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NONCOMMUTATIVE MOTIVES OF AZUMAYA ALGEBRAS

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Abstract Let k be a base commutative ring, R a commutative ring of coefficients, X a quasi-compact quasi-separated k-scheme, and A a sheaf of Azumaya algebras over X of rank r. Under the assumption that $1/r \in R$, we prove that the noncommutative motives with R-coefficients of X and A are isomorphic. As an application, we conclude that a similar isomorphism holds for every R-linear additive invariant. This leads to several computations. Along the way we show that, in the case of finite-dimensional algebras of finite global dimension, all additive invariants are nilinvariant.

Keywords: algebraic K-theory; Azumaya algebras; cyclic homology; nilinvariance; noncommutative algebraic geometry; noncommutative motives

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1. Introduction

Azumaya algebras

Sheaves of Azumaya algebras over schemes X were introduced in the late 1960s by Grothendieck [17]. Formally, a sheaf A of $\mathcal{O}_X$-algebras is *Azumaya* if it is locally free of finite rank over $\mathcal{O}_X$ and the canonical morphism

$$A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(A, A)$$

is an isomorphism. Locally, for the étale topology, A is simply a matrix algebra. This

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generalizes the notion of an Azumaya algebra over a commutative ring [2, 3] and consequently the notion of a central simple algebra over a field.

Noncommutative motives

A *differential graded* $(= dg)$ *category* $\mathcal{A}$, over a base commutative ring k, is a category enriched over complexes of k-modules; see § 4. Every (dg) k-algebra S gives rise to a dg category $\underline{S}$ with a single object and (dg) k-algebra of endomorphisms S. In the same vein, every quasi-compact quasi-separated k-scheme X gives rise to a canonical dg category $\mathsf{perf}_{\mathsf{dg}}(X)$ which enhances the category $\mathsf{perf}(X)$ of perfect complexes of $\mathcal{O}_X$-modules; consult § 6 for details. Let us denote by $\mathsf{dgcat}(k)$ the category of (small) dg categories.

Classical invariants such as algebraic K-theory (K), nonconnective algebraic K-theory $(\mathbb{K})$, Hochschild homology (HH), cyclic homology (HC), periodic cyclic homology (HP), negative cyclic homology (HN), topological Hochschild homology (THH), and topological cyclic homology (TC), extend naturally from k-algebras to dg categories. In order to study all these invariants simultaneously, the notion of an additive invariant was introduced in [40]. Let us now recall it. Given a dg category $\mathcal{A}$, let $T(\mathcal{A})$ be the dg category of pairs (i, x), where $i \in \{1, 2\}$ and x is an object of $\mathcal{A}$. The complex of morphisms in $T(\mathcal{A})$ from (i, x) to (i', x') is given by $\mathcal{A}(x, x')$ if $i' \geqslant i$ and is zero otherwise. Composition is induced by $\mathcal{A}$; consult [40, § 4] for details. Intuitively speaking, $T(\mathcal{A})$ 'dg categorifies' the notion of an upper triangular matrix. Note that we have two inclusion dg functors, $i_1 : \mathcal{A} \hookrightarrow T(\mathcal{A})$ and $i_2 : \mathcal{A} \hookrightarrow T(\mathcal{A})$. A functor $E : \mathsf{dgcat}(k) \to \mathsf{D}$, with values in an additive category, is called an *additive invariant* if it satisfies the following two conditions:

(i) it sends *Morita equivalences* (see § 4) to isomorphisms;

(ii) given a dg category $\mathcal{A}$, the inclusion dg functors induce an isomorphism[1]

$$[E(i_1)\ E(i_2)] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(T(\mathcal{A})).$$

Thanks to the work of Blumberg and Mandell, Keller, Quillen, Schlichting, Thomason and Trobaugh, Waldhausen, and others (see [8, 20, 21, 34, 36, 38, 42, 43]), all the above invariants are additive. Moreover, when applied to $\underline{S}$ (respectively, to $\mathsf{perf}_{\mathsf{dg}}(X)$), they agree with the classical invariants of (dg) k-algebras (respectively, of k-schemes).

Let R be a commutative ring of coefficients. In [40], the *universal additive invariant* (with R-coefficients) was constructed:

$$U(-)_R : \mathsf{dgcat}(k) \longrightarrow \mathsf{Hmo}_0(k)_R. \tag{1.1}$$

Given any R-linear additive category D, there is an induced equivalence of categories,

$$U(-)_R^* : \mathrm{Fun}_{R\text{-linear}}(\mathsf{Hmo}_0(k)_R, \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Additive}}(\mathsf{dgcat}(k), \mathsf{D}), \tag{1.2}$$

where the left-hand side denotes the category of additive R-linear functors and the right-hand side the category of additive invariants. Because of this universal property,

[1] Condition (ii) can be equivalently formulated in terms of semi-orthogonal decompositions in the sense of Bondal and Orlov [9]; see [40, Theorem 6.3(4)].

which is reminiscent from motives, $\mathsf{Hmo}_0(k)_R$ is called the category of *noncommutative motives* (with R-coefficients); consult § 5 for further details. The tensor product of k-algebras also extends naturally to dg categories, thus giving rise to a symmetric monoidal structure $-\otimes-$ on $\mathsf{dgcat}(k)$. After deriving it, this structure descends to $\mathsf{Hmo}_0(k)_R$ and makes the functor 1.1 symmetric monoidal.

Motivation

Let X be a quasi-compact quasi-separated k-scheme and A a sheaf of Azumaya algebras over X. Similarly to $\mathrm{perf}_{\mathsf{dg}}(X)$, one can construct the dg category $\mathrm{perf}_{\mathsf{dg}}(A)$ of perfect complexes of A-modules; see § 6. This dg category reduces to $\mathrm{perf}_{\mathsf{dg}}(X)$ when $A=\mathcal{O}_X$, and comes endowed with a canonical dg functor $-\otimes_{\mathcal{O}_X} A : \mathrm{perf}_{\mathsf{dg}}(X) \to \mathrm{perf}_{\mathsf{dg}}(A)$. One obtains in this way two well-defined noncommutative motives:

$$U(\mathrm{perf}_{\mathsf{dg}}(X))_R \quad U(\mathrm{perf}_{\mathsf{dg}}(A))_R. \tag{1.3}$$

As mentioned above, A is étale-locally a matrix algebra. Hence, up to an étale covering of X, A and $\mathcal{O}_X$ are Morita equivalent. This leads naturally to the following motivating question: *How 'close' are the above noncommutative motives* 1.3?

In this article, we provide a precise and complete answer to this question. As a by-product, we obtain several applications of general interest; see § 3.

2. Statement of results

Let k be a base commutative ring and R a commutative ring of coefficients. Recall that a scheme X is *quasi-compact* if it admits a finite covering by affine open subschemes, and *quasi-separated* if the intersection of any two affine open subschemes is quasi-compact. Note that every such scheme X always has a finite number of connected components. Our main result, which answers the above motivating question, is the following.

THEOREM 2.1. *Let X be a quasi-compact quasi-separated k-scheme with m connected components, and A a sheaf of Azumaya algebras over X of rank $(r_1,\ldots,r_m)$. Assume that $1/r \in R$ with $r := r_1 \times \cdots \times r_m$. Under these assumptions, one has the following isomorphism:*

$$U(-\otimes_{\mathcal{O}_X} A)_R : U(\mathrm{perf}_{\mathsf{dg}}(X))_R \xrightarrow{\sim} U(\mathrm{perf}_{\mathsf{dg}}(A))_R. \tag{2.2}$$

When k is a field and $X=\mathrm{Spec}(k)$, 2.2 is an isomorphism if and only if $1/r\in R$.

Theorem 2.1 shows us that the difference between the noncommutative motives 1.3 is simply an r-torsion phenomenon. In order to prove Theorem 2.1, we have established a K-theoretical result, which is of independent interest. Recall from [44, p. 71] that, given a scheme X with m connected components, one has a well-defined (split surjective) ring homomorphism $\mathsf{rank} : K_0(X) \twoheadrightarrow \mathbb{Z}^m$. Let us write I_X for its kernel. Whenever X is Noetherian[2], of Krull dimension d, and admits an ample sheaf, we have $I_X^{d+1}=0$; see [14, § V Corollary 3.10]. If one does not require a uniform bound on the order of nilpotency of the elements in I_X, then this result may be generalized as follows.

[2]Note that every Noetherian scheme is quasi-compact and quasi-separated.

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Theorem 2.3. *Let X be a quasi-compact quasi-separated scheme X. Under this assumption, every element in $K_0(X)$ of rank zero is nilpotent.*

This statement appears not to exist in the literature. The affine case was proved by Gabber in [15, p. 188] using absolute Noetherian approximation.[3] Making use of Theorem 2.3, one obtains the following useful invertibility result.

Corollary 2.4. Let X be as in Theorem 2.3, and let α be an element in $K_0(X)$ of rank $(r_1, \ldots, r_m)$. Assume that $1/r \in R$ with $r := r_1 \times \cdots \times r_m$. Under these assumptions, the image of α in $K_0(X)_R$ is invertible.

3. Applications

Additive invariants

Let k be a base commutative ring. As explained above, all the classical invariants of quasi-compact quasi-separated k-schemes X can be recovered from the dg category $\mathsf{perf}_{\mathsf{dg}}(X)$. Hence, given a sheaf A of Azumaya algebras over X and an additive invariant $E : \mathsf{dgcat}(k) \to \mathsf{D}$, let us write $E(A)$ for the value of E at $\mathsf{perf}_{\mathsf{dg}}(A)$. By combining Theorem 2.1 with the above equivalence 1.2 of categories, one obtains the following result.

Corollary 3.1. Let X, A, r, R be as in Theorem 2.1, and let $E : \mathsf{dgcat}(k) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under these assumptions, one has an isomorphism $E(X) \simeq E(A)$.

When applied to the above examples of additive invariants, Corollary 3.1 gives rise to the following (concrete) isomorphisms:

$$K_*(X)_{1/r} \simeq K_*(A)_{1/r} \quad \mathbb{K}_*(X)_{1/r} \simeq \mathbb{K}_*(A)_{1/r} \tag{3.2}$$

$$THH_*(X)_{1/r} \simeq THH_*(A)_{1/r} \quad TC_*(X)_{1/r} \simeq TC_*(A)_{1/r}, \tag{3.3}$$

where $(-)_{1/r} := (-)_{\mathbb{Z}[1/r]}$. When $1/r \in k$, one has moreover the following isomorphisms:

$$HH_*(X) \simeq HH_*(A) \quad HC_*(X) \simeq HC_*(A) \tag{3.4}$$

$$HP_*(X) \simeq HP_*(A) \quad HN_*(X) \simeq HN_*(A). \tag{3.5}$$

Remark 3.6. By considering $K, \mathbb{K}, THH$, and TC as spectra-valued functors, one observes that the isomorphisms 3.2–3.2 can be lifted to the homotopy category of spectra localized at the $\mathbb{Z}[1/r]$-linear stable equivalences. In the same vein, the isomorphisms 3.4–3.4 admit a lifting to the derived category of mixed complexes; consult Keller's survey [19, § 5.3] for details.

