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IMAGES OF QUANTUM REPRESENTATIONS OF MAPPING CLASS GROUPS AND DUPONT–GUICHARDET–WIGNER QUASI-HOMOMORPHISMS

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Abstract We prove that either the images of the mapping class groups by quantum representations are not isomorphic to higher rank lattices or else the kernels have a large number of normal generators. Further, we show that the images of the mapping class groups have non-trivial 2-cohomology, at least for small levels. For this purpose, we considered a series of quasi-homomorphisms on mapping class groups extending the previous work of Barge and Ghys (*Math. Ann.* **294** (1992), 235–265) and of Gambaudo and Ghys (*Bull. Soc. Math. France* **133**(4) (2005), 541–579). These quasi-homomorphisms are pull-backs of the Dupont–Guichardet–Wigner quasi-homomorphisms on pseudo-unitary groups along quantum representations.

Keywords: symplectic group; pseudo-unitary group; Dupont–Guichardet–Wigner cocycle; quasi-homomorphism; group homology; mapping class group; central extension; quantum representation

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1. Introduction and statements

The main motivation of this paper is to obtain new information about the images of mapping class groups by quantum representations, by analyzing their 2-cohomology. McMullen [40] addressed the question of the arithmeticity of Burau representations of braid groups at roots of unity and Venkataramana [46, 47] solved it affirmatively in the case where the order of the root is bounded by twice the number of strands. Burau representations are particular examples of quantum representations in genus zero. Whether the image of quantum representations of mapping class groups of higher genus is arithmetic or thin seems to be a challenging problem with possible implications for the fine structure of mapping class groups. One additional difficulty in both the present case and the Burau representation at roots of unity of higher order is the absence of unipotents. Another, seemingly unrelated, question that arose recently is the determination of the kernel of the quantum representations at a fixed level, to be compared with the normal subgroup generated by given powers of Dehn twists. It is known that the intersection of infinitely many of these kernels is trivial, according to the asymptotic faithfulness result by Andersen ([2]; see also [19, 35] for different proofs). Our aim is to prove first that the two questions above are directly related; in particular, arithmeticity implies a large number of normal generators for the kernel, hence many others besides powers of Dehn twists. Our second result shows that in infinitely many cases, the real 2-cohomology of the image of the quantum representations is non-trivial and hence these images are not virtually free.

One ingredient in this work is the relation between Burau and quantum representations, which we use to estimate the signature of Hermitian forms invariant by the mapping class groups. As a consequence, quantum representations are Zariski dense within semi-simple groups with a large number of pseudo-unitary factors (see also [21]). We then apply Matsushima's vanishing theorem to prove that either the images of quantum representations are not higher rank irreducible lattices, or else the number of normal generators of the kernels of the quantum representations is bounded from below by linear functions on the level of the representation. In the second part, we consider the family of quasi-homomorphisms on mapping class groups defined in [21], extending and inspired by the previous work of Barge and Ghys [5] and of Gambaudo and Ghys [25]. These quasi-homomorphisms are constructed as trivializations of pull-backs of Dupont–Guichardet–Wigner cocycles along quantum representations of mapping class groups M_g of oriented surfaces of genus $g \geqslant 2$ into pseudo-unitary groups. Although Bestvina and Fujiwara proved in [6] that there are uncountably many quasi-homomorphisms on mapping class groups, which could be derived using the action of mapping class groups on curve complexes, it seems that there are very few explicit ones. Explicit computations using arithmetic properties of the signatures from the first part give then the non-triviality of 2-cohomology classes on the image of the quantum representations, at least for small levels.

1.1. Quantum representations

In [7], Blanchet, Habegger, Masbaum and Vogel defined the topological quantum field theory (TQFT) functor $\mathcal{V}_p$, for every integer $p \geqslant 3$ and a primitive root of unity ζ of order $2p$. These TQFTs should correspond to the so-called $SU(2)$-TQFT, for even p and to the $SO(3)$-TQFT, for odd p (see also [34] for another version of $SO(3)$-TQFT). It is known that these TQFTs determine and are determined by a series of projective representations of the mapping class groups.

