Perverse bundles and Calogero–Moser spaces

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Abstract

We present a simple description of moduli spaces of torsion-free $\mathcal{D}$-modules ($\mathcal{D}$-*bundles*) on general smooth complex curves, generalizing the identification of the space of ideals in the Weyl algebra with Calogero–Moser quiver varieties. Namely, we show that the moduli of $\mathcal{D}$-bundles form twisted cotangent bundles to moduli of torsion sheaves on X, answering a question of Ginzburg. The corresponding (untwisted) cotangent bundles are identified with moduli of *perverse vector bundles* on T^*X, which contain as open subsets the moduli of framed torsion-free sheaves (the Hilbert schemes $T^*X^{[n]}$ in the rank-one case). The proof is based on the description of the derived category of $\mathcal{D}$-modules on X by a noncommutative version of the Beilinson transform on $\mathbf{P}^1$.

1. Introduction

The starting point for this paper is the following. Let

$$M = \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^* = T^*(\mathfrak{gl}_n \times \mathbb{C}^n).$$

Define a map $\mu : M \to \mathfrak{gl}_n$ by $\mu(X, Y, i, j) = [X, Y] + ij$. It is well known (see, for example, [Nak99]) that the geometric invariant theory quotient (at a nontrivial character χ, the determinant character of GL_n)

$$(\mathbb{C}^2)^{[n]} = \mu^{-1}(0) /\!\!/_\chi \mathrm{GL}_n$$

is the Hilbert scheme of n points on $\mathbb{C}^2$. Indeed, stability forces j to vanish and $i \in \mathbb{C}^n$ to generate $\mathbb{C}^n$ under multiplication by X and Y, so $\mu^{-1}(0)$ parametrizes commuting $n \times n$ matrices X, Y with a cyclic vector for $\mathbb{C}[X, Y]$; hence, $\mu^{-1}(0) /\!\!/ \mathrm{GL}_n$ parametrizes length n quotients of $\mathbb{C}[x, y]$.

Now consider the spaces

$$\mathsf{CM}_n = \mu^{-1}(I) / \mathrm{GL}_n,$$

known as the *n-particle rational Calogero–Moser (CM) spaces*. The CM spaces arise in the study of integrable systems and soliton equations [Wil98]; they also play a central role in the representation theory of Cherednik algebras [EG02]. We write $A_1 = \mathcal{D}_{\mathbf{A}^1} = \mathbb{C}[x, \partial/\partial x]$ for the *first Weyl algebra*. There is the following appealing generalization by Berest and Wilson [Wil98, BW00, BW02] (following the earlier work of Cannings–Holland [CH94] and Le Bruyn [Leb95]) of the above description of the Hilbert scheme to the classification of ideals in A_1.

THEOREM 1.1 (Berest and Wilson [BW00, BW02]). *The space* CM_n *parametrizes (isomorphism classes of) right ideals in* A_1 *that 'have second Chern class n'.*

It is tempting to think that the meaning of the theorem is transparent, since the CM relation $\mu(X, Y, i, j) = I$, that is, $[X, Y] + ij = I$, is very close to the defining relation $[x, \partial/\partial x] = 1$ for A_1.

However, the equation $[X, Y] = I$ for $n \times n$ matrices has no solutions, so the naive generalization of the description of the Hilbert scheme fails.

The relations between Hilbert schemes, CM spaces and the Weyl algebra have been further explored in many works. These include [BGK02, BGK03, BN06, KKO01] and, most notably for the present paper, [BC07], where CM_n was described via A_∞-modules over A_1. In [KKO01], the moduli spaces of (*framed*) modules of any rank over A_1 were described and related to the classical (Atiyah–Drinfeld–Hitchin–Manin (ADHM) quiver) description of moduli of torsion-free sheaves on $\mathbb{C}^2$ (framed at infinity), the Berest–Wilson theorem being the case of rank one; in [BGK02, BGK03], the corresponding moduli spaces for a more general class of 'noncommutative planes' were identified with quiver varieties.

In light of the importance of the CM spaces, one naturally wants to generalize the description of ideals in $A_1 = \mathcal{D}_{\mathbf{A}^1}$ to that of torsion-free modules over differential operators on a higher genus curve (the special case of elliptic curves was treated in [BN06] by methods that do not generalize to higher genus). As we indicated already, the space M with which we started is the cotangent bundle of $Q_n = \mathfrak{gl}_n \times \mathbb{C}^n$. Furthermore, Q_n is identified with a type of Quot scheme: it is the moduli space of quotients $\mathcal{O}_{\mathbf{A}^1}^n \to Q$, where Q is a torsion sheaf on the affine line of length n equipped with a section $i \in H^0(Q)$. Finally, μ is identified with a moment map for the GL_n-action on $M = T^* Q_n$. With this in mind, Ginzburg proposed the following definition.

DEFINITION 1.2 (Ginzburg). Let X be a smooth curve and let $Q_n(X)$ be the variety parametrizing pairs $(\mathcal{O}_X^n \to Q, i \in H^0(Q))$, where Q is a length n torsion sheaf on X. The *nth CM space of X*, denoted by $\mathsf{CM}_n(X)$, is the Hamiltonian reduction of $T^* Q_n(X)$ at $I \in \mathfrak{gl}_n = \mathfrak{gl}_n^*$.

The following theorem answers a question of Ginzburg asking for the higher-genus generalization of Theorem 1.1.

THEOREM 1.3. *The space $\mathsf{CM}_n(X)$ is the 'Hilbert scheme of the quantized cotangent bundle of X', that is, the moduli space for trivially framed $\mathcal{D}$-line bundles (Definition 2.11 and Proposition 2.12) on X with second Chern class n.*

Our method, which we outline below, gives as a byproduct a quite transparent description of the procedures that identify rank-one torsion-free A_1-modules with quadruples (X, Y, i, j) satisfying the CM relation as an application of Koszul duality.

In fact, we prove a significantly more general result. Let V denote a fixed vector bundle on X. Let $\mathsf{PS}(X, V)$ denote the *perverse symmetric power* of the curve X; this is the moduli stack of framed torsion sheaves on X, that is, pairs (Q, i), where Q is a torsion coherent sheaf on X and $i : V \to Q$ is a homomorphism. The substack where the length of Q is n is denoted by $\mathsf{PS}_n(X, V)$. The terminology is motivated by the embedding of the symmetric powers $\mathsf{Sym}^n X \hookrightarrow \mathsf{PS}(X, \mathcal{O})$ as the locus where i is surjective (that is, where the complex $\mathcal{O} \to Q$ has no first cohomology).

A *perverse vector bundle* on the compactification $S = \mathbb{P}(T_X \oplus \mathcal{O}_X)$ of $T^* X$ is a coherent complex $\mathcal{F}$ on S with $\mathrm{H}^0(\mathcal{F})$ torsion-free, $\mathrm{H}^1(\mathcal{F})$ zero-dimensional and all other cohomology sheaves vanishing (Definition 2.11); it is *V-framed* if its restriction to the divisor $\mathbb{P}(T_X \oplus \mathcal{O}_X) \setminus T^* X \cong X$ is equipped with a quasi-isomorphism to V. A *$\mathcal{D}$-bundle on X* is a torsion-free $\mathcal{D}$-module on X, which one may consider as a torsion-free sheaf on the quantized cotangent bundle to X. A V-framing on a $\mathcal{D}$-bundle is the noncommutative analogue of fixing the behavior of a torsion-free sheaf on $T^* X$ near the curve at infinity in S. Technically it amounts to a filtration of a prescribed form: a precise definition is given in Definition 2.11 (which, by Proposition 2.12, agrees with our previous definition in [BN06]).

THEOREM 1.4. *We have the following.*

(1) *The cotangent bundle of the perverse symmetric power $\mathsf{PS}(X, V)$ is the moduli stack $\mathsf{PB}(X, V)$ of V-framed perverse vector bundles on $T^* X$.*

(2) *The CM space* $\mathsf{CM}(X,V)$, *defined as the twisted cotangent bundle (see [BB93]) of* $\mathsf{PS}(X,V)$ *associated to the dual determinant line bundle, is isomorphic to the moduli stack of V-framed $\mathcal{D}$-bundles on X.*

Moreover, the components of CM *and* PB *over* $\mathsf{PS}_n(X,V)$ *parametrize $\mathcal{D}$-bundles (respectively, perverse bundles) with second Chern class $c_2 = n$.*

The reader may wish to view Theorem 1.4 in analogy with a well-known description of the moduli of flat vector bundles on X. Let $\mathrm{Bun}_n\, X$ denote the moduli stack of rank n vector bundles on X. The cotangent bundle $T^*\mathrm{Bun}_n\, X$ is identified with the moduli stack of Higgs bundles on X, which are coherent sheaves on T^*X which are finite of degree n over X. On the other hand, the moduli space of rank-n bundles with flat connection is the twisted cotangent bundle of $\mathrm{Bun}_n\, X$ corresponding to the dual of the determinant line bundle, that is, the space of connections on this bundle (see, for example, [BB03, Fal93]). This twisted cotangent bundle is not 'supported' everywhere: for a bundle to admit a connection, all of its indecomposable summands must have degree zero, so the image in $\mathrm{Bun}_n(X)$ of this twisted cotangent bundle is a proper subset of $\mathrm{Bun}_n(X)$. However, it carries a canonical action of the cotangent bundle and is a torsor for $T^*\mathrm{Bun}_n\, X$ over its image in $\mathrm{Bun}_n(X)$, which we refer to as its *support*, making it a *pseudo-torsor* over its support.

Let us spell out more explicitly the meaning of part (2) of Theorem 1.4. Let $\widetilde{\mathsf{PS}}(X,V)$ denote the (Quot-type) scheme parametrizing (Q,i) as above together with an identification $\Gamma(Q) = \mathbb{C}^n$ (this scheme is actually a smooth variety). The CM space $\mathsf{CM}(X,V)$ is given by hamiltonian reduction by GL_n of the cotangent bundle of $\widetilde{\mathsf{PS}}(X,V)$, with moment map given by the dual to the determinant character $\det \in \mathfrak{gl}_n^*$. This is a pseudo-torsor for the cotangent bundle of $\mathsf{PS}(X,V)$, that is, a torsor over the locus of its support. One can show that the support of $\mathsf{CM}(X,V) \to \mathsf{PS}(X,V)$ is the substack of indecomposable framed torsion sheaves.

The moduli stack $\mathsf{PB}_n(X, \mathcal{O}^k)$ contains the moduli of framed rank-k torsion-free sheaves on T^*X, as the open subset of perverse bundles with vanishing first cohomology. In particular, in the rank-one case we find that the Hilbert scheme of n points on T^*X is an open subset of the moduli of the *perverse Hilbert scheme*:

$$(T^*X)^{[n]} \subset \mathsf{PB}_n(X) = T^*\mathsf{PS}_n(X)$$

(where we drop the trivial framing from the notation). In the case of $\mathbf{A}^1$, the perverse Hilbert scheme is the Hamiltonian reduction $\mu^{-1}(0)/\mathrm{GL}_n$, that is, we have an open embedding

$$(\mathbb{C}^2)^{[n]} \subset \mathsf{PB}_n(\mathbf{A}^1) = T^*[(\mathfrak{gl}_n \times \mathbb{C}^n)/\mathrm{GL}_n]$$

obtained by dropping the stability condition for the geometric invariant theory quotient.

On the other hand, Theorem 1.4 asserts in particular that the moduli of (trivially framed $c_2 = n$) $\mathcal{D}$-bundles is the pseudo-torsor $\mathsf{CM}_n(X)$ over the perverse Hilbert scheme $\mathsf{PB}_n(X) = T^*\mathsf{PS}_n(X)$. Moreover, if $\mathrm{Pic}(X)$ is trivial, rank-one $\mathcal{O}_X$-framed $\mathcal{D}$-bundles up to isomorphism correspond to isomorphism classes of ideals in $\mathcal{D}$. For $X = \mathbf{A}^1$, we thus recover the intimate relationship between CM spaces, ideals in the Weyl algebra and the Hilbert scheme of points in the plane that appears in the numerous works cited above.

1.1 Techniques and outline

We deduce Theorem 1.4 from a general description of arbitrary framed $\mathcal{D}$-modules on (not necessarily projective) smooth curves X (in particular, of general holonomic $\mathcal{D}$-modules and $\mathcal{D}$-bundles), generalizing a classical description of flat vector bundles (which are precisely $\mathcal{D}$-modules framed by 0). As we review in § 2, framed $\mathcal{D}$-modules on X and framed sheaves on T^*X are both examples of sheaves on noncommutative $\mathbf{P}^1$-bundles (that is, noncommutative ruled surfaces) over X,

in the sense of [Van01b]. In order to describe sheaves on a $\mathbf{P}^1$-bundle, it is natural to imitate the Beilinson transform description of complexes on projective spaces or, equivalently, to apply Koszul duality. This technique was first applied to give a description of $\mathcal{D}$-bundles on projective curves by Katzarkov *et al.* [KOP]. We build on this idea to prove a general derived equivalence (Theorem 1.5) from which we deduce descriptions of various classes of objects (Theorems 1.6 and 1.7).

As an important technical note, we work throughout not with derived categories of modules as triangulated categories, but rather with differential graded (dg) categories which are 'enhancements' of underlying (triangulated) derived categories. See [Kel06] for an excellent overview and § 2.2 for further discussion of the framework we need.

