The Journal of Symbolic Logic
Volume 86, Number 4, December 2021

$G_{\delta\sigma}$ GAMES AND INDUCTION ON REALS

J. P. AGUILERA AND P. D. WELCH

Abstract. It is shown that the determinacy of $G_{\delta\sigma}$ games of length ω^2 is equivalent to the existence of a transitive model of $\mathsf{KP} + \mathsf{AD} + \Pi_1\text{-MI}_{\mathbb{R}}$ containing $\mathbb{R}$. Here, $\Pi_1\text{-MI}_{\mathbb{R}}$ is the axiom asserting that every monotone Π_1 operator on the real numbers has an inductive fixpoint.

§1. Introduction. The purpose of this article is to compute the reverse-mathematical strength of the assertion that all $\mathbf{\Sigma}^0_3$ Gale-Stewart games on $\mathbb{N}$ of length ω^2 are determined (Theorem 1.1).

That $\mathbf{\Sigma}^0_3$ games on $\mathbb{N}$ of length ω are determined is a theorem of Davis [7] from 1964. This was also the last natural pointclass within the arithmetical hierarchy of such sets the determinacy of which could be proved in analysis, that is, second order number theory, for Martin (improving a result of Friedman [8]) showed that the determinacy of ω-length games with payoff sets in the class $\mathbf{\Sigma}^0_4$ could not be proven in analysis. Games of length ω played with real moves are a different matter, where we identify a real with an element of Baire space $\mathbb{N}^{\mathbb{N}}$. Essentially then these games are equivalent to a particular kind of games of ω^2-many moves on $\mathbb{N}$. By the proof of Martin's Borel determinacy theorem [14], $\mathbf{\Sigma}^0_3$-determinacy for games on $\mathbb{R}$ is provable in third-order arithmetic, and similarly by adapting the arguments for games on $\mathbb{N}$ one can show that $\mathbf{\Sigma}^0_4$-determinacy for games on $\mathbb{R}$ is not. Determinacy for all games of length ω^2 with moves in $\mathbb{N}$ and payoffs in these pointclasses is stronger, however. The first author has established equivalences for *open* (or $\mathbf{\Sigma}^0_1$), F_σ (or $\mathbf{\Sigma}^0_2$; see [3]), and *Borel* (or $\mathbf{\Delta}^1_1$; see [1]) games of length ω^2, and this article provides the analogue for $G_{\delta\sigma}$ (or $\mathbf{\Sigma}^0_3$) games.

Winning strategies (for either player) for the effective (so lightface) versions of the Σ^0_1 and Σ^0_2 classes were shown to be definable over $L_{\omega_1^{ck}}$ (Kleene and Moschovakis), and to belong to the next admissible set after the closure ordinal of monotone-Σ^1_1 inductive definitions (Solovay) respectively, in the 1970s. The second author located [19] the strategies for Σ^0_3-games as being definable over the least level L_β which supported a *nesting* (Definition 3.1). Hachtman, [9], then took this nesting concept and showed that the L_β that supported nestings were models of $\mathbf{\Pi}^1_2$-monotone induction. It is a feature of the argument below for the longer games, that we use in addition Hachtman's characterisation. A fourth and final equivalence to the least such nesting ordinal can be given in terms of the definability of the complete

Received March 29, 2020.
2020 *Mathematics Subject Classification*. 03B30, 03D60, 03D70, 03E60, 91A44.
Key words and phrases. determinacy, games on reals, inductive definitions.

0022-4812/21/8604-0016
DOI:10.1017/jsl.2021.66

semi-decidable set for a notion of higher type recursion in ${}^2\mathsf{E}$. This is due to the second author and shall appear elsewhere.

The main theorem we prove is:

THEOREM 1.1. *The following are equivalent over* ZFC:

(1) $\mathbf{\Sigma}^0_3$ *games of length* ω^2 *with moves in* $\mathbb{N}$ *are determined.*
(2) *There is a transitive model of* $\mathsf{KP} + \mathsf{AD} + \Pi_1\text{-MI}_{\mathbb{R}}$ *which contains* $\mathbb{R}$.

Below, we adopt the convention that AD includes a clause asserting the existence of $\mathbb{R}$.

Here, $\Pi_1\text{-MI}_{\mathbb{R}}$ is the axiom stating that every monotone Π_1 operator on $\mathbb{R}$ has a least fixed point. Let us be more precise: a Π_1 *operator* on $\mathbb{R}$ is a function

$$\Phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}),$$

which is given by

$$X \mapsto \{x \in \mathbb{R} : \phi(x, X, a)\}$$

for some Π_1 formula ϕ in the language of set theory and some set a. We say Φ is *monotone* if

$$X \subset Y \text{ implies } \Phi(X) \subset \Phi(Y).$$

For such a Φ, we may inductively define

$$\Phi^0 = \Phi(\varnothing),$$

$$\Phi^\alpha = \Phi\left(\bigcup_{\beta<\alpha} \Phi^\beta\right).$$

Let Φ^∞ be the least fixed point of the operator Φ and let $|\Phi|$ denote the least ordinal α such that $\Phi^\infty = \Phi^\alpha$. The axiom of $\Pi_1\text{-MI}_{\mathbb{R}}$ says that for every monotone Π_1 operator on $\mathbb{R}$, the sequence $\{\Phi^\alpha : \alpha \leq |\Phi|\}$ exists. The axiom $\Pi_1\text{-MI}_{\mathbb{N}}$ is defined analogously, but in terms of operators on $\mathbb{N}$; over $\mathsf{KP} + \mathrm{V} = \mathrm{L}$, it is equivalent to $\mathbf{\Pi}^1_2$-MI, as long as $\mathcal{P}(\mathbb{N})$ does not exist (this is because $\mathbf{\Pi}^1_2$-MI admits only real parameters and we allowed arbitrary parameters in the definition of $\Pi_1\text{-MI}_{\mathbb{N}}$ – this simplifies some arguments but does not increase the consistency strength).

From Theorem 1.1, we can deduce that the theory ZFC+ "$\mathbf{\Sigma}^0_3$-determinacy for games of length ω^2" lies in consistency strength between the theories

$$\mathsf{KP} + \mathsf{AD} + \Sigma_1\text{-Separation}$$

and

$$\mathsf{KP} + \mathsf{AD} + \Sigma_2\text{-Separation}.$$

Theorem 1.1 should be regarded as an analogue of the corresponding result for games of length ω:

THEOREM 1.2 (Hachtman [9]). *The following are equivalent over* $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$:

(1) Σ^0_3 *games of length* ω *with moves in* $\mathbb{N}$ *are determined.*
(2) *There is a* β*-model of* $\mathsf{KP} + \Pi^1_2\text{-MI}$.