[3] Ben Antieau [1] has indicated to us how Gabber's approach can be extended to the general case by using absolute Noetherian approximation for quasi-compact quasi-separated schemes and the local global spectral sequence for nonconnective K-theory. Our argument uses simply the Mayer–Vietoris property and does not depend on absolute Noetherian approximation.

The isomorphism $HH_*(X) \simeq HH_*(A)$ is well known and holds without the assumption that $1/r \in k$. In what concerns cyclic homology, the isomorphism $HC_*(X) \simeq HC_*(A)$ was established by Cortiñas and Weibel [13] in the affine case. The algebraic K-theory isomorphism $K_*(X)_{1/r} \simeq K_*(A)_{1/r}$ was obtained recently by Hazrat and Hoobler [18] under the assumption that X is either regular Noetherian or Noetherian of finite Krull dimension with an ample sheaf. Besides these particular cases, all the remaining isomorphisms provided by Corollary 3.1 are, to the best of the authors' knowledge, new in the literature.

Differential operators in positive characteristic

Let k be an algebraically closed field of characteristic $p > 0$, X a smooth k-scheme[4], $T^*X^{(1)}$ the Frobenius twist of the total cotangent bundle of X, and $\mathcal{D}_X$ the sheaf of (crystalline) differential operators on X; consult [7, § 1] for details. As proved by Bezrukavnikov, Mirković, and Rumynin in [7, Theorem 2.2.3], $\mathcal{D}_X$ is a sheaf of Azumaya algebras over $T^*X^{(1)}$ of rank $p^{2\dim(X)}$. In the particular case where X is the affine space $\mathbb{A}^n := \mathsf{Spec}(k[x_1, \ldots, x_n])$, $\mathcal{D}_X$ reduces to the Weyl algebra ($\partial_i := \partial/\partial x_i$)

$$k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle \quad [\partial_i, x_j] = \delta_{ij}$$

and $T^*X^{(1)}$ to polynomials in $2n$ variables $k[x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p]$; consult [7, p. 951] as well as Revoy's work [35]. Thanks to Theorem 2.1 (with $X = T^*X^{(1)}$ and $A = \mathcal{D}_X$), one hence obtains a motivic isomorphism

$$U(\mathrm{perf}_{\mathsf{dg}}(T^*X^{(1)}))_R \simeq U(\mathrm{perf}_{\mathsf{dg}}(\mathcal{D}_X))_R$$

for every commutative ring R containing $1/p$.

Corollary 3.7. Let k, X, R be as above, and let $E : \mathsf{dgcat}(k) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under these assumptions, one has an isomorphism $E(T^*X^{(1)}) \simeq E(\mathcal{D}_X)$.

Cubic fourfolds containing a plane

Let $k = \mathbb{C}$, and let X be a (generic) cubic fourfold, i.e. a smooth complex hypersurface of degree 3 in $\mathbb{P}^5$. In the case where X contains a plane, Kuznetsov constructed in [29] a semi-orthogonal decomposition

$$\mathrm{perf}(X) = \langle \mathrm{perf}(B_S), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle,$$

where S is a smooth projective complex $K3$-surface and B_S a sheaf of Azumaya algebras over S of rank 4. By combining Theorem 2.1 (with $X = S$ and $A = B_S$) with [33, Lemma 5.1], one hence obtains the following motivic decomposition

$$U(\mathrm{perf}_{\mathsf{dg}}(X))_R \simeq U(\mathrm{perf}_{\mathsf{dg}}(S))_R \oplus U(\mathbb{C})_R^{\oplus 3}$$

for every commutative ring R containing $1/2$.

[4]In particular, X is quasi-compact and separated.

Corollary 3.8. Let X, S, R be as above, and let $E : \mathsf{dgcat}(\mathbb{C}) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under these assumptions, one has an isomorphism $E(X) \simeq E(S) \oplus E(\mathbb{C})^{\oplus 3}$.

Severi–Brauer varieties

Let k be a field, A a central simple k-algebra of degree $\sqrt{\dim(A)} = d$, and $SB(A)$ the associated Severi–Brauer variety. As proved in [5, Proposition 2.8], one has the following motivic decomposition

$$U(\mathsf{perf}_{\mathsf{dg}}(SB(A)))_R \simeq U(\underline{k})_R \oplus U(\underline{A})_R \oplus U(\underline{A})_R^{\otimes 2} \oplus \cdots \oplus U(\underline{A})_R^{\otimes d-1} \tag{3.9}$$

for every commutative ring R. Consequently, Theorem 2.1 (with $X = \mathrm{Spec}(k)$), combined with the fact that $U(\underline{k})_R$ is the $\otimes$-unit of $\mathsf{Hmo}_0(k)_R$, allows us to conclude that, whenever $1/d \in R$, 3.9 reduces to

$$U(\mathsf{perf}_{\mathsf{dg}}(SB(A)))_R \simeq \underbrace{U(\underline{k})_R \oplus \cdots \oplus U(\underline{k})_R}_{d\text{-copies}}.$$

Corollary 3.10. Let A, R be as above, and let $E : \mathsf{dgcat}(k) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under these assumptions, one has an isomorphism $E(SB(A)) \simeq E(k)^{\oplus d}$.

Remark 3.11. As pointed out by the referee, the combination of [33, Lemma 5.1] with Bernardara's semi-orthogonal decomposition [4] leads to a generalization of the above motivic decomposition 3.9 to every Severi–Brauer variety $X \to S$ over a smooth projective k-scheme S.

Clifford algebras

Let k be a field of characteristic $\neq 2$, V a finite-dimensional k-vector space of dimension n, and $q : V \to k$ a nondegenerate quadratic form. Recall from [31, § V] that out of this data one can construct the Clifford algebra $C(q)$, the even Clifford algebra $C_0(q)$, and the signed determinant $\delta(q) \in k^\times/(k^\times)^2$. The k-algebra $C(q)$ has dimension 2^n and the k-algebra $C_0(q)$ has dimension 2^{n-1}. When n is odd, we have the following structure results (see [31, § V Theorem 2.4]).

(i) $C_0(q)$ is a central simple k-algebra.

(ii) When $\delta(q) \notin (k^\times)^2$, $C(q)$ is a central simple algebra over its center $k(\sqrt{\delta(q)})$.

(iii) When $\delta(q) \in (k^\times)^2$, $C(q)$ is a product of two isomorphic central simple algebras over the center $k \times k$.

Using Theorem 2.1, we then obtain the following motivic decompositions

$$U(\underline{C_0(q)})_R \simeq U(\underline{k})_R \quad U(\underline{C(q)})_R \simeq \begin{cases} U(\underline{k(\sqrt{\delta(q)})})_R & \text{when } \delta(q) \notin (k^\times)^2 \\ U(\underline{k})_R \oplus U(\underline{k})_R & \text{when } \delta(q) \in (k^\times)^2 \end{cases}$$

for every commutative ring R containing $1/2$. When n is even, we have the (opposite) structure results (see [31, § V Theorem 2.5]).

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(i′) $C(q)$ is a central simple k-algebra.
(ii′) When $\delta(q) \notin (k^\times)^2$, $C_0(q)$ is a central simple algebra over its center $k(\sqrt{\delta(q)})$.
(iii′) When $\delta(q) \in (k^\times)^2$, $C_0(q)$ is a product of two isomorphic central simple algebras over the center $k \times k$.

Using Theorem 2.1 once again, we obtain the motivic decompositions

$$U(\underline{C(q)})_R \simeq U(\underline{k})_R \quad U(\underline{C_0(q)})_R \simeq \begin{cases} U(\underline{k(\sqrt{\delta(q)})})_R & \text{when } \delta(q) \notin (k^\times)^2 \\ U(\underline{k})_R \oplus U(\underline{k})_R & \text{when } \delta(q) \in (k^\times)^2 \end{cases}$$

for every commutative ring R containing $1/2$. Thanks to Corollary 3.1, the above four isomorphisms also hold with U replaced by any additive invariant E with values in an R-linear category.

Quadrics

Let k, q be as in the previous subsection (with $n \geqslant 3$), and let $Q_q \subset \mathbb{P}(V)$ be the associated smooth projective quadric of dimension $n-2$. As explained in the proof of [5, Proposition 2.3], one has the following motivic decomposition

$$U(\mathrm{perf}_{\mathsf{dg}}(Q_q))_R \simeq U(\underline{C_0(q)})_R \oplus \underbrace{U(\underline{k})_R \oplus \cdots \oplus U(\underline{k})_R}_{(n-2)\text{-copies}} \tag{3.12}$$

for every commutative ring R. By combing it with Corollary 3.1 and with the four isomorphisms of the previous subsection, we obtain the following result.

Corollary 3.13. Let k, q be as above, let R be a commutative ring containing $1/2$, and let $E : \mathsf{dgcat}(k) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under these assumptions, one has an isomorphism between $E(Q_q)$ and

(i) $E(k)^{\oplus n-1}$ when n is odd;
(ii) $E(k)^{\oplus n}$ when n is even and $\delta(q) \in (k^\times)^2$;
(iii) $E(C_0(q)) \oplus E(k)^{\oplus n-1}$ when n is even and $\delta(q) \notin (k^\times)^2$.

Remark 3.14. As pointed out by Bernardara, the combination of [33, Lemma 5.1] with Kuznetsov's semi-orthogonal decomposition [30, Theorem 4.2] leads to a generalization of the above motivic decomposition 3.12 to every flat quadric fibration $Q \to S$ over a smooth k-scheme S.

Finite-dimensional algebras of finite global dimension

Let k be a field of characteristic $p \geqslant 0$ and R a commutative ring. We start by describing the behavior of the universal additive invariant with respect to nilpotent extensions.

THEOREM 3.15 (Nilinvariance). *Let S be a finite-dimensional k-algebra of finite global dimension, and let $I \subset S$ be a nilpotent (two-sided) ideal. Assume that*

(i) *the k-algebra S/I has finite global dimension;*

(ii) *the quotient of S by its Jacobson radical $J(S)$ is k-separable (e.g. k perfect) or that $1/p \in R$.*

Under the above assumptions, one has an induced isomorphism $U(\underline{S})_R \xrightarrow{\sim} U(\underline{S/I})_R$.

In the particular case where $I = J(S)$, the above assumption (i) holds automatically since $S/J(S)$ is semi-simple. Hence, modulo assumption (ii), Theorem 3.15 shows us that the noncommutative motives of a finite-dimensional algebra of finite global dimension and of its largest semi-simple quotient are isomorphic.

Corollary 3.16. Let k, S, I, R be as in Theorem 3.15, and let $E : \mathsf{dgcat}(k) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under these assumptions, one has an isomorphism $E(S) \simeq E(S/I)$. In particular, $E(S) \simeq E(S/J(S))$.

The isomorphisms $K_n(S) \simeq K_n(S/I)$, $n \geqslant 0$, are well known. The case $n = 0$ follows from idempotent lifting; see [44, § II Lemma 2.2]. Since S and S/I are regular Noetherian, the cases $n > 0$ follow from the combination of the fundamental theorem (see [44, § V Theorem 3.3]) with dévissage (see [44, § V Corollary 4.2]). All the remaining isomorphisms provided by Corollary 3.16 are, to the best of the authors' knowledge, new in the literature.

