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NILPOTENT-BY-NOETHERIAN FACTORIZED GROUPS

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ABSTRACT. It is shown that a soluble-by-finite product $G = AB$ of a nilpotent-by-noetherian group A and a noetherian group B is nilpotent-by-noetherian. Moreover, a bound for the torsion-free rank of the Fitting factor group of G is given, in terms of the torsion-free rank of the Fitting factor group of A and the torsion-free rank of B.

1. **Introduction.** A group G is *noetherian* if it satisfies the maximum condition on subgroups. It is well-known that a soluble-by-finite group is noetherian if and only if it is polycyclic-by-finite. The group G is called *nilpotent-by-noetherian* if it contains a nilpotent normal subgroup with noetherian factor group. In particular in such a group G the Fitting subgroup Fit(G) is nilpotent and the Fitting factor group G/Fit(G) is noetherian. From this it follows that every group which contains a nilpotent-by-noetherian normal subgroup with noetherian factor group is likewise nilpotent-by-noetherian.

Lennox and Roseblade in [5] and independently Zaicev in [9] have shown that every soluble product $G = AB$ of two noetherian subgroups A and B is noetherian. The proof in [5] even holds when the group $G = AB$ is soluble-by-finite. Also the torsion-free rank of $G = AB$ satisfies the following formula

$$r_0(G) = r_0(A) + r_0(B) - r_0(A \cap B),$$

so that in particular $r_0(G)$ is bounded by $r_0(A) + r_0(B)$ (see [1], Satz 5.2).

In this paper we use these results to establish the following theorem.

THEOREM A. *Let the soluble-by finite group $G = AB$ be the product of a nilpotent-by-noetherian subgroup A and a noetherian subgroup B. Then the following holds:*
(a) *G is nilpotent-by-noetherian,*
(b) *If the soluble radical of G has derived length $n > 1$, then the torsion-free rank of the Fitting factor group of G satisfies the following inequality:*

$$r_0(G/\mathrm{Fit}(G)) \leqq (2n-3)(r_0(B) + r_0(A/\mathrm{Fit}(A))).$$

When A and B are abelian and B is finitely generated, this was obtained by Heineken in [3]; note that in this case $G = AB$ is metabelian by the well-known theorem of Itô

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[4]. The special case of Theorem A when G is soluble and A is nilpotent was already pointed out in [2].

Recently Wilson has shown that every soluble product $G = AB$ of two minimax subgroups A and B is likewise a minimax group (see [8]). In this case for the minimax rank and the p^∞-rank of G the following formulas hold:

$$m(G) = m(A) + m(B) - m(A \cap B),$$
$$m_p(G) = m_p(A) + m_p(B) - m_p(A \cap B) \quad \text{for each prime } p,$$

so that in particular $m(G)$ is bounded by $m(A) + m(B)$ and $m_p(G)$ is bounded by $m_p(A) + m_p(B)$ for each p (see [7], Theorem 1 and Theorem 3).

Here we use these results to obtain our second theorem.

THEOREM B. *Let the soluble group $G = AB$ with derived length n be the product of a hypercentral-by-minimax subgroup A and a minimax subgroup B. Then the following holds:*

(a) G is an extension of a Gruenberg group by a minimax group,

(b) If G is not abelian, the minimax rank and the p^∞-rank of the Gruenberg factor group satisfy the following inequalities:

$$m(G/K(G)) \leqq (2n-3)(m(B) + m(A/H)),$$
$$m_p(G/K(G)) \leqq (2n-3)(m_p(B) + m_p(A/H)) \quad \textit{for each prime } p,$$

where H is any hypercentral normal subgroup of A with minimax factor group A/H. If, in addition, B is artinian, then G is an extension of a Gruenberg group by an artinian group.

Note that it follows from Theorem B that if A is hypercentral-by-noetherian and B is noetherian, then the group G is an extension of a Gruenberg group by a noetherian group.

Again the special case of Theorem B when A is hypercentral is already contained in [2]. There it was also mentioned that there is no corresponding inequality for the torsion-free rank of the Gruenberg factor group in Theorem B. This theorem also becomes false if “minimax” is replaced by “finite (Prüfer) rank”. Similarly Theorem A becomes false when “noetherian” is replaced by “artinian” or “minimax” (see [2]).