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Hachtman's theorem plays an important role in the proof of Theorem 1.1. As in previous proofs of determinacy for definable games of length ω^2 (see e.g., [1, 3, 4]), part of the argument consists in "lifting" the reversals for determinacy of games of length ω to locate the winning strategies for games on $\mathbb{R}$ in the $L(\mathbb{R})$-hierarchy. The main novelty here is that the stage by which these strategies appear is characterized in terms of a certain non-standard model of KP which cannot exist in V, but only in a generic extension of it. As part of the proof of the main theorem, we shall state a general upper bound for the complexity of these strategies, as well as for winning strategies for games of length ω^2 in some sufficiently closed subclass of the Borel sets (see Claim 4.5 within the proof of Theorem 4.1, as well as Remark 4.7).

§2. Monotone Induction in $L(\mathbb{R})$. We refer the reader to Barwise [5] for general background in admissible sets, to Moschovakis [17] for general background in descriptive set theory, and to Jech [10] for general background in forcing and constructibility.

The first step towards proving Theorem 1.1 is reducing the assertion that there is a transitive model of $\mathsf{KP} + \mathsf{AD} + \Pi_1\text{-}\mathsf{MI}_{\mathbb{R}}$ to this theory holding in a specific model:

Lemma 2.1 (KP). *Suppose Π_1-$MI_{\mathbb{R}}$ holds. Then,* $\mathsf{KP} + \Pi_1$-$MI_{\mathbb{R}}$ *holds in $L(\mathbb{R})$.*

Proof. That $L(\mathbb{R}) \models \mathsf{KP}$ is a theorem of KP. Let Φ be a Π_1 monotone operator in $L(\mathbb{R})$, say, given by a Π_1 formula ϕ and a set a. Consider the operator Φ' given by

$$X \mapsto \{x \in \mathbb{R} : L(\mathbb{R}) \models \phi(x, X, a)\},$$

i.e., by

$$X \mapsto \{x \in \mathbb{R} : \forall \beta \in \mathsf{Ord}\, L_\beta(\mathbb{R}) \models \phi(x, X, a)\}.$$

This is an operator of the form

$$X \mapsto \{x \in \mathbb{R} : \phi'(x, X, a, \mathbb{R})\}$$

for some Π_1 formula ϕ' and is monotone, so the sequence $\{(\Phi')^\alpha : \alpha \leq |\Phi'|\}$ exists. Moreover, the operator Φ' belongs to $L(\mathbb{R})$ and has the same least fixed point as Φ. Since all ordinals belong to $L(\mathbb{R})$ and $L(\mathbb{R}) \models \mathsf{KP}$, $\{(\Phi')^\alpha : \alpha \leq |\Phi'|\}$ belongs to $L(\mathbb{R})$, so $\Phi^\infty \in L(\mathbb{R})$, as desired. ⊣

Below, recall that DC denotes the Principle of Dependent Choices, stating that every tree with no terminal nodes has a branch. For a set X, DC_X denotes the restriction of DC to trees whose nodes are indexed by finite sequences of elements of X. In the case that $X = \mathbb{R}$ and $\mathsf{DC}_{\mathbb{R}}$ holds, branches through such trees can themselves be coded by reals, so $\mathsf{DC}_{\mathbb{R}}$ relativizes downwards to transitive inner models that contain $\mathbb{R}$.

Corollary 2.2. *The following are equivalent over* ZFC:

(1) *There is a transitive model of* $\mathsf{KP} + \mathsf{AD} + \Pi_1\text{-}\mathsf{MI}_{\mathbb{R}}$ *which contains all reals.*
(2) *Let β be least such that $L_\beta(\mathbb{R}) \models \Pi_1\text{-}\mathsf{MI}_{\mathbb{R}}$. Then $L_\beta(\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}$.*

Proof. To see that (2) implies (1), it suffices to show that $L_\beta(\mathbb{R})$ satisfies KP. First, by a reflection argument, we see that $L_\beta(\mathbb{R})$ satisfies "$\mathcal{P}(\mathbb{R})$ does not exist," so in $L_\beta(\mathbb{R})$ every set is a surjective image of $\mathbb{R}$. Then, we see that $L_\beta(\mathbb{R})$ satisfies

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(boldface) Σ_1-Separation for sets of reals and thus for arbitrary sets. Σ_1-Separation is true because it is equivalent to Π_1-Separation, and this follows from Π_1–$\mathsf{MI}_\mathbb{R}$ (instances of separation are inductions of length 1). To prove Δ_0-Collection, suppose that

$$L_\beta(\mathbb{R}) \models \forall x \in A\, \exists y\, \phi(x, y, p)$$

for some Δ_0 formula ϕ. By Σ_1-Separation, there is a transitive set $B \in L_\beta(\mathbb{R})$ such that $A, p \in B$ and

$$B \prec_1 L_\beta(\mathbb{R}).$$

(This follows e.g., from the argument of Steel [18, Lemma 1.11].) It follows that

$$L_\beta(\mathbb{R}) \models \forall x \in A\, \exists y \in B\, \phi(x, y, p),$$

so $L_\beta(\mathbb{R})$ satisfies Δ_0-Collection.

Conversely, suppose there is a transitive model M of $\mathsf{KP} + \mathsf{AD} + \Pi_1\text{-}\mathsf{MI}_\mathbb{R}$ which contains all reals. By Lemma 2.1,

$$L(\mathbb{R})^M \models \Pi_1\text{-}\mathsf{MI}_\mathbb{R}.$$

Since M contains all reals (hence all strategies for Gale-Stewart games on $\mathbb{N}$), it follows that $L(\mathbb{R})^M = L_{\mathsf{Ord}\cap M}(\mathbb{R})$ and that

$$L_{\mathsf{Ord}\cap M}(\mathbb{R}) \models \mathsf{AD}.$$

Letting β be least such that $L_\beta(\mathbb{R}) \models \Pi_1\text{-}\mathsf{MI}_\mathbb{R}$, we have $\beta \leq \mathsf{Ord} \cap M$, so

$$L_\beta(\mathbb{R}) \models \mathsf{AD}.$$

Since $L_\beta(\mathbb{R})$ contains all reals, it satisfies $\mathsf{DC}_\mathbb{R}$. By minimality, no $L_\gamma(\mathbb{R})$ is a model of separation, for $\gamma < \beta$, and, in fact, for all such γ, there is a subset of $\mathbb{R}$ not in $L_\gamma(\mathbb{R})$ which is definable over $L_\gamma(\mathbb{R})$. Thus, for every $\gamma < \beta$, there is a surjection

$$\rho : \mathbb{R} \twoheadrightarrow L_\gamma(\mathbb{R}),$$

which is definable over $L_\gamma(\mathbb{R})$ (see e.g., Steel [18]). Hence, every set in $L_\beta(\mathbb{R})$ is coded by a set of reals in $L_\beta(\mathbb{R})$, so, by $\mathsf{DC}_\mathbb{R}$,

$$L_\beta(\mathbb{R}) \models \mathsf{DC},$$

as desired. ⊣

§3. Generic Σ_2-Extendibility. The next step for proving Theorem 1.1 is locating the simplest winning strategies for $\boldsymbol{\Sigma}^0_3$ games on reals. The result requires some definitions:

Definition 3.1. Let $x \in \mathbb{R}$ and M be a model of $\mathsf{KP} + V = L[x]$. An *x-nesting on an ordinal β and M* is a sequence of pairs (ζ_n, s_n) for $n \in \mathbb{N}$ such that

(1) $\zeta_n < \zeta_{n+1} < \beta$ for all n,
(2) $L_\beta[x]$ is the wellfounded part of M,
(3) $M \models s_{n+1} < s_n$ for all n and each s_n belongs to the illfounded part of M, and
(4) $M \models L_{\zeta_n}[x] \prec_{\Sigma_2} (L_{s_n}[x])^M$ for all n.

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Given an ordinal β and a real x, if such a sequence exists for some model M, we say that β *admits an x-nesting*. Observe that if

$$\{(\zeta_n, s_n) : n \in \mathbb{N}\}$$

is an x-nesting on β and M, then

$$\beta \geq \sup_{n \in \mathbb{N}} \zeta_n.$$

Here we shall have that taking a least such β results in the supremum of the lower ends of the Σ_2-end-extensions being β itself.

LEMMA 3.2. *Suppose that β is least that supports an infinite x-nesting $\{(\zeta_n, s_n) : n \in \mathbb{N}\}$. Then $\beta = \sup_{n \in \mathbb{N}} \zeta_n$.*

PROOF. Suppose otherwise for a contradiction and that $\zeta := \sup_{n \in \mathbb{N}} \zeta_n < \beta$. (We shall drop mention of the uniform parameter x.) Note first that for such ζ_n. $L_{\zeta_n} \prec_{\Sigma_1} L_{\zeta}$.

Then, in M, for arbitrarily large $\tilde{\zeta} < \zeta$, also with $L_{\tilde{\zeta}} \prec_{\Sigma_1} L_{\zeta}$, there are $\tilde{s} > \zeta$ with $L_{\tilde{\zeta}} \prec_{\Sigma_2} L_{\tilde{s}}$. We note that any such $\tilde{s}$ must be in the illfounded part of M. Suppose not. By definition of ζ there will be some $\bar{\zeta} > \tilde{\zeta}$ with a Σ_2-end-extension to some $\bar{s}$ where $L_{\bar{\zeta}} \prec_{\Sigma_2} L_{\bar{s}}$ and $\bar{s}$ certainly in the illfounded part of M (just take some sufficiently large ζ_m). However now we have two overlapping Σ_2-extendible pairs with $\tilde{\zeta} < \bar{\zeta} < \tilde{s} < \bar{s}$.

Claim: $L_{\tilde{\zeta}} \prec_{\Sigma_2} L_{\bar{\zeta}} \prec_{\Sigma_2} L_{\bar{s}}$.

PROOF. We only need to justify the first substructure relation. This is an exercise. (Any Σ_2 statement about $\vec{x}$ from $L_{\tilde{\zeta}}$ true in $L_{\bar{\zeta}}$ upwardly persists in being true in L_τ for any $\tau \in [\bar{\zeta}, \bar{s}]$ and so in particular is true in $L_{\tilde{s}}$. By downward Σ_2-reflection it is true in $L_{\tilde{\zeta}}$.) ⊣

Claim: $\bar{\zeta}$ *admits an infinite nesting.*

PROOF. By a Barwise Compactness argument, let M be any illfounded model with wellfounded part equal to $L_{\bar{\zeta}}$. By the double Σ_2-end-extension configuration in the last claim we can repeatedly use Σ_2-reflection and have that $L_{\tilde{\zeta}}$ has arbitrarily large Σ_2-end-extensions L_τ with $\tau < \bar{\zeta}$. By overspill it has such end-extensions L_t for $t \in On^M$ with t in the illfounded part of M. However then it is easy to see that $\bar{\zeta}$ has an infinite nesting supported in M. ⊣

But the last Claim contradicts the choice of β. Hence we can recursively define an infinite sequence of nested pairs (ζ_n, s_n) in M confident that the upper end s_n is in the illfounded part of M, and hence the recursion is successful. ⊣

DEFINITION 3.3. Let $x \in \mathbb{R}$. We denote by β_x the least ordinal β such that all $\Sigma^0_3(x)$ games on $\mathbb{N}$ have a winning strategy definable over $L_\beta[x]$; we denote by α_x the least ordinal such that

$$L_{\alpha_x}[x] \prec_{\Sigma_1} L_{\beta_x}[x].$$

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THEOREM 3.4 [20]. *For every $x \in \mathbb{R}$, β_x is the least ordinal that admits an x-nesting.*

If $\{(\zeta_n, s_n) : n \in \mathbb{N}\}$ is an x-nesting on β and M, then $M \models L_{\zeta_n}[x] \prec_{\Sigma_2} (L_{s_n}[x])^M$ and hence $M \models L_{\zeta_n}[x] \prec_{\Sigma_1} (L_{s_n}[x])^M$. It follows that

$$L_{\zeta_n}[x] \prec_{\Sigma_1} L_\beta[x].$$

From this and Theorem 3.4, we see that $\alpha_x < \beta_x$ for all x.

We would like to generalise Theorem 3.4 to games on $\mathbb{R}$ and $L(\mathbb{R})$. For this, a natural candidate is the least ordinal β that satisfies Definition 3.1, but with M a model of $V = L(\mathbb{R})$ with $\mathbb{R}^M = \mathbb{R}$, and with condition (4) replaced by

$$L_{\zeta_n}(\mathbb{R}) \prec_{\Sigma_2} L_{s_n}(\mathbb{R})^M.$$

The problem with this approach is that such an ordinal cannot exist, because of a very general fact. Here, recall that Θ denotes the least ordinal such that there is no surjection from $\mathbb{R}$ onto Θ.