Remark 3.17. Theorem 3.15 is *false* when S is of infinite global dimension. An example is given by the k-algebra $S := k[\epsilon]/\epsilon^2$ of dual numbers and by the ideal $I := \epsilon S$. Since S and $S/\epsilon S \simeq k$ are local k-algebras, one has the following isomorphisms:

$$K_1(k[\epsilon]/\epsilon^2) \simeq (k[\epsilon]/\epsilon^2)^* = k^* + k\epsilon \quad K_1(S/\epsilon S) \simeq K_1(k) \simeq k^*; \tag{3.18}$$

see [44, § III Lemma 1.4]. This implies that the induced map $K_1(S) \overset{\epsilon=0}{\to} K_1(S/\epsilon S)$ is not an isomorphism, and so, using Corollary 3.16, one concludes that $U(\underline{S}) \overset{\epsilon=0}{\to} U(S/\epsilon S)$ is not an isomorphism. Note that, in the particular case where k is a finite field, the groups 3.18 are not even abstractly isomorphic, because they have different cardinality. In this case we hence have $U(\underline{S}) \not\simeq U(S/\epsilon S)$.

Let $V_1, \dots, V_m$ be the simple (right) S-modules, and let $D_1 := \mathrm{End}_S(V_1), \dots, D_m := \mathrm{End}_S(V_m)$ be the associated division k-algebras. Thanks to the Artin–Wedderburn theorem, the quotient $S/J(S)$ is Morita equivalent to $D_1 \times \cdots \times D_m$. The center Z_i of D_i is a finite field extension of k, and D_i is a central simple Z_i-algebra. Let $r_i := [D_i : Z_i]$ and $r := r_1 \times \cdots \times r_m$. Using Theorem 2.1 (with $X = \mathrm{Spec}(Z_i)$ and $A = D_i$), one then obtains the following isomorphism

$$U(\underline{S/J(S)})_R \simeq U(\underline{Z_i})_R \oplus \cdots \oplus U(\underline{Z_m})_R \tag{3.19}$$

for every commutative ring R containing $1/r$ or $1/(rp)$ depending on whether we assume that $S/J(S)$ is k-separable or not. The combination of 3.19 with the above Corollary 3.16 gives rise to the following result.

Corollary 3.20. Let k, S, Z_i, R be as above, and let $E : \mathsf{dgcat}(k) \to \mathsf{D}$ be an additive invariant with values in an R-linear category. Under the assumptions of Theorem 3.15, one has an isomorphism $E(S) \simeq E(Z_1) \oplus \cdots \oplus E(Z_m)$.

Intuitively speaking, Corollary 3.20 shows us that up to some torsion all additive invariants of finite-dimensional k-algebras of finite global dimension can be computed using *solely* finite field extensions of k.

Remark 3.21. When k is algebraically closed, we have $D_1 = \cdots = D_m = k$, and consequently $Z_1 = \cdots = Z_m = k$. Corollary 3.20 then reduces (for every commutative ring R) to an isomorphism $E(S) \simeq E(k)^{\oplus m}$. This isomorphism was also obtained by Keller in [22, § 2.3] using different arguments.

Notation

Throughout the article we will reserve the letter k for a base commutative ring, the letter R for a commutative ring of coefficients, the letters A, B, C for sheaves of Azumaya algebras over schemes, the letters $\mathcal{A}, \mathcal{B}$ for dg categories, the letters S, T for (dg) k-algebras, and finally the letters X, Y for k-schemes. All schemes will be assumed to be quasi-compact and quasi-separated. Given a small category $\mathcal{C}$, we will write $\mathrm{Iso}\,\mathcal{C}$ for its set of isomorphism classes of objects.

4. Background on dg categories

Let $\mathcal{C}(k)$ be the category of cochain complexes of k-modules; we use cohomological notation. A *differential graded* (= *dg*) *category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ (morphisms sets $\mathcal{A}(x, y)$ are complexes) in such a way that composition fulfills the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [19]. In what follows, we will write $\mathsf{dgcat}(k)$ for the category of (small) dg categories and dg functors.

Modules

Let $\mathcal{A}$ be a dg category. The category $\mathrm{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by $\mathrm{H}^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y))$, where H^0 denotes degree-zero cohomology. The *opposite* dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ and complexes of morphisms given by $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right $\mathcal{A}$-module* is a dg functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathsf{dg}}(k)$ of cochain complexes of k-modules. Let us denote by $\mathcal{C}(\mathcal{A})$ the category of right $\mathcal{A}$-modules. Recall from [19, § 3.2] that the *derived category* $\mathcal{D}(\mathcal{A})$ *of* $\mathcal{A}$ is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of objectwise quasi-isomorphisms. Its full subcategory of compact objects will be denoted by $\mathcal{D}_c(\mathcal{A})$.

Morita equivalences

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the induced restriction of scalars $\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories; see [19, § 4.6]. As proved in

[40, Theorem 5.3], $\mathsf{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are precisely the Morita equivalences. Let us denote by $\mathsf{Hmo}(k)$ the homotopy category hence obtained.

Tensor product

The *tensor product* $\mathcal{A}\otimes\mathcal{B}$ of two dg categories $\mathcal{A}$ and $\mathcal{B}$ is defined by the Cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and by the complexes of morphisms $(\mathcal{A}\otimes\mathcal{B})((x,z),(y,w)) := \mathcal{A}(x,y)\otimes\mathcal{B}(z,w)$. As explained in [19, § 2.3], this gives rise to a symmetric monoidal structure on $\mathsf{dgcat}(k)$ with $\otimes$-unit the dg category $\underline{k}$. After deriving it $-\otimes^{\mathbb{L}}-$, this symmetric monoidal structure descends to $\mathsf{Hmo}(k)$; consult [19, § 4.3] for details.

Bimodules

Let $\mathcal{A},\mathcal{B}\in\mathsf{dgcat}(k)$. A $\mathcal{A}$-$\mathcal{B}$-*bimodule* B is a right $(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$-module, i.e. a dg functor $\mathsf{B}:\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathsf{dg}}(k)$. Standard examples are the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}\to\mathcal{C}_{\mathsf{dg}}(k)\quad(x,y)\mapsto\mathcal{A}(y,x)\tag{4.1}$$

as well as the $\mathcal{A}$-$\mathcal{B}$-bimodule

$${}_F\mathsf{Bi}:\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathsf{dg}}(k)\quad(x,z)\mapsto\mathcal{B}(z,F(x))\tag{4.2}$$

associated to a dg functor $F:\mathcal{A}\to\mathcal{B}$.

Smoothness and properness

Recall from Kontsevich [23–26] that a dg category $\mathcal{A}$ is called *smooth* if the $\mathcal{A}$-$\mathcal{A}$-bimodule 4.1 belongs to $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}}\otimes^{\mathbb{L}}\mathcal{A})$ and *proper* if for each ordered pair of objects (x,y) we have $\sum_i \operatorname{rank} H^i\mathcal{A}(x,y)<\infty$.

5. Background on noncommutative motives

In this section, we recall from [40] the construction of the category of noncommutative motives; consult also the survey article [37]. Let $\mathcal{A},\mathcal{B}\in\mathsf{dgcat}(k)$. As proved in [40, Corollary 5.10], one has a bijection

$$\mathrm{Hom}_{\mathsf{Hmo}(k)}(\mathcal{A},\mathcal{B})\simeq\mathrm{Iso}\,\mathrm{rep}(\mathcal{A},\mathcal{B}),\tag{5.1}$$

where $\mathrm{rep}(\mathcal{A},\mathcal{B})$ denotes the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes^{\mathbb{L}}\mathcal{B})$ consisting of those $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every $x\in\mathcal{A}$ the right $\mathcal{B}$-module $\mathsf{B}(x,-)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Under 5.1, the composition law in $\mathsf{Hmo}(k)$ corresponds to the (derived) tensor product of bimodules, and the identity of an object $\mathcal{A}$ is given by the isomorphism class of the $\mathcal{A}$-$\mathcal{A}$-bimodule 4.1. Since the $\mathcal{A}$-$\mathcal{B}$-bimodules 4.2 belong to $\mathrm{rep}(\mathcal{A},\mathcal{B})$, we hence obtain a well-defined symmetric monoidal functor:

$$\mathsf{dgcat}(k)\longrightarrow\mathsf{Hmo}(k)\quad F\mapsto{}_F\mathsf{Bi}.\tag{5.2}$$

The *additivization* of $\mathsf{Hmo}(k)$ is the additive category $\mathsf{Hmo}_0(k)$ with the same objects as $\mathsf{Hmo}(k)$ and abelian groups of morphisms given by

$$\mathrm{Hom}_{\mathsf{Hmo}_0(k)}(\mathcal{A},\mathcal{B}):=K_0\,\mathrm{rep}(\mathcal{A},\mathcal{B}),$$

where $K_0\,\mathsf{rep}(\mathcal{A},\mathcal{B})$ is the Grothendieck group of the triangulated category $\mathsf{rep}(\mathcal{A},\mathcal{B})$. The composition law is induced by the (derived) tensor product of bimodules. Note that we also have a canonical functor:

$$\mathsf{Hmo}(k) \longrightarrow \mathsf{Hmo}_0(k) \quad \mathsf{B} \mapsto [\mathsf{B}]. \tag{5.3}$$

Finally, given a commutative ring of coefficients R, the *R-linearization* of $\mathsf{Hmo}_0(k)$ is the R-linear additive category $\mathsf{Hmo}_0(k)_R$ obtained by tensoring each abelian group of morphisms of $\mathsf{Hmo}_0(k)$ with R. This gives rise to a functor

$$\mathsf{Hmo}_0(k) \longrightarrow \mathsf{Hmo}_0(k)_R \quad [\mathsf{B}] \mapsto [\mathsf{B}] \otimes_{\mathbb{Z}} R. \tag{5.4}$$

As proved in [40], the symmetric monoidal structure on $\mathsf{Hmo}(k)$ descends first to a bilinear symmetric monoidal structure on $\mathsf{Hmo}_0(k)$ and then to a R-linear bilinear symmetric monoidal structure on $\mathsf{Hmo}_0(k)_R$, making 5.3–5.4 into symmetric monoidal functors. The universal additive invariant with R-coefficients 1.1 is then defined by the following composition:

$$U(-)_R : \mathsf{dgcat}(k) \xrightarrow{5.2} \mathsf{Hmo}(k) \xrightarrow{5.3} \mathsf{Hmo}_0(k) \xrightarrow{5.4} \mathsf{Hmo}_0(k)_R.$$

Finally, given dg categories $\mathcal{A},\mathcal{B} \in \mathsf{dgcat}(k)$ with $\mathcal{A}$ smooth and proper, the triangulated category $\mathsf{rep}(\mathcal{A},\mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$ identifies with $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$; see [12, § 5]. As a consequence, we obtain the following isomorphism:

$$\mathrm{Hom}_{\mathsf{Hmo}_0(k)_R}(U(\mathcal{A})_R, U(\mathcal{B})_R) \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})_R. \tag{5.5}$$

6. Perfect complexes

Let k be a base commutative ring, X a quasi-compact quasi-separated k-scheme, and A a sheaf of $\mathcal{O}_X$-algebras. We introduce some notation and concepts that are standard in the particular case where $A = \mathcal{O}_X$ (consult [10, § 3] [19, § 4.4] and the references therein) and whose generalization to an arbitrary sheaf A is immediate.