Also Theorems A and B do not hold when the subgroups A and B are both nilpotent-by-noetherian or hypercentral-by-minimax, respectively. For, there exist groups which are the product of two abelian subgroups, but which are not even (locally nilpotent)-by-minimax. Let, for example, B be the additive group of rational numbers and let A be its automorphism group. Then the semidirect product $G = AB$ is not (locally nilpotent)-by-mimimax, since B is the Hirsch-Plotkin radical of G.

Moreover this example shows that Theorems A and B cannot be extended to a soluble product $G = A_1 \ldots A_t$ of finitely many pairwise permutable nilpotent-by-noetherian (hypercentral-by-minimax) subgroups $A_1, \ldots, A_t$, where more than one factor is merely nilpotent-by-noetherian but not noetherian (hypercentral-by-minimax but not minimax). On the other hand the following statements can be deduced immediately from our theorems.

COROLLARY A*. *Let the soluble-by-finite group $G = A_1 \ldots A_t$ be the product of finitely many pairwise permutable subgroups $A_1, \ldots, A_t$, where A_1 is nilpotent-by-noetherian and $A_2, \ldots, A_t$ are noetherian. Then G is nilpotent-by-noetherian and*

$$r_0(G/\mathrm{Fit}(G)) \leqq (2n-3)(r_0(A_1/\mathrm{Fit}(A_1)) + r_0(A_2) + \ldots + r_0(A_t)),$$

where $n > 1$ is the derived length of the soluble radical of G.

COROLLARY B*. *Let the soluble group $G = A_1 \ldots A_t$ with derived length n be the product of finitely many pairwise permutable subgroups $A_1, \ldots, A_t$, where A_1 is hypercentral-by-minimax and $A_2, \ldots, A_t$ are minimax. Then G is an extension of a Gruenberg group by a minimax group and, if G is not abelian, the following inequalities hold:*

$$m(G/K(G)) \leqq (2n-3)(m(A_1/H) + m(A_2) + \ldots + m(A_t)),$$
$$m_p(G/K(G)) \leqq (2n-3)(m_p(A_1/H) + m_p(A_2) + \ldots + m_p(A_t)) \quad \textit{for each prime } p,$$

where H is any hypercentral normal subgroup of A_1 with minimax factor group A_1/H. If, in addition, $A_2, \ldots, A_t$ are artinian, then G is an extension of a Gruenberg group by an artinian group.

NOTATION. The notation is standard and can for instance be found in [6]. We note in particular:

$\mathrm{Fit}(G)$ is the *Fitting subgroup* of the group G, i.e. the product of all nilpotent normal subgroups of G,

$K(G)$ is the *Gruenberg radical* of G, i.e. the subgroup generated by all abelian ascendant subgroups of G,

$\mathrm{Soc}(G)$ is the *socle* of G, i.e. the subgroup generated by all minimal normal subgroups of G.

A normal subgroup H of G is *hypercentrally embedded* in G if for every G-invariant proper subgroup K of G the intersection $(H/K) \cap Z(G/K)$ is non-trivial.

A group G is *artinian* if it satisfies the minimum condition on subgroups,

A soluble-by finite group G is a *minimax* group if it has a finite series whose factors are finite or infinite cyclic or quasicyclic of type p^∞; the number of infinite cyclic factors in such a series is the *torsion-free rank* $r_0(G)$ of G, the number of factors of type p^∞ for the prime p is the *p^∞-rank* $m_p(G)$ of G, and the *minimax rank* of G is

$$m(G) = r_0(G) + \sum_p m_p(G).$$

If N is a normal subgroup of a factorized group $G = AB$, the *factorizer* of N in G is the subgroup $X(N) = AN \cap BN$.

2. **Auxiliary results.** The following lemma – which ultimately depends on Dirichlet's Unit Theorem of algebraic number theory – will be essential for the proof of part (b) of Theorem A.