LEMMA 3.5. *Let M be an illfounded model of* KP $+$ *$V = L(\mathbb{R})$ with $\mathbb{R}^M = \mathbb{R}$ and let $L_\alpha(\mathbb{R})$ be the wellfounded part of M. If $\alpha < \Theta^M$, then $\omega <$ cof(α).*

PROOF. Assume towards a contradiction that $\mathrm{cof}(\alpha) = \omega$. Since $\alpha < \Theta^M$, for every M-ordinal β with $\alpha < \beta < \Theta^M$, there is a surjection

$$f : \mathbb{R} \twoheadrightarrow (L_\beta(\mathbb{R}))^M$$

in M. Choose such a β and such an f. Letting

$$\{\alpha_n : n \in \mathbb{N}\}$$

be cofinal in α and $n \in \mathbb{N}$, there is $x_n \in \mathbb{R}$ such that $f(x_n) = \alpha_n$. Because $\mathbb{R}^M = \mathbb{R}$, and the Axiom of Dependent Choices holds in V, there is some $x \in \mathbb{R}$ coding each x_n. By Δ_0-collection, the image of $\{x_n : n \in \mathbb{N}\}$ under f, i.e., the sequence

$$\{\alpha_n : n \in \mathbb{N}\},$$

belongs to M, which is a contradiction. ⊣

The solution is to look for nestings in generic extensions of V. Below, we denote by $\Sigma_1^{L_\alpha(\mathbb{R})}$ the pointclass of all sets which are Σ_1-definable over $L_\alpha(\mathbb{R})$, with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$. Given a pointclass Γ, we denote by $\Game^{\mathbb{R}}\Gamma$ the pointclass of all sets of the form

$$\{x : \text{Player I has a winning strategy in the game on } \mathbb{R} \text{ with payoff } \{y : (x, y) \in A\}\}$$

with $A \in \Gamma$. In the following lemma, we speak of filters $g \subset \mathrm{Coll}(\omega, \mathbb{R})$ which are $L(\mathbb{R})$-generic. Such a g may be identified with a real number which codes $\mathbb{R}$ and belongs to $L[g]$ and thus one may define the ordinals α_g and β_g.

LEMMA 3.6. *Let $g \subset Coll(\omega, \mathbb{R})$ be $L(\mathbb{R})$-generic. Then,*

$$\Sigma_1^{L_{\alpha_g}(\mathbb{R})} \subset \Game^{\mathbb{R}} \mathbf{\Sigma}_3^0,$$

where α_g is computed in $L[g]$ and $\mathbb{R} = \mathbb{R}^V$.

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PROOF. Let $\gamma \leq \alpha_g$ be admissible, but Σ_1-projectible to $\mathbb{R}$[1] and let $A \subset \mathbb{R}$ be Π_1-definable over $L_\gamma(\mathbb{R})$. Given $a \in \mathbb{R}$, we define a $\mathbf{\Sigma}^0_3$ game $G(a, A)$ (uniformly in a) on $\mathbb{R}$ with the property that Player II has a winning strategy in it if, and only if, $a \in A$. Since the winning condition for Player II is $\mathbf{\Pi}^0_3$, this implies that all $\Pi_1^{L_{\alpha_g}(\mathbb{R})}$ subsets of $\mathbb{R}$ are $\Game^{\mathbb{R}}\mathbf{\Pi}^0_3$. Since $\mathbf{\Sigma}^0_3$ games on reals are determined, the dual class of $\Game^{\mathbb{R}}\mathbf{\Pi}^0_3$ is $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_3$, so this in turn implies that all $\Sigma_1^{L_{\alpha_g}(\mathbb{R})}$ subsets of $\mathbb{R}$ are $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_3$, which yields the result. This game we define combines the Σ^0_3 games from [19] with the games on reals of Martin-Steel [15]. The new feature in this game is the appearance of the generic g—-this seems unavoidable, by Lemma 3.5.

Let φ be the formula defining A over $L_\gamma(\mathbb{R})$ from a parameter $x_A \in \mathbb{R}$ and let φ_γ be a Σ_1 formula with one free variable such that for some real x_γ, $\varphi_\gamma(x_\gamma)$ holds in $L_\gamma(\mathbb{R})$ but in no proper initial segment of $L_\gamma(\mathbb{R})$.

Let $\mathcal{L}$ be the language of set theory with an additional collection of constants $\{\dot{x}_i : i \in \mathbb{N}\}$ and let $\{\psi_i : i \in \mathbb{N}\}$ enumerate all formulae in this language in such a way that ψ_i contains only constants among $\dot{x}_0, \dot{x}_1, \dots, \dot{x}_i$. We also fix a ternary formula $\theta(\cdot,\cdot,\cdot)$ such that in any model of $\mathsf{KP} + V = L(\mathbb{R})$, θ defines the graph of an $\mathbb{R}$-parametrized family of wellorders the union of whose fields include all of V. More precisely, we assume that, provably in $\mathsf{KP} + V = L(\mathbb{R})$,

- θ holds of a triple (x, α, a) only if $x \in \mathbb{R}$ and α is an ordinal and
- for every set a there is a real x and an ordinal α such that $\theta(x, \alpha, a)$.

Such a formula is found easily e.g., by appealing to Steel [18, Section 1]. Finally, we fix two recursive injections $n(\cdot)$ and $m(\cdot)$ from $\{\psi_i : i \in \mathbb{N}\}$ into the set of odd integers and whose ranges are disjoint.

In the game $G(A, x)$, Players I and II together build a structure in the language of set theory containing certain real numbers and a partial function

$$f : \mathbb{N} \times \mathbb{N} \to \{\psi_i : i \in \mathbb{N}\}.$$

More specifically,

(1) During turn $2i + 1$, Player II plays a real number x_{2i+1} and either "accepts" or "rejects" the formula ψ_i.
(2) During turn $2i$, Player I plays a real number x_{2i}. In addition, Player I may play a pair (j, k) and select a formula ψ as the value of $f(j, k)$, but only if the following conditions are met:
   (a) Player II has previously accepted the formula "the unique x satisfying ψ is an ordinal" and
   (b) either $k = 0$, or $f(j, k-1)$ has been defined previously, and, moreover, Player II has previously accepted the formula "the unique x satisfying ψ is smaller than the unique x satisfying $f(j, k-1)$."

Let T be the collection of all formulae accepted by Player II. The rules of the game are:

(3) During the course of the game, Player II must play a, x_γ, and x_A. If so, let i_a, i_γ, and i_A be such that $x_{i_a} = a$, $x_{i_\gamma} = x_\gamma$, and $x_{i_A} = x_A$.