In § 2 we develop the general algebraic setting we need for noncommutative $\mathbf{P}^1$-bundles, by localizing the category of graded modules for the Rees algebra $\mathcal{R}(\mathcal{D})$ with respect to bounded modules (the usual construction of Qgr $\mathcal{R}(\mathcal{D})$). We also introduce the dg derived category, denoted by $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$, of this localized module category. In § 3, we adapt the Beilinson transform to our setting and develop an analogue of Čech cohomology for computing its output. We then prove, in § 4, that the Beilinson transform gives a concise description of $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$: it is equivalently described as a dg category of *Koszul data*[1] (§ 4.1). An object $M \in D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ has a well-defined 'restriction to the curve at infinity' $i_\infty^* M$, and we may specialize the general Koszul description to framed complexes.

THEOREM 1.5 (Theorem 4.3). *There is a natural quasi-equivalence of dg categories between $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ and the dg category* Kos *of Koszul data (see § 4.1) whose objects are triples $\mathcal{C} = (C_{-1}, C_0, a : \mathcal{D}^1 \otimes C_{-1} \to C_0)$ consisting of objects C_{-1} and C_0 of $D_{\mathrm{dg,qcoh}}(X)$ together with a morphism $a : \mathcal{D}^1 \otimes C_{-1} \to C_0$ of complexes.*

Under this quasi-equivalence, a choice of V-framing of an object M of $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ corresponds to a choice of quasi-isomorphism $\mathrm{Cone}(C_{-1} = \mathcal{O} \otimes C_{-1} \xrightarrow{a} C_0) \simeq V$ for the corresponding object (C_{-1}, C_0, a) of Kos.

An identical result holds in the commutative case, that is, for complexes on $\overline{T^*X}$, with $\mathcal{D}^1$ replaced by $\mathcal{O} \oplus \mathcal{T}$: this is just the families version of the usual description of the derived category of $\mathbf{P}^1$, and the same methods can be extended to the noncommutative setting. Furthermore, the difference between two $\mathcal{D}^1$-action maps a_1, a_2 as above gives their commutative analogue, thereby explicitly exhibiting a pseudotorsor structure for framed $\mathcal{D}$-complexes over framed complexes on T^*X. This is one of the main points of our paper: that the standard Beilinson description of sheaves on the projective line, with straightforward modifications, provides an explicit description of the moduli of $\mathcal{D}$-modules on curves generalizing and clarifying the much-studied case of the Weyl algebra.

Next, in § 4, we identify which objects in Kos correspond to honest framed $\mathcal{D}$-modules, rather than complexes. Derived categories of surfaces carry a natural perverse coherent t-structure, obtained from the standard t-structure by tilting the torsion sheaves of dimension zero into cohomological degree one.[2] A similar definition makes sense for the noncommutative ruled surface defined by $\mathcal{D}$-modules; however, since there are no $\mathcal{D}$-modules with zero-dimensional support, this recovers the standard t-structure. As might be expected from the interpretation as a Koszul duality or de Rham functor, it is the perverse t-structure which is compatible with the Beilinson transform description above. In particular we find that the commutative analogue of the data describing $\mathcal{D}$-modules parametrizes (framed) perverse coherent sheaves on T^*X.

The equivalence of Theorem 1.5 simplifies considerably in the case of $\mathcal{D}$-bundles, that is, the case of pure two-dimensional support. Suppose that the framing V is a vector bundle. Then, under the

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[1] Alternatively, one may use the Koszul duality between modules for the Rees algebra and for the de Rham algebra [Kap91] to describe $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$.

[2] A t-structure on a dg category is just a t-structure on its derived category (§ 2.2).

quasi-equivalence of Theorem 1.5, a V-framed perverse $\mathcal{D}$-bundle corresponds to a triple (C_{-1}, C_0, a) in which $C_{-1} = Q[-1]$ where Q is a torsion sheaf on X, and $C_0 = \mathrm{Cone}(V \xrightarrow{i} Q)$ for a map i. We thus obtain the following result.

THEOREM 1.6 (Theorem 4.6). *For a vector bundle V on X, the moduli stack $\mathsf{CM}(X, V)$ is isomorphic to the stack of triples (Q, i, a) where Q is a torsion sheaf, $i : V \to Q$ is a homomorphism, and*

$$a : Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$$

is a map in $D_{\mathrm{dg,qcoh}}(X)$.

The corresponding commutative data give perverse vector bundles on T^*X. More generally, let

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{E}/\mathcal{O}_X \to 0$$

be an exact sequence of vector bundles on X, with $\mathrm{rk}(\mathcal{E}) = 2$. Let $S = \mathbf{P}(\mathcal{E})$ with the section σ determined by $\mathcal{O}_X \subset \mathcal{E}$.

THEOREM 1.7. *We have the following.*

(1) *The moduli stack of V-framed perverse vector bundles on S (framed along the section σ) is isomorphic to the moduli stack of triples (Q, i, a) where (Q, i) is a framed torsion sheaf and*

$$a : \mathcal{E} \otimes Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$$

is a map whose restriction to $\mathcal{O} \otimes Q[-1]$ is the natural 'inclusion'.

(2) *In particular, the conclusion of Theorem 4.6 holds for $\mathsf{PB}(X, V)$, with $\mathcal{D}^1$ replaced by $\mathcal{O} \oplus \mathcal{T}_X$.*

Remark 1.8 (Holonomic modules). It is easy to read off from Theorem 1.5 the analogue of the above classification in the case of holonomic $\mathcal{D}$-modules, that is, the case of pure one-dimensional support. More precisely, for a torsion sheaf V on X, the moduli stack of V-framed (hence, holonomic) $\mathcal{D}$-modules is isomorphic to the stack of triples (C_{-1}, C_0, a) as in Theorem 1.5, where C_{-1} is a vector bundle (in degree zero) and C_0 is an extension of V by C_{-1}. The same data, with $\mathcal{D}^1$ replaced by $\mathcal{O} \oplus \mathcal{T}_X$, describes the moduli stack of V-framed coherent sheaves on T^*X with pure one-dimensional support: these are precisely the *framed spectral sheaves* on T^*X, as described in [BN07, KOP].

The resulting descriptions are evidently functorial and local in X (in other words, we describe framed $\mathcal{D}$-modules and perverse coherent sheaves as stacks over X). When the curve is projective, Serre duality then identifies the perverse coherent data (Q, i, a) with the cotangent bundle to the underlying coherent data (Q, i), so that the moduli stacks of framed $\mathcal{D}$-modules all form pseudotorsors over the cotangent bundles to certain moduli spaces of complexes of coherent sheaves on the curve.

A key calculation takes place in §5, where we identify the space of Koszul data for $\mathcal{D}$-bundles with a *specific* twisted cotangent bundle to the stack of framed torsion sheaves. We show that the relevant twist comes from the dual determinant bundle (resulting in Theorem 1.3). To prove this, we use the natural description of the cotangents to quote schemes to identify the CM moment condition with the Leibniz rule governing the action map a in the Koszul data. This calculation can be carried over to describe the moduli space of flat vector bundles or general holonomic $\mathcal{D}$-modules as twisted cotangent bundles associated to dual determinant bundles; we hope to investigate this realization further in future work.

We conclude in §6 by spelling out very explicitly the dictionary between our data and the usual CM data on the affine line.

2. Noncommutative $\mathbf{P}^1$-bundles and perverse bundles

2.1 Preliminaries on algebras and noncommutative $\mathbf{P}^1$-bundles

Fix a smooth quasiprojective complex curve X.

NOTATION 2.1. *Let $\mathcal{D}^1$ denote a bimodule extension of the tangent sheaf $T = T_X$ of X by $\mathcal{O}_X$ of one of the following two sorts:*

(1) *$\mathcal{D}^1$ is a Lie algebroid on X, that is, a sheaf of twisted first-order differential operators $\mathcal{D}^1(L^{\otimes c})$ on some line bundle L on X (where $c \in \mathbb{C}$);*

(2) *$\mathcal{D}^1$ is supported scheme-theoretically on the diagonal $\Delta \subset X \times X$, so it is just given by an extension of vector bundles on X.*

Remark 2.2. Our choice of $\mathcal{D}^1$ above is more restrictive than required for Theorem 1.7. However, it is easy to see that all of the proofs needed for Theorem 1.7 work with $\mathcal{D}^1$ replaced by a commutative bimodule $\mathcal{E}$ as in the statement of the theorem. The hypotheses on $\mathcal{D}^1$ chosen above are intended to make statements and proofs easier to follow in the cases of greatest interest.

In case (1), we let $\mathcal{D}$ denote the universal enveloping algebroid of $\mathcal{D}^1$, that is, $\mathcal{D}(L^{\otimes c})$, a sheaf of twisted differential operators [BB93]; in case (2), we let $\mathcal{D}$ denote the quotient $\mathrm{Sym}(\mathcal{D}^1)/(1-\mathbf{1})$ of the symmetric algebra of $\mathcal{D}^1$ by the relation that $1 \in \mathrm{Sym}^0(\mathcal{D}^1) = \mathcal{O}$ equals $\mathbf{1} \in \mathrm{Sym}^1(\mathcal{D}^1) = \mathcal{D}^1$. This is a filtered $\mathcal{O}_X$-algebra (warning: $\mathcal{O}_X$ is not necessarily a central subalgebra of $\mathcal{D}$) with associated graded algebra isomorphic to $\mathrm{Sym}(T_X)$ and with $\mathcal{D}^1$ as the first term in its filtration. Some examples: if $\mathcal{D}^1$ is the sheaf of first-order differential operators, then $\mathcal{D}$ is the sheaf of differential operators $\mathcal{D}_X$; if $\mathcal{D}^1 = \mathcal{D}^1(L)$, then $\mathcal{D}$ is the sheaf $\mathcal{D}_X(L)$ of differential operators acting on sections of L; if $\mathcal{D}^1 = \mathcal{O}_X \oplus T_X$ (the split extension) then $\mathcal{D}$ is the sheaf $\mathcal{O}_{T^*X}$ of functions on the cotangent bundle of X.

Let

$$\mathcal{R} = \mathcal{R}(\mathcal{D}) = \sum_n \mathcal{D}^n t^n \subset \mathcal{D}[t]$$

denote the Rees algebra of $\mathcal{D}$; this is a graded algebra with a central element $t \in \mathcal{R}$ of degree one such that $\mathcal{R}/t\mathcal{R} \cong \mathrm{gr}(\mathcal{D})$. In particular, $\mathcal{R}$ is a graded $\mathbb{C}[t]$-algebra. The localization $\mathcal{R}[t^{-1}] \cong \mathcal{D} \otimes \mathbb{C}[t, t^{-1}]$ is also a graded ring and we have the localization functor $\ell : \mathcal{R} - \mathrm{mod} \to \mathcal{R}[t^{-1}] - \mathrm{mod}$. Any modules over graded rings that we consider are always assumed to be graded modules.

We let $\mathcal{E}$ denote the (formal) microlocalization of $\mathcal{D}$, see, for example, [AVV89]; it is obtained by formally adding to $\mathcal{D}$ power series in negative powers of vector fields. For example, if T_X is trivial with nonvanishing section ∂, then $\mathcal{E} \cong \mathcal{O}_X((\partial^{-1}))$. The ring $\mathcal{E}$ is also a filtered ring that contains $\mathcal{D}$ as a filtered subring. Note that $\mathcal{E}$ is not quasicoherent. Given a (left) $\mathcal{D}$-module M, we let $M_{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{D}} M$, which we call (slightly abusively) the *microlocalization of M*; similarly, given an $\mathcal{R}$-module N, we let $N_{\mathcal{R}(\mathcal{E})} = \mathcal{R}(\mathcal{E}) \otimes_{\mathcal{R}} N$ denote its microlocalization, which is a graded $\mathcal{R}(\mathcal{E})$-module.

LEMMA 2.3. *The localization and microlocalization functors ℓ and $(-)_{\mathcal{R}(\mathcal{E})}$ are exact; their right adjoints are the forgetful functors back to $\mathcal{R} - \mathrm{mod}$.*

Proof. For microlocalization, this is Theorem 3.19(2) and Corollary 3.20 of [AVV89]. The statements about adjoints are standard. $\qquad\square$

Remark 2.4. As in §7 of [BN06], if M is a filtered $\mathcal{D}$-module, then $M_{\mathcal{E}}$ comes equipped with a 'canonical' filtration.

Example 2.5. Suppose that $\mathcal{D}^1 = \mathcal{O}_X \oplus T_X$ with the central $\mathcal{O}_X$-bimodule structure. Then $\mathcal{D} = \pi_* \mathcal{O}_{T^* X}$ (where $\pi : T^* X \to X$ the projection) is the symmetric algebra of the tangent sheaf and $\mathcal{R}$ is the homogeneous coordinate ring of the ruled surface

$$\mathbb{P}_{\mathcal{D}} := \operatorname{Proj} \mathcal{R} = \mathbb{P}(T^* X \oplus \mathcal{O}).$$

This is the union of $\underline{\operatorname{Spec}} \mathcal{D} = T^* X$ and of a copy of the curve $X = \operatorname{Proj}(\mathcal{R}/t\mathcal{R})$ 'at infinity'. The ring $\mathcal{E}$ is the ring of formal Laurent series with poles along the section $X \subset \mathbb{P}_{\mathcal{D}}$.

More generally, we use $\mathcal{R}$ to construct a noncommutative $\mathbf{P}^1$-bundle [Van01b] as follows. Let $\operatorname{Gr} \mathcal{R}$ denote the category of graded left $\mathcal{R}$-modules that are quasicoherent as $\mathcal{O}$-modules, and let $\operatorname{gr} \mathcal{R}$ denote its full subcategory of locally finitely generated modules. We let $\operatorname{Tors} \mathcal{R}$ denote the full subcategory of $\operatorname{Gr} \mathcal{R}$ whose objects are *irrelevant* or *locally bounded modules*: a module M is irrelevant if for every local section m of M there is some $n_0 \in \mathbb{Z}$ such that for all $n \geqslant n_0$ and all local sections r of $\mathcal{R}_n$, one has $r \cdot m = 0$. Geometrically, irrelevant modules are supported set-theoretically at the irrelevant ideal of $\mathcal{R}$. Similarly, $\operatorname{tors} \mathcal{R}$ is the full subcategory of $\operatorname{gr} \mathcal{R}$ consisting of irrelevant modules.