LEMMA 2.1. *If N is a finitely generated abelian normal subgroup of a soluble-by finite group G with nilpotent-by-noetherian factor group G/N, then the Fitting factor group $G/\mathrm{Fit}(G)$ is noetherian and*

$$r_0(G/\mathrm{Fit}(G)) \leqq r_0(N) + r_0((G/N)/\mathrm{Fit}(G/N)).$$

PROOF. If M/N is the Fitting subgroup of G/N, then G/M is noetherian. Let $F = \mathrm{Fit}(M)$. Application of Lemma 2.3 of [2] to M yields that M/F is polycyclic and $r_0(M/F) \leqq r_0(N)$. Since G/M is noetherian, also G/F is noetherian and

$$r_0(G/F) = r_0(M/F) + r_0(G/M) \leqq r_0(N) + r_0((G/N)/\mathrm{Fit}(G/N)).$$

As a soluble group of automorphisms of a polycyclic group, $M/C_M(N)$ is polycyclic (see [6] Part 1, p. 82). Since N is contained in the centre of $C_M(N)$ and $C_M(N)/N \leqq M/N$ is nilpotent, also $C_M(N)$ is nilpotent. Therefore the subgroup M is nilpotent-by-noetherian, so that $F = \mathrm{Fit}(M)$ is nilpotent and $F \leqq \mathrm{Fit}(G)$. The lemma is proved. □

The next lemma plays a similar role in the proof of statement (b) of Theorem B.

LEMMA 2.2. *Let N and M be normal subgroups of a soluble-by finite group G with $N \leqq M$ and such that N is an abelian minimax group, M/N is hypercentral and G/M is a minimax group. Then the Gruenberg factor group $G/K(G)$ is a minimax group and*

$$m(G/K(G)) \leqq m(N) + m(G/M),$$
$$m_p(G/K(G)) \leqq m_p(N) + m_p(G/M) \quad \textit{for each prime } p.$$

PROOF. If $L = K(M)$ is the Gruenberg radical of M, it follows from Lemma 2.3 of [2] applied to M that M/L is a finitely generated nilpotent group with $r_0(M/L) = m(M/L) \leqq m(N)$. Since G/M is a minimax group, also G/L is minimax and

$$m(G/L) = m(M/L) + m(G/M) \leqq m(N) + m(G/M),$$
$$m_p(G/L) = m_p(M/L) + m_p(G/M) \leqq m_p(N) + m_p(G/M) \quad \text{for each prime } p.$$

Since L is contained in $K(G)$, the result follows. □

3. **Proof of Statements (a) of Theorems A and B.** We begin with the proof of statement (a) of Theorem A.

The first lemma gives a criterion for a soluble-by finite factorized group to be nilpotent-by-noetherian.

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LEMMA 3.1. *Let the soluble-by-finite group $G = AB$ be the product of a subgroup A and a noetherian subgroup B, and let K be a normal subgroup of G with nilpotent-by-noetherian factor group G/K. If $A \cap K$ is noetherian, then also K is noetherian and G is nilpotent-by-noetherian.*

PROOF. The factor group

$$(A \cap BK)/(A \cap K) \simeq (A \cap BK)K/K \leqq BK/K$$

is noetherian, and hence $A \cap BK$ is noetherian. Therefore the factorized group

$$BK = B(A \cap BK)$$

is noetherian by the theorem of Lennox-Roseblade-Zaicev (see [5]). In particular K is noetherian.

As a soluble-by finite group of automorphisms of K also the factor group $G/C_G(K)$ is noetherian (see [6] Part 1, p. 82). Let L/K be a nilpotent normal subgroup of G/K with noetherian factor group G/L. Then $G/(L \cap C_G(K))$ is noetherian. Since $Z(K)$ is contained in the centre of $L \cap C_G(K)$ and

$$(L \cap C_G(K))/Z(K) \simeq (L \cap C_G(K))K/K \leqq L/K$$

is nilpotent, also $L \cap C_G(K)$ is nilpotent. Therefore G is nilpotent-by-noetherian. □

The next lemma is a special case of Theorem A(a).