Definition 1.1. Let $p \in \mathbb{Z}_+$, $p \geqslant 5$ and ζ be a primitive $2p$th root of unity.

1. The quantum representation $\rho_{p,\zeta}$ is the projective representation of the mapping class group associated to $\mathcal{V}_p$, the TQFT at the root of unity ζ.
2. We denote by $\widetilde{\rho}_{p,\zeta}$ the linear representation of the central extension $\widetilde{M}_g$ of the mapping class groups M_g of the genus g closed oriented surface, which resolves the projective ambiguity of $\rho_{p,\zeta}$ (see [26, 39]).
3. Furthermore, $N(g, p)$ denotes the dimension of the space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the closed oriented surface of genus g.

Recall now that M_g is perfect when $g \geqslant 3$ and that the universal central extension $\widetilde{M_g}^{\mathrm{u}}$ of $\mathcal{M}_g$ is a subgroup of index 12 in the central extension $\widetilde{M_g}$ (see [39]). We will often consider the restriction of $\widetilde{\rho}_{p,\zeta}$ to the perfect subgroup $\widetilde{M_g}^{\mathrm{u}}$ since the latter has no other central extensions than itself.

Remark 1.1. The TQFT $\mathcal{V}_p$ is unitary in the case $\zeta = A_p$, where

$$A_p = \begin{cases} -\exp\left(\dfrac{2\pi i}{2p}\right), & \text{if } p \equiv 0 \pmod 2; \\ (-1)^{\frac{p-1}{2}} \exp\left(\dfrac{(p+1)\pi i}{2p}\right), & \text{if } p \equiv 1 \pmod 2. \end{cases}$$

Notice a slight change with respect to the convention [21] where a typo arose in the expression for odd p.

For prime $p \geqslant 5$ we denote by $\mathcal{O}_p$ the ring of cyclotomic integers $\mathcal{O}_p = \mathbb{Z}[\zeta_p]$, if $p \equiv -1 \pmod 4$ and $\mathcal{O}_p = \mathbb{Z}[\zeta_{4p}]$, if $p \equiv 1 \pmod 4$ respectively, where ζ_r denotes a primitive rth root of unity. The main result of [28] states that there exists a free $\mathcal{O}_p$-lattice $S_{g,p}$ in the $\mathbb{C}$-vector space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the genus g closed orientable surface and a non-degenerate Hermitian $\mathcal{O}_p$-valued form on $S_{g,p}$ both invariant under the action of $\widetilde{M_g}$ via the representation $\widetilde{\rho}_{p,\zeta}$. Therefore the image of the mapping class group consists of unitary matrixes (with respect to the Hermitian form) with entries in $\mathcal{O}_p$. Let $\mathbb{U}_{p,g}(\mathcal{O}_p)$ and $P\mathbb{U}_{p,g}(\mathcal{O}_p)$ be the group of all such matrixes and respectively its quotient by scalars.

When p is prime, $p \geqslant 5$ and $g \geqslant 3$, then it is known that $\widetilde{\rho}_{p,A_p}$ takes values in $S\mathbb{U}_{p,g}$ (see [15, 24]). It is known that $S\mathbb{U}_{p,g}(\mathcal{O}_p)$ is an irreducible lattice in a semi-simple algebraic group $\mathbb{G}_{p,g}$ obtained by the so-called restriction of scalars construction from the totally real cyclotomic field $\mathbb{Q}(\zeta_p + \overline{\zeta_p})$ to $\mathbb{Q}$. Specifically, the group $\mathbb{G}_{p,g}$ is a product $\prod_{\sigma \in S(p)} S\mathbb{U}^{\sigma}_{p,g}$. Here $S(p)$ stands for a set of representatives of the classes of complex embeddings σ of $\mathcal{O}_p$ modulo complex conjugacy. The factor $S\mathbb{U}^{\sigma}_{p,g}$ is the special unitary group associated to the Hermitian form conjugated by σ, thus corresponding to some Galois conjugate root of unity.