The subcategories $\operatorname{Tors} \mathcal{R}$ and $\operatorname{tors} \mathcal{R}$ are Serre subcategories of $\operatorname{Gr} \mathcal{R}$ and $\operatorname{gr} \mathcal{R}$, respectively, and hence the quotient categories, denoted $\mathcal{Q}coh(\mathbb{P}_{\mathcal{D}}) = \operatorname{Qgr} \mathcal{R}$ and $\mathcal{C}oh(\mathbb{P}_{\mathcal{D}}) = \operatorname{qgr} \mathcal{R}$, exist and have reasonable properties [AZ94, Van01a].

Remark 2.6. We use the notation $\mathcal{Q}coh, \mathcal{C}oh$ to emphasize the relationship with derived categories in the commutative world, since this relationship is absolutely central to our point of view.

As usual, there is an adjoint pair of functors

$$(\pi : \operatorname{Gr} \mathcal{R} \to \mathcal{Q}coh(\mathbb{P}_{\mathcal{D}}), \quad \omega : \mathcal{Q}coh(\mathbb{P}_{\mathcal{D}}) \to \operatorname{Gr} \mathcal{R})$$

such that the adjunction $\pi \omega \to \operatorname{Id}$ is an isomorphism. Here π is exact and ω is left-exact.

By [AZ94, Theorem 8.8] (using the element $t \in \mathcal{R}$), $\mathcal{R}$ satisfies the χ-condition; it follows that Serre's finiteness theorem for cohomology [AZ94, Theorem 7.4] holds for $\mathcal{R}$. Moreover, $\mathcal{R}$ is strongly noetherian.

All of the constructions above sheafify over X; in particular, $\operatorname{Qgr} \mathcal{R}$ forms a sheaf of abelian categories in the Zariski topology of X.

Remark 2.7. It is occasionally useful to work algebraically over affine open subsets of X: given an affine open subset $U \subset X$, we may work with the ring of global sections $\mathcal{R}(U)$ and carry out the usual constructions for this ring. In particular, we use this at one point later (the proof of Proposition 3.12).

2.2 dg derived categories

As we mentioned in the introduction, we work throughout not with derived categories of modules as triangulated categories, but rather with dg categories which are 'enhancements' of underlying derived categories. For an excellent overview of dg categories we refer the reader to [Kel06]; detailed treatments of localization of dg categories can also be found in [Dri04] and [Toe07]. See also [Lur06] for the foundations (including a discussion of t-structures) of the theory of stable ∞-categories, of which dg categories are the rational case, and [BLL04] for properties of pretriangulated dg categories (those whose homotopy categories are triangulated).

There are natural dg versions of essentially all natural constructions with derived categories, the main technical difference being the need to sometimes consider dg categories up to quasi-equivalence rather than equivalence (see the above references as well as [Tam07]). The localization construction for dg categories of Keller, Drinfeld and Toën (which is a dg analogue of Dwyer–Kan

simplicial localization) allows one, for example, to take the dg category of complexes in an abelian category and localize it with respect to quasi-isomorphisms, resulting in a *canonical* dg category whose homotopy category is the usual derived category; we refer to this canonical dg enhancement as 'the dg derived category'. Another quasi-equivalent dg enhancement may be obtained from the dg category of injective complexes (see the discussion in [BLL04]).

There are numerous technical advantages in working with dg enhancements of derived categories rather than with triangulated categories. Among these we single out three that we use below. First, the dg enhancements of derived categories satisfy a homotopical form of descent (or sheaf property) over the Čech nerve of a covering (see, for example, [BD, Section 7.4] or [HS98, Section 21]). Second, one can recover moduli spaces of sheaves from the dg enhanced derived category [TV05]. Finally, dg categories are amenable to explicit description as modules over the endomorphisms of a compact generator (see [Kel] and the discussion of Theorem 4.3 below). Since we work uniformly throughout the paper with dg enhancements, we often abuse terminology and refer simply to derived categories.

DEFINITION 2.8. Let $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ denote the dg enhancement of the bounded derived category $D^b(\mathrm{Qgr}\,\mathcal{R})$ obtained by localizing the dg category of complexes in $\mathrm{Qgr}\,\mathcal{R}$ with respect to the class of quasi-isomorphisms [Dri04, Kel06, Toe07]. We let $D_{\mathrm{dg,coh}}(\mathbb{P}_{\mathcal{D}})$ denote the full dg subcategory of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ whose objects have cohomologies in qgr $\mathcal{R}$.

Remark 2.9 (Calculating in dg categories). The reader who is unfamiliar with the yoga of dg categories may rest assured that such familiarity is mostly unnecessary for this paper. Indeed, although the dg derived category is essential for Theorem 4.3 and for the 'families' part of Theorem 4.6, the proofs, and, in particular, the calculations of objects, depend only on calculations involving cohomologies and so can be understood in the usual (triangulated) derived category.

Note that one has an equivalence $\mathrm{Qgr}(\mathrm{Sym}\,T_X) \simeq \mathcal{Q}coh(X)$. It follows that one has a base-change functor

$$i_{\infty}^* : \mathcal{Q}coh(\mathbb{P}_{\mathcal{D}}) = \mathrm{Qgr}\,\mathcal{R} \to \mathrm{Qgr}\,\mathcal{R}/t\mathcal{R} \simeq \mathcal{Q}coh(X)$$

and an induced base-change functor i_{∞}^* on derived categories. Both of these functors may be applied to objects of the abelian category $\mathrm{Qgr}\,\mathcal{R}$, and we write Li_{∞}^* for the derived functor applied to such objects. There is also a 'direct image' functor $(i_{\infty})_*$.

2.3 Perverse bundles

We continue to assume that X and $\mathcal{D}$ are as in the previous section. An object M of $\mathrm{coh}(\mathbb{P}_{\mathcal{D}})$ is said to be *zero-dimensional* if:

(a) the graded $\mathcal{R}$-module ωM has zero-dimensional support on X; and

(b) the Hilbert function $h_{\omega M}(k) = \mathrm{length}(\omega(M)_k)$ is a bounded function of k.

We let $\mathcal{T}$ denote the full subcategory of $\mathrm{coh}(\mathbb{P}_{\mathcal{D}})$ consisting of zero-dimensional objects. We let $\mathcal{F}$ denote the full subcategory of $\mathrm{coh}(\mathbb{P}_{\mathcal{D}})$ consisting of objects that have no nonzero zero-dimensional subobjects. For every object N of $\mathrm{coh}(\mathbb{P}_{\mathcal{D}})$, there is a unique maximal zero-dimensional subobject $t(N)$ obtained by forming the sum $T(\omega N) \subset \omega N$ of all graded $\mathcal{R}$-submodules of ωN that have properties (a) and (b) above and taking the subobject $t(N) = \pi(T(\omega N)) \subset N$. We thus obtain an exact sequence in qgr $\mathcal{R}$,

$$0 \to t(N) \to N \to N/t(N) \to 0$$

with $t(N) \in \mathcal{T}$ and $N/t(N) \in \mathcal{F}$. The pair $(\mathcal{T}, \mathcal{F})$ forms a torsion pair in qgr $\mathcal{R}$, then, and we have the following result.

PROPOSITION 2.10 ([HRS96, Proposition I.2.1] or [Bri05, Proposition 2.5]). *The full subcategory* $\mathcal{P}$ *of* $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ *given by*

$$\mathcal{P} = \{E \in D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D}) \mid H^i(E) = 0 \text{ for } i \notin \{0,1\}, H^0(E) \in \mathcal{F}, H^1(E) \in \mathcal{T}\}$$

is the heart of a bounded t-structure on $D_{\mathrm{dg,coh}}(\mathbb{P}_\mathcal{D})$.

Note that, by definition, a t-structure on a pretriangulated dg category (that is, one whose homotopy category is triangulated [BLL04]) is a t-structure on its homotopy category.

We will refer to an object of $\mathcal{P}$ as a *perverse* $\mathcal{R}$-*module*. An object N of $\mathcal{Q}coh(\mathbb{P}_\mathcal{D})$ is said to be *torsion-free* if it has the form $\pi(M)$ for some torsion-free module M in $\mathrm{Gr}\,\mathcal{R}$. It follows from [AZ94, S2, p. 252] that in this case ωN is a torsion-free graded $\mathcal{R}$-module.

Let V be a vector bundle on X.

DEFINITION 2.11. A V-*framed* complex of $\mathcal{D}$-modules is an object M of the dg derived category $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ equipped with an isomorphism $i^*_\infty(M) \to V$. A V-*framed perverse* $\mathcal{D}$-*bundle* is a V-framed object M of $\mathcal{P}$ such that:

(1) $H^0(M)$ is a torsion-free object of $\mathcal{C}oh(\mathbb{P}_\mathcal{D})$; and

(2) the kth term in the grading $\omega(H^0(M))_k$ is a locally free $\mathcal{O}_X$-module of rank $(k+1)\,\mathrm{rk}(V)$ for all $k \geqslant -1$.

The second condition is an open condition on perverse $\mathcal{D}$-bundles. If $\mathcal{D}$ is commutative, this condition just means that the cohomology sheaf $H^0(M)$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}^{\mathrm{rk}(V)}$ on the generic fiber of our $\mathbf{P}^1$-bundle over X: in other words, $H^0(M)$ has 'trivial generic splitting type'. This condition is important when studying M using the Beilinson transform, but it is also worth noting that it is the right condition 'in nature': that is, it agrees with the definition in [BN06] that appears in the study of the Kadomtsev–Petviashvili (KP) hierarchy. Since we will not need this in the rest of the paper, we do not give the definition of [BN06]: we only state the following without proof.

PROPOSITION 2.12. *Suppose that* $\mathcal{D} = \mathcal{D}_X(L^{\otimes c})$ *is a sheaf of twisted differential operators. Then the moduli stack of* V-*framed perverse* $\mathcal{D}$-*bundles is isomorphic to the moduli stack of* V-*framed* $\mathcal{D}$-*bundles in the sense of* [BN06, Definition 3.2].

We need the following.

LEMMA 2.13. *We have the following.*

(1) *If* $N \in \mathcal{C}oh(\mathbb{P}_\mathcal{D})$ *is zero-dimensional, then* $h_{\omega N}(k)$ *is constant for all* $k \gg 0$.

(2) *Suppose that* $N \in \mathcal{C}oh(\mathbb{P}_\mathcal{D})$ *is zero-dimensional. If* $i^*_\infty N = 0$, *then* $Li^*_\infty N = 0$.

(3) *Suppose that* $M \in \mathcal{P}$. *If* $H^0(M)$ *is torsion-free, then* $i^*_\infty H^0(M) = Li^*_\infty H^0(M)$.

In particular, if M *is a* V-*framed perverse* $\mathcal{D}$-*bundle, then* $i^*_\infty M = i^*_\infty H^0(M) \cong V$.

Proof. A standard argument using finite generation of N proves part (1).

We use the complex

$$\mathbf{R} : \mathcal{R}(-1) \to \mathcal{R}, \tag{2.1}$$

which is quasi-isomorphic to $\mathcal{R}/t\mathcal{R}$. Thus, $i^*_\infty N \simeq \pi(\mathbf{R} \otimes \omega N)$. Using part (1), it follows that, for all k sufficiently large, multiplication by t from ωN_k to ωN_{k+1} is injective if and only if it is surjective. In particular, the homomorphism $\omega N(-1) \to \omega N$ has kernel that is a bounded $\mathcal{R}$-module if and only if its cokernel is a bounded $\mathcal{R}$-module. This proves part (2).

Note that $\omega H^0(M)$ is torsion-free. It follows that multiplication by t is injective on $\omega H^0(M)$, and using (2.1) to compute $i^*_\infty H^0(M)$ gives part (3). $\qquad\square$

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LEMMA 2.14. *Suppose that $N \in \mathcal{C}oh(\mathbb{P}_\mathcal{D})$ is zero-dimensional and $Li_\infty^* N = 0$. Then (with notation as in § 2.1) $\ell((\omega N)_{\mathcal{R}(\mathcal{E})}) = 0$.*

Proof. If $\mathcal{D}$ is commutative this is well known. If $\mathcal{D}$ is a ring of twisted differential operators, then the question reduces to the same question for the ring of differential operators $\mathcal{D}_X$. However, $\mathcal{D}_X$ has no zero-dimensional modules. $\square$

A perverse $\mathcal{D}$-bundle M has a local numerical invariant called the *(local) second chern class* $c_2(M)$, defined as follows. Since $\omega H^0(M)$ is torsion-free, the V-framing determines an injective homomorphism

$$\omega H^0(M)/t\omega H^0(M) \hookrightarrow \mathcal{R}/t\mathcal{R} \otimes V \cong \mathrm{Sym}(T_X) \otimes V.$$

The cokernel of this homomorphism has finite length c by part (2) of Definition 2.11. Similarly, $\omega H^1(M)_k$ has finite length as an $\mathcal{O}_X$-module, which is constant for $k \gg 0$; we let c' denote this length for $k \gg 0$.

DEFINITION 2.15. *With notation as above, $c_2(M) = c + c'$.*

In the commutative case, this definition reproduces the 'local contribution to the second Chern class of M'. For example, suppose that M is given by a complex $\mathcal{O}_{\mathbb{P}_\mathcal{D}} \to Q$, where Q is a torsion sheaf of length n on the commutative $\mathbf{P}^1$-bundle $\mathbb{P}_\mathcal{D}$. Then $c_2(M) = n$ regardless of the map in the complex. In general, such a 'local c_2', which measures the failure of $H^\bullet(M)$ to be locally free, exists, regardless of whether X is projective (hence, our use of the word 'local').