LEMMA 3.2. *Let the soluble-by-finite group $G = AB$ be the product of a nilpotent-by-noetherian subgroup A and a noetherian subgroup B. If there exists an abelian normal subgroup K of G such that $G = AK$, then G is nilpotent-by-noetherian.*

PROOF. Let N be a nilpotent normal subgroup of A with noetherian factor group A/N. Then $N \cap K$ is normal in $AK = G$ and the factor group

$$(A \cap K)/(N \cap K) \simeq (A \cap K)N/N \leqq A/N$$

is noetherian. By Lemma 3.1 it follows that the factorized group

$$G/(N \cap K) = (A/(N \cap K))(B(N \cap K)/(N \cap K))$$

is nilpotent-by-noetherian.

If c is the nilpotency class of N, then

$$N \cap K = Z_c(N) \cap K \leqq Z_c(NK).$$

and hence NK is nilpotent-by-noetherian. Since NK is normal in G and $G/NK \simeq A/(A \cap NK)$ is noetherian, G is nilpotent-by-noetherian.

We can now prove Theorem A(a).

Assume that Theorem A(a) is false and let $G = AB$ be a counterexample such that the derived length n of the soluble radical G_0 of G is minimal. Clearly $n > 1$. If K is the last non-trivial term of the derived series of G_0, there exists a nilpotent normal subgroup L/K of G/K with noetherian factor group G/L.

By Lemma 3.2 the factorized group

$$AK = A(B \cap AK)$$

is nilpotent-by-noetherian. Put $H = AK \cap L$. Then $K \leqq H \leqq L$ and since L/K is nilpotent, H is a nilpotent-by-noetherian subnormal subgroup of G. Let

$$H = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_t = G$$

be the standard series of H in G. Since A normalizes H, it is well-known that A also normalizes each H_i. Then $A \leqq N_G(H_i)$ and so $N_G(H_i) = A(B \cap N_G(H_i))$. Consider the factorized group

$$N_G(H_i)/H_i = (AH_i/H_i)((B \cap N_G(H_i))H_i/H_i).$$

The group AH_i/H_i is an image of

$$A/(A \cap H) = A/(A \cap L) \simeq AL/L \leqq G/L,$$

and hence it is noetherian. By the theorem of Lennox-Roseblade-Zaicev (see [5]) the group $N_G(H_i)/H_i$ is also noetherian. Hence, if H_i is nilpotent-by-noetherian for some i, also $N_G(H_i)$ and its subgroup H_{i+1} are nilpotent-by-noetherian. Since $H_0 = H$ is nilpotent-by-noetherian, $G = H_t$ is nilpotent-by-noetherian.

This proves Theorem A(a). □

The next two lemmas will be used in the proof of Theorem B(a).

LEMMA 3.3. *Let the soluble group $G = AB$ be the product of a subgroup A and a minimax subgroup B, and let K and L be normal subgroups of G with $K \leqq L$ and such that L/K is a Gruenberg group and G/L is a minimax group. If $A \cap K$ is a minimax group, then K and the Gruenberg factor group $G/K(G)$ are minimax groups.*

PROOF. The factor group

$$(A \cap BK)/(A \cap K) \simeq (A \cap BK)K/K \leqq BK/K$$

is a minimax group, so that $A \cap BK$ is a minimax group. Therefore

$$BK = B(A \cap BK)$$

is a minimax group by Wilson's theorem [8]. In particular K is a minimax group.

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Let n be the derived length of K. For every $i \leqq n-1$ let $T_i/K^{(i+1)}$ be the torsion subgroup of $K^{(i)}/K^{(i+1)}$. Put

$$C_1 = \cap_i C_L(\mathrm{Soc}(T_i/K^{(i+1)}))$$

and

$$C_2 = \cap_i C_L(K^{(i)}/T_i).$$

Since every $T_i/K^{(i+1)}$ satisfies the minimum condition, the group L/C_1 is finite. As a soluble group of automorphisms of a torsion-free abelian minimax group each $L/C_L(K^{(i)}/T_i)$ is a minimax group (see [6] Part 2, p. 171–173). Hence also L/C_2 is a minimax group. It follows that $G/(C_1 \cap C_2)$ is a minimax group.