Denote by $\widetilde{\rho}_p$ and ρ_p the representations $\prod_{\sigma \in S(p)} \widetilde{\rho}_{p,\sigma(A_p)}$ and $\prod_{\sigma \in S(p)} \rho_{p,\sigma(A_p)}$, respectively. Notice that the real Lie group $\mathbb{G}_{p,g}$ is a semi-simple algebraic group defined over $\mathbb{Q}$.

In [21], the first author proved that $\widetilde{\rho}_p(\widetilde{M_g})$ is a discrete Zariski dense subgroup of $\mathbb{G}_{p,g}(\mathbb{R})$ whose projections onto the simple factors of $\mathbb{G}_{p,g}(\mathbb{R})$ are topologically dense, for $g \geqslant 3$ and $p \geqslant 5$ prime, $p \equiv -1 \pmod 4$.

Remark 1.2. 1. Notice that, when $p \equiv 1 \pmod 4$ the image of $\widetilde{\rho}_p(\widetilde{M_g})$ is contained in $\mathbb{G}_{p,g}(\mathbb{Z}[i])$ and thus it is a discrete Zariski dense subgroup of $\mathbb{G}_{p,g}(\mathbb{C})$. Thus we have to replace each factor $SU(m,n)$ of $\mathbb{G}_{p,g}(\mathbb{R})$ by its complexification $SL(m+n,\mathbb{C})$. There are a number of essential changes to be made if we wish to extend Theorem 1.1 to this case, contrary to the situation in [21]. However, for Theorem 1.2 the discreteness is not an issue.

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2. When $p=2r$, for a prime $r\geqslant 5$, according to [7, Theorem 1.5] there is an isomorphism of TQFTs between $\mathcal{V}_{2r}$ and $\mathcal{V}_2'\otimes\mathcal{V}_r$. Furthermore, the image of the TQFT representation associated to $\mathcal{V}_2'$ is finite. Thus, the restriction of $\widetilde{\rho}_p$ to the finite index subgroup $\ker\widetilde{\rho}_2'\subset\widetilde{M}_g$ is the tensor product of a trivial representation and $\widetilde{\rho}_r$, hence is a direct sum of copies of $\widetilde{\rho}_r$. The projection on a factor gives us a homomorphism $\pi:\rho_{2r}(\ker\widetilde{\rho}_2')\to\mathbb{G}_{r,g}$. Therefore, up to passing to a finite index subgroup of $\widetilde{M}_g$ the image $\pi\circ\widetilde{\rho}_{2r}$ is a discrete Zariski dense subgroup of $\mathbb{G}_{r,g}$.

1.2. Main results

The questions addressed here concern the description of the image of $\rho_{p,\zeta}$ and its kernel. The first problem is whether the image of $\rho_{p,\zeta}$ is of finite index in $P\mathbb{U}_{p,g}(\mathcal{O}_p)$, and in particular a higher rank lattice. Let $M_g[p]$ denote the (normal) subgroup of M_g generated by the pth powers of all Dehn twists. It is known that $M_g[p]\subset\ker\rho_{p,\zeta}$, and the second problem is whether this inclusion is strict. This was stated in [37] and in unpublished notes by Jørgen Andersen. For instance, this inclusion is an equality when the surface is a 1-holed torus and the representations are 2-dimensional (see [22, 37]) or a 4-holed sphere (see [3]). Notice that $M_g[p]$ has a small normally generating system.

Our first result states that whenever $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is isomorphic to a higher rank lattice the group $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ should be the quotient of $\widetilde{M_g}^{\mathrm{u}}$ by a *large* number of relations, growing linearly with p.