3. Resolution of the diagonal and Beilinson transform

We next develop the analogue, for our noncommutative $\mathbf{P}^1$-bundles, of the 'fiberwise Beilinson transform'. Although this follows the standard method, it does not seem to appear in the literature in the form we need.

We also explain the main tool for computing the Beilinson transform, namely, an analogue of Čech cohomology. More precisely, we want to compute direct images from the $\mathbf{P}^1$-bundle $\mathbb{P}_\mathcal{D}$ to X, which we carry out using Čech cohomology: essentially, this amounts to covering each $\mathbf{P}^1$ by its cover consisting of the affine line (which corresponds to the ring $\mathcal{D}$) and the 'formal neighborhood of the point at infinity' (which is captured using the microlocalization $\mathcal{E}$ of $\mathcal{D}$). The Čech cohomology is easy to calculate in some examples (see Corollary 3.15), which in Section 4 allow us to explicitly compute the Beilinson transform of a perverse bundle.

3.1 Resolution of the diagonal

Fix X, $\mathcal{D}^1$, and $\mathcal{R}$ as in § 2.1. Let Δ denote the *diagonal bigraded $\mathcal{R}$-bimodule*, which is given by $\Delta_{i,j} = \mathcal{R}^{i+j}$ when $i,j \geqslant 0$ and $\Delta_{i,j} = 0$ otherwise. There is also a larger bigraded bimodule $\widetilde{\Delta}$ given by $\widetilde{\Delta}_{i,j} = \mathcal{R}^{i+j}$ for all i,j.

LEMMA 3.1. *For any object M of $\mathrm{Gr}(\mathcal{R})$, let $C(M)$ denote its Čech complex (Definition 3.10). Then the images of the complexes of left $\mathcal{R}$-modules*

$$\pi(\Delta \otimes_\mathcal{R} C(M)) \quad and \quad \pi(\widetilde{\Delta} \otimes_\mathcal{R} C(M))$$

are equal as complexes in $\mathcal{Q}coh(\mathbb{P}_\mathcal{D})$.

Proof. Note that $(\widetilde{\Delta}/\Delta)_{i,j} = 0$ whenever $i,j \geqslant 0$. Thus, for any $\mathcal{R}[t^{-1}]$-module N or $\mathcal{R}(\mathcal{E})$-module N, we have $\pi(\widetilde{\Delta}/\Delta \otimes_\mathcal{R} N) = 0$. The lemma follows. $\square$

Suppose that $M_{\bullet,\bullet}$ is a bigraded sheaf. We call $M_{\bullet,\bullet}$ a *bigraded left $\mathcal{R}$-module* if, for each k, $M_{\bullet,k}$ comes equipped with a structure of graded left $\mathcal{R}$-module.

DEFINITION 3.2. Given a bigraded left $\mathcal{R}$-module $M_{\bullet,\bullet}$, we write $(p_1)_* M = M_{\bullet,0}$, which is a singly-graded left $\mathcal{R}$-module.

LEMMA 3.3. *The functor* $\mathbf{F} : \mathrm{Gr}\,\mathcal{R} \to \mathrm{Gr}\,\mathcal{R}$ *defined by*

$$\mathbf{F}(M) = (p_1)_*(\widetilde{\Delta} \otimes_{\mathcal{R}} M) = (\widetilde{\Delta} \otimes_{\mathcal{R}} M)_{\bullet,0}$$

is isomorphic to the identity functor.

Proof. This is immediate from the identity $\widetilde{\Delta}_{i,\bullet} = \mathcal{R}(i)$ of graded right $\mathcal{R}$-modules. $\square$

There is an exact sequence of $\mathcal{R}$-bimodules on X:

$$\mathcal{R}(-1) \otimes_{\mathcal{O}} \mathcal{D}^1 \otimes_{\mathcal{O}} \mathcal{R}(-1) \xrightarrow{\alpha} \mathcal{R} \otimes_{\mathcal{O}} \mathcal{R} \xrightarrow{m} \Delta \to 0. \tag{3.1}$$

Here m is the usual multiplication map and $\alpha = \alpha_1 \otimes t - t \otimes \alpha_2$ is the difference of the two maps given by

$$\alpha_1 : \mathcal{R}(-1) \otimes \mathcal{D}^1 \to \mathcal{R}, \quad \alpha_2 : \mathcal{D}^1 \otimes \mathcal{R}(-1) \to \mathcal{R} \tag{3.2}$$

defined by

$$\alpha_1(D_1 \otimes Z) = D_1 Z, \quad \alpha_2(Z \otimes D_2) = Z D_2.$$

The kernel of α is easily checked to be the image of $\mathcal{R}(-1) \otimes \mathcal{O} \otimes \mathcal{R}(-1)$ under the natural inclusion, and so we obtain the following lemma.

LEMMA 3.4. *The complexes of $\mathcal{R}$-bimodules*

$$\delta' = [\mathcal{R}(-1) \otimes \mathcal{R}(-1) \to \mathcal{R}(-1) \otimes \mathcal{D}^1 \otimes \mathcal{R}(-1) \xrightarrow{\alpha} \mathcal{R} \otimes \mathcal{R}]$$

and

$$\delta = [\mathcal{R}(-1) \otimes T_X \otimes \mathcal{R}(-1) \xrightarrow{\overline{\alpha}} \mathcal{R} \otimes \mathcal{R}]$$

are quasi-isomorphic to Δ.

It is important to note that, although α is defined as the difference between two maps induced from $\mathcal{O}$-module maps α_1 and α_2, there is, in general, no such description of $\overline{\alpha}$.

3.2 Beilinson transform

For any complex of graded $\mathcal{R}$-modules $P_\bullet$, the double complex $\delta \otimes_{\mathcal{R}} P_\bullet$ is a double complex of bigraded left $\mathcal{R}$-modules, and we may form

$$\mathbb{B}(P_\bullet) = (p_1)_*(\delta \otimes_{\mathcal{R}} C(P_\bullet)) \overset{\mathrm{def}}{=} \mathrm{Tot}((\delta \otimes_{\mathcal{R}} C(P_\bullet))_{\bullet,0}); \tag{3.3}$$

here Tot denotes the total complex. We call $\mathbb{B}(P_\bullet)$ the *Beilinson transform* of $P_\bullet$. Similarly we may replace δ by δ' in (3.3) and obtain a functor denoted by $\mathbb{B}'$, also called a *Beilinson transform*.

LEMMA 3.5. *The Beilinson transforms $\mathbb{B}$ and $\mathbb{B}'$ are exact functors.*

Proof. Since δ and δ' are complexes of projective right $\mathcal{R}$-modules, $\mathbb{B}$ and $\mathbb{B}'$ are composites of exact functors $\delta \otimes_{\mathcal{R}} -$ (respectively, $\delta' \otimes_{\mathcal{R}} -$), $(p_1)_*$, and $C(-)$. $\square$

PROPOSITION 3.6. *For any complex $P_\bullet$ of quasicoherent graded $\mathcal{R}$-modules, there are natural morphisms of complexes*

$$\mathbb{B}'(P_\bullet) \to \mathbb{B}(P_\bullet) \to C(P_\bullet). \tag{3.4}$$

If $P_\bullet$ is a complex of flat $\mathcal{R}$-modules, then the cones of (3.4) have irrelevant (that is, locally bounded) cohomologies.

Proof. There are natural maps

$$\delta' \otimes_{\mathcal{R}} C(P_\bullet) \to \delta \otimes_{\mathcal{R}} C(P_\bullet) \to \Delta \otimes_{\mathcal{R}} C(P_\bullet). \tag{3.5}$$

Suppose that $P_\bullet$ is a complex of flat $\mathcal{R}$-modules; then the maps (3.5) are quasi-isomorphisms by Lemma 3.4, as are the natural maps

$$(p_1)_*(\delta' \otimes_{\mathcal{R}} C(P_\bullet)) \to (p_1)_*(\delta \otimes_{\mathcal{R}} C(P_\bullet)) \to (p_1)_*(\Delta \otimes_{\mathcal{R}} C(P_\bullet)).$$

Now, the cone of the natural map

$$(p_1)_*(\Delta \otimes C(P_\bullet)) \to (p_1)_*(\widetilde{\Delta} \otimes C(P_\bullet)) \tag{3.6}$$

is naturally identified with $(p_1)_*(\widetilde{\Delta}/\Delta \otimes C(P_\bullet))$ which, by Lemma 3.1, has irrelevant cohomologies. □

If M is an object of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$, the *Beilinson transform* of M is defined to be $\pi(\mathbb{B}(M))$, respectively $\pi(\mathbb{B}'(M))$: by Proposition 3.12, this is well-defined.

LEMMA 3.7. *Suppose that $M \to M'$ is a morphism of complexes in $\mathrm{Gr}\,\mathcal{R}$ whose cone has irrelevant cohomologies. Then the natural morphisms $\pi\mathbb{B}(M) \to \pi\mathbb{B}(M')$, $\pi\mathbb{B}'(M) \to \pi\mathbb{B}'(M')$ are isomorphisms in $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$.*

Proof. It suffices to check that $\pi\mathbb{B}(N) = 0 = \pi\mathbb{B}'(N)$ when N is a module with irrelevant cohomologies. Resolving N by locally free $\mathcal{R}$-modules and using Lemma 3.5, this follows from the second part of Proposition 3.6. □

THEOREM 3.8. *The Beilinson transforms $\mathbb{B}$ and $\mathbb{B}'$ are isomorphic to the identity functor on $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$.*

Proof. We only give the proof for $\mathbb{B}$, as the proof for $\mathbb{B}'$ is identical. By Proposition 3.6, there is a natural morphism

$$\pi((p_1)_*(\delta \otimes C(P_\bullet))) \to \pi \circ C(P_\bullet) \simeq P_\bullet.$$

If $P_\bullet$ is an object of $\mathcal{Q}coh(\mathbb{P}_{\mathcal{D}})$ of the form $\pi\mathcal{R}(k)$, then the cone of the adjunction $\mathcal{R}(k) \to R\omega \circ \pi\mathcal{R}(k)$ has irrelevant (that is, locally bounded) cohomologies [AZ94, Proposition 7.2]. Hence, by Lemma 3.7 and Corollary 3.14, we have an isomorphism $\pi(\mathbb{B}(\mathcal{R}(k))) \to \mathbb{B}(P_\bullet)$ in $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$. Now, by Proposition 3.6, we have an isomorphism $\pi\mathbb{B}(\mathcal{R}(k)) \to C(\pi\mathcal{R}(k)) \simeq P_\bullet$. It follows that the natural map $\mathbb{B}(P_\bullet) \to P_\bullet$ is an isomorphism in this case. Since every object of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ is isomorphic to a complex of direct sums of objects $\pi\mathcal{R}(k)$ as above, the theorem follows by Lemma 3.5. □

Remark 3.9. Everything in this and the previous section seems to work without complications for an arbitrary noncommutative $\mathbf{P}^1$-bundle in the sense of Van den Bergh [Van01b]. This is interesting to work out in the case of difference operators, which we plan to carry out in [BN3].

3.3 Čech resolution

In this section, we define an analogue of the Čech resolution for $\mathcal{Q}coh(\mathbb{P}_{\mathcal{D}})$. We use this to compute Beilinson transforms.

We begin with some general observations. For any filtered ring S and its associated Rees ring $\mathcal{R}(S)$, one has two functors between the categories of filtered S-modules and graded $\mathcal{R}(S)$-modules. The first functor takes a filtered S-module M with filtration $\{M_k\}$ to the graded $\mathcal{R}(S)$-module $\bigoplus_k M_k$. In the other direction, given a graded $\mathcal{R}(S)$-module N, one may invert t and take the degree-zero term, $f(N) = \ell(N)_0$. This is naturally a module over $\mathcal{R}(S)[t^{-1}]_0 = S$; moreover, $f(N)$ has a filtration given by the images of $N_k t^{-k}$ in $f(N)$. This filtration makes $f(N)$ into a filtered S-module. These functors are studied in [LV96].

Suppose now that M is a quasicoherent graded $\mathcal{R}$-module. Recall that $\ell(M)$ means the localization of M given by inverting t, and $M_{\mathcal{R}(\mathcal{E})} = \mathcal{R}(\mathcal{E}) \otimes_{\mathcal{R}(\mathcal{D})} M$ is the *microlocalization*.

DEFINITION 3.10. The *Čech complex* of M is the complex

$$C(M) = [\ell(M) \oplus M_{\mathcal{R}(\mathcal{E})} \xrightarrow{\delta_M} \ell(M_{\mathcal{R}(\mathcal{E})})] \tag{3.7}$$

of $\mathcal{R}$-modules (implicitly, we compose localization or microlocalization with the forgetful functor back to $\mathcal{R}$). Here the map δ_M is the difference of the two natural maps $\ell(M) \to \ell(M_{\mathcal{R}(\mathcal{E})})$ and $M_{\mathcal{R}(\mathcal{E})} \to \ell(M_{\mathcal{R}(\mathcal{E})})$ determined by the adjunctions of Lemma 2.3.

See the proof of Corollary 3.15 below for the description of this complex in an example.