Clearly $K \cap C_1 \cap C_2$ is hypercentrally embedded in $C_1 \cap C_2$. Since

$$(C_1 \cap C_2)/(C_1 \cap C_2 \cap K) \simeq (C_1 \cap C_2)K/K$$

is a Gruenberg group, also $C_1 \cap C_2$ is a Gruenberg group. This proves the lemma. □

Using Lemma 3.3 in the place of Lemma 3.1, our last lemma can be proved similarly as Lemma 3.2.

LEMMA 3.4. *Let the soluble group $G = AB$ be the product of a hypercentral-by-minimax subgroup A and a minimax subgroup B. If there exists an abelian normal subgroup K of G such that $G = AK$, the Gruenberg factor group $G/K(G)$ is a minimax group.*

We are now ready to prove Theorem B(a).

Assume that Theorem B(a) is false and let $G = AB$ be a counterexample with minimal derived length n. Clearly $n > 1$. If K is the last non-trivial term of the derived series of G, there exists a Gruenberg normal subgroup L/K of G/K with minimax factor group G/L.

By Lemma 3.4 the factorized group

$$AK = A(B \cap AK)$$

is an extension of a Gruenberg group by a minimax group. Put $H = AK \cap L$. If V is the Gruenberg radical of H, then H/V is a minimax group. For each element x of V the subgroup $\langle x \rangle$ is ascendant in $\langle x, K \rangle \leqq H$. Since $\langle x, K \rangle/K \leqq L/K$ is ascendant in G/K, it follows that $\langle x \rangle$ is ascendant in G. Hence V is contained in the Gruenberg radical of G and so V^G is a Gruenberg group.

The factor group

$$(AV^G \cap L)/V^G = (A \cap L)V^G/V^G \simeq (A \cap L)/(A \cap V^G)$$

is a minimax group as an image of the minimax group

$$(A \cap L)/(A \cap V) \simeq (A \cap L)V/V \leqq H/V.$$

Since also

$$AV^G/(AV^G \cap L) \simeq AL/L \leqq G/L$$

is a minimax group, AV^G/V^G is minimax. Now by the theorem of Wilson [8] the factorized group

$$G/V^G = (AV^G/V^G)(BV^G/V^G)$$

is a minimax group.

This proves Theorem B(a).

4. **Proof of Statements (b) of Theorems A and B.** Clearly it is enough to prove that

$$r_0(G/\mathrm{Fit}(G)) \leqq (2m-3)(r_0(B) + r_0(A/\mathrm{Fit}(A)))$$

where $m > 1$ is the derived length of an arbitrary (non-abelian) soluble normal subgroup S of finite index of G.

Assume that this is false. Among all the counterexamples $G = AB$ for which the torsion-free rank of B is minimal, consider those for which the index of the soluble subgroup S is minimal and among these choose one for which the derived length m of S is minimal.

The factorizer $X = X(S)$ of S in $G = AB$ has the triple factorization

$$X = S(A \cap BS) = S(B \cap AS) = (A \cap BS)(B \cap AS).$$

If X is properly contained in G, then

$$\begin{aligned} r_0(X/\mathrm{Fit}(X)) &\leqq (2m-3)(r_0(B \cap AS) + r_0((A \cap BS)/\mathrm{Fit}(A \cap BS))) \\ &\leqq (2m-3)(r_0(B) + r_0(A/\mathrm{Fit}(A))). \end{aligned}$$

Since $\mathrm{Fit}(X)$ is nilpotent, $\mathrm{Fit}(X) \cap S$ is a nilpotent normal subgroup of S, so that

$$\mathrm{Fit}(X) \cap S \leqq \mathrm{Fit}(S) \leqq \mathrm{Fit}(G).$$

Therefore

$$\begin{aligned} r_0(G/\mathrm{Fit}(G)) &\leqq r_0(G/\mathrm{Fit}(S)) = r_0(S/\mathrm{Fit}(S)) \\ &\leqq r_0(S/(\mathrm{Fit}(X) \cap S)) = r_0(X/\mathrm{Fit}(X)) \\ &\leqq (2m-3)(r_0(B) + r_0(A/\mathrm{Fit}(A))). \end{aligned}$$

This contradiction shows that

$$G = X = AS = BS = AB.$$

In the following K denotes the last non-trivial term of the derived series of S. If $A_0 = A \cap S$, then A/A_0 is finite. Let F be the Fitting subgroup of A_0. Since $A \cap K$ is an abelian normal subgroup of A_0, we have that $A \cap K = F \cap K$. Note also that, if c is the nilpotency class of F, then

$$F \cap K = Z_c(F) \cap K \leqq Z_c(FK).$$

The proof of Theorem A(b) will now be accomplished in a series of steps.