To state this properly we need more notation. Set $s_{p,g}$ for the number of simple non-compact factors of the semi-simple Lie group $\mathbb{G}_{p,g}(\mathbb{R})$. We also write $s^*_{p,g}$ for the number of such factors of non-zero signature, i.e. of the form $SU(m,n)$ with $m\neq n$, $mn\neq 0$. Each simple factor is associated to a primitive root of unity ζ of order $2p$ having a positive imaginary part. Those ζ corresponding to non-compact simple factors or non-compact with non-zero signature will be called *non-compact roots* and respectively *non-compact roots of non-zero signature*. Denote also by $r_{p,g}$ the minimal number (possibly infinite) of normal generators of $\ker\widetilde{\rho}_p$ within $\widetilde{M_g}^{\mathrm{u}}$, namely the minimum number of relators to be added in order to obtain the quotient $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$.

Theorem 1.1. *Let $g\geqslant 4$, p prime, $p\equiv -1\pmod 4$. Either $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is not isomorphic to a higher rank lattice, or else $r_{p,g}\geqslant s_{p,g}$. Moreover,*

$$s_{p,g}\geqslant\left\lceil\frac{g-3}{2(g-1)}p+\frac{3}{2}\right\rceil,\quad\text{for } p\geqslant 2g-1,\ g\geqslant 4,$$

where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

A consequence of our theorem above is the following.

Corollary 1.1. *Let $g\geqslant 4$, p prime, $p\equiv -1\pmod 4$ such that $p\geqslant 2g-1$. Then the quotient $M_g/M_g[p]$ is not isomorphic to a higher rank lattice.*

The way one proves this theorem is by finding an upper bound for the dimension of the cohomology group $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}),\mathbb{R})$ in terms of the number of normal generators. This is carried on in § 2.1. The necessary estimates for $s_{p,g}$ and the real rank of $\mathbb{G}_{p,g}$ are provided in §§ 3.2 and 3.3, after having set the notation for the skein TQFT in § 3.1.

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Lower bounds for these dimensions are more difficult to obtain, and this is the subject of the second part of the article. Here we use the aforementioned family of quasi-homomorphisms on mapping class groups arising as trivializations of pull-backs of Dupont–Guichardet–Wigner cocycles along quantum representations. We first need an explicit formula for these quasi-homomorphisms, which will be stated in Proposition 4.2 § 4. Then computations of signatures arising in non-unitary TQFTs obtained in § 5.1 for small values of the level provide the necessary ingredients for the following result.

Theorem 1.2. *For* $p \in \{5, 7, 9\}$ *and infinitely many values of* g*, we have* $\dim H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R}) \geqslant 1$.

Since $\rho_p(M_g)$ is of finite index within $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$, from the 5-term exact sequence in cohomology, Corollary 1.2 follows.

Corollary 1.2. *For* $p \in \{5, 7, 9\}$ *and infinitely many values of* g*, we have* $\dim H^2(\rho_p(M_g), \mathbb{R}) \geqslant 1$.

An immediate consequence is the fact that $\rho_p(M_g)$ is *not* a virtually free group. This can be improved, as follows. For a group Γ that is virtually torsion-free we denote by $\mathrm{vcd}(\Gamma)$ its virtual cohomological dimension, i.e. the cohomological dimension of any of its finite index torsion-free subgroups (see [9, VIII.11]).

Proposition 1.1. *If* $p \notin \{2, 3, 4, 6, 8, 12\}$ *and* $g \geqslant 2$*,* $(p, g) \neq (10, 2)$ *then we have:*

$$\mathrm{vcd}(\widetilde{\rho}_p(\widetilde{\Gamma}_g)) \geqslant g + \left[\frac{g-2}{2}\right].$$

In particular, $\widetilde{\rho}_p(\widetilde{\Gamma}_g)$ *is not virtually a free product of finite groups.*

Moreover the cohomology classes in Theorem 1.2 are not related to known classes on mapping class groups.