[1]This means that there is a surjection from $\mathbb{R}$ to γ which is Σ_1-definable over $L_\gamma(\mathbb{R})$.

(4) T must be a complete, consistent theory containing the axioms:
  (a) $\mathsf{KP} + V = L(\mathbb{R})$;
  (b) $\dot{x}_i \in \mathbb{R}$, for each $i \in \mathbb{R}$;
  (c) $\dot{x}_i(m) = k$, but only if $x_i(m) = k$;
  (d) $\varphi_\gamma(\dot{x}_{i_\gamma})$ + "no proper initial segment of me satisfies $\varphi_\gamma(\dot{x}_{i_\gamma})$"; and
  (e) the Skolem axioms

$$\exists x \in \mathbb{R}\, \chi(x) \to \chi(\dot{x}_{n(\chi)}),$$

  as well as

$$\exists x\, \chi(x) \to \exists x\, \exists \alpha \in \mathsf{Ord}\, \Big(\theta(\dot{x}_{m(\chi)}, \alpha, x) \wedge \chi(x)\Big),$$

  for every formula χ;
  (f) $\varphi(\dot{x}_{i_a}, \dot{x}_{i_A})$.
  Otherwise, Player II loses the game.
(5) If there is $j \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, the value of $f(j, k)$ is specified by Player I at some point in the game, then she wins; otherwise, Player II wins.

This is a game in which Player II attempts to construct a model that looks like $L_\gamma(\mathbb{R})$ and incorporates the reals specified by Player I. Because we demand that the theory contain the Skolem axioms, it will have a minimal model containing all reals played during the course of the game (and none other). We shall denote by M_p the model associated to a run p of the game. Condition (3) states that, during this game, Player I simultaneously carries out infinitely many attempts to find an infinite descending sequence in Player II's model, and wins if one of those attempts is successful.

Clearly this is a $\mathbf{\Sigma}^0_3$ game on $\mathbb{R}$ for Player I (so a $\mathbf{\Pi}^0_3$ game for Player II). We need to show that Player II has a winning strategy in the game if, and only if, $a \in A$ i.e., if, and only if,

$$L_\gamma(\mathbb{R}) \models \varphi(a). \tag{3.1}$$

If (3.1) in fact holds, then Player II has a simple winning strategy obtained by playing the theory of $L_\gamma(\mathbb{R})$ and making sure that if x is a real played during the course of the game, then every real definable from x in $L_\gamma(\mathbb{R})$ is also played at some point in the game. The point is that *any* winning strategy for Player II requires that she actually play the theory of $L_\gamma(\mathbb{R})$; we verify this now.

To see this, let σ be a winning strategy for Player II. Suppose that there is a run of the game consistent with σ in which Player II accepts a formula not satisfied by $L_\gamma(\mathbb{R})$. Let X be a countable elementary substructure of some sufficiently large V_κ such that $\sigma \in X$ and let

$$\pi : W \to X$$

be the collapse embedding. Write $\bar{\sigma} = \pi^{-1}(\sigma)$ and $\bar{\gamma} = \pi^{-1}(\gamma)$. Let p_0 be a partial play in W which is consistent with $\bar{\sigma}$ and in which Player II has accepted a formula not satisfied by $L_{\bar{\gamma}}(\mathbb{R}^W)$. Observe that $\bar{\sigma}$ is simply the restriction of σ to partial plays belonging to W. Let $h \subset \mathrm{Coll}(\omega, \mathbb{R}^W)$ be W-generic, with $h \in V$. Thus, $\bar{\gamma}$ admits no h-nesting in $L[h]$. (This is because of the elementarity between V and W: γ admits no g-nesting in $V[g]$ and this is forced over V by a condition g_0. Without loss of

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generality, g_0 belongs to X and thus to W, and h extends g_0. The existence of a g-nesting over γ is easily seen to be $\Sigma^1_1(g, x_\gamma)$ where x_γ is a real coding γ, and thus first-order expressible over the next (g, x_γ)-admissible.) Working in $L[h]$, extend p_0 by having Player II play according to $\bar{\sigma}$ and having Player I play recursive reals, and selecting values for f as in the proof of [19, Theorem 4] (see the proof of (4) on pp. 426–428). The values of f are natural numbers, so every move by Player I results in an extension of p_0 which belongs to W and, hence, to the domain of $\bar{\sigma}$, so that this procedure is well defined. Although it is not immediately clear whether the full play p belongs to W, it certainly belongs to $L[h]$ and thus to V. In addition, all initial segments of p are consistent with $\bar{\sigma}$ and hence with σ, so that p is a play won by Player II. However, the proof of [20, Theorem 2] (appealing to that of [19, Theorem 4]) shows that p is a winning play for Player I unless $\bar{\gamma}$ admits an h-nesting in $L[h]$, so we have reached a contradiction, thus proving the claim.

Suppose that σ is a winning strategy for Player II in the game. To each $b \in \mathcal{P}_{\omega_1}(\mathbb{R})$, we shall associate the play p_b of the game that results from Player II playing according to σ, and Player I enumerating the reals of b (in any fixed order) and not specifying any values of f. Denote by M_b the minimal model of the theory whose reals are precisely the reals played in the course of p_b (so in particular $b \subset \mathbb{R}^{M_b}$). Because σ demands that Player II play the theory of $L_\gamma(\mathbb{R})$, M is wellfounded and thus of the form $L_{\gamma_b}(\mathbb{R}^{M_b})$. The proof of the subclaim on p. 116 of Martin–Steel [15] shows that if $c \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and $b \subset c$, then $\mathbb{R}^{M_b} \subset \mathbb{R}^{M_c}$ and there is a (unique) elementary embedding from $L_{\gamma_b}(\mathbb{R}^{M_b})$ into $L_{\gamma_c}(\mathbb{R}^{M_c})$ obtained by mapping each real to itself and each element of $L_{\gamma_b}(\mathbb{R}^{M_b})$ (which is definable from some real $x \in \mathbb{R}^{M_b}$) to the element of $L_{\gamma_c}(\mathbb{R}^{M_c})$ that has the same definition. The directed system

$$\mathcal{D} = \{L_{\gamma_b}(\mathbb{R}^{M_b}) : b \in \mathcal{P}_{\omega_1}(\mathbb{R})\}$$

is countably closed, so its direct limit is wellfounded and hence of the form $L_{\gamma^*}(\mathbb{R})$. Because of rule (4d) of the game,

$$L_{\gamma^*}(\mathbb{R}) \models \varphi_\gamma(x_\gamma) \wedge \text{"no proper initial segment of me satisfies } \varphi_\gamma(x_\gamma)\text{,"}$$

so $\gamma^* = \gamma$. By rule (4f), $L_\gamma(\mathbb{R}) \models \varphi(a)$. This completes the proof of the lemma. $\dashv$