(1) *$K/(F \cap K)$ is finitely generated and $r_0(K/(F \cap K)) \leqq r_0(B)$.*

Suppose first that $F \cap K = 1$. Clearly the subgroup

$$A \cap BK \simeq (A \cap BK)K/K \leqq BK/K$$

is noetherian and hence also $BK = B(A \cap BK)$ is noetherian by the theorem of Lennox-Roseblade-Zaicev (see [5]). In particular K is finitely generated and

$$r_0(BK) \leqq r_0(B) + r_0(A \cap BK)$$

(see [1], Satz 5.2). It is also clear that

$$r_0(K) = r_0(BK) - r_0(B) + r_0(B \cap K)$$

and

$$r_0(A \cap BK) \leqq r_0(B) - r_0(B \cap K).$$

Therefore

$$r_0(K) \leqq r_0(B) + r_0(A \cap BK) - r_0(B) + r_0(B \cap K) \leqq r_0(B).$$

In the general case consider the factorized group $AK = A(B \cap AK)$ and its factor group

$$AK/(F \cap K) = (A/(F \cap K))((B \cap AK)(F \cap K)/(F \cap K)).$$

Since

$$(A/(F \cap K)) \cap (K/(F \cap K)) = 1,$$

it follows that $K/(F \cap K)$ is finitely generated and $r_0(K/(F \cap K)) \leqq r_0(B)$.

(2) *S is not metabelian.*

Assume that S is metabelian. Then FK/K is normal in $G/K = (AK/K)(S/K)$, so that FK is normal in G and therefore

$$N = (F \cap K)^G \leqq Z_c(FK).$$

It follows from (1) that K/N is finitely generated and $r_0(K/N) \leqq r_0(B)$. Application of Lemma 2.1 to G/N yields that

$$\begin{aligned} r_0((G/N)/\mathrm{Fit}(G/N)) &\leqq r_0(K/N) + r_0((G/K)/\mathrm{Fit}(G/K)) \\ &= r_0(K/N) \leqq r_0(B). \end{aligned}$$

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because G/K is abelian-by-finite.

Let E/N be the Fitting subgroup of G/N and $E_0 = E \cap FK$. Since $N \leqq Z_c(FK)$ we have also that $N \leqq Z_c(E_0)$, so that E_0 is a nilpotent normal subgroup of G. Moreover

$$r_0(G/E_0) \leqq r_0(G/E) + r_0(G/FK) \leqq r_0(B) + r_0(G/FK).$$

Since S is metabelian, A_0K is normal in $G = AS$, and it follows that

$$\begin{aligned} r_0(G/FK) &= r_0(G/A_0K) + r_0(A_0K/FK) \\ &= r_0(G/A_0K) + r_0(A_0/(A_0 \cap FK)) \\ &\leqq r_0(G/A_0K) + r_0(A_0/F) \leqq r_0(G/A_0K) + r_0(A/\mathrm{Fit}(A)). \end{aligned}$$

If G/A_0K is finite, this implies

$$r_0(G/E_0) \leqq r_0(B) + r_0(A/\mathrm{Fit}(A)).$$

This contradiction shows that G/A_0K must be infinite. Then also $|G : AK| = |B : B \cap AK|$ is infinite and

$$r_0(B \cap AK) < r_0(B).$$

Therefore for the factorized group

$$AK = A(B \cap AK)$$

the rank inequality holds, so that

$$\begin{aligned} r_0(A_0K/\mathrm{Fit}(A_0K)) &\leqq r_0(AK/\mathrm{Fit}(AK)) \\ &\leqq r_0(B \cap AK) + r_0(A/\mathrm{Fit}(A)). \end{aligned}$$