Proposition 1.2. *For any* $g \geqslant 2$*, the map induced in cohomology in degree* 2

$$\rho_p^* : H^2(\rho_p(M_g), \mathbb{R}) \to H^2(M_g, \mathbb{R})$$

is the trivial (zero) map.

Remark 1.3. The restriction to $p \in \{5, 7, 9\}$ comes from our inability to obtain modular properties for the signatures of TQFTs for general p. A general theory for these is beyond the scope of this paper and partial results in this direction will appear in [12]. We expect the result to hold for all primes p. However, these cases with small p are already interesting since the representations $\widetilde{\rho}_p$ are known to be Zariski dense in the corresponding semi-simple Lie groups $\mathbb{G}_{p,g}$. Our method could improve this lower bound for specific values of p and g, but could not do better than $[\frac{g}{2}] + 1$ without additional information about the group $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$. The arithmetic progressions above are rather explicit; for instance, $g \equiv 1 \pmod{24}$ is convenient for $p \in \{5, 7\}$.

2. Quasi-homomorphisms on mapping class group quotients

2.1. Restriction homomorphisms and proof of Theorem 1.1

Proposition 2.1. *We have* $\dim H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R}) \leqslant r_{p,g}$, *if* $g \geqslant 3$.

Proof. The 5-term exact sequence in cohomology associated to the exact sequence

$$1 \to \ker \widetilde{\rho}_p \to \widetilde{M_g}^{\mathrm{u}} \to \widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}) \to 1,$$

gives us:

$$0 = H^1(\widetilde{M_g}^{\mathrm{u}}, \mathbb{R}) \to \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}} \xrightarrow{\iota} H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R}) \to H^2(\widetilde{M_g}^{\mathrm{u}}, \mathbb{R}) = 0.$$

By exactness of the sequence above, ι is an isomorphism and hence identifies $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$ with $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$. The next lemma shows that $\dim \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}} \leqslant r_{p,g}$ and Proposition 2.1 follows. □

Lemma 2.1. *Assume that* $r_{p,g}$ *is finite and let* $\{a_1, a_2, \ldots, a_{r_{p,g}}\}$ *be a minimal system of normal generators for* $\ker \widetilde{\rho}_p$ *within* $\widetilde{M_g}^{\mathrm{u}}$. *Then the evaluation homomorphism* E : $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}} \to \mathbb{R}^{r_{p,g}}$, *given by* $E(f) = (f(a_1), f(a_2), \ldots, f(a_n))$ *is injective.*

Proof. Any element $x \in \ker \widetilde{\rho}_p$ is a product $x = \prod_i g_i a_i g_i^{-1}$, for some $g_i \in \widetilde{M_g}^{\mathrm{u}}$. Since $f \in \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$ is conjugacy invariant we have $f(x) = \sum_i f(g_i a_i g_i^{-1}) = \sum_i f(a_i)$ and the lemma follows. □

Proposition 2.2. *If* $s_{p,g} > r_{p,g}$ *then* $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ *is not a lattice in* $\mathbb{G}_{p,g}$.

Proof. Recall from [21] that $\mathbb{G}_{p,g}$ is a real semi-simple linear algebraic group defined over $\mathbb{Q}$. Since $\mathbb{G}_{p,g}$ is obtained by restriction of scalars from an anisotropic unitary group it follows that all elements of $\mathbb{G}_{p,g}(\mathbb{Z})$ are semi-simple, as being obtained as Galois conjugates of unitary and hence diagonalizable matrixes. Therefore, by Borel's theorem, $\mathbb{G}_{p,g}(\mathbb{Z})$ is a cocompact lattice in $\mathbb{G}_{p,g}(\mathbb{R})$. This was also noticed in [38].