Let $\mathrm{Mod}(A)$ be the Grothendieck category of sheaves of (right) A-modules, $\mathrm{Qcoh}(A)$ the full subcategory of quasi-coherent A-modules, $\mathcal{D}(A) := D(\mathrm{Mod}(A))$ the derived category of A, and $\mathcal{D}_{\mathsf{Qcoh}}(A) \subset \mathcal{D}(A)$ the full triangulated subcategory of those complexes of A-modules with quasi-coherent cohomology. When X is separated, we have $\mathcal{D}_{\mathsf{Qcoh}}(A) \simeq D(\mathrm{Qcoh}(A))$.

Definition 6.1. A complex of A-modules $\mathcal{F} \in \mathcal{D}(A)$ is called *perfect* if there exists a covering $X = \bigcup_i V_i$ of X by affine open subschemes $V_i \subset X$ such that for every i the restriction $\mathcal{F}_{|V_i}$ of $\mathcal{F}$ to V_i is quasi-isomorphic to a bounded complex of finitely generated projective $A_{|V_i}$-modules. Let us denote by $\mathrm{perf}(A)$ the triangulated category of perfect complexes. Note that by construction we have the inclusions $\mathrm{perf}(A) \subset \mathcal{D}_{\mathsf{Qcoh}}(A) \subset \mathcal{D}(A)$.

Let $\mathcal{E}$ be an abelian (or exact) category. As explained in [19, § 4.4], the derived dg category $\mathcal{D}_{\mathsf{dg}}(\mathcal{E})$ of $\mathcal{E}$ is defined as the dg quotient $\mathcal{C}_{\mathsf{dg}}(\mathcal{E})/\mathcal{A}c_{\mathsf{dg}}(\mathcal{E})$ of the dg category of complexes over $\mathcal{E}$ by its full dg subcategory of acyclic complexes. Note that every exact functor $\mathcal{E} \to \mathcal{E}'$ (or more generally every dg functor $\mathcal{C}_{\mathsf{dg}}(\mathcal{E}) \to \mathcal{C}_{\mathsf{dg}}(\mathcal{E}')$ which restricts to $\mathcal{A}c_{\mathsf{dg}}(\mathcal{E}) \to \mathcal{A}c_{\mathsf{dg}}(\mathcal{E}')$) gives rise to a dg functor $\mathcal{D}_{\mathsf{dg}}(\mathcal{E}) \to \mathcal{D}_{\mathsf{dg}}(\mathcal{E}')$.

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Notation 6.2. Let us write $\mathcal{D}_{\mathsf{dg}}(A)$ for the dg category $\mathcal{D}_{\mathsf{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}(A)$, $\mathcal{D}_{\mathrm{Qcoh},\mathsf{dg}}(A)$ for the full dg category of those complexes of A-modules with quasi-coherent cohomology, and $\mathrm{perf}_{\mathsf{dg}}(A)$ for the full dg subcategory of perfect complexes. By construction, we have inclusions $\mathrm{perf}_{\mathsf{dg}}(A) \subset \mathcal{D}_{\mathrm{Qcoh},\mathsf{dg}}(A) \subset \mathcal{D}_{\mathsf{dg}}(A)$ of dg categories and canonical equivalences

$$\mathrm{H}^0(\mathrm{perf}_{\mathsf{dg}}(A)) \simeq \mathrm{perf}(A) \quad \mathrm{H}^0(\mathcal{D}_{\mathrm{Qcoh},\mathsf{dg}}(A)) \simeq \mathcal{D}_{\mathrm{Qcoh}}(A) \quad \mathrm{H}^0(\mathcal{D}_{\mathsf{dg}}(A)) \simeq \mathcal{D}(A).$$

When $A = \mathcal{O}_X$, we will write in what follows X instead of $\mathcal{O}_X$.

Note that we have a well-defined forgetful functor $\mathcal{D}(A) \to \mathcal{D}(X)$ which restricts to $\mathrm{perf}(A) \to \mathrm{perf}(X)$ when A is perfect as a complex of $\mathcal{O}_X$-modules. Since the forgetful functor $\mathrm{Mod}(A) \to \mathrm{Mod}(X)$ is exact, one has similar forgetful dg functors $\mathcal{D}_{\mathsf{dg}}(A) \to \mathcal{D}_{\mathsf{dg}}(X)$ and $\mathrm{perf}_{\mathsf{dg}}(A) \to \mathrm{perf}_{\mathsf{dg}}(X)$. The following (well-known) fact will be used in what follows.

LEMMA 6.3. *Let X and A be as above, with A a sheaf of Azumaya algebras over X. Under these assumptions, the following square is Cartesian:*

$$\begin{array}{ccc} \mathrm{perf}(A) & \longrightarrow & \mathcal{D}(A) \\ {\scriptstyle \mathrm{forget}}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{forget}} \\ \mathrm{perf}(X) & \longrightarrow & \mathcal{D}(X), \end{array}$$

i.e. a complex $\mathcal{F} \in \mathcal{D}(A)$ belongs to $\mathrm{perf}(A)$ *if an only if it belongs to* $\mathrm{perf}(X)$.

Proof. Thanks to the above Definition 6.1, it suffices to prove the affine case where $X = \mathrm{Spec}(S)$ and A is an Azumaya algebra over S. Recall from [28, III § 5] that

(i) A is finitely generated and projective as a right S-module;

(ii) A is *separable*, i.e. A is projective as a A-A-bimodule.

If by hypothesis $\mathcal{F}$ belongs to $\mathrm{perf}(A)$, then condition (i) allows us to conclude that $\mathcal{F}$ also belongs to $\mathrm{perf}(X)$. In order to prove the converse implication, consider the base-change functor $-\otimes_S A : \mathcal{D}(S) \to \mathcal{D}(A)$. By construction, it preserves perfect complexes. Hence, if by hypothesis $\mathcal{F}$ belongs to $\mathrm{perf}(X)$, $\mathcal{F} \otimes_S A$ belongs to $\mathrm{perf}(A)$. Now, consider the following short exact sequence of A-A-bimodules:

$$0 \longrightarrow \mathrm{Ker}(m) \longrightarrow A \otimes_S A \xrightarrow{m} A \longrightarrow 0, \tag{6.4}$$

where m stands for the multiplication of A. Thanks to the above condition (ii), 6.4 splits, and hence A becomes a direct summand of the A-A-bimodule $A \otimes_S A$. Using the canonical isomorphism $\mathcal{F} \otimes_S A \simeq \mathcal{F} \otimes_A (A \otimes_S A)$, one concludes that $\mathcal{F}$ is a direct summand of $\mathcal{F} \otimes_S A$. Since $\mathcal{F} \otimes_S A$ belongs to $\mathrm{perf}(A)$ and this category is idempotent complete, $\mathcal{F}$ also belongs to $\mathrm{perf}(A)$. This completes the proof. □

Every sheaf A of $\mathcal{O}_X$-algebras gives rise to the following dg functor:

$$-\otimes^{\mathbb{L}}_{\mathcal{O}_X} A : \mathcal{C}_{\mathsf{dg}}(\mathrm{Mod}(X)) \longrightarrow \mathcal{C}_{\mathsf{dg}}(\mathrm{Mod}(A)) \quad \mathcal{F} \mapsto \mathcal{F}_{\mathrm{flat}} \otimes_{\mathcal{O}_X} A, \tag{6.5}$$

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where $\mathcal{F}_{\mathsf{flat}}$ denotes a (functorial) $\mathcal{O}_X$-flat resolution of $\mathcal{F}$. Note that, when A is $\mathcal{O}_X$-flat (e.g. A is locally free of finite rank over $\mathcal{O}_X$), 6.5 identifies with $-\otimes_{\mathcal{O}_X} A$. Since 6.5 preserves acyclic and perfect complexes, it induces a dg functor

$$-\otimes^{\mathbb{L}}_{\mathcal{O}_X} A : \mathrm{perf}_{\mathsf{dg}}(X) \longrightarrow \mathrm{perf}_{\mathsf{dg}}(A).$$

7. Proof of Theorem 2.3

Since X is quasi-compact and quasi-separated, the proof can be reduced to the affine case and to a Mayer–Vietoris argument; see [10, Proposition 3.3.1]. The following result is due to Thomason and ,Trobaugh [42, Theorem 8.1]. We have nevertheless decided to include its proof because we will make use of it in Lemma 7.5 below.

LEMMA 7.1. *Let X be a quasi-compact quasi-separated scheme, and U_1, U_2 two Zariski open subschemes. Assume that $X = U_1 \cup U_2$ and write $U_{12} := U_1 \cap U_2$. Under these assumptions, one has the exact sequence.*

$$K_1(U_1) \oplus K_1(U_2) \to K_1(U_{12}) \xrightarrow{\partial} K_0(X) \xrightarrow{\pm} K_0(U_1) \oplus K_0(U_2) \to K_0(U_{12}). \tag{7.2}$$

Proof. Let us write $\iota_1 : U_1 \hookrightarrow X$ and $\iota_2 : U_2 \hookrightarrow X$ for the two open inclusions. Consider the following commutative diagram in $\mathsf{Hmo}(k)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{perf}_{\mathsf{dg}}(X)_Z & \longrightarrow & \mathrm{perf}_{\mathsf{dg}}(X) & \xrightarrow{\mathbb{L}\iota_1^*} & \mathrm{perf}_{\mathsf{dg}}(U_1) & \longrightarrow & 0 \\ & & \sim\downarrow & & \downarrow{\scriptstyle \mathbb{L}\iota_2^*} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{perf}_{\mathsf{dg}}(U_2)_{Z'} & \longrightarrow & \mathrm{perf}_{\mathsf{dg}}(U_2) & \longrightarrow & \mathrm{perf}_{\mathsf{dg}}(U_{12}) & \longrightarrow & 0\,, \end{array} \tag{7.3}$$

where Z (respectively, Z') is the closed set $X - U_1$ (respectively, $U_2 - U_{12}$) and $\mathrm{perf}_{\mathsf{dg}}(X)_Z$ (respectively, $\mathrm{perf}_{\mathsf{dg}}(U_2)_{Z'}$) the dg category of those perfect complexes of $\mathcal{O}_X$-modules (respectively, of $\mathcal{O}_{U_2}$-modules) that are supported on Z (respectively, on Z'). Recall from [19, § 4.6] the notion of a short exact sequence of dg categories. Roughly speaking, it consists of a sequence of dg categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ for which $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{C})$ is exact in the sense of Verdier. As explained in [42, § 5], both rows in 7.3 are short exact sequences of dg categories; see also [19, § 4.6]. Furthermore, as proved in [42, Theorem 2.6.3], the induced dg functor $\mathrm{perf}_{\mathsf{dg}}(X)_Z \xrightarrow{\sim} \mathrm{perf}_{\mathsf{dg}}(U_2)_{Z'}$ is a Morita equivalence and hence an isomorphism in $\mathsf{Hmo}(k)$.