§4. Integrating Borel Games. In this section, we prove Theorem 4.1, a general result on games of length ω^2 and the complexity of winning strategies of games on reals. It is obtained by "integrating" winning strategies for games on $\mathbb{N}$, provided they are obtained in a sufficiently uniform way. All the various parts of the proof of Theorem 4.1 are already present in the literature, but the abstraction might not be immediately obvious, so we lay out all the main points for the reader's convenience. We mention that, although we state the theorem for Borel pointclasses, it can be extended to larger classes under large-cardinal assumptions. We leave these generalizations to the reader. Below, KPi denotes the extension of KP by the axiom asserting that every set is contained in an admissible set. Admissible ordinals which are limits of admissibles are called *recursively inaccessible* ordinals; x-recursively inaccessible ordinals and $\mathbb{R}$-recursively inaccessible ordinals are defined similarly. For other undefined notions below, we refer the reader to Moschovakis [17].

THEOREM 4.1 (ZFC). *Let $\Delta^0_1 \subset \Gamma \subset \Delta^1_1$ be an ω-parametrized pointclass and let $\boldsymbol{\Gamma}$ be the associated boldface pointclass. Suppose that Γ is closed under recursive substitutions, finite unions, and finite intersections. Let ϕ be some fixed formula in the language of set theory and let*

$$\gamma_x = \text{ least } \gamma \text{ such that } L_{\gamma_x}[x] \models \mathsf{KPi} + \phi(x),$$

if such exists.

Suppose that for a Turing cone of x, every $\Gamma(x)$ game for which Player I has a winning strategy has a winning strategy in $L_{\gamma_x}[x]$, and let $\gamma_{\mathbb{R}}$ be the least $\mathbb{R}$-recursively inaccessible ordinal such that

$$\Vdash^{L(\mathbb{R})}_{Coll(\omega,\mathbb{R})} L_{\gamma_{\mathbb{R}}}[g] \models \phi(g).$$

Then,

(1) $⅁^{\mathbb{R}}\boldsymbol{\Gamma} \subset \Sigma_1^{L_{\gamma_{\mathbb{R}}}(\mathbb{R})}$ *and*
(2) *Suppose $\boldsymbol{\Gamma}$ has the scale property and that $L_{\gamma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$. Then, all games of length ω^2 with payoff in $\boldsymbol{\Gamma}$ and moves in $\mathbb{N}$ are determined.*

PROOF. Before proving the theorem, let us mention the following lemma which will be used repeatedly without mention:

LEMMA 4.2. *Let α be an ordinal. Then, the following are equivalent:*

(1) $L_\alpha(\mathbb{R}) \models \mathsf{KP}$.
(2) $L_\alpha(\mathbb{R}) \models$"$\Vdash_{Coll(\omega,\mathbb{R})}$ KP."
(3) *If $g \subset Coll(\omega,\mathbb{R})$ is $L(\mathbb{R})$-generic, then $L_\alpha[g] \models \mathsf{KP}$.*

And similarly for KPi.

PROOF. We prove the part of the lemma which concerns KP. Suppose $g \subset \mathrm{Coll}(\omega,\mathbb{R})$ is $L(\mathbb{R})$-generic and $L_\alpha[g] \models \mathsf{KP}$. Since $\mathbb{R} \in L_\alpha[g]$, $L_\alpha(\mathbb{R}) \models \mathsf{KP}$. It is verified in the course of the proof of Mathias [16, Theorem 10.17] that this implies

$$L_\alpha(\mathbb{R}) \models \text{“} \Vdash_{\mathrm{Coll}(\omega,\mathbb{R})} \mathsf{KP}.\text{”}$$

Now, suppose that g is $L(\mathbb{R})$-generic and

$$L_\alpha(\mathbb{R}) \models \text{“} \Vdash_{\mathrm{Coll}(\omega,\mathbb{R})} \mathsf{KP}.\text{”}$$

Since g is $L(\mathbb{R})$-generic, it decides every formula in the forcing language, in the sense that for every φ, there is a condition in g forcing φ or $\neg\varphi$. By combining Mathias [16, Theorem 10.11] and Mathias [16, Proposition 10.12], we see that $L_\alpha[g] \models \mathsf{KP}$.

To obtain the part of the lemma which concerns KPi, we simply apply the first part to every $\mathbb{R}$-admissible ordinal $\leq \alpha$. ⊣

We begin with the proof of (1). It is based on the argument from [4] for showing that the $⅁^{\mathbb{R}}\Pi^1_{n+1}$ subsets of $\mathbb{R}$ are all Σ_1-definable over the least inner model of ZF with n Woodin cardinals containing the reals. The details are made simpler by the fact that $\Gamma \subset \Delta^1_1$, but harder by the fact that we work with fragments of ZFC. We shall use the notation from [4].

Let G be a game of length ω on $\mathbb{R}$ with payoff in $\boldsymbol{\Gamma}$. For simplicity, let us assume the payoff is in Γ. Given $A \subset \mathbb{R}$, denote by G_A the variant of the game G in which each player is required to choose each individual move from the elements of A. If

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A is countable and x_A is a real coding A, then G_A can be simulated by a game on natural numbers, which we denote by $G(x_A)$. We first show:

CLAIM 4.3. The following are equivalent:

(1) Player I has a winning strategy in G and
(2) There is a closed, cofinal $C \subset \mathcal{P}_{\omega_1}(\mathbb{R})$ such that Player I has a winning strategy in G_A for all $A \in C$.

PROOF. Suppose Player I has a winning strategy σ in G and let C be the set of all countable sets of reals closed under σ. Then C is closed and cofinal and Player I has a winning strategy in G_A for all $A \in C$. The argument is symmetric for Player I and II and G is a Borel game, so it is determined, by Martin's theorem [14]. Therefore, the claim follows.

Thus if Player I has a winning strategy in G, then she has one in G_A for almost every A. It follows that if x_A codes A, then Player I has a winning strategy in $G(x_A)$. By hypothesis, there is one such strategy in $L_{\gamma_{x_A}}[x_A]$. Let $\psi(x_A)$ be the Σ_1 formula expressing in KPi that there is a winning strategy for Player I for the game $G(x_A)$.