We know as part of Matsushima's vanishing theorem that for cocompact lattices Γ in semi-simple Lie groups $\mathbb{G}$ the restriction homomorphism $H^j(\mathbb{G}, \mathbb{R}) \to H^j(\Gamma, \mathbb{R})$ is an isomorphism as long as $j \leqslant \mathrm{rk}_{\mathbb{R}}\mathbb{G} - 1$ (see [8, Ch. 7, Proposition 4.3]). We will show in Proposition 3.2, § 3.3 that $\mathbb{G}_{p,g}(\mathbb{R})$ is of rank at least 3 for any odd $p \geqslant 5$, and hence $H^2(\mathbb{G}_{p,g}(\mathbb{R}), \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is an isomorphism for any lattice Γ in $\mathbb{G}_{p,g}(\mathbb{R})$.

Now, $\mathbb{G}_{p,g}(\mathbb{R})$ is a product of $s_{p,g}$ pseudo-unitary groups of type $SU(m, n)$, each factor being a simple group of isometries of some irreducible Hermitian space. Then by [31] we have that $H^2(\mathbb{G}_{p,g}(\mathbb{R}), \mathbb{R}) = \mathbb{R}^{s_{p,g}}$ is the vector space generated by the set of Dupont–Guichardet–Wigner classes of the simple factors. In particular, if $s_{p,g} > r_{p,g}$ then the restriction map $H^2(\mathbb{G}_{p,g}(\mathbb{R}), \mathbb{R}) \to H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ cannot be an isomorphism by dimensional reasons and so $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ cannot be isomorphic to a lattice in $\mathbb{G}_{p,g}(\mathbb{R})$. □

Proof of Theorem 1.1. Assume that $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is isomorphic to a higher rank irreducible lattice. For p as in the hypothesis one knows that $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is a discrete subgroup of $\mathbb{G}_{p,g}(\mathbb{R})$. Then, by the Margulis super-rigidity theorem (see [36]) and the arithmeticity

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of lattices in higher rank Lie groups there exists a finite index subgroup of $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$, which is a lattice in a product P of simple factors of $\mathbb{G}_{p,g}(\mathbb{R})$. Therefore the Zariski closure of $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is contained in the subgroup P. On the other hand, as observed before, $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ is Zariski dense in $\mathbb{G}_{p,g}$ and hence $P = \mathbb{G}_{p,g}(\mathbb{R})$, so that $\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}})$ must be a lattice in $\mathbb{G}_{p,g}(\mathbb{R})$. Now Proposition 2.2 settles the first part of the theorem. We postpone the proof of the lower bound $s_{p,g} \geqslant \left\lceil \frac{g-3}{2(g-1)} p + \frac{3}{2} \right\rceil$, for $p \geqslant 2g+1$, until § 3.2; see Proposition 3.1. □

Proof of Corollary 1.1. First, $M_g[p]$ is normally generated by the pth powers of Dehn twists along a set of curves containing one simple closed curve for each integer $1 \leqslant h \leqslant \frac{g}{2}$, which is bounding a subsurface of genus h along with one non-separating simple curve. This gives an upper bound of $t_g = 1 + [\frac{g}{2}]$ for the number of normal generators of $M_g[p]$, which is independent on p.

Assume that $M_g/M_g[p]$ is a higher rank lattice Γ in the semi-simple Lie group H. We know that there exists a surjection of Γ onto $\rho_p(M_g)$ which is a discrete Zariski dense subgroup of $P\mathbb{G}_{p,g}$. By the Margulis super-rigidity theorem (see [36]) there exists a surjective continuous homomorphism $H \to P\mathbb{G}_{p,g}(\mathbb{R})$ covering this surjection. Therefore the number of virtual Hermitian simple non-compact factors of H is at least the number $s_{p,g}$ associated to $P\mathbb{G}_{p,g}(\mathbb{R})$.