Nonconnective algebraic K-theory gives rise to a functor $\mathbb{K} : \mathsf{Hmo}(k) \to \mathrm{Ho(Spt)}$ with values in the homotopy category of spectra. Among other properties, it sends short exact sequences of dg categories to distinguished triangles of spectra; see [36] [39, Theorem 10.9]. Hence, by applying it to 7.3 we obtain the following morphism between distinguished triangles:

$$\begin{array}{ccccccc} \mathbb{K}\mathrm{perf}_{\mathsf{dg}}(X)_Z & \longrightarrow & \mathbb{K}\mathrm{perf}_{\mathsf{dg}}(X) & \xrightarrow{\mathbb{K}(\mathbb{L}\iota_1^*)} & \mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_1) & \longrightarrow & \Sigma\mathbb{K}\mathrm{perf}_{\mathsf{dg}}(X)_Z \\ \sim\downarrow & & \downarrow{\scriptstyle \mathbb{K}(\mathbb{L}\iota_2^*)} & & \downarrow & & \downarrow\sim \\ \mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_2)_{Z'} & \longrightarrow & \mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_2) & \longrightarrow & \mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_{12}) & \longrightarrow & \Sigma\mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_2)_{Z'}\,. \end{array}$$

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Since the outer left and right vertical maps are isomorphisms, we hence obtain a Mayer–Vietoris long exact sequence:

$$\cdots \to K_{n+1}(U_1) \oplus K_{n+1}(U_2) \to K_{n+1}(U_{12}) \xrightarrow{\partial} K_n(X) \xrightarrow{\pm} K_n(U_1) \oplus K_n(U_2) \to \cdots,$$

where the boundary maps ∂ are obtained from the composition

$$\mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_{12}) \longrightarrow \Sigma\mathbb{K}\mathrm{perf}_{\mathsf{dg}}(U_2)_{Z'} \xrightarrow{\sim} \Sigma\mathbb{K}\mathrm{perf}_{\mathsf{dg}}(X)_Z. \tag{7.4}$$

The exact sequence 7.2 is a chunk of the above one, and so the proof is finished. □

Recall now from [44, § IV Remark 6.6.4] that the pairing

$$-\otimes^{\mathbb{L}}_{\mathcal{O}_X} - : \mathrm{perf}(X) \times \mathrm{perf}(X) \longrightarrow \mathrm{perf}(X)$$

endows $K_*(X)$ with a graded-commutative ring structure.

Lemma 7.5. *Let X, U_1, U_2, U_{12} be as in the above Lemma 7.1. Given an element α in $K_0(X)$, we have the following commutative diagram:*

$$\begin{array}{ccccccccc}
K_1(U_1)\oplus K_1(U_2) & \longrightarrow & K_1(U_{12}) & \xrightarrow{\partial} & K_0(X) & \xrightarrow{\pm} & K_0(U_1)\oplus K_0(U_2) & \longrightarrow & K_0(U_{12}) \\
\downarrow{\scriptstyle -\cdot\alpha_1\oplus -\cdot\alpha_2} & & \downarrow{\scriptstyle -\cdot\alpha_{12}} & & \downarrow{\scriptstyle -\cdot\alpha} & & \downarrow{\scriptstyle -\cdot\alpha_1\oplus -\cdot\alpha_2} & & \downarrow{\scriptstyle -\cdot\alpha_{12}} \\
K_1(U_2)\oplus K_1(U_2) & \longrightarrow & K_1(U_{12}) & \xrightarrow[\partial]{} & K_0(X) & \xrightarrow[\pm]{} & K_0(U_1)\oplus K_0(U_2) & \longrightarrow & K_0(U_{12}),
\end{array}$$

where α_1, α_2, and α_{12} denote the images of α in $K_0(U_1), K_0(U_2)$, and $K_0(U_{12})$, respectively.

Proof. Recall first from Thomason [41, § 1.6] that every element α in $K_0(X)$ is of the form $\alpha = [\mathcal{F}]$ for some perfect complex $\mathcal{F}$. Note that we have the following commutative cube in the homotopy category $\mathsf{Hmo}(k)$:

$$\begin{array}{ccccccc}
\mathrm{perf}_{\mathsf{dg}}(X) & & & \xrightarrow{\mathbb{L}\iota_1^*} & & & \mathrm{perf}_{\mathsf{dg}}(U_1) \\
 & \nwarrow{\scriptstyle -\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{F}} & & & & \nearrow{\scriptstyle -\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{F}_1} & \\
 & & \mathrm{perf}_{\mathsf{dg}}(X) & \xrightarrow{\mathbb{L}\iota_1^*} & \mathrm{perf}_{\mathsf{dg}}(U_1) & & \\
\downarrow{\scriptstyle \mathbb{L}\iota_2^*} & & \downarrow{\scriptstyle \mathbb{L}\iota_2^*} & & \downarrow & & \downarrow \\
 & & \mathrm{perf}_{\mathsf{dg}}(U_2) & \longrightarrow & \mathrm{perf}_{\mathsf{dg}}(U_{12}) & & \\
 & \swarrow{\scriptstyle -\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{F}_2} & & & & \searrow{\scriptstyle -\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{F}_{12}} & \\
\mathrm{perf}_{\mathsf{dg}}(U_2) & & & \longrightarrow & & & \mathrm{perf}_{\mathsf{dg}}(U_{12}),
\end{array} \tag{7.6}$$

where $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_{12}$ denote the restriction of $\mathcal{F}$ to U_1, U_2, and U_{12}, respectively. Following the proof of Lemma 7.1, one observes that the commutative cube 7.6 gives

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rise to a morphism between Mayer–Vietoris long exact sequences:

$$\begin{array}{ccccccc}
\cdots K_{n+1}(U_1)\oplus K_{n+1}(U_2) & \longrightarrow & K_{n+1}(U_{12}) & \xrightarrow{\partial} & K_n(X) & \xrightarrow{\pm} & K_n(U_1)\oplus K_n(U_2)\cdots \\
{\scriptstyle -\cdot\alpha_1\oplus-\cdot\alpha_2}\downarrow & & {\scriptstyle -\cdot\alpha_{12}}\downarrow & & {\scriptstyle -\cdot\alpha}\downarrow & & {\scriptstyle -\cdot\alpha_1\oplus-\cdot\alpha_2}\downarrow \\
\cdots K_{n+1}(U_2)\oplus K_{n+1}(U_2) & \longrightarrow & K_{n+1}(U_{12}) & \xrightarrow[\partial]{} & K_n(X) & \xrightarrow[\pm]{} & K_n(U_1)\oplus K_n(U_2)\cdots ,
\end{array}$$

where the commutativity of the middle square follows from composition 7.4. The diagram of Lemma 7.5 is a chunk of this one, and so the proof is finished. □

LEMMA 7.7. *Let X, U_1, U_2, U_{12}, and $\alpha, \alpha_1, \alpha_2, \alpha_{12}$ be as in Lemma 7.5. Whenever α_1, α_2, and α_{12} are nilpotent, α is also nilpotent.*

Proof. Since by hypothesis α_1, α_2, and α_{12} are nilpotent, there exists an integer $N > 0$ for which the homomorphisms

$$K_0(U_1) \xrightarrow{-\cdot\alpha_1^N} K_0(U_1) \quad K_0(U_2) \xrightarrow{-\cdot\alpha_2^N} K_0(U_2) \quad K_1(U_{12}) \xrightarrow{-\cdot\alpha_{12}^N} K_1(U_{12})$$

are all trivial. Consequently, using the above Lemma 7.5 (with α replaced by α^N), one obtains the following commutative diagram:

$$\begin{array}{ccccc}
K_1(U_{12}) & \xrightarrow{\partial} & K_0(X) & \xrightarrow{\pm} & K_0(U_1)\oplus K_0(U_2) \\
{\scriptstyle -\cdot\alpha_{12}^N}\downarrow & & {\scriptstyle -\cdot\alpha^N}\downarrow & & {\scriptstyle -\cdot\alpha_1^N\oplus-\cdot\alpha_2^N}\downarrow{\scriptstyle \mathbf{0}} \\
K_1(U_{12}) & \xrightarrow{\partial} & K_0(X) & \xrightarrow{\pm} & K_0(U_1)\oplus K_0(U_2) \\
{\scriptstyle -\cdot\alpha_{12}^N}\downarrow{\scriptstyle \mathbf{0}} & & {\scriptstyle -\cdot\alpha^N}\downarrow & & {\scriptstyle -\cdot\alpha_1^N\oplus-\cdot\alpha_2^N}\downarrow \\
K_1(U_{12}) & \xrightarrow[\partial]{} & K_0(X) & \xrightarrow[\pm]{} & K_0(U_1)\oplus K_0(U_2)\,.
\end{array}$$

A simple diagram chasing argument then shows that the composition of the middle vertical arrows is zero, i.e. $-\cdot\alpha^{2N} = 0$. This implies that $\alpha^{2N} = 0$, and hence that α is nilpotent. □

We now have all the ingredients needed for the proof of Theorem 2.3. Recall that X has m connected components and that I_X stands for the kernel of the ring homomorphism $\mathsf{rank} : K_0(X) \twoheadrightarrow \mathbb{Z}^m$. Let α be an element in $I_X \subset K_0(X)$. One needs to show that α is nilpotent. This will be done in three steps.

Step 1

We claim that $X = \bigcup_{i=1}^n V_i$, where $V_i \subset X$ is an affine open subscheme such that the image of α in $K_0(V_i)$ is zero. Let $x \in X$, and let V_x be an affine open neighborhood of x. The image $\alpha_{|V_x}$ of α in $K_0(V_x)$ can be written as $[P]-[Q]$, with P and Q two vector bundles of the same rank. By shrinking V_x we can assume that P and Q are free of the same rank, and hence are isomorphic. As a consequence, $\alpha_{|V_x} = 0$. Finally, using quasi-compactness, we may take a finite subcover $\{V_i\}_{i=1}^n$ of $\{V_x\}_{x\in X}$, which yields the above claim.

Step 2

Assume that X is a quasi-compact *separated* scheme. We prove Theorem 2.3 using induction on the number of affine open subschemes in a covering trivializing α, as in Step 1. The case $n = 1$ is clear. Let us then assume that $n > 1$, and write $U_1 := \cup_{i=1}^{n-1} V_i$, $U_2 := V_n$, and $U_{12} := U_1 \cap U_2$. Since by hypothesis X is separated, $V_i \cap V_j$ is affine for all i, j, and so U_1 and U_{12} are covered by $n-1$ affine open subschemes on which the restriction of α is trivial. By our induction hypothesis, $\alpha_1, \alpha_2, \alpha_{12}$ are nilpotent. Hence, using the above Lemma 7.7, we conclude that α is also nilpotent.

Step 3

Assume that X is a quasi-compact quasi-separated scheme. Let U_1, U_2, and U_{12} be as in Step 2. Note that U_1 is covered by $n-1$ affine open subschemes on which α is trivial, and that U_2 and U_{12} are separated (since U_2 is affine and $U_{12} \subset U_2$). Therefore, using induction and Step 2, we can again assume that $\alpha_1, \alpha_2, \alpha_{12}$ are nilpotent. We finish the proof by invoking Lemma 7.7 once again.

