recall model (1)–(2), and define the parameter $u = \mu - \lambda$. We assume throughout that $u > 0$, which ensures stability properties of the process (cf. [15] and [14]).

**Theorem 1.** *Let $0 < H < 1$. For $T > 0$, let $\hat{u}_T = T^{-1}L(T)$. Then*

$$\hat{u}_T \to u \quad \text{as } T \to \infty \text{ a.s.} \tag{3}$$

*Proof.* We begin with the definition of the standard one-dimensional reflection mapping (also known as the Skorokhod map) $\Gamma \colon C([0, \infty), \mathbb{R}) \to C([0, \infty), \mathbb{R})$, which is defined as

$$\Gamma(f)(t) = f(t) + \max\Big\{0, \max_{s\in[0,t]}(-f(s))\Big\}$$

for $f \in C([0, \infty), \mathbb{R})$ and $t \geq 0$. Here, $C([0, \infty), \mathbb{R})$ denotes the space of continuous real-valued functions defined on $[0, \infty)$. Then we have

$$Z(t) = \Gamma(\sigma W_H - ue)(t), \quad \text{where} \quad e(t) \equiv t \quad \text{for all } t \geq 0.$$

Since $ut$ is nonnegative, we have $\sigma W_H(t) - ut \le \sigma W_H(t)$. Therefore, from the basic monotonicity and Lipschitz properties of the Skorokhod map (see, for instance, [14]), we have

$$0 \le Z(t) \le \Gamma(\sigma W_H)(t) \le 2\Big(\max_{s\in[0,t]} \sigma |W_H(s)|\Big). \tag{4}$$

Since

$$L(T) = Z(T) - \sigma W_H(T) + uT$$

and

$$\lim_{T\to\infty} \frac{1}{T} W_H(T) = 0 \quad \text{a.s.}, \tag{5}$$

it is easy to see that the desired result (3) would follow if we prove that $T^{-1}Z(T) \to 0$ as $T \to \infty$ a.s. To show such a claim, we note that, from (4),

$$0 \le \frac{Z(T)}{T} \le \frac{2}{T}\Big(\max_{s\in[0,T]} \sigma |W_H(s)|\Big),$$

and then it suffices to prove the following basic property of the FBM:

$$\frac{1}{T} \max_{s\in[0,T]} |W_H(s)| \to 0 \quad \text{as } T \to \infty \text{ a.s.} \tag{6}$$

For $t > 0$, let $M(t) = \max_{s\in[0,t]} |W_H(s)|$ and let $\varepsilon > 0$ be arbitrary. Recall the law of large numbers in (5). Take $\omega \in \Omega$ such that $\lim_{t\to\infty} t^{-1} W_H(t,\omega) = 0$. Then, there exists $T_0(\omega) > 0$ such that $|W_H(t,\omega)| < \varepsilon t$ for all $t \ge T_0(\omega)$. Consequently,

$$\max_{t\in[T_0(\omega),T]} |W_H(t,\omega)| < \varepsilon T \quad \text{for all } T > T_0(\omega). \tag{7}$$

On the other hand, $M(T_0(\omega)) < +\infty$ and, therefore, there exists $T_1(\omega)$ (say, $T_1(\omega) > \varepsilon^{-1} M(T_0(\omega)))$ such that

$$M(T_0(\omega)) < \varepsilon T \quad \text{for all } T > T_1(\omega). \tag{8}$$

Choose $T_2(\omega) = \max\{T_0(\omega), T_1(\omega)\}$. Note that

$$M(T) \le M(T_0(\omega)) + \max_{t\in[T_0(\omega),T]} |W_H(t,\omega)|$$

for $T > T_0(\omega)$. Therefore, for each $T > T_2(\omega)$, we obtain $0 < M(T) < 2\varepsilon T$ by using (7) and (8). Consequently, $0 < T^{-1}M(T) < 2\varepsilon$ and, since $\varepsilon > 0$ is arbitrary, the claim in (6) follows.

**Remark 1.** We note that the above strong consistency result continues to hold for any reflected processes driven by continuous noise processes with a mild property (5) (or (6)).

Next, we present the asymptotic normality result for the estimator $\hat{u}_T \equiv L(T)/T$.