The proof of Proposition 2.2 applied to the surjection $M_g \to M_g/M_g[p]$ shows that

$$\dim H^2(M_g/M_g[p], \mathbb{R}) \leqslant t_g.$$

Finally, by Matsushima's vanishing theorem we also have $\dim H^2(\Gamma, \mathbb{R}) \geqslant s_{p,g}$. This leads to a contradiction for p large enough, as stated. □

3. Estimates concerning the TQFT Hermitian form

3.1. The setting of the skein TQFT

We briefly review the properties of the TQFT $\mathcal{V}_p$ and refer to [7] for more details. A TQFT is a functor from the category of surfaces into the category of finite-dimensional vector spaces. Specifically, the objects of the first category are closed oriented surfaces endowed with colored banded points and morphisms between two objects are cobordisms decorated by uni-trivalent ribbon graphs compatible with the banded points. A banded point on a surface is a point with a tangent vector at that point, or equivalently a germ of an oriented interval embedded in the surface. There is a corresponding surface with colored boundary obtained by deleting a small neighborhood of the banded points and letting the boundary circles inherit the colors of the respective points.

The vector space associated by the functor $\mathcal{V}_p$ to a surface is called the *space of conformal blocks*. Let Σ_g denote the genus g closed orientable surface and H_g be a genus g handlebody with $\partial H_g = \Sigma_g$. Assume that there is a finite set $\mathcal{Y}$ of banded points on Σ_g. Let G be a uni-trivalent ribbon graph embedded in H_g in such a way that H_g retracts onto G, its univalent vertexes are the banded points $\mathcal{Y}$ and it has no other intersections with Σ_g.

For an odd number $p \geqslant 5$, called the *level* of the TQFT, we consider the *set of colors* in level p to be $\{0, 2, 4, \ldots, p-3\}$. An edge coloring of G is called *p-admissible* if the

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triangle inequality is satisfied at any trivalent vertex of G and the sum of the three colors around a vertex is bounded by $2(p-2)$. There is a similar description of p-admissibility for even p.

Fix a coloring of the banded points $\mathcal{Y}$. Then there exists a basis of the space of conformal blocks associated to the surface $(\Sigma_g, \mathcal{Y})$ with the colored banded points (or the corresponding surface with colored boundary), which is indexed by the set of all p-admissible colorings of G extending the boundary coloring. We denote by W_g the vector space associated to the closed surface Σ_g without banded points, or equivalently, where all banded points are given the color 0.

In fact, an admissible p-coloring of G provides an element of the skein module $S_\zeta(H_g)$ of the handlebody evaluated at a primitive $2p$th root of unity ζ. This skein element is obtained by cabling the edges of G by the Jones–Wenzl idempotents prescribed by the coloring. Let $\overline{H}_g$ denote the complementary handlebody in the 3-sphere S^3. Then there is a sesquilinear form:

$$\langle\ ,\ \rangle : S_\zeta(H_g) \times S_\zeta(\overline{H}_g) \to \mathbb{C}$$

defined by

$$\langle x, y\rangle = \langle x \sqcup y\rangle.$$

Here $x \sqcup y$ is the element of $S_\zeta(S^3)$ obtained by the disjoint union of x and y in $H_g \cup \overline{H}_g = S^3$, and $\langle\ \rangle : S_\zeta(S^3) \to \mathbb{C}$ is the Kauffman bracket invariant.

Eventually the space of conformal blocks W_g is the quotient $S_\zeta / \ker\langle\ ,\ \rangle$ by the left kernel of the sesquilinear form above. It follows that W_g is endowed with an induced *Hermitian form* H_ζ. The projections of skein elements associated to the p-admissible colorings of a trivalent graph G as above form an orthogonal basis of W_g with respect to H_ζ.

Let $G' \subset G$ be a uni-trivalent subgraph whose degree-one vertexes are colored, corresponding to a subsurface Σ' of Σ_g with colored boundary. The projections in W_g of skein elements associated to the p-admissible colorings of G' form an orthogonal basis of the space of conformal blocks associated to the surface Σ' with colored boundary components.