The Fitting subgroup $\mathrm{Fit}(A_0K)$ of the normal subgroup A_0K of G is a nilpotent normal subgroup of G and so it is contained in $\mathrm{Fit}(G)$. Hence

$$\begin{aligned} r_0(G/\mathrm{Fit}(G)) &\leqq r_0(G/\mathrm{Fit}(A_0K)) \\ &= r_0(G/A_0K) + r_0(A_0K/\mathrm{Fit}(A_0K)) \\ &\leqq r_0(G/A_0K) + r_0(B \cap AK) + r_0(A/\mathrm{Fit}(A)) \\ &= r_0(B/(B \cap A_0K)) + r_0(B \cap A_0K) + r_0(A/\mathrm{Fit}(A)) \\ &= r_0(B) + r_0(A/\mathrm{Fit}(A)). \end{aligned}$$

This contradiction proves (2).

(3) *If L/K is the Fitting subgroup of G/K and W is the Fitting subgroup of $J = A_0K \cap L$, then W^G is nilpotent with noetherian factor group G/W^G and*

$$r_0(J/W) \leqq r_0(A_0K/\mathrm{Fit}(A_0K)).$$

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If Y is the Fitting subgroup of A_0K, then $Y \cap L$ is contained in W and so

$$\begin{aligned} r_0(J/W) &\leqq r_0(J/(Y\cap L)) = r_0((A_0K\cap L)/(Y\cap L)) \\ &= r_0((A_0K\cap L)Y/Y) \leqq r_0(A_0K/Y). \end{aligned}$$

As $K \leqq J \leqq L$ and L/K is nilpotent, J is a subnormal subgroup of G. Let

$$J = J_0 \lhd J_1 \lhd \ldots \lhd J_t = G$$

be the standard series of J in G. Since G is nilpotent-by noetherian, $\mathrm{Fit}(J_i) \leqq \mathrm{Fit}(J_{I+1})$ for each i. Hence W is contained in $\mathrm{Fit}(G)$ and so W^G is nilpotent. The factor group

$$(A_0W^G\cap L)/W^G = (A_0\cap L)W^G/W^G \simeq (A_0\cap L)/(A_0\cap W^G)$$

is noetherian as an image of the noetherian group

$$(A_0\cap L)/(A_0\cap W) \simeq (A_0\cap L)W/W \leqq J/W.$$

Since also

$$A_0W^G/(A_0W^G\cap L) \simeq A_0L/L \leqq G/L$$

is noetherian, it follows that A_0W^G/W^G and also AW^G/W^G are noetherian. Hence

$$G/W^G = (AW^G/W^G)(BW^G/W^G)$$

is noetherian by the Lennox-Roseblade-Zaicev theorem (see [5]).

(4) *We have that*

$$r_0(A_0K/\mathrm{Fit}(A_0K)) \leqq r_0(B) + 2r_0(A/\mathrm{Fit}(A)).$$

By (1) $K/(F\cap K)$ is finitely generated and $r_0(K/(F\cap K)) \leqq r_0(B)$. Consider the factorized group $AK = A(B\cap AK)$. Application of Lemma 2.1 to the group $AK/(F\cap K)$ yields that

$$\begin{aligned} & r_0((A_0K/(F\cap K))/\mathrm{Fit}(A_0K/(F\cap K))) \\ &\quad \leqq r_0((AK/(F\cap K))/\mathrm{Fit}(AK/(F\cap K))) \\ &\quad \leqq r_0(K/(F\cap K)) + r_0((AK/K)/\mathrm{Fit}(AK/K)) \\ &\quad \leqq r_0(K/(F\cap K)) + r_0(A/\mathrm{Fit}(A)) \leqq r_0(B) + r_0(A/\mathrm{Fit}(A)). \end{aligned}$$