CLAIM 4.4. The following are equivalent:

(1) Player I has a winning strategy in G and
(2) $\Vdash^{L_{\gamma_\mathbb{R}}(\mathbb{R})}_{\mathrm{Coll}(\omega,\mathbb{R})} \psi(g)$.

PROOF. Suppose Player I has a winning strategy in G and let W be the transitive collapse of a countable elementary substructure of some large V_κ. Let $A = \mathbb{R}^W$. By the previous claim, Player I has a winning strategy in G_A. Let $h \subset \mathrm{Coll}(\omega, A)$ be W-generic. Thus, h codes A, so

$$L_{\gamma_h}[h] \models \psi(h).$$

Now, ψ is a Σ_1 formula. By a theorem of Mathias [16], every formula true in $L_{\gamma_h}[h]$ is forced; it follows that

$$L_{\gamma_h}(A) \models \text{“} \Vdash_{\mathrm{Coll}(\omega,A)} \psi(h).\text{”}$$

The elementarity between W and V yields that

$$L_{\gamma_\mathbb{R}}(\mathbb{R}) \models \text{“} \Vdash_{\mathrm{Coll}(\omega,\mathbb{R})} \psi(g),\text{”}$$

as desired. Suppose now that Player I does not have a winning strategy in G. Then, by Borel determinacy, Player II has a winning strategy in G. Repeating the above argument, we see that

$$\Vdash^{L(\mathbb{R})}_{\mathrm{Coll}(\omega,\mathbb{R})} \text{“Player II has a winning strategy in } G(g).\text{”}$$

It follows that we cannot possibly have

$$\Vdash^{L_{\gamma_\mathbb{R}}(\mathbb{R})}_{\mathrm{Coll}(\omega,\mathbb{R})} \psi(g),$$

for ψ is a Σ_1 formula, so otherwise by forcing over $L(\mathbb{R})$ we should be able to construct winning strategies for both players in $G(g)$, which is absurd.

Since ψ is Σ_1, this last claim completes the proof of (1).

Let us proceed now to the proof of (2); we sketch it. Let $\gamma = \gamma_{\mathbb{R}}$. The argument is similar to the one used in [4] to prove Projective Determinacy for games of length ω^2. The difference is that we have not shown in general that

$$\Sigma_1^{L_\gamma(\mathbb{R})} \subset \Game^{\mathbb{R}}\mathbf{\Gamma},$$

although we conjecture that it is true, under the hypotheses of the theorem.

Suppose that $L_\gamma(\mathbb{R}) \models \mathsf{AD}$. By the Kechris–Woodin determinacy transfer theorem [13], all subsets of $\mathbb{R}$ which are Σ_1-definable over $L_\gamma(\mathbb{R})$ are determined.[2] By (1), all sets in $\Game^{\mathbb{R}}\mathbf{\Gamma}$ are determined. Applying Steel [18, Theorem 2.1] to $L_\gamma(\mathbb{R})$, we see that $\Sigma_1^{L_\gamma(\mathbb{R})}$ has the scale property. Because $L_\gamma(\mathbb{R})$ is admissible by hypothesis, $\Sigma_1^{L_\gamma(\mathbb{R})}$ is closed under real universal quantification, so, by a well-known theorem of Moschovakis (see [17, 4E.7]), every relation in $\Sigma_1^{L_\gamma(\mathbb{R})}$ (hence every relation in $\Game^{\mathbb{R}}\mathbf{\Gamma}$) can be uniformized by a function in $\Sigma_1^{L_\gamma(\mathbb{R})}$.

Claim 4.5. Every $\mathbf{\Gamma}$ game on reals has a winning strategy which is Σ_1-definable over $L_\gamma(\mathbb{R})$.

Proof Sketch. This follows from adapting the proof of Moschovakis' third periodicity theorem [17, Theorem 6E.1] to games on $\mathbb{R}$ using the assumption that $\mathbf{\Gamma}$ has the scale property. The strategy is built from a scale on a $\Game^{\mathbb{R}}\mathbf{\Gamma}$ set and a uniformizing function, both of which can be taken to belong to $\Sigma_1^{L_\gamma(\mathbb{R})}$. We refer the reader to [2, Lemma 3] for a few more details. ⊣

Now, the determinacy of $\mathbf{\Gamma}$ games of length ω^2 follows by the following general fact:

Lemma 4.6. *Let Λ and Π be pointclasses closed under continuous preimages and finite unions and intersections. Suppose that $\Lambda \subset \Pi$ and every game on reals in Λ has a winning strategy in Π. Suppose moreover that Π is closed under real quantifiers and countable unions and has the uniformization property, and every game on $\mathbb{N}$ in Π is determined. Then, every game of length ω^2 with moves in $\mathbb{N}$ and payoff in Λ is determined.*

Proof Sketch. This can be proved by localising the argument of Blass [6] that $\mathsf{AD}_{\mathbb{R}}$ implies that every game of length ω^2 is determined. We refer the reader to [1, Lemma 3.11] for a few more details. ⊣

Let $\Lambda = \mathbf{\Gamma}$ and $\Pi = \Sigma_1^{L_\gamma(\mathbb{R})}$. Since γ is $\mathbb{R}$-admissible, $\Sigma_1^{L_\gamma(\mathbb{R})}$ is closed under real quantifiers and countable unions. Thus, the hypotheses of the lemma are satisfied, so every game of length ω^2 on $\mathbb{N}$ with payoff in $\mathbf{\Gamma}$ is determined. This completes the proof of the theorem. ⊣

[2]It has been pointed out to us by the reviewer that this fact also follows by an adaptation of Martin's (1974, unpublished) argument that Δ^1_{2n}-determinacy implies Σ^1_{2n}-determinacy. A similar type of argument was used by Kechris and Solovay [11, Theorem 5.1] to prove their determinacy transfer theorem which also implies $\Sigma_1^{L_\gamma(\mathbb{R})}$-determinacy in our context.

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REMARK 4.7. Although we did not need this fact, we point out that the proof of Lemma 4.6 gives a bound on the complexity of winning strategies for Λ games of length ω^2 with moves in $\mathbb{N}$. These games are obtained from the strategies for games on $\mathbb{R}$ with payoff Λ, together with appropriate uniformizing functions. The hypotheses of the lemma imply that these strategies and functions belong to Π, and the closure properties on Π will imply that the strategies for Λ games of length ω^2 with moves in $\mathbb{N}$ also belong to Π.