As described above, we cannot simply apply the functor π to obtain a Čech complex $\pi C(M)$ of objects of $\mathcal{Q}coh(\mathbb{P}_{\mathcal{D}})$: indeed, the 'microlocal' terms in $C(M)$ are typically not quasicoherent over $\mathcal{O}_X$. To remedy this, we may replace $C(M)$ with the complex

$$[\ell(M) \to \ell(M_{\mathcal{R}(\mathcal{E})})/M_{\mathcal{R}(\mathcal{E})}] \tag{3.8}$$

which does consist of objects of $\operatorname{Gr}\mathcal{R}$. Since the kernel of $M_{\mathcal{R}(\mathcal{E})} \to \ell(M_{\mathcal{R}(\mathcal{E})})$ is a locally bounded module, the images in $\mathcal{Q}coh(\mathbb{P}_{\mathcal{D}})$ of these complexes are quasi-isomorphic functorially in M. Hence, we usually ignore the distinction between (the results of applying π to) $C(M)$ and (3.8).

Remark 3.11. As we alluded to above, when $\mathcal{D}$ is commutative, $\ell(M)$ corresponds to the restriction of the quasicoherent sheaf (corresponding to) M on $\mathbb{P}_{\mathcal{D}}$ to $\operatorname{Spec}(\mathcal{D})$. Moreover, $M_{\mathcal{R}(\mathcal{E})}$ corresponds to the restriction of M to the formal neighborhood of the 'curve at infinity' $X_\infty \subset \mathbb{P}_{\mathcal{D}}$, and $\ell(M_{\mathcal{R}(\mathcal{E})})$ corresponds to the restriction of M to the 'punctured formal neighborhood' of X_∞, that is, the formal neighborhood minus X_∞ itself. Thus, $C(M)$ is exactly the Čech complex corresponding to a cover of the $\mathbf{P}^1$-bundle, fiber-by-fiber, by affine lines and formal disks around the points at infinity in each fiber.

The Čech complex comes equipped with a natural morphism $M \to C(M)$ given by the adjunction $M \to \ell(M)$ plus the adjunction $M \to M_{\mathcal{R}(\mathcal{E})}$.

PROPOSITION 3.12. *We have the following.*

(1) *The Čech complex functor is exact.*
(2) *The Čech complex functor descends to an exact endofunctor of* $\operatorname{Qgr}\mathcal{R}$.
(3) $C(M)$ *is a complex of acyclic objects for the functor* $\omega : \operatorname{Qgr}\mathcal{R} \to \operatorname{Gr}\mathcal{R}$.
(4) *The natural morphism* $M \to C(M)$ *is a quasi-isomorphism of complexes in* $\operatorname{Qgr}\mathcal{R}$.

Proof. Part (1) follows from Lemma 2.3. Part (2) follows from Corollaries 4.3.11 and 4.3.12 of [Pop73], once one observes that inverting t and microlocalization both kill locally bounded modules.

For (3), we first restrict attention to an affine open subset U of X and work as in Remark 2.7: that is, we work with the ring $\mathcal{R}(U)$ rather than the sheaf of rings $\mathcal{R}$. In this case the Čech complex as defined in (3.7) is a complex in $\operatorname{Qgr}\mathcal{R}(U)$. We now make several applications (for the functors ℓ, $(-)_{\mathcal{R}(\mathcal{E})}$, and $\ell(-)_{\mathcal{R}(\mathcal{E})}$) of the following general fact.

LEMMA 3.13. *Let* $\mathcal{C}, \mathcal{D}$ *be abelian categories and* $\mathcal{T} \subset \mathcal{C}$ *a dense subcategory. Let* $\ell : \mathcal{C} \to \mathcal{D}$ *be an exact functor with exact right adjoint* $f : \mathcal{D} \to \mathcal{C}$ *so that* $\ell(X) = 0$ *for all* $X \in \operatorname{ob}(\mathcal{T})$. *Let* $(\pi : \mathcal{C} \leftrightarrow \mathcal{C}/\mathcal{T} : \omega)$ *denote the 'standard' adjoint pair of functors. Let* $\overline{\ell} : \mathcal{C}/\mathcal{T} \to \mathcal{D}$ *be the functor determined by* $\overline{\ell} \circ \pi = \ell$ *and* $\overline{f}$ *its right adjoint. Then* $\overline{f}$ *is exact and* $R\omega \circ \overline{f} = \omega \circ \overline{f}$.

It follows that (3.7) consists of acyclic objects for ω, and consequently that (3.8) also consists of acylic objects. However, (3.8) is now the complex of sections over U of the (sheafified) Čech complex, and (3) follows.

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To prove (4) it suffices, by (2), to prove (4) for objects $\pi\mathcal{R}(k)$, for which see (3.9) below. $\qquad\square$

COROLLARY 3.14. *If $P_\bullet \in D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$, then $C(P_\bullet) \simeq R\omega(P_\bullet)$.*

COROLLARY 3.15 (Calculations of Čech cohomology). *We have the following.*

(1) *We have*

$$H^\bullet((p_1)_* R\omega(\pi\mathcal{R}(n))) = \begin{cases} \mathcal{D}^n & \text{if } n \geqslant -1, \\ (\mathcal{E}^{-1}/\mathcal{E}^n)[-1] & \text{if } n \leqslant -2. \end{cases} \tag{3.9}$$

In particular, $(p_1)_* R\omega(\pi\mathcal{R}(-2)) \cong \Omega^1_X[-1]$.

(2) *For objects $A, B \in D_{\mathrm{dg,qcoh}}(X)$ and $\ell \geqslant k$, we have*

$$R\operatorname{Hom}_{\operatorname{Qgr}\mathcal{R}}(\pi\mathcal{R}(k) \otimes A, \pi\mathcal{R}(\ell) \otimes B) = R\operatorname{Hom}_X(A, \mathcal{D}^{\ell-k} \otimes B). \tag{3.10}$$

Proof. Assertion (1) follows from a computation of the Čech cohomology of $\pi\mathcal{R}(n)$. Indeed, the Čech complex of $\pi\mathcal{R}(n)$ is just

$$\mathcal{D}[t](n) \oplus \mathcal{R}(\mathcal{E})(n) \to \mathcal{E}[t](n).$$

Applying $(p_1)_*$ to this complex (that is, taking the part in graded degree zero) gives

$$\mathcal{D} \oplus \mathcal{E}^n \to \mathcal{E}, \tag{3.11}$$

where $\mathcal{E}^n$ is the nth term in the filtration of $\mathcal{E}$. If $n \geqslant -1$, then (3.11) is surjective, with kernel $\mathcal{D} \cap \mathcal{E}^n = \mathcal{D}^n$. If $n \leqslant -2$, then (3.11) is injective, with cokernel $\mathcal{E}/(\mathcal{D} + \mathcal{E}^n) \cong \mathcal{E}^{-1}/\mathcal{E}^n$.

Equation (3.10) then follows immediately from (3.9) using the usual 'induction-restriction adjunction'. $\qquad\square$

4. Equivalence and isomorphism

In this section we prove two main consequences of the Beilinson transform of the previous section.

First, we prove the standard consequence of the Beilinson procedure: that the derived category of our noncommutative $\mathbf{P}^1$-bundle, $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$, is quasi-equivalent to a dg category described completely in terms of 'linear algebra' or 'quiver data', which we call the dg category Kos of *Koszul data*. Namely, the Beilinson transform shows that $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ is generated by the analogue of the exceptional collection $\{\mathcal{O}(-1), \mathcal{O}\}$ and, hence, may be described via modules over the endomorphism algebra of this collection. In our setting, when everything is done in families over the curve X, this amounts to describing an object of $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ in terms of a pair of quasicoherent complexes on the curve X together with the data encoding the lift to the $\mathbf{P}^1$-bundle, that is, an action of $\mathcal{D}^1$ from one complex on X to the other.

Second, we describe explicitly (Theorem 4.6) which Koszul data (objects of Kos) correspond to perverse bundles in $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$. This amounts to performing a Čech calculation starting from a perverse bundle and, conversely, computing the cohomologies of an object of $D_{\mathrm{dg}}(\mathbb{P}_\mathcal{D})$ starting from an object of Kos. As we explain below, once one has carried through these calculations at the level of objects, the framework of dg categories gives the equivalence of the moduli essentially 'for free'.

All tensor products in this section are taken over $\mathcal{O}_X$.

4.1 dg category of Koszul data

Let P denote the sheaf of rings on X consisting of 2×2 upper triangular matrices whose diagonal coefficients lie in $\mathcal{O}_X$ and whose upper-left entry lies in $\mathcal{D}^1_X$:

$$P = \left\{ \begin{pmatrix} f_1 & D \\ 0 & f_2 \end{pmatrix} \Big| f_1, f_2 \in \mathcal{O}_X, D \in \mathcal{D}^1 \right\}.$$

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This algebra is the version, for our noncommutative $\mathbf{P}^1$ bundle $\mathbb{P}_{\mathcal{D}}$, of the endomorphism (or self-Ext) algebra of the Beilinson generator $\mathcal{O}(-1) \oplus \mathcal{O}$ on $\mathbf{P}^1$ (see below).

We let Kos denote the abelian category consisting of quasicoherent sheaves M together with a map $P \otimes_{\mathcal{O}_X} M \to M$ that defines a P-module structure (and whose morphisms are P-linear maps). It is immediate that specifying such a module is the same as giving a pair of quasicoherent $\mathcal{O}_X$-modules M_{-1}, M_0 (our choice of labelling will seem more natural in light of Theorem 4.3) together with an $\mathcal{O}_X$-linear map $\mathcal{D}^1 \otimes_{\mathcal{O}_X} M_{-1} \xrightarrow{a} M_0$, an *action map*.

DEFINITION 4.1. The dg category of *Koszul data* is the dg derived category Kos of the abelian category Kos.

Objects of Kos are thus given by pairs (M_{-1}, M_0) of quasicoherent complexes on X together with an action map $a : \mathcal{D}^1 \otimes M_{-1} \to M_0$. Let $V \in D(X)$ be a coherent complex on X. An object (M_{-1}, M_0, a) of Kos is *V-framed* if it comes equipped with a quasi-isomorphism $\mathrm{Cone}(M_{-1} \xrightarrow{a|_{M_{-1}}} M_0) \simeq V$; here $a|_{M_{-1}}$ means the restriction of a to $M_{-1} \subset \mathcal{D}^1 \otimes M_{-1}$.

4.2 General equivalence theorem

Let $G = \pi\mathcal{R} \oplus \pi\mathcal{R}(1)$ in $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ (of course, this is the image of an object of $\mathcal{C}oh(\mathbb{P}_{\mathcal{D}})$). It follows from Corollary 3.15 that the sheaf of dg algebras $\mathrm{Hom}^{\bullet}_{D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})}(G, G)$ is canonically quasi-isomorphic to the sheaf of algebras P defined above (that is, concentrated in cohomological degree zero).

Given an object $\mathcal{M}$ of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$, we get a complex of (left) P-modules M by setting $M = \mathrm{Hom}^{\bullet}_{D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})}(G, \mathcal{M})$. It is instructive to apply this procedure to an output of the Beilinson functor $\mathbb{B}$, that is, to the total complex associated to a double complex

$$\mathcal{M} : \pi\mathcal{R}(-1) \otimes T_X \otimes M_{-1} \to \pi\mathcal{R} \otimes M_0 \tag{4.1}$$

(that is, M_{-1} and M_0 are quasicoherent complexes on X). From Corollary 3.15 we obtain the following result.

LEMMA 4.2. *For $\mathcal{M}$ as in (4.1),*

$$\mathrm{Hom}^{\bullet}(\pi\mathcal{R}(1), \mathcal{M}) \simeq M_{-1} \quad and \quad \mathrm{Hom}^{\bullet}(\pi\mathcal{R}, \mathcal{M}) \simeq M_0.$$

We now have the following theorem.

THEOREM 4.3. *We have the following.*

(1) *There is a quasi-equivalence of differential graded categories from $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ to Kos that takes an object $\mathcal{M}$ of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ to $\mathrm{Hom}^{\bullet}_{D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})}(G, \mathcal{M})$.*

(2) *Under this quasi-equivalence, if an object $\mathcal{M}$ of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ is identified with a triple $M = (M_{-1}, M_0, a)$ of Kos, then a choice of V-framing of $\mathcal{M}$ corresponds to a choice of V-framing of M.*

Proof. By definition of quasi-equivalence [Kel06], it suffices to define a functor of dg categories that induces an equivalence of the homotopy categories (as triangulated categories). The functor described above at the level of objects defines a functor from $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ to Kos by a standard procedure (see [Kel94, § 4] or [Kel, § 8.7]). Moreover, by Theorem 4.3 of [Kel94] (see also the theorem in [Kel, § 8.7]), if X is affine (so that the category Kos is a (derived) module category for a ring, not just a sheaf of rings) then, to conclude that this functor induces an equivalence of the homotopy categories, it suffices to check that the compact object G is a generator of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$. However, this is immediate from Theorem 3.8. Finally, for projective X, we obtain a compatible collection of such functors for all open subsets of X. By the above discussion, these functors are quasi-equivalences for all

affine open subsets of X. It then follows from effective descent for dg categories that the functor $\mathrm{Hom}^\bullet(G,-)$ gives a quasi-equivalence over X as well. More precisely, the dg categories $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ and Kos are homotopy limits of the categories over affine open subsets of X – they are obtained by totalizing the cosimplicial dg categories associated to the Čech nerve of an affine cover. (See [BD, §7.4] or [HS98, §21] for descent of dg categories of sheaves.)

We now prove the second part of the theorem. We first note the following. Consider the complex of bigraded $\mathcal{R}$-bimodules δ in §3.1. Take the associated graded complex with respect to the left-hand grading. The description of the map $\overline{\alpha}$ in terms of α_1 and α_2 shows that, after passing to associated grades with respect to the left-hand grading, $\overline{\alpha}$ is identified with $\alpha_1 \otimes t$.