Let $V/(F\cap K)$ be the Fitting subgroup of $A_0K/(F\cap K)$ and $V_0 = V\cap FK$. Since $F\cap K \leqq Z_c(FK)$, it follows that $F\cap K$ is also contained in $Z_c(V_0)$, and so V_0 is a nilpotent normal subgroup of A_0K. Hence

$$\begin{aligned} r_0(A_0K/\mathrm{Fit}(A_0K)) &\leqq r_0(A_0K/V_0) \leqq r_0(A_0K/V) + r_0(A_0K/FK) \\ &\leqq r_0(B) + r_0(A/\mathrm{Fit}(A)) + r_0(A_0/(A_0\cap FK)) \\ &\leqq r_0(B) + 2r_0(A/\mathrm{Fit}(A)). \end{aligned}$$

(5) *The final contradiction.*

By the minimality of the derived length m of S we have that

$$r_0(G/L) \leqq (2(m-1)-3)(r_0(B)+r_0(A/\mathrm{Fit}(A))).$$

Application of (3) and (4) yields

$$\begin{aligned} r_0(G/\mathrm{Fit}(G)) &\leqq r_0(G/W^G) \leqq r_0(AW^G/W^G)+r_0(BW^G/W^G) \\ &\leqq r_0(A_0W^G/W^G)+r_0(B) \\ &= r_0(A_0W^G/(A_0W^G\cap L))+r_0((A_0W^G\cap L)/W^G)+r_0(B) \\ &\leqq r_0(G/L)+r_0(J/W)+r_0(B) \leqq r_0(G/L)+r_0(A_0K/\mathrm{Fit}(A_0K))+r_0(B) \\ &\leqq (2(m-1)-3)(r_0(B)+r_0(A/\mathrm{Fit}(A)))+2r_0(B)+2r_0(A/\mathrm{Fit}(A)) \\ &= (2m-3)(r_0(B)+r_0(A/\mathrm{Fit}(A))). \end{aligned}$$

This contradiction proves Theorem A(b).

The proof of Theorem B(b) is very similar.

Assume that Statement (b) of Theorem B is false. Among all the counterexamples $G = AB$ for which the minimax rank of B is minimal choose one such that the derived length n of the soluble group G is minimal. Let K be the last non-trivial term of the derived series of G. If H is any hypercentral normal subgroup of A with minimax factor group A/H, then also $H(A\cap K)$ is a hypercentral normal subgroup of A with minimax factor group, and of course $m(A/H(A\cap K)) \leqq m(A/H)$. Hence we may assume that $A\cap K$ is contained in H.

Now the proof of Theorem B(b) is accomplished in a series of steps, which are similar to the corresponding steps in the proof of Theorem A(b). Here the torsion-free rank has to be replaced by the minimax rank and by the p^∞-rank, the Fitting subgroup of G by the Gruenberg radical of G and the Fitting subgroup F of A_0 by the subgroup H of A. Then one obtains:

(1) *$K/(H\cap K)$ is a minimax group and*

$$\begin{aligned} &m(K/(H\cap K)) \leqq m(B), \\ &m_p(K/(H\cap K)) \leqq m_p(B) \quad \textit{for each prime } p. \end{aligned}$$

(2) *G is not metabelian.*

(3) *If L/K is the Gruenberg radical of G/K and W is the Gruenberg radical of $J = AK\cap L$, then W^G is a Gruenberg group with minimax factor group G/W^G and*

$$\begin{aligned} &m(J/W) \leqq m(AK/K(AK)), \\ &m_p(J/W) \leqq m_p(AK/K(AK)) \quad \textit{for each prime } p. \end{aligned}$$

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(To see that W^G is a Gruenberg group observe that, for any element x of W, the subgroup $\langle x \rangle$ is ascendant in $\langle x, K \rangle \leqq J$ and $\langle x, K \rangle / K \leqq L/K$ is ascendant in G/K. Hence $\langle x \rangle$ is ascendant in G and W is contained in the Gruenberg radical of G, so that W^G is a Gruenberg group).

(4) *We have that*

$$m(AK/K(AK)) \leqq m(B) + 2m(A/H) \quad \textit{and}$$
$$m_p(AK/K(AK)) \leqq m_p(B) + 2m_p(A/H) \quad \textit{for each prime } p.$$

Now we reach a contradiction in the same way as in the final step (5) of the proof of Theorem A(b). This proves Theorem B(b).

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