**Theorem 2.** *Let* $\frac{1}{2} \le H < 1$. *Then*

$$\frac{T^{1-H}(\hat{u}_T - u)}{\sigma} \Rightarrow N(0,1) \quad \text{as } T \to \infty. \tag{9}$$

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*Proof.* We first consider the case $H = \frac{1}{2}$. Note that, for all $s \in \mathbb{R}$, we have

$$\mathbb{P}\biggl(\frac{\sqrt{T}}{\sigma}\biggl(\frac{L(T)}{T} - u\biggr) \le s\biggr) = \mathbb{P}\biggl(L(T) \le \sigma\sqrt{T}\biggl(s + \frac{u\sqrt{T}}{\sigma}\biggr)\biggr). \tag{10}$$

The distribution of $L(T)$ follows from the one-sided first passage time distribution for standard Brownian motion with drift (cf. [7, p. 14]). More precisely, we have, for $y > 0$,

$$\mathbb{P}(L(T) \le y) = \Phi\biggl(\frac{y - uT}{\sigma\sqrt{T}}\biggr) - \mathrm{e}^{2uy/\sigma^2}\Phi\biggl(\frac{-y - uT}{\sigma\sqrt{T}}\biggr), \tag{11}$$

where $\Phi(\cdot)$ is the $N(0, 1)$ distribution function. Then the probability in (10) can be calculated explicitly as

$$\mathbb{P}\biggl(\frac{\sqrt{T}}{\sigma}\biggl(\frac{L(T)}{T} - u\biggr) \le s\biggr) = \Phi(s) - \mathrm{e}^{2u(\sigma\sqrt{T}s + uT)/\sigma^2}\Phi\biggl(-s - \frac{2u}{\sigma}\sqrt{T}\biggr). \tag{12}$$

To obtain the desired result in (9), it suffices to show that the second term on the right-hand side of (12) converges to 0 as $T \to \infty$. We will use the following well-known result on $\Phi(\cdot)$ to achieve this goal: $1 - \Phi(x) \sim x^{-1}\phi(x)$ as $x \to \infty$, where $\phi(\cdot)$ denotes the density function of the $N(0, 1)$ random variable and '$\sim$' stands for asymptotic equivalence, i.e. $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$ (cf. [6, p. 175]). Therefore, we obtain

$$\begin{aligned}
&\mathrm{e}^{2u(\sigma\sqrt{T}s + uT)/\sigma^2}\Phi\biggl(-s - \frac{2u}{\sigma}\sqrt{T}\biggr)\\
&\quad = \mathrm{e}^{2u(\sigma\sqrt{T}s + uT)/\sigma^2}\biggl(1 - \Phi\biggl(s + \frac{2u}{\sigma}\sqrt{T}\biggr)\biggr)\\
&\quad \sim \mathrm{e}^{2u(\sigma\sqrt{T}s + uT)/\sigma^2}\biggl(s + \frac{2u}{\sigma}\sqrt{T}\biggr)^{-1}\exp\biggl\{-\frac{1}{2}\biggl(s^2 + \frac{4us}{\sigma}\sqrt{T} + \frac{4u^2}{\sigma^2}T\biggr)\biggr\}\frac{1}{\sqrt{2\pi}};
\end{aligned}$$

note that the last term converges to 0 as $T \to \infty$ for any fixed $s \in \mathbb{R}$. This completes the proof for the case $H = \frac{1}{2}$.

Next, we consider the case $\frac{1}{2} < H < 1$. The asymptotic distribution of $L(T)$ can be deduced from the result for the tail distribution of the transient FBM queue length at time $T$. More precisely, from Theorem 2 of [5] (see also the discussion in Section 4 therein), the dominant component of the tail distribution of $\mathbb{P}(L(T) \le y)$ for $y > 0$ is given by $\exp\{-(2\sigma^2)^{-1}(y - uT)^2/T^{2H}\}$ for $\frac{1}{2} < H < 1$. This also coincides with result (11) with $H$ formally substituted by $\frac{1}{2}$ in the expression. Therefore, we obtain, for $s \in \mathbb{R}$,

$$\mathbb{P}\biggl(\frac{T^{1-H}}{\sigma}\biggl(\frac{L(T)}{T} - u\biggr) \le s\biggr) = \mathbb{P}\biggl(L(T) \le \frac{s\sigma T}{T^{1-H}} + uT\biggr) = \Phi\biggl(\frac{s\sigma T^{1-H}}{\sigma T^{1-H}}\biggr)(1 + o(1))$$

as $T \to \infty$. This completes the proof.

**Remark 2.** It would be of interest to extend the above result to the case of more general Gaussian driven queues. One needs sharp asymptotic results for the tail distribution of the running supremum of the Gaussian process with nonzero drift.

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