Consider now a double complex of the form (4.1), an output of the functor $\mathbb{B}$; by Lemma 4.2, the corresponding Koszul data are (M_{-1}, M_0, a). Applying i^*_∞ to (4.1) gives

$$[\pi(\mathrm{Sym}\, T_X)(-1) \otimes T_X \otimes M_{-1} \longrightarrow \pi\,\mathrm{Sym}(T_X) \otimes M_0].$$

By the conclusion of the previous paragraph, this map is just the tensor product of the identity map $\pi\,\mathrm{Sym}\, T_X(-1) \otimes T_X \to \mathrm{Sym}\, T_X$ with $a|_{M_{-1}}$. Under the identification of $\mathcal{Q}coh(X)$ with $\mathrm{Qgr}\,\mathrm{Sym}(T_X)$ (more precisely, of their derived categories), this is then identified with $a|_{M_{-1}}$. Part (2) of the theorem is then immediate from the two definitions of V-framing. $\square$

4.3 Calculating in Kos

Let V be a sheaf on X. We want to calculate some V-framed objects of Kos that have a particularly simple form: their components M_{-1} are (quasi-isomorphic to) sheaves in cohomological degree one.

Suppose that M is a V-framed object of Kos. As we have remarked above, since our sheaf of rings P has two projections onto $\mathcal{O}_X$, M gives two quasicoherent complexes M_{-1} and M_0; more precisely, we obtain two functors to $D_{\mathrm{dg}}(X)$. Suppose that M_{-1} is quasi-isomorphic to a sheaf in cohomological degree 1. Letting $Q = H^1(M_{-1})$, we then obtain a canonical quasi-isomorphism $M_{-1} \to Q[-1]$. Furthermore, the V-framing of M, that is, the choice of quasi-isomorphism $\mathrm{Cone}(M_{-1} \to M_0) \simeq V$, yields a choice of quasi-isomorphism $M_0 \to \mathrm{Cone}(V \to Q)$ (with V in cohomological degree zero), that is, up to quasi-isomorphism M_0 corresponds to a choice of morphism $V \xrightarrow{i} Q$. The action map $a : \mathcal{D}^1 \otimes M_{-1} \to M_0$ then yields a unique map $a : \mathcal{D}^1 \otimes Q[-1] \to \mathrm{Cone}(V \to Q)$ in $D_{\mathrm{dg}}(X)$. This map has the property that the composite

$$Q[-1] \to \mathcal{D}^1 \otimes Q[-1] \to \mathrm{Cone}(V \to Q) \tag{4.2}$$

equals the 'natural' map $Q[-1] \to \mathrm{Cone}(V \to Q)$.

Suppose, for the moment, that Q is a quasicoherent sheaf on X, $V \xrightarrow{i} Q$ is a choice of homomorphism, and $a : \mathcal{D}^1 \otimes Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$ is a choice of map in $D_{\mathrm{dg}}(X)$ whose composite (4.2) equals the natural map $Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$. We refer to (Q, i, a) as a *Koszul triple* on X. We define the set of *morphisms* of triples $(Q, i, a) \to (Q', i', a')$ to consist of morphisms $Q \xrightarrow{\phi} Q'$ so that the diagram

$$
\begin{array}{ccc}
\mathcal{D}^1 \otimes Q[-1] & \xrightarrow{\ a\ } & \mathrm{Cone}(V \xrightarrow{i} Q) \\
\Big\downarrow{\scriptstyle 1_{\mathcal{D}^1} \otimes \phi} & & \Big\downarrow{\scriptstyle 1_V \times \phi} \\
\mathcal{D}^1 \otimes Q'[-1] & \xrightarrow{\ a'\ } & \mathrm{Cone}(V \xrightarrow{i'} Q)
\end{array}
$$

commutes in $D_{\mathrm{dg}}(X)$. This defines a category of triples (Q, i, a). Similarly, we let $\mathsf{Kos}^0(V)$ denote the category of V-framed objects M in Kos such that M_{-1} is quasi-isomorphic to a sheaf in cohomological degree one; Homs are morphisms in Kos that are compatible with the V-framing.

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PROPOSITION 4.4. *There is an equivalence of categories between:*

(1) *the category* $\mathsf{Kos}^0(V)$;

(2) *The category of triples* (Q, i, a).

Proof. The above constructions define a functor from $\mathsf{Kos}^0(V)$ to the category of Koszul triples. It is faithful, since our analysis of M_{-1} and M_0 shows that any map of Koszul data $M \to M'$ is determined by the induced map $Q = H^1(M_{-1}) \to H^1(M'_{-1}) = Q'$.

To see that this functor is full and essentially surjective, we do the following. Given (Q, i, a), choose representatives M_{-1} and M_0 of $Q[-1]$ and $V \to Q$ in $D_{\mathrm{dg}}(X)$, respectively, for which a lift $a : \mathcal{D}^1 \otimes M_{-1} \to M_0$ exists (for example, choose a complex of injectives representing M_0). Applying the functor to the Koszul data (M_{-1}, M_0, a) returns (Q, i, a), that is, the functor is essentially surjective. Furthermore, given another triple (Q', i', a'), we may choose complexes of injectives M'_{-1} and M'_0 quasi-isomorphic to Q' and $V \to Q'$, so that a' lifts to an action map $\mathcal{D}^1 \otimes M'_{-1} \xrightarrow{a'} M'_0$ and (M'_{-1}, M'_0, a') is mapped to (Q', i', a'). Then any map $Q \to Q'$ lifts to a map $M_{-1} \xrightarrow{\phi} M'_{-1}$. It then suffices to choose a map $M_0 \to M'_0$ compatible with ϕ and the given quasi-isomorphisms to V, but the existence of such a map is guaranteed by injectivity of M'_0 using an exact sequence argument. $\square$

We use this description in Theorem 4.6.

4.4 Moduli of perverse bundles

Fix a vector bundle V on X.

DEFINITION 4.5. Let $\mathsf{PS}(X, V)$ denote the moduli stack of pairs (Q, i) consisting of a coherent torsion sheaf Q on X and a homomorphism $i : V \to Q$; we call the component parametrizing pairs (Q, i) with Q of length n the *nth perverse symmetric power* of X.

In the case $\mathcal{D}^1 = \mathcal{O}_X \oplus T_X$ (commutative), we let $\mathsf{PB}(X, V)$ denote the moduli stack of V-framed perverse $\mathcal{D}$-bundles. In the case when $\mathcal{D}^1 = \mathcal{D}^1_X$ consists of first-order differential operators, we let $\mathsf{CM}(X, V)$ denote the moduli stack of V-framed perverse $\mathcal{D}$-bundles.

We now describe the moduli stacks $\mathsf{PB}(X, V)$ and $\mathsf{CM}(X, V)$ in terms of stacks that 'live over' $\mathsf{PS}(X, V)$.

THEOREM 4.6. *The moduli stacks for the following data are isomorphic:*

(1) *V-framed perverse $\mathcal{D}$-bundles $\mathcal{M}$ with second Chern class c_2;*

(2) *triples (Q, i, a) where Q is a torsion sheaf on X of length c_2, $i : V \to Q$ is a homomorphism of $\mathcal{O}$-modules, and*

$$a : \mathcal{D}^1 \otimes Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$$

is a morphism in the dg derived category of $\mathcal{O}$-modules such that the induced morphism

$$\mathcal{O} \otimes Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$$

is natural.

Remark 4.7. It is instructive to consider what the theorem says in case $\mathcal{R}$ is commutative and $\mathcal{M}$ is a rank-one torsion-free sheaf. Let X_∞ denote the 'divisor at infinity' in $S = \mathrm{Proj}(\mathcal{R})$ and $i_\infty : X_\infty \to S$ the inclusion. Let $p : S \to X$ denote the projection. Note that $\mathrm{Qgr}(\mathcal{R}) \simeq \mathcal{Q}coh(S)$.

By condition (2) of Definition 2.11, $\mathcal{M}$ comes equipped with an injective homomorphism $\mathcal{M} \hookrightarrow p^*V$. We let $\mathcal{Q}$ denote the cokernel; this gives an exact sequence

$$0 \to \mathcal{M} \to p^*V \to \mathcal{Q} \to 0. \tag{4.3}$$

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We have $M_0 = \mathbb{R}p_*(\mathcal{M})$ and $M_{-1} = \mathbb{R}p_*(\mathcal{M}(-X_\infty))$. We also set $Q = p_*\mathcal{Q}$; this is a torsion sheaf of length $c_2(\mathcal{M})$. Using the sequence (4.3), one finds that $M_0 \simeq [V \xrightarrow{i} p_*\mathcal{Q}]$ and $M_{-1} \simeq p_*\mathcal{Q}(-X_\infty) \cong Q$.

Proof of Theorem 4.6. In light of Proposition 4.4, it suffices to check that, under the correspondence of Theorem 4.3 above, the perverse $\mathcal{D}$-bundles correspond to Koszul data (M_{-1}, M_0, a) in which $M_{-1} = (p_1)_*R\omega(M(-1))$ is quasi-isomorphic to $Q[-1]$ for Q a torsion sheaf of length c_2.

The strategy for the proof, then, is simple: we start with a perverse bundle $\mathcal{M}$ and compute

$$\text{Hom}^\bullet(G, \mathcal{M}) = M_{-1} \oplus M_0 \simeq (p_1)_*R\omega(\mathcal{M}) \oplus (p_1)_*R\omega(\mathcal{M}(-1))$$

and show that it has the desired form. Then, going the other way, we start with a triple (Q, i, a) (equivalently, by Proposition 4.4, with framed Koszul data) and compute the corresponding object of $D_{\text{dg}}(\mathbb{P}_\mathcal{D})$.

Correspondence for $\mathbb{C}$-points. We first give the proof at the level of $\mathbb{C}$-valued points. Later we explain that this is already enough to give the equivalence in families, that is, an isomorphism of stacks.

Suppose first that $\mathcal{M}$ is a V-framed perverse $\mathcal{D}$-bundle. In order to check that the corresponding triple (M_{-1}, M_0, a) of Koszul data has the desired property, it suffices to compute the 'derived direct image' $(p_1)_*R\omega\mathcal{M}(-1)$ and prove that it is a torsion sheaf (in cohomological degree one) of length c_2. To carry out this calculation, we use the exact triangle

$$H^1(\mathcal{M})(-1)[-1] \to \mathcal{M}(-1) \to H^0(\mathcal{M})(-1) \xrightarrow{[+1]} \qquad (4.4)$$

in the triangulated derived category. By Lemma 2.13, the natural map

$$V \simeq i_\infty^*\mathcal{M}(-1) \to Li_\infty^* H^0(\mathcal{M})(-1)$$

is a quasi-isomorphism, and it follows that $Li_\infty^* H^1(\mathcal{M})(-1) = 0$: in other words, 'the zero-dimensional object $H^1(\mathcal{M})(-1)$ is supported away from X_∞'.

Write $H = \omega H^1(\mathcal{M})(-1)$. By Lemma 2.14 and the conclusion of the previous paragraph, $\ell(H_{\mathcal{R}(\mathcal{E})}) = 0$. Thus, using Formula (3.8) for the Čech complex $C(H)$, $C(H)$ is given by $C(H) \simeq H[t^{-1}]$ and $(p_1)_*C(H)$ is a torsion coherent sheaf on X of length c' (notation as in Definition 2.15). For $L = H^0(\mathcal{M})(-1)$, the argument of [BN06, Proposition 5.8] (more specifically, the exact sequence (5.2)) applies here to prove that $H^0((p_1)_*C(L)) = 0$ and $H^1((p_1)_*C(L))$ is a coherent $\mathcal{O}_X$-torsion sheaf of length c (notation as in Definition 2.15).

Taking account of the cohomological degrees of the 'direct images' of the left- and right-hand terms in (4.4), it is now immediate that $(p_1)_*R\omega(\mathcal{M}(-1))[1]$ is concentrated in a single cohomological degree. It follows that $(p_1)_*R\omega(\mathcal{M}(-1))[1]$ is a torsion $\mathcal{O}_X$-module of length $c + c' = c_2$, as desired. Summarizing, we have proven that a V-framed perverse $\mathcal{D}$-bundle $\mathcal{M}$ gives a triple of Koszul data as claimed in the statement of the theorem.

We now prove the converse: Koszul data that reduce, via Proposition 4.4, to a triple (Q, i, a) as in part (2) of the theorem correspond, under Theorem 4.3, to a perverse bundle with the prescribed c_2.

Suppose that we are given Koszul data (M_{-1}, M_0, a) that correspond to a triple (Q, i, a) as in part (2) of the present theorem. We set

$$\mathcal{M} = \text{Tot}[\pi\mathcal{R}(-1) \otimes T_X \otimes M_{-1} \longrightarrow \pi\mathcal{R} \otimes M_0], \qquad (4.5)$$

the object of $D_{\text{dg}}(\mathbb{P}_\mathcal{D})$ corresponding to (M_{-1}, M_0, a) (see Lemma 4.2, and note that $\pi\mathcal{R} \otimes M_0$ lies in cohomological degree zero, not one). To compute whether $\mathcal{M}$ is a perverse bundle, it suffices

to compute in the triangulated category, that is, we may compute cohomologies of $\mathcal{M}$ using the quasi-isomorphic object

$$\overline{\mathcal{M}} = \text{Tot}[\pi\mathcal{R}(-1) \otimes T_X \otimes Q[-1] \longrightarrow \pi\mathcal{R} \otimes \text{Cone}(V \to Q)] \qquad (4.6)$$

in the triangulated derived category (again, note the normalization, that the right-hand term $\pi\mathcal{R} \otimes \text{Cone}(V \to Q)$ of the double complex lies in 'horizontal cohomological degree zero'). In particular, one sees from this description of $\overline{\mathcal{M}}$ that $H^k(\mathcal{M}) = 0$ for $k \neq 0, 1$. Furthermore, over $U = X \setminus \text{supp}(Q)$, (4.6) immediately reduces to

$$\overline{\mathcal{M}}|_U = V \otimes \pi\mathcal{R}|_U; \qquad (4.7)$$

in particular, this implies the rank condition of part (2) of Definition 2.11: the 'locally free' part of the condition is immediate provided that $H^0(\mathcal{M})$ is a torsion-free object of $\mathcal{Q}coh(\mathbb{P}_\mathcal{D})$, which we prove below.