Proof of Corollary 2.4

Every ring homomorphism $R \to R'$ gives rise to a well-defined ring homomorphism $K_0(X)_R \to K_0(X)_{R'}$. Hence, since $\mathbb{Z}[1/r]$ is initial among the rings containing $1/r$, it suffices to prove the particular case $R := \mathbb{Z}[1/r]$. Note that since $\mathbb{Z}[1/r]$ is torsion free we have the following short exact sequence:

$$0 \to I_X \otimes \mathbb{Z}[1/r] \subset K_0(X)_{\mathbb{Z}[1/r]} \overset{\text{rank}}{\twoheadrightarrow} \mathbb{Z}^m \otimes \mathbb{Z}[1/r] \to 0.$$

Moreover, thanks to Theorem 2.3, every element in $I_X \otimes \mathbb{Z}[1/r]$ is nilpotent. The rank homomorphism is surjective, and so there exists an element $\beta \in K_0(X)_{\mathbb{Z}[1/r]}$ of rank $(1/r_1, \ldots, 1/r_m)$. Therefore, $\alpha \cdot \beta$ is of rank $(1, \ldots, 1)$, and consequently $([\mathcal{O}_X] - \alpha \cdot \beta) \in I_X \otimes \mathbb{Z}[1/r]$. There exists then an integer $N > 0$ such that $([\mathcal{O}_X] - \alpha \cdot \beta)^{N+1} = 0$. This implies that the following element

$$[\mathcal{O}_X] + ([\mathcal{O}_X] - \alpha \cdot \beta) + ([\mathcal{O}_X] - \alpha \cdot \beta)^2 + \cdots + ([\mathcal{O}_X] - \alpha \cdot \beta)^N \in K_0(X)_{\mathbb{Z}[1/r]}$$

is the inverse of $\alpha \cdot \beta$, and hence that α is invertible in $K_0(X)_{\mathbb{Z}[1/r]}$.

8. Proof of Theorem 2.1

The proof is divided into two steps. First, we introduce an auxiliary $\mathbb{Z}[1/r]$-linear category $\mathsf{Az}_0(X)_{1/r}$ of sheaves of Azumaya algebras over X and prove the analog of Theorem 2.1 therein; see Proposition 8.3. Then, we construct a $\mathbb{Z}[1/r]$-linear functor from $\mathsf{Az}_0(X)_{1/r}$ to the category $\mathsf{Hmo}_0(k)_{\mathbb{Z}[1/r]}$ of noncommutative motives.

Note that every ring homomorphism $R \to R'$ gives rise to a well-defined additive functor $\mathsf{Hmo}_0(k)_R \to \mathsf{Hmo}_0(k)_{R'}$. Hence, since $\mathbb{Z}[1/r]$ is initial among the rings containing $1/r$, it suffices to prove the case $R = \mathbb{Z}[1/r]$.

Auxiliary category $\mathsf{Az}_0(X)$

Given two sheaves A and B of Azumaya algebras over X, let $\mathrm{rep}(A, B)$ be the full triangulated subcategory of $\mathcal{D}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$ consisting of those A-B-bimodules ${}_A\mathsf{B}_B$ such that $\mathsf{B}_B \in \mathrm{perf}(B)$.

LEMMA 8.1. *The full triangulated subcategories* $\mathrm{rep}(A, B)$ *and* $\mathrm{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$ *of* $\mathcal{D}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$ *are the same.*

Proof. We start with the inclusion $\mathrm{rep}(A, B) \subseteq \mathrm{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$. Let ${}_A\mathsf{B}_B$ be an object of $\mathrm{rep}(A, B) \subset \mathcal{D}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$. By definition, $\mathsf{B}_B \in \mathrm{perf}(B)$. Hence, Lemma 6.3 (with $A = B$) shows us that $B \in \mathrm{perf}(X)$. Using Lemma 6.3 (with $A = A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B$) again, we conclude that ${}_A\mathsf{B}_B \in \mathrm{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$.

We now show the converse inclusion. Let ${}_A\mathsf{B}_B$ be an object of $\mathrm{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B) \subset \mathcal{D}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B)$. Lemma 6.3 (with $A = A^{\mathrm{op}} \otimes_{\mathcal{O}_X} B$) shows us that $\mathsf{B} \in \mathrm{perf}(X)$. Using Lemma 6.3 (with $A = B$) again, we conclude that $\mathsf{B}_B \in \mathrm{perf}(B)$. By definition, this implies that ${}_A\mathsf{B}_B \in \mathrm{rep}(A, B)$, and so the proof if finished. □

Let $\mathsf{Az}(X)$ be the category whose objects are the sheaves of Azumaya algebras over X, whose morphisms are given by $\mathrm{Hom}_{\mathsf{Az}(X)}(A, B) := \mathrm{Iso}\,\mathrm{rep}(A, B)$, and whose composition law is induced by

$$\mathrm{rep}(A, B) \times \mathrm{rep}(B, C) \longrightarrow \mathrm{rep}(A, C) \quad ({}_A\mathsf{B}_B, {}_B\mathsf{B}'_C) \mapsto {}_A\mathsf{B} \otimes^{\mathbb{L}}_B \mathsf{B}'_C. \tag{8.2}$$

Note that the identity of an object $A \in \mathsf{Az}(X)$ is given by the isomorphism class of the A-A-bimodule ${}_AA_A$. The *additivization* of $\mathsf{Az}(X)$ is the additive category $\mathsf{Az}_0(X)$ with the same objects as $\mathsf{Az}(X)$ and with abelian groups of morphisms given by $\mathrm{Hom}_{\mathsf{Az}_0(X)}(A, B) := K_0\,\mathrm{rep}(A, B)$, where $K_0\,\mathrm{rep}(A, B)$ is the Grothendieck group of the triangulated category $\mathrm{rep}(A, B)$. The composition law is induced by the above bi-triangulated functor 8.2. Note that we have a functor:

$$\mathsf{Az}(X) \to \mathsf{Az}_0(X) \quad {}_A\mathsf{B}_B \mapsto [{}_A\mathsf{B}_B].$$

Finally, the $\mathbb{Z}[1/r]$-*linearization* of $\mathsf{Az}_0(X)$ is the $\mathbb{Z}[1/r]$-linear category $\mathsf{Az}_0(X)_{1/r}$ obtained by tensoring each abelian group of morphisms of $\mathsf{Az}_0(X)$ with $\mathbb{Z}[1/r]$. This gives rise to the functor

$$\mathsf{Az}_0(X) \to \mathsf{Az}_0(X)_{1/r} \quad [{}_A\mathsf{B}_B] \mapsto [{}_A\mathsf{B}_B]_{1/r} := [{}_A\mathsf{B}_B] \otimes_{\mathbb{Z}} \mathbb{Z}[1/r].$$

PROPOSITION 8.3. *Let X, A be as in Theorem 2.1. Under these assumptions and using the above notation, one has the isomorphism $[{}_{\mathcal{O}_X}A_A]_{1/r} : \mathcal{O}_X \xrightarrow{\sim} A$ in $\mathsf{Az}_0(X)_{1/r}$.*

Proof. By definition, A is locally free of finite rank over $\mathcal{O}_X$. Consequently, the A-$\mathcal{O}_X$-bimodule ${}_AA_{\mathcal{O}_X}$ belongs to $\mathrm{rep}(A, \mathcal{O}_X)$, and so one obtains a well-defined morphism $[{}_AA_{\mathcal{O}_X}]_{1/r} : A \to \mathcal{O}_X$ in $\mathsf{Az}_0(X)_{1/r}$. The proof will consist in showing that both compositions

$$[{}_{\mathcal{O}_X}A_A]_{1/r} \circ [{}_AA_{\mathcal{O}_X}]_{1/r} \quad [{}_AA_{\mathcal{O}_X}]_{1/r} \circ [{}_{\mathcal{O}_X}A_A]_{1/r} \tag{8.4}$$

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are isomorphisms. Thanks to the above Lemma 8.1 (with $A = B = \mathcal{O}_X$), one has the following $\mathbb{Z}[1/r]$-algebra isomorphism:

$$\mathsf{End}_{\mathsf{Az}_0(X)_{1/r}}(\mathcal{O}_X) := K_0(\mathsf{rep}(\mathcal{O}_X, \mathcal{O}_X))_{1/r} \simeq K_0(\mathsf{perf}(X))_{1/r} =: K_0(X)_{1/r}, \tag{8.5}$$

where the right-hand side is endowed with the multiplication induced by $- \otimes^{\mathbb{L}}_{\mathcal{O}_X} -$. Since $A \otimes_A A \simeq A$, the composition $[{}_{\mathcal{O}_X}A_A]_{1/r} \circ [{}_A A_{\mathcal{O}_X}]_{1/r}$ equals $[{}_{\mathcal{O}_X}A_{\mathcal{O}_X}]_{1/r}$. Hence, since by hypothesis A is of rank $(r_1, \dots, r_m)$, we conclude from Corollary 2.4 and from isomorphism 8.5 that $[{}_{\mathcal{O}_X}A_{\mathcal{O}_X}]_{1/r}$ is invertible in $\mathsf{End}_{\mathsf{Az}_0(X)_{1/r}}(\mathcal{O}_X)$. The first composition in 8.4 is then an isomorphism.

Let us now prove that the second composition in 8.4 is also an isomorphism. Thanks to Lemma 8.1 (with $A = B$), one has the $\mathbb{Z}[1/r]$-algebra isomorphism

$$\mathsf{End}_{\mathsf{Az}_0(X)_{1/r}}(A) := K_0(\mathsf{rep}(A, A))_{1/r} \simeq K_0(\mathsf{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A))_{1/r}, \tag{8.6}$$

where the right-hand side is endowed with the multiplication induced by $- \otimes^{\mathbb{L}}_A -$. On the other hand, Lemma 8.10 below furnishes us with the following ring isomorphism:

$$K_0(X) \xrightarrow{\sim} K_0(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A) \quad \mathcal{F} \mapsto \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} A. \tag{8.7}$$

Now, note that the composition $[{}_A A_{\mathcal{O}_X}]_{1/r} \circ [{}_{\mathcal{O}_X}A_A]_{1/r}$ is equal to $[{}_A A \otimes_{\mathcal{O}_X} A_A]_{1/r}$. There exists then a unique element α in $K_0(X)$ which is mapped to $[{}_A A \otimes_{\mathcal{O}_X} A_A]$ via the above isomorphism 8.7. We claim that $\mathsf{rank}(\alpha) = (r_1, \dots, r_m)$. In order to prove this claim, consider the composed functor

$$\mathsf{perf}(X) \xrightarrow{8.11} \mathsf{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A) \xrightarrow{\text{forget}} \mathsf{perf}(X). \tag{8.8}$$

Since the $\mathcal{O}_X$-rank of A is $(r_1, \dots, r_m)$, 8.8 gives rise to the commutative square

$$\begin{array}{ccc} K_0(X) & \xrightarrow{-\cdot[A]} & K_0(X) \\ {\scriptstyle \mathsf{rank}}\downarrow & & \downarrow{\scriptstyle \mathsf{rank}} \\ \mathbb{Z}^m & \xrightarrow[-\cdot(r_1,\dots,r_m)]{} & \mathbb{Z}^m . \end{array} \tag{8.9}$$

The equalities $\mathsf{rank}(\alpha \cdot [A]) = \mathsf{rank}([A \otimes_{\mathcal{O}_X} A]) = (r_1, \dots, r_m)^2$, combined with the commutativity of 8.9 and the injectivity of the homomorphism $- \cdot (r_1, \dots, r_m)$, allow us then to conclude that $\mathsf{rank}(\alpha) = (r_1, \dots, r_m)$. Thanks to Corollary 2.4, the element α then becomes invertible in $K_0(X)_{1/r}$, and so, using 8.6 and the $\mathbb{Z}[1/r]$-linearization of 8.7, one concludes that $[{}_A A \otimes_{\mathcal{O}_X} A_A]_{1/r}$ is invertible in $\mathsf{End}_{\mathsf{Az}_0(X)_{1/r}}(A)$. This implies that the second composition in 8.4 is also an isomorphism, and so the proof is finished. □

LEMMA 8.10. *Let X, A be as in Theorem 2.1. Under these assumptions, one has the following equivalence of monoidal triangulated categories:*

$$\mathsf{perf}(X) \xrightarrow{\sim} \mathsf{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A) \quad \mathcal{F} \mapsto \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} A, \tag{8.11}$$

where the monoidal structure on $\mathsf{perf}(X)$ (respectively, on $\mathsf{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A)$) is induced by $- \otimes^{\mathbb{L}}_{\mathcal{O}_X} -$ (respectively, by $- \otimes^{\mathbb{L}}_A -$).