§5. Putting Everything Together. Before proving the main theorem, we need a lemma that characterizes β_g without referring to generic extensions:

LEMMA 5.1. *Let $g \subset Coll(\omega, \mathbb{R})$ be $L(\mathbb{R})$-generic. Let $\beta = \beta_g$, as computed in $L[g]$. Then, β is the least ordinal such that*

$$L_\beta(\mathbb{R}) \models \Pi_1\text{-}\mathsf{MI}_\mathbb{R}.$$

PROOF. Putting together Theorems 2.1, 3.11, and 4.2 of Hachtman [9], we see that β is least such that

$$L_\beta[g] \models \boldsymbol{\Pi}^1_2\text{-}\mathsf{MI}_\mathbb{N}.$$

Since $\boldsymbol{\Pi}^1_2$ is the same as Π_1 over $H(\omega_1)$ (with parameters), provably in KPi, β is least such that

$$L_\beta[g] \models \Pi_1\text{-}\mathsf{MI}_\mathbb{N}.$$

Using the fact that in $L_\beta[g]$ there is a bijection between $\mathbb{N}$ and $\mathbb{R}^V$, it is easy to show, by an argument like the one of Lemma 2.1, that

$$L_\beta(\mathbb{R}) \models \Pi_1\text{-}\mathsf{MI}_\mathbb{R}.$$

To prove the converse, let η be least such that

$$L_\eta(\mathbb{R}) \models \Pi_1\text{-}\mathsf{MI}_\mathbb{R}.$$

We show that

$$L_\eta[g] \models \Pi_1\text{-}\mathsf{MI}_\mathbb{N}.$$

Thus, let Φ be a Π_1 operator in $L_\eta[g]$, say, given by

$$x \mapsto \{n \in \mathbb{N} : \phi(n, x, a)\}$$

for some set $a \in L_\eta[g]$. Working in $L_\eta(\mathbb{R})$, let τ be a $\mathrm{Coll}(\omega, \mathbb{R})$-name for a and let p_0 be a condition forcing that Φ is monotone. We consider the following operator Ψ that inductively constructs a name for a set of natural numbers:

$$X \mapsto \{(p, n) : p \leq p_0 \,\&\, p \Vdash \phi(n, X[g], \tau[g])\}.$$

Since ϕ is Π_1, Ψ is a Π_1 operator using τ, p_0 and $\mathrm{Coll}(\omega, \mathbb{R})$ as parameters. It is monotone, for if $X \subset Y$, and $(p, n) \in \Psi(X)$, then $p \leq p_0$ and $p \Vdash \phi(n, X[g], \tau[g])$. But then p forces that ϕ is monotone, so that $p \Vdash \phi(n, Y[g], \tau[g])$; thus, $(p, n) \in Y$.

It follows that Ψ has a least fixed point, Ψ^∞. Since $\Psi^\infty \in L_\eta[g]$, $L_\eta[g]$ can compute $\Psi^\infty[g]$ and it is easy to see that

$$\Psi^\infty[g] = \Phi^\infty,$$

so $L_\eta[g]$ satisfies Π_1-$\mathsf{MI}_\mathbb{N}$, as claimed. ⊣

We are now ready to prove the main theorem:

THEOREM 5.2. *The following are equivalent over* ZFC:

(1) $\boldsymbol{\Sigma}^0_3$ *games of length* ω^2 *with moves in* $\mathbb{N}$ *are determined.*
(2) *There is a transitive model of* $\mathsf{KP} + \mathsf{AD} + \Pi_1$-$MI_\mathbb{R}$ *containing* $\mathbb{R}$.

PROOF. Let $g \subset \mathrm{Coll}(\omega, \mathbb{R})$ be $L(\mathbb{R})$-generic. By Corollary 2.2 and Lemma 5.1, it suffices to prove that the following are equivalent:

(1) $\boldsymbol{\Sigma}^0_3$ games of length ω^2 with moves in $\mathbb{N}$ are determined and
(2) $L_{\beta_g}(\mathbb{R}) \models \mathsf{AD}$, where β_g is computed in $L[g]$.

Since

$$L_{\alpha_g}[g] \prec_{\Sigma_1} L_{\beta_g}[g],$$

it follows that

$$L_{\alpha_g}(\mathbb{R}) \prec_{\Sigma_1} L_{\beta_g}(\mathbb{R}),$$

and thus $L_{\beta_g}(\mathbb{R}) \models \mathsf{AD}$ if, and only if, $L_{\alpha_g}(\mathbb{R}) \models \mathsf{AD}$. If $\boldsymbol{\Sigma}^0_3$ games of length ω^2 are determined, then $\Game^\mathbb{R}\boldsymbol{\Sigma}^0_3$ games of length ω are determined, so $L_{\beta_g}(\mathbb{R}) \models \mathsf{AD}$ by Lemma 3.6.

Conversely, suppose $L_{\beta_g}(\mathbb{R}) \models \mathsf{AD}$. We apply Theorem 4.1 with $\Gamma = \Sigma^0_3$. $\boldsymbol{\Sigma}^0_3$ has the scale property by a theorem of Kechris [12]. Using a universal Π^1_2 formula, there is a first-order formula expressing $\boldsymbol{\Pi}^1_2$-MI over every model of KP, and we choose this as the formula ϕ in the statement of Theorem 4.1. All $\Sigma^0_3(x)$ games for which Player I has a winning strategy have winning strategies in $L_{\alpha_x+1}[x]$ by [19], and in particular in $L_{\gamma_x}[x]$, since $\gamma_x = \beta_x$ by Hachtman [9].

By Lemma 5.1, β_g does not depend on g, so it is least such that

$$\Vdash^{L(\mathbb{R})}_{\mathrm{Coll}(\omega,\mathbb{R})} L_{\beta_g}[g] \models \boldsymbol{\Pi}^1_2\text{-MI}.$$

We have verified all hypotheses, so all $\boldsymbol{\Sigma}^0_3$ games of length ω^2 on $\mathbb{N}$ are determined by Theorem 4.1. ⊣

Acknowledgements. The first author was partially supported by FWF grant I4513-N and FWO grant 3E017319.

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DEPARTMENT OF MATHEMATICS
GHENT UNIVERSITY
KRIJGSLAAN 281-S8 B9000 GHENT BELGIUM

and

INSTITUTE OF DISCRETE MATHEMATICS AND GEOMETRY
VIENNA UNIVERSITY OF TECHNOLOGY
1040 VIENNA, AUSTRIA
E-mail: aguilera@logic.at

SCHOOL OF MATHEMATICS
UNIVERSITY OF BRISTOL
CLIFTON, BRISTOL BS8 1UG, UK
E-mail: p.welch@bristol.ac.uk