We next prove that $H^1(\mathcal{M})$ is zero-dimensional. By (4.7), $\omega H^1(\mathcal{M})$ is supported over $\text{supp}(Q) \subset X$. Moreover, applying i_∞^* to (4.6), we find that $i_\infty^* H^1(\mathcal{M})$ is the cokernel of a map

$$\pi(\text{Sym}\, T_X \otimes V) \oplus ((\text{Sym}\, T_X(-1)) \otimes T_X \otimes Q) \longrightarrow \pi\,\text{Sym}\, T_X \otimes Q. \qquad (4.8)$$

By an argument similar to the proof of part (2) of Theorem 4.3, this map is surjective; in particular, $i_\infty^* H^1(\mathcal{M}) = 0$. It follows that $\omega H^1(\mathcal{M})$ is $\mathcal{O}$-coherent. Since $H^1(\mathcal{M})$ is also $\mathcal{O}_X$-torsion (supported on $\text{supp}(Q)$), it is zero-dimensional.

By the previous paragraph and Lemma 2.13, $Li_\infty^* H^1(\mathcal{M}) = 0$ and, hence, since (4.8) comes equipped with a quasi-isomorphism to $\text{Sym}\, T_X \otimes V$, we obtain a quasi-isomorphism of $i_\infty^* \mathcal{M}$ with V. Hence, $\mathcal{M}$ is V-framed.

To see that $\mathcal{M}$ is a perverse bundle, it remains only to prove that $H^0(\mathcal{M})$ is torsion-free. The description (4.6) gives us an exact triangle

$$\pi\mathcal{R} \otimes \text{Cone}(V \to Q) \longrightarrow \overline{\mathcal{M}} \longrightarrow \pi\mathcal{R}(-1) \otimes T_X \otimes Q \xrightarrow{[1]}$$

(the lack of shifts may look unexpected, but results from the shifts implicit in our description of $\overline{\mathcal{M}}$ via (4.6)). The associated long exact sequence takes the form

$$0 \to H^0(\pi\mathcal{R} \otimes \text{Cone}(V \to Q)) \to H^0(\mathcal{M}) \to H^0(\pi\mathcal{R}(-1) \otimes T_X \otimes Q).$$

Applying ω, this remains left exact. Moreover, the left-hand term $\omega H^0(\pi\mathcal{R} \otimes \text{Cone}(V \to Q))$ is a subobject of $\mathcal{R} \otimes V$, hence is torsion-free. Consequently, if $\omega H^0(\mathcal{M})$ is not torsion-free, then its torsion submodule τ maps injectively to a submodule of

$$\omega H^0(\pi\mathcal{R}(-1) \otimes T_X \otimes Q) = \mathcal{R}(-1) \otimes T_X \otimes Q;$$

in particular, the torsion submodule is a direct summand of $\omega H^0(\mathcal{M})$ and is isomorphic to a submodule of $\mathcal{R}(-1) \otimes T_X \otimes Q$. Now $i_\infty^* H^0(\mathcal{M})$ is torsion-free, so $i_\infty^* \pi(\tau) = 0$. It follows that $t \cdot \tau_k = \tau_{k+1}$ for $k \gg 0$. However, it is clear that no nonzero submodule of $\mathcal{R} \otimes T_X \otimes Q$ has this property. This proves that $H^0(\mathcal{M})$ is torsion-free, thus completing the proof that $\mathcal{M}$ is a V-framed perverse $\mathcal{D}$-bundle.

We thus have the desired equivalence at the level of $\mathbb{C}$-valued points of the stack.

Correspondence in families. This is essentially a formal consequence of what we have already proven. More precisely, following [Lie06, Definition 2.1.8 and Proposition 2.1.9] (or, more generally, [TV05]) there is an intrinsic notion of S-object in a dg category (where S is a locally noetherian scheme). A quasi-equivalence of dg categories induces an equivalence of the associated categories fibered in groupoids over schemes. The only additional information that one would like is that, if one has a 'classical' notion of flat family, the two notions coincide.

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For families $\mathcal{M}$ of V-framed perverse $\mathcal{D}$-bundles for which $H^0(\mathcal{M}_s) = 0$ for every geometric point $s \in S$ this is straightforward (note, for example, that this applies to *every* family when $\mathcal{D}$ is a TDO algebra). Such a family certainly satisfies the condition $\mathrm{Ext}^i(\mathcal{M}_s, \mathcal{M}_s) = 0$ for $s \in S$ and $i < 0$ that appears in the definition [Lie06] of a universally gluable family. Conversely, suppose that $\mathcal{M}$ is a universally gluable S-object of $D_{\mathrm{dg}}(\mathbb{P}_{\mathcal{D}})$ all of whose geometric fibers $\mathcal{M}_s$ are (V-framed) perverse $\mathcal{D}$-bundles and such that $H^i(\mathcal{M}_s) = 0$ for $i \neq 0$ (note that here $\mathcal{M}_s$ means derived restriction). It follows by [Bri99, Lemma 4.3] that $H^i(\mathcal{M}) = 0$ for $i \neq 0$ and that $H^0(\mathcal{M})$ is S-flat. $\qquad\square$

5. CM spaces as twisted cotangent bundles

In this section we identify precisely the moduli stacks of perverse bundles PB on T^*X and $\mathcal{D}$-bundles CM and the relation between them. Recall (Definition 4.5) that $\mathsf{PS}(X, V)$ denotes the moduli stack of pairs (Q, i) consisting of a coherent torsion sheaf Q on X and a homomorphism $i : V \to Q$. For fixed (Q, i) we may now consider its possible extensions to data (Q, i, a) which by Theorem 4.6 describe either perverse bundles or $\mathcal{D}$-bundles. In the first case, a is completely determined by a map $\overline{a} : T_X \otimes Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$, while in the latter case a is a map $\mathcal{D}^1 \otimes Q[-1] \to \mathrm{Cone}(V \xrightarrow{i} Q)$ with fixed restriction to $\mathcal{O} \otimes Q[-1]$. It is clear that the latter data form a pseudo-torsor over the former: in other words, there is a simply transitive action of the perverse bundle structures on (Q, i) on the (possibly empty) set of $\mathcal{D}$-bundle structures on the same (Q, i) (both over $\mathbb{C}$ and in families).

It is easy to compute from the deformation theory of pairs (Q, i) that the data $\overline{a}$ are dual to the tangent space to $\mathsf{PS}(X, V)$ at (Q, i). However we would like to be more precise in the identification of the moduli of $\mathcal{D}$-bundles as (the stack analogue of) a twisted cotangent bundle in the sense of [BB93], that is, as (pseudo)torsor for the cotangent bundle equipped with a compatible symplectic form. Recall that two standard constructions of a twisted cotangent bundle are as the affine bundle of connections on a line bundle [BB93], and as a 'magnetic cotangent bundle' [OPRS93], a Hamiltonian reduction at a one-point coadjoint orbit. The two constructions agree in the case of reduction at a coadjoint orbit coming from a character of the group, which produces the affine bundle of connections on the line bundle obtained by descending the trivial line bundle with equivariant structure given by the character.

We describe both the perverse bundle and the $\mathcal{D}$-bundle spaces as Hamiltonian reductions at (zero and nonzero, respectively) integral one-point orbits from the cotangent bundle of the relevant Quot-type scheme. Recall from [BD, Section 1.2] that the very definition of the cotangent bundle of a stack is naturally given as a Hamiltonian reduction (at zero) of the cotangent bundle of a cover. Thus, perverse bundles naturally form the cotangent bundle of a Quot scheme, while $\mathcal{D}$-bundles form the space of connections of a line bundle which we identify with the dual of the determinant line bundle.

Remark 5.1. In addition to its concrete flavor, the description of PB and CM via Hamiltonian reduction from a Quot scheme also has an implicit advantage. Namely, it is easily adapted to capture the cotangent and twisted cotangent stacks PB and CM as derived (or dg) stacks, rather than naively as stacks.

Let $\widetilde{\mathsf{PS}}_n(X, V)$ denote the moduli space of triples $(Q, i, \overline{q})$ where $(Q, i) \in \mathsf{PS}_n(X, V)$ and $\overline{q} : \mathbb{C}^n \simeq H^0(Q)$ is a basis of sections of Q. The datum of $\overline{q}$ is equivalent to that of a surjection $q : \mathbb{C}^n \otimes \mathcal{O} \to Q$ (that induces an isomorphism $\mathbb{C}^n \to H^0(Q)$), so that $\widetilde{\mathsf{PS}}_n(X, V)$ is the fiber product (over length n sheaves Q) of $\mathsf{PS}_n(X, V)$ with an open set of the Quot scheme parametrizing n-dimensional quotients q. As an immediate consequence of this description one has the following result.

LEMMA 5.2. *The space $\widetilde{\mathsf{PS}}_n(X, V)$ is a smooth variety.*

Note that

$$\mathsf{PS}_n(X, V) = \widetilde{\mathsf{PS}}_n(X, V)/\operatorname{GL}_n,$$

where GL_n acts by changing the basis of $H^0(Q)$.

LEMMA 5.3. *The cotangent bundle of $\widetilde{\mathsf{PS}}_n(X, V)$ is the stack of quadruples (Q, i, q, r) with (Q, i, q) as above and*

$$r : Q[-1] \to \operatorname{Cone}(V \oplus \mathcal{O}^n \xrightarrow{(i,q)} Q) \otimes \Omega. \tag{5.1}$$

Proof. First suppose that X is projective. Then a standard calculation gives

$$T_{(Q,i)}\widetilde{\mathsf{PS}}_n(X, V) = \operatorname{Hom}(V \oplus \mathcal{O}^n \xrightarrow{(i,q)} Q, Q)$$

and the lemma follows by Serre duality. If X is only quasiprojective, we complete X to a projective curve Y (and extend V) and apply the above argument to Y; the lemma then follows by noting that all of our data (Q, i, q, r) are, in fact, supported on $X \subseteq Y$. $\square$

Remark 5.4. From this point forward, we may always assume that X is projective. Indeed, if not, then we may complete X to a projective curve Y and extend V to a vector bundle $\overline{V}$ on Y. Since $\mathsf{PS}(X, V)$ and $\widetilde{\mathsf{PS}}(X, V)$ are open substacks of $\mathsf{PS}(Y, \overline{V})$ and $\widetilde{\mathsf{PS}}(Y, \overline{V})$, respectively, the conclusions of Proposition 5.6 and Theorem 5.9 will apply to X as well.

LEMMA 5.5. *The moment map for the action of GL_n on $T^*\widetilde{\mathsf{PS}}_n(X, V)$ lifting the action on $\widetilde{\mathsf{PS}}_n(X, V)$ is the map*

$$T^*\widetilde{\mathsf{PS}}_n(X, V) \to \operatorname{Hom}(\mathcal{O}^n[-1], \mathcal{O}^n \otimes \Omega) \simeq \mathfrak{gl}_n^*$$

assigning to (Q, i, q, r) the composition

$$\mathcal{O}^n[-1] \xrightarrow{q[-1]} Q[-1] \xrightarrow{r} \operatorname{Cone}(V \oplus \mathcal{O}^n \xrightarrow{(i,q)} Q) \otimes \Omega \xrightarrow{\operatorname{can}} \mathcal{O}^n \otimes \Omega,$$

where can is the canonical map from the cone of a morphism to (a summand of) its source.

PROPOSITION 5.6. *The stack $\mathsf{PB}_n(X, V)$ is canonically identified with the Hamiltonian reduction*

$$T^*\widetilde{\mathsf{PS}}_n(X, V)/\!\!/_0 \operatorname{GL}_n$$

of $T^*\widetilde{\mathsf{PS}}_n(X, V)$ at the moment value zero, that is, to the cotangent stack to $\mathsf{PS}_n(X, V)$.

Proof. Maps r as in (5.1) that satisfy the vanishing moment condition define and are defined by maps

$$\overline{a} : Q[-1] \to \operatorname{Cone}(V \xrightarrow{i} Q) \otimes \Omega.$$

The proposition then follows from Theorem 4.6. $\square$

LEMMA 5.7. *There are canonical morphisms*

$$\operatorname{Ext}^1(Q, Q \otimes \Omega) \leftarrow \operatorname{Ext}^1(Q, \mathcal{O}^n \otimes \Omega) \to \operatorname{Ext}^1(\mathcal{O}^n, \mathcal{O}^n \otimes \Omega). \tag{5.2}$$

The left-hand map is surjective; the right-hand map is an isomorphism.

Proof. First observe that the two right-hand vector spaces have dimension n^2. The left arrow is surjective due to the vanishing of Ext^2. The right arrow is dual to the map $\operatorname{Hom}(\mathcal{O}^n, Q) \leftarrow \operatorname{Hom}(\mathcal{O}^n, \mathcal{O}^n)$, which is injective (hence, an isomorphism) since the kernel of $\mathcal{O}^n \to Q$ has no global sections. $\square$

Let $\mathcal{J}^1$ denote the sheaf of one-jets of functions on X. Let $[X] \in H^1(\Omega^1)$ be the canonical nonzero class (that is, the fundamental class of X).