Remark 8.12. Since the monoidal structure on $\mathrm{perf}(X)$ is symmetric, we conclude from 8.11 that the monoidal structure on $\mathrm{perf}(A^{\mathrm{op}} \otimes_{\mathcal{O}_X} A)$ is also symmetric.

Proof. The fact that 8.11 is monoidal follows from the canonical isomorphisms

$$(\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} A) \otimes^{\mathbb{L}}_A (\mathcal{F}' \otimes^{\mathbb{L}}_{\mathcal{O}_X} A) \simeq (\mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{F}') \otimes^{\mathbb{L}}_{\mathcal{O}_X} A.$$

In order to prove that 8.11 is moreover an equivalence, it suffices from Definition 6.1 to show the affine case where $X = \mathrm{Spec}(S)$ and A is an Azumaya algebra over S. As explained in [28, § III Theorem 5.1 3], one has an equivalence of categories:

$$\mathrm{Mod}(S) \xrightarrow{\sim} \mathrm{Mod}(A^{\mathrm{op}} \otimes_S A) \quad \mathcal{F} \mapsto \mathcal{F} \otimes_S A. \tag{8.13}$$

Since 8.13 preserves finitely generated projective modules, one concludes from the definition of a perfect complex that 8.11 is an equivalence in the affine case. This completes the proof. □

Remark 8.14. Using Proposition 8.3, one observes that the category $\mathsf{Az}_0(X)_{\mathbb{Q}}$ (obtained by tensoring each abelian group of morphisms of $\mathsf{Az}_0(X)$ with $\mathbb{Q}$) has a single isomorphism class. Kontsevich calls such categories 'algebroids' [27, § 1.1]. Intuitively speaking, all the complexity of $\mathsf{Az}_0(X)$ is torsion.

From $\mathsf{Az}_0(X)$ to noncommutative motives

Let $A, B \in \mathsf{Az}_0(X)$. Note that every A-B-bimodule ${}_A\mathsf{B}_B \in \mathrm{rep}(A, B)$ gives rise to a dg functor

$$- \otimes^{\mathbb{L}}_A \mathsf{B} : \mathrm{perf}_{\mathsf{dg}}(A) \longrightarrow \mathrm{perf}_{\mathsf{dg}}(B) \quad \mathcal{F} \mapsto \mathcal{F}_{\mathrm{flat}} \otimes_A \mathsf{B}$$

and consequently to a bimodule ${}_{(-\otimes^{\mathbb{L}}_A \mathsf{B})}\mathsf{Bi}$ which belongs to $\mathrm{rep}(\mathrm{perf}_{\mathsf{dg}}(A), \mathrm{perf}_{\mathsf{dg}}(B))$; recall from 4.2 the notation ${}_{-}\mathsf{Bi}$. Similarly, every morphism $f : {}_A\mathsf{B}_B \to {}_A\mathsf{B}'_B$ of A-B-bimodules gives rise to a morphism of dg functors $\nu_f : - \otimes^{\mathbb{L}}_A \mathsf{B} \Rightarrow - \otimes^{\mathbb{L}}_A \mathsf{B}'$ (see [19, § 2.3]), and consequently to a morphism of bimodules ${}_{\nu_f}\mathsf{Bi} : {}_{-\otimes^{\mathbb{L}}_A \mathsf{B}}\mathsf{Bi} \Rightarrow {}_{-\otimes^{\mathbb{L}}_A \mathsf{B}'}\mathsf{Bi}$.

Lemma 8.15. *The above constructions give rise to a triangulated functor:*

$$\mathrm{rep}(A, B) \longrightarrow \mathrm{rep}(\mathrm{perf}_{\mathsf{dg}}(A), \mathrm{perf}_{\mathsf{dg}}(B)). \tag{8.16}$$

Proof. Note that, when f is a quasi-isomorphism, $\mathrm{H}^0(\nu_f)$ is a natural isomorphism between triangulated functors. Using [6, Lemma 9.8], one then concludes that ${}_{\nu_f}\mathsf{Bi}$ is a quasi-isomorphism. This implies that 8.16 is well defined. The fact that it is triangulated is clear. □

Proposition 8.17. *The assignment $A \mapsto U(\mathrm{perf}_{\mathsf{dg}}(A))$ on objects and ${}_A\mathsf{B}_B \mapsto U({}_{(-\otimes^{\mathbb{L}}_A \mathsf{B})}\mathsf{Bi})$ on morphisms gives rise to a well-defined functor:*

$$\mathsf{Az}_0(X) \longrightarrow \mathsf{Hmo}_0(k). \tag{8.18}$$

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Proof. We start by verifying that the assignment $A \mapsto \mathsf{perf}_{\mathsf{dg}}(A)$ on objects and ${}_A\mathsf{B}_B \mapsto {}_{(-\otimes^{\mathbb{L}}_A \mathsf{B})}\mathsf{Bi}$ on morphisms gives rise to a well-defined functor from $\mathsf{Az}_0(X)$ to $\mathsf{Hmo}(k)$. Thanks to 8.16, one has well-defined morphisms:

$$\mathrm{Hom}_{\mathsf{Az}(X)}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{Hmo}(k)}(\mathsf{perf}_{\mathsf{dg}}(A), \mathsf{perf}_{\mathsf{dg}}(B)).$$

Given bimodules ${}_A\mathsf{B}_B \in \mathrm{rep}(A, B)$ and ${}_B\mathsf{B}'_C \in \mathrm{rep}(B, C)$, the associativity of the (derived) tensor product gives rise to a canonical isomorphism of dg functors (and consequently to an isomorphism of bimodules):

$$(-\otimes^{\mathbb{L}}_A \mathsf{B}) \otimes^{\mathbb{L}}_B \mathsf{B}' \simeq -\otimes^{\mathbb{L}}_A (\mathsf{B} \otimes^{\mathbb{L}}_B \mathsf{B}') \quad {}_{(-\otimes^{\mathbb{L}}_A \mathsf{B})\otimes^{\mathbb{L}}_B \mathsf{B}'}\mathsf{Bi} \simeq {}_{-\otimes^{\mathbb{L}}_A (\mathsf{B}\otimes^{\mathbb{L}}_B \mathsf{B}')}\mathsf{Bi}.$$

This shows that the assignment ${}_A\mathsf{B}_B \mapsto {}_{(-\otimes^{\mathbb{L}}_A \mathsf{B})}\mathsf{Bi}$ preserves the composition operation. The identities are also preserved, since the A-A-bimodule ${}_A A_A$ is mapped to the identity bimodule ${}_{\mathsf{id}}\mathsf{Bi} = \mathsf{id}_{\mathsf{perf}_{\mathsf{dg}}(A)}$. In conclusion, we obtain a functor:

$$\mathsf{Az}(X) \longrightarrow \mathsf{Hmo}(k). \tag{8.19}$$

Now, from Lemma 8.15 and from the construction of the categories $\mathsf{Az}_0(X)$ and $\mathsf{Hmo}_0(k)$, one concludes that the searched functor 8.18 is the additivization of 8.19. This completes the proof. ∎

We now have all the ingredients needed for the conclusion of the proof of Theorem 2.1. By $\mathbb{Z}[1/r]$-linearizing the above functor 8.18, one obtains the following commutative diagram:

$$\begin{array}{ccc} \mathsf{Az}_0(X) & \xrightarrow{(8.18)} & \mathsf{Hmo}_0(k) \\ \big\downarrow{\scriptstyle (-)_{1/r}} & & \big\downarrow{\scriptstyle (-)_{\mathbb{Z}[1/r]}} \\ \mathsf{Az}_0(X)_{1/r} & \xrightarrow[(8.18)]{} & \mathsf{Hmo}_0(k)_{\mathbb{Z}[1/r]}\,. \end{array}$$

Hence, the image of the isomorphism $[_{\mathcal{O}_X} A_A]_{1/r} : \mathcal{O}_X \xrightarrow{\sim} A$ of Proposition 8.3 under 8.18 identifies with the following isomorphism:

$$U(-\otimes_{\mathcal{O}_X} A)_{\mathbb{Z}[1/r]} : U(\mathsf{perf}_{\mathsf{dg}}(X))_{\mathbb{Z}[1/r]} \xrightarrow{\sim} U(\mathsf{perf}_{\mathsf{dg}}(A))_{\mathbb{Z}[1/r]}.$$

This concludes the proof of the first claim of Theorem 2.1.

Assume now that k is a field and that $X = \mathrm{Spec}(k)$. In this case, A is a central simple k-algebra and $r = \dim(A)$. Let us prove that, if by hypothesis $U(\underline{k})_R \simeq U(\underline{A})_R$, then $1/\dim(A) \in R$. Thanks to the Wedderburn theorem (see [16, Theorem 2.1.3]), $A \simeq M_{n\times n}(D)$ for some integer $n \geqslant 1$ and division algebra $D \supseteq k$. This implies that A and D are Morita equivalent, and consequently that $U(\underline{k})_R \simeq U(\underline{A})_R \simeq U(\underline{D})_R$. Consider now the additive invariant

$$K_0(-)_R : \mathsf{dgcat}(k) \longrightarrow \mathsf{Mod}(R) \tag{8.20}$$

with values in the category of R-modules. Thanks to the equivalence of categories 1.2, 8.20 descends to $\mathsf{Hmo}_0(k)_R$. Hence, it induces an R-linear homomorphism:

$$\mathrm{Hom}_{\mathsf{Hmo}_0(k)_R}(U(\underline{D})_R, U(\underline{k})_R) \longrightarrow \mathrm{Hom}_{\mathsf{Mod}(R)}(K_0(D)_R, K_0(k)_R). \tag{8.21}$$

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Since k is a field, we have $K_0(k)_R \simeq R$. Similarly, since D is a division k-algebra, $K_0(D)_R \simeq R$. As a consequence, the right-hand side of 8.21 identifies with the R-linear endomorphisms of R. In what concerns the left-hand side, we have

$$\mathrm{Hom}_{\mathsf{Hmo}_0(k)_R}(U(\underline{D})_R, U(\underline{k})_R) \overset{5.5}{\simeq} K_0(D^{\mathrm{op}})_R \simeq K_0(D)_R \simeq R$$

with the class $[D]$ of the $\underline{D}$-$\underline{k}$-bimodule D corresponding to $1 \in R$. Under these isomorphisms, 8.21 sends $1 \in R$ to the endomorphism $- \cdot \dim(D) : R \to R$. Therefore, if by hypothesis $U(\underline{D})_R \simeq U(\underline{k})_R$, we conclude that $1/\dim(D) \in R$. The proof follows now automatically from the equality $\dim(A) = \dim(D)^{n^2}$.