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Lemma 5.8. *The image of the class* $[X] \otimes \mathcal{O}^n \in \operatorname{Ext}^1(\mathcal{O}^n, \mathcal{O}^n \otimes \Omega)$ *under the map* (5.2) *equals* $[\mathcal{J}^1(Q)] \in \operatorname{Ext}^1(Q, Q \otimes \Omega)$.

Proof. In the case $Q = \mathcal{O}_p$, $p \in X$ the assertion follows immediately from the observation that the tautological extension of $\mathcal{O}_p$ by $\mathcal{O}_p \otimes \Omega$, Serre dual to the identity map of $\mathcal{O}_p$, is given by one-jets. This class is identified using (5.2) (and the functoriality of Serre duality) with the Serre dual to the identity map of $\mathcal{O}_X$, that is, the fundamental class $[X]$. The statement of the lemma is now immediate for $Q = \bigoplus_i \mathcal{O}_{p_i}$, where $p_1, \ldots, p_n$ are distinct points of X, and follows for any cyclic module Q by continuity (note that the dimension of $\operatorname{Ext}^1(Q, Q \otimes \Omega)$ is n for any such Q). The general case now follows by splitting Q as a direct sum of cyclic modules. $\square$

We now perform hamiltonian reduction from $T^*\widetilde{\mathsf{PS}}_n(X, V)$ at a nontrivial moment value, corresponding to the dual of the determinant character of GL_n. More precisely, the extension class

$$[X]^n := [X] \otimes \mathcal{O}^n \in \operatorname{Hom}(\mathcal{O}^n[-1], \mathcal{O}^n \otimes \Omega) = \mathfrak{gl}_n^*$$

corresponds to the determinant character; we take the hamiltonian reduction of $T^*\widetilde{\mathsf{PS}}_n(X, V)$ at $-[X]^n$.

Theorem 5.9. *The CM stack* $\mathsf{CM}_n(X, V)$ *is isomorphic, as a pseudotorsor over* $\mathsf{PB}_n(X, V) = T^*\mathsf{PS}_n(X, V)$, *to the hamiltonian reduction*

$$T^*\widetilde{\mathsf{PS}}_n(X, V)/\!\!/_{-[X]^n} \operatorname{GL}_n,$$

that is, to the twisted cotangent bundle to $\mathsf{PS}_n(X, V)$ *associated to the dual determinant line bundle.*

Proof. Let $(Q, i, q) \in \widetilde{\mathsf{PS}}_n(X, V)$ be a framed torsion sheaf with a basis. We wish to show the equivalence of CM data $a : Q \otimes \mathcal{D}^1[-1] \to \operatorname{Cone}(V \to Q)$, satisfying the condition on its restriction to zeroth-order differential operators, and of cotangent data r as above, satisfying the moment condition.

Since $\mathcal{J}^1$ is the dual of $\mathcal{D}^1$, the data of the map a is equivalent to that of a map

$$a_1 : Q[-1] \to \operatorname{Cone}(\mathcal{J}^1 V \xrightarrow{\mathcal{J}^1(i)} \mathcal{J}^1 Q)$$

such that the composite map

$$Q[-1] \to \operatorname{Cone}(\mathcal{J}^1 \otimes (V \to Q)) \to \operatorname{Cone}(V \to Q)$$

is the standard one ('inclusion of Q'). The datum of a_1 is similarly equivalent to the choice of a map

$$a_2 : Q[-1] \to \operatorname{Cone}(V \otimes \Omega \xrightarrow{\mathcal{J}^1(i)|_{V \otimes \Omega}} \mathcal{J}^1 Q)$$

such that the composite

$$a_2 : Q[-1] \to \operatorname{Cone}(V \otimes \Omega \xrightarrow{\mathcal{J}^1(i)|_{V \otimes \Omega}} \mathcal{J}^1 Q) \to Q[-1]$$

is the identity.

Next, note that our condition on a_2 implies that the composite

$$Q[-1] \xrightarrow{a_2} \operatorname{Cone}(V \otimes \Omega \xrightarrow{\mathcal{J}^1(i)|_{V \otimes \Omega}} \mathcal{J}^1 Q) \to \operatorname{Cone}(Q \otimes \Omega \to \mathcal{J}^1(Q)) \tag{5.3}$$

is inverse to the canonical quasi-isomorphism $\operatorname{Cone}(Q \otimes \Omega \to \mathcal{J}^1(Q)) \simeq Q[-1]$. Consequently, if we compose (5.3) with the further projection to $Q \otimes \Omega$, the resulting class in $\operatorname{Hom}(Q[-1], Q \otimes \Omega)$ is exactly $[\mathcal{J}^1(Q)]$. Let $\overline{a}_2 : Q[-1] \to V \otimes \Omega$ be the result of composing a_2 with the projection on $V \otimes \Omega$. The above then shows that the composite

$$Q[-1] \xrightarrow{\overline{a}_2} V \otimes \Omega \xrightarrow{i \otimes \Omega} Q \otimes \Omega$$

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also gives $[\mathcal{J}^1(Q)] \in \text{Hom}(Q[-1], Q \otimes \Omega) = \text{Ext}^1(Q, Q \otimes \Omega)$.

We now define a map

$$\widetilde{r} : Q[-1] \to (V \otimes \Omega) \oplus (\mathcal{O}^n \otimes \Omega)$$

by $\widetilde{r} = (\overline{a}_2, -q^\vee)$, where $q^\vee \in \text{Ext}^1(Q, \mathcal{O}^n \otimes \Omega)$ corresponds to $[X] \otimes \mathcal{O}^n$ and maps to $\mathcal{J}^1 Q$ under the first map of (5.2). It follows that the composite

$$Q[-1] \xrightarrow{\widetilde{r}} (V \otimes \Omega) \oplus (\mathcal{O}^n \otimes \Omega) \xrightarrow{(i+q)\otimes\Omega} Q \otimes \Omega$$

is

$$(i \otimes \Omega) \circ \overline{a}_2 - (q \otimes \Omega) \circ q^\vee = [\mathcal{J}^1(Q)] - [\mathcal{J}^1(Q)] = 0.$$

As a result, we obtain a lift r of $\widetilde{r}$ to $\text{Cone}((i+q)\otimes\Omega)$. To see that r satisfies the moment condition, we compute that

$$\text{can} \circ r \circ q[-1] = -q^\vee \circ q[-1],$$

which equals $-[X]^n$ by Lemma 5.8.

Conversely, given r as above, we find that its restriction to a map $Q[-1] \to V \otimes \Omega$ composes with $q \otimes \Omega$ to define the one-jet map $Q[-1] \to Q \otimes \Omega$. It follows that we may use it to define a map a_2 as above, and hence a map a satisfying the restriction condition as desired.

Clearly all of the identifications are GL_n-equivariant and preserve the torsor structures over the cotangent bundle of $\mathsf{PS}_n(X, V)$. The theorem follows. $\square$

6. Explicit description of CM matrices for $\mathbf{A}^1$

In this section we illustrate in detail how our description of $\mathcal{D}$-bundles specializes, in the case of the affine line, to the familiar CM space description. In other words we want to spell out the correspondence between:

(1) quadruples

$$(X, Y, i, j) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^*$$

satisfying the CM relation $[X, Y] - ij + I = 0$ (which we call 'CM quadruples') up to simultaneous conjugation; and

(2) our data for $\mathcal{O}$-framed $\mathcal{D}$-bundles on $\mathbf{A}^1$ with local second Chern class n, consisting of a length n torsion sheaf Q, a framing map $\mathcal{O} \to Q$, and an action map $\mathcal{D}^1 \otimes Q[-1] \to \text{Cone}(\mathcal{O} \to Q)$ (defined in the derived category, that is, up to homotopy) that agrees with the usual map on $\mathcal{O} \otimes Q \subset \mathcal{D}^1 \otimes Q$.

To summarize the relationship concisely: the matrix X describes the sheaf Q, the vector i describes the framing map (justifying our general notation for the latter), and Y, j provide the components of the action morphism a.

Remark 6.1. It is important to note that a CM quadruple corresponds to a unique triple (Q, i, a) but where a is only well defined *up to homotopy*. What we find below is that, once we replace $Q[-1]$ by a resolution, we can produce a canonical representative of the homotopy class of (Q, i, a), and from this we can read off the CM quadruple completely explicitly.

Choosing a basis for $H^0(Q)$, we may identify Q with $\mathbb{C}^n$. A choice of $\mathbb{C}[x]$-module structure on $\mathbb{C}^n$ is given by an $n \times n$ matrix X, and we then obtain a presentation of Q by

$$0 \to \mathbb{C}[x]^n \xrightarrow{x-X} \mathbb{C}[x]^n \to Q \to 0 \tag{6.1}$$

where the last map takes $f(x) \otimes v$ to $f(X)v$ for $v \in \mathbb{C}^n = Q$. Our Koszul data, then, take the following form.

$$
\begin{array}{ccc}
\mathcal{D}^1 \otimes \mathbb{C}[x]^n & \xrightarrow{\underline{a}_0} & \mathbb{C}[x] \\
\downarrow{\scriptstyle 1 \otimes (x-X)} & {\scriptstyle \underline{i}} & \downarrow \\
\mathcal{D}^1 \otimes \mathbb{C}[x]^n & \xrightarrow{\underline{a}_1} & \mathbb{C}^n
\end{array}
\tag{6.2}
$$

Since all maps are required to be $\mathbb{C}[x]$-linear, the map $\underline{a}_0$ is determined by its values on vectors in $\mathbb{C}^n$ and elements of the form $\partial \otimes v$ for $v \in \mathbb{C}^n$. Since we require the morphism of complexes to give the standard one on $\mathcal{O} \otimes Q$, we find that $\underline{a}_0(v) = 0$ for $v \in \mathbb{C}^n$; we write $\underline{a}_0(\partial \otimes v) = j(x)v$ where $j(x) : \mathbb{C}^n \to \mathbb{C}[x]$ is a $\mathbb{C}$-linear map. This completely determines $\underline{a}_0$. Similarly, we may write $\underline{i} = \underline{i}(1)$, which determines $\underline{i}$. By (6.1), $\underline{a}_1(f(x) \otimes v) = f(X)v$. We may write $\underline{a}_1(\partial \otimes v) = Yv$ for all $v \in \mathbb{C}^n$, where Y is an appropriate $n \times n$ matrix. Applying $\underline{a}_1 \circ (1 \otimes (x - X))$ and $\underline{i} \circ \underline{a}_0$ to an element of the form $\partial \otimes v$, we may compute that the square (6.2) commutes if and only if the following equation is satisfied:

$$
I + XY - YX = i \cdot j(x).
\tag{6.3}
$$

Hence, our Koszul data are determined by a choice of $(X, Y, i, j(x))$ satisfying this equation. It is immediate that one has the following converse: given a CM quadruple (X, Y, i, j) satisfying the CM relation, the diagram of complexes defined as above (with $j(x) = j$) gives our $\mathcal{D}$-bundle data.

It remains to show that, the diagram corresponding to a quadruple $(X, Y, i, j(x))$ is homotopic to a unique diagram corresponding to a CM quadruple. A homotopy of our complex is a diagonal 'lower left to upper right' map $h : \mathcal{D}^1 \otimes \mathbb{C}[x]^n \to \mathbb{C}[x]$. We first choose this map to vanish on elements $f(x) \otimes v$, $v \in \mathbb{C}^n$, and to take the form $h(\partial \otimes v) = h_\partial(x)(v)$ for a linear map $h_\partial(x) : \mathbb{C}^n \to \mathbb{C}[x]$. A computation then shows that, modifying our original complex by the homotopy, the data $(X, Y, i, j(x))$ are replaced by

$$
(X, Y + h_\partial(X), i, j(x) + xh_\partial(x) - h_\partial(x) \cdot X).
$$

In particular, one may check that, writing $j(x) = j + xj'$ where $j' : \mathbb{C}^n \to \mathbb{C}[x]$ is a linear map, one can solve the equation $j(x) + xh_\partial - h_\partial(x) \cdot X = j$ for $h_\partial(x)$. Consequently, up to homotopy, we can replace $(X, Y, i, j(x))$ by a CM quadruple giving the same homotopy class of Koszul data.

To complete the proof, we start from a CM quadruple (X, Y, i, j) and consider an arbitrary homotopy h. As before, we write $h_\partial(x)(v)$ for $h(\partial \otimes v)$. The formula above shows that, after modifying by the homotopy, j is replaced by $j + xh_\partial(x) - h_\partial(x) \cdot X$, which, in particular, has degree greater than zero in the variable x unless $h_\partial = 0$. So if modifying (X, Y, i, j) by h gives another CM quadruple (X, Y', i, j'), then $h_\partial = 0$. Moreover, it is easy to check that the new map j' obtained after applying h satisfies $j'(v) = xh(v) - h(Xv)$. Writing the linear map $h : \mathbb{C}^n \to \mathbb{C}[x]$ as a polynomial $\sum_{i=0}^{k} x^i h_i$ with covector coefficients h_i, we find that if $h_k(v) \neq 0$, then $\deg xh(v) = k + 1 > \deg h(Xv)$. Since we require $j'(v) = 0$ for all $v \in \mathbb{C}^n$, we conclude that $h(v) = 0$ for all $v \in \mathbb{C}^n$. Together with the vanishing of h_∂, this proves that $h = 0$, that is, there is a unique CM quadruple in a given homotopy class of our Koszul data. This completes the correspondence.

over free algebras). Indeed, one of the goals of this paper was to place their work in a general dg context and bypass the explicit use of free algebras, making possible our generalization to arbitrary (complexes of) $\mathcal{D}$-modules on curves. The authors are grateful to Yuri Berest for very helpful discussions regarding [BC07].

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