9. Proof of Theorem 3.15

Recall that S is a finite-dimensional k-algebra of finite global dimension and that $I \subset S$ is a nilpotent (two-sided) ideal. Let us write $\pi : S \twoheadrightarrow S/I$ for the quotient map. One needs to show that it yields an isomorphism $U(\underline{\pi})_R : U(\underline{S})_R \xrightarrow{\sim} U(\underline{S/I})_R$. By the Yoneda lemma for the full subcategory of $\mathsf{Hmo}_0(k)_R$ containing the objects $U(\underline{S})_R$ and $U(\underline{S/I})_R$, one observes that it suffices to show that the induced homomorphism

$$(U(\underline{\pi})_R)_* : \mathrm{Hom}_{\mathsf{Hmo}_0(k)_R}(U(\underline{T})_R, U(\underline{S})_R) \longrightarrow \mathrm{Hom}_{\mathsf{Hmo}_0(k)_R}(U(\underline{T})_R, U(\underline{S/I})_R)$$

is an isomorphism when $T = S$ and $T = S/I$. Concretely, is suffices to show that

$$[- \otimes^{\mathbb{L}}_{\underline{S}} {}_{\pi}\mathsf{Bi}] : K_0(\mathrm{rep}(\underline{T}, \underline{S}))_R \longrightarrow K_0(\mathrm{rep}(\underline{T}, \underline{S/I}))_R \tag{9.1}$$

is an isomorphism. The proof is now divided into two cases.

Case 1 ($S/J(S)$ ***k*-separable)**

Assume that S/I has finite global dimension and that $S/J(S)$ is k-separable. Since $(S/I)/J(S/I) = S/J(S)$, one concludes then from [11, p. 2] that the dg categories $\underline{S}$ and $\underline{S/I}$ are smooth. They are also proper, and so, thanks to description 5.5, the induced homomorphism 9.1 reduces to

$$[- \otimes^{\mathbb{L}}_{\underline{S}} {}_{\pi}\mathsf{Bi}] : K_0(T^{\mathrm{op}} \otimes S)_R \longrightarrow K_0(T^{\mathrm{op}} \otimes S/I)_R. \tag{9.2}$$

Now, recall that by assumption I is nilpotent. As a consequence, the (two-sided) ideal of the quotient map $T^{\mathrm{op}} \otimes S \twoheadrightarrow T^{\mathrm{op}} \otimes (S/I)$ is also nilpotent. Using the invariance of the Grothendieck group functor with respect to nilpotent extensions (see [44, § II Lemma 2.2]), we hence conclude that 9.2 is an isomorphism.

Case 2 ($1/p \in R$)

Assume that S/I has finite global dimension and that k is a field of characteristic $p > 0$ such that $1/p \in R$. Note that, since S and S/I are finite dimensional and of finite global dimension, we have the equivalences

$$\mathrm{rep}(\underline{T}, \underline{S}) \simeq \mathcal{D}^b(\mathrm{mod}(T^{\mathrm{op}} \otimes S)) \quad \mathrm{rep}(\underline{T}, \underline{S/I}) \simeq \mathcal{D}^b(\mathrm{mod}(T^{\mathrm{op}} \otimes S/I)),$$

where $\mathcal{D}^b(\mathrm{mod}(-))$ stands for the bounded derived category of finitely generated modules. The above homomorphism 9.1 identifies then with

$$[-\otimes^{\mathbb{L}}_{S}{}_{\pi}\mathsf{Bi}] : G_0(T^{\mathrm{op}}\otimes S)_R \longrightarrow G_0(T^{\mathrm{op}}\otimes S/I)_R.$$

Now, consider the following quotient maps:

$$r_S : S \twoheadrightarrow S/J(S) \quad q_T : T \twoheadrightarrow T/J(T) \quad q_{S/I} : S/I \twoheadrightarrow S/J(S/I) = S/J(S).$$

Using the following commutative diagram[5]

$$\begin{array}{ccc} G_0(T^{\mathrm{op}}\otimes S)_R & \xrightarrow{[-\otimes^{\mathbb{L}}_{S}{}_{\pi}\mathsf{Bi}]} & G_0(T^{\mathrm{op}}\otimes (S/I))_R \\ & \searrow^{[-\otimes^{\mathbb{L}}_{T^{\mathrm{op}}\otimes S}({}_{q_T}\mathsf{Bi}\otimes {}_{q_S}\mathsf{Bi})]} & \downarrow {\scriptstyle [-\otimes^{\mathbb{L}}_{T^{\mathrm{op}}\otimes(S/I)}({}_{q_T}\mathsf{Bi}\otimes {}_{q_{S/I}}\mathsf{Bi})]} \\ & & G_0((T/J(T))^{\mathrm{op}}\otimes (S/J(S)))_R \end{array}$$

one observes that it suffices to prove that

$$[-\otimes^{\mathbb{L}}_{T^{\mathrm{op}}\otimes S}({}_{q_T}\mathsf{Bi}\otimes {}_{q_S}\mathsf{Bi})] : G_0(T^{\mathrm{op}}\otimes S)_R \longrightarrow G_0((T/J(T))^{\mathrm{op}}\otimes (S/J(S)))_R \tag{9.3}$$

is an isomorphism. Moreover, it is sufficient by base change to treat the case $R = \mathbb{Z}[1/p]$. Note that the kernel of the quotient map $q_T^{\mathrm{op}}\otimes q_S$ is nilpotent. Hence, $G_0(T^{\mathrm{op}}\otimes S)$ and $G_0(T/J(T))^{\mathrm{op}}\otimes (S/J(S))$ are free $\mathbb{Z}$-modules with a basis given by the simple $(T/J(T))^{\mathrm{op}}\otimes (S/J(S))$-modules. In particular, they have the same rank. As a consequence, it suffices to prove that 9.3 (with $R = \mathbb{Z}[1/p]$) is a surjection. In order to do so, we consider the following commutative diagram:

$$\begin{array}{ccc} K_0(T^{\mathrm{op}}\otimes S)_{\mathbb{Z}[1/p]} & \xrightarrow[\sim]{-\otimes^{\mathbb{L}}_{T^{\mathrm{op}}\otimes S}({}_{q_T}\mathsf{Bi}\otimes {}_{q_S}\mathsf{Bi})} & K_0((T/J(T))^{\mathrm{op}}\otimes (S/J(S)))_{\mathbb{Z}[1/p]} \\ \downarrow & & \downarrow \wr \\ G_0(T^{\mathrm{op}}\otimes S)_{\mathbb{Z}[1/p]} & \xrightarrow[(9.3)]{} & G_0((T/J(T))^{\mathrm{op}}\otimes (S/J(S)))_{\mathbb{Z}[1/p]}\,. \end{array}$$

As in the proof of Case 1, the upper horizontal map is an isomorphism. Thanks to Proposition 9.4 below (with $U = T/J(T)$ and $U' = S/J(S)$), the right vertical map is also an isomorphism. Using these isomorphisms and the commutativity of the above diagram, we conclude that 9.3 is a surjection. This finishes the proof.

PROPOSITION 9.4. *Given a field k of characteristic $p > 0$, the induced map*

$$K_0(U\otimes U')_{\mathbb{Z}[1/p]} \longrightarrow G_0(U\otimes U')_{\mathbb{Z}[1/p]} \tag{9.5}$$

is an isomorphism for any two finite-dimensional semi-simple k-algebras U and U'.

[5] Note that although $T^{\mathrm{op}}\otimes S$ may have infinite global dimension, it is still true that $(T/J(T))^{\mathrm{op}}\otimes (S/J(S))$ has finite projective dimension over $T^{\mathrm{op}}\otimes S$.

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Proof. Since $U \otimes U'$ is finite dimensional, $K_0(U \otimes U')_{\mathbb{Z}[1/p]}$ and $G_0(U \otimes U')_{\mathbb{Z}[1/p]}$ are free $\mathbb{Z}$-modules with the same rank. Hence, it suffices to show that 9.5 is surjective. Let us assume without loss of generality that U and U' are indecomposable. Let Z (respectively, Z') be the center of U (respectively, of U') and Z_0 (respectively, Z'_0) the separable closure of k in Z (respectively, in Z'). Under this notation, one has $Z_0 \otimes Z'_0 = \bigoplus_i W_i$ with W_i/k a separable field extension. As a consequence, one obtains the following equalities:

$$\begin{aligned} U \otimes U' &= U \otimes_{Z_0} (Z_0 \otimes Z'_0) \otimes_{Z'_0} U' \\ &= \bigoplus_i (U \otimes_{Z_0} W_i \otimes_{Z'_0} U') \\ &= \bigoplus_i (U \otimes_{Z'_0} W_i) \otimes_{W_i} (U' \otimes_{W_0} W_i). \end{aligned}$$

Replacing k by W_i and U (respectively, U') by $U \otimes_{Z_0} W_i$ (respectively, by $U' \otimes_{Z'_0} W_i$), one can (and will) assume that Z and Z' are purely inseparable k-algebras. We hence have

$$U \otimes U' = (U \otimes Z') \otimes_{(Z \otimes Z')} (U' \otimes Z').$$

Note that $D := U \otimes U'$ is the tensor product of two Azumaya algebras over $W := Z \otimes Z'$, and hence is itself an Azumaya algebra. Thanks to Lemma 9.6 below, W is a local k-algebra. By lifting idempotents and invoking Morita equivalence, the problem of showing that 9.5 is surjective can be reduced to the case that $D/J(D)$ is a division algebra. In this case, $D/J(D) = W/J(W) \otimes_W D$ is the unique simple D-module. Invoking Lemma 9.6 again, we find that $D = W \otimes_W D$ is an extension of p^n copies (for some n) of $W/J(W) \otimes_W D$. Hence, $p^n[D/J(D)]$ is in the image of 9.5. This finishes the proof. □

LEMMA 9.6. *Let k be a field of characteristic $p > 0$, and let $Z/k, Z'/k$ be two purely inseparable field extensions. Under these assumptions, $Z \otimes Z'$ is a local k-algebra, and its length (as a module over itself) is a power of p.*

Proof. Note that, if $e \in Z \otimes Z'$, then $e^{p^n} \in k$ for some $n \gg 0$. In the case where e is an idempotent we then conclude that $e = 0, 1$. This implies that $Z \otimes Z'$ is a local k-algebra. It is also clear that the length of $Z \otimes Z'$ must divide $\dim_k(Z \otimes Z')$. Hence, it is necessarily a power of p. □

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