There is a geometric action of the mapping class groups of the handlebodies H_g and $\overline{H}_g$ respectively on their skein modules and hence on the space of conformal blocks. Moreover, these actions extend to the projective action $\rho_{p,\zeta}$ of M_g on W_g respecting the Hermitian form H_ζ. Notice that the mapping class group of an essential (i.e. without annuli or disk complements) subsurface $\Sigma' \subset \Sigma_g$ is a subgroup of M_g, which preserves the subspace of conformal blocs associated to Σ' with colored boundary. This kind of restriction to subsurfaces is an essential ingredient in § 3.2.

The functor $\mathcal{V}_p$ associates to a handlebody H_g the projection of the skein element corresponding to the trivial coloring of the trivalent graph G by 0. The invariant associated to a closed 3-manifold is given by pairing the two vectors associated to handlebodies in a Heegaard decomposition of some genus g and taking into account the twisting by the gluing mapping class action on W_g.

One should notice that the skein TQFT $\mathcal{V}_p$ is unitary, in the sense that H_ζ is a positive definite Hermitian form when $\zeta = A_p$, as chosen in § 1. The main concern of the present article is the case of a general primitive $2p$th root of unity, in which case the isometries of H_ζ form a pseudo-unitary group.

3.2. Estimations on $s_{p,g}$

Proposition 3.1. *If $g \geqslant 4$ and $p \equiv -1 \pmod 4$, then*

$$s_{p,g} \geqslant \left\lceil \frac{g-3}{2(g-1)} p + \frac{3}{2} \right\rceil, \quad \text{for } p \geqslant 2g+1,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof. This statement is essentially combinatorial, as the Hermitian form H_ζ on the space of conformal blocks is given rather explicitly in [7] in its diagonal form. Nevertheless the combinatorial-arithmetic problem of counting the roots of unity for which the entries of H_ζ are all positive seems rather complicated. We propose here an alternative way to bound from below $s_{p,g}$ by restricting the problem from mapping class groups to braids, where computations are immediate. Although not sharp, our estimates are linear in p.

There is an obvious injection of the pure braid group PB_{g-1} on $(g-1)$ strands into M_g, when $g \geqslant 3$. Specifically, if the g-holed sphere is embedded in Σ_g, in such a way that its complement consists of g 1-holed tori, then the map induced at the level of their mapping class groups is injective. Now, the pure mapping class group of the $(g-1)$-holed disk is an extension of PB_{g-1} by the free abelian group $\mathbb{Z}^{g-1}$ of Dehn twists along $(g-1)$ boundary components. This extension splits non-canonically, thus providing an embedding of PB_{g-1} into M_g.

The restriction of the representation $\rho_{p,\zeta}$ of M_g to PB_{g-1} is not irreducible. Set $W_{0,g}$ for the space of conformal blocks associated to the disk with $(g-1)$ holes, whose boundary circles are labeled by the colors $(2g-4, 2, 2, \ldots, 2)$, the first label corresponding to the disk boundary. In order to admit an extension to a p-admissible coloring we need to impose the condition $p \geqslant 2g-1$. Then the restriction $\rho_{p,\zeta}|_{PB_{g-1}}$ leaves invariant the subspace $W_{0,g} \subset W_g$. Moreover this representation naturally extends to one of the full braid group B_{g-1}, since the colors of $(g-1)$ boundary circles of the subsurface coincide. Eventually the projective representation of B_{g-1} lifts to a linear representation of B_{g-1}. Indeed, central extensions by $\mathbb{Z}$ of the braid groups B_{g-1} are trivial, as Arnold [4] proved that $H^2(B_n, \mathbb{Z}) = 0$. We will still denote this linear lift by $\rho_{p,\zeta}|_{B_{g-1}}$.

Recall now that the (reduced) Burau representation $\beta_k : B_k \to GL(k-1, \mathbb{Z}[q, q^{-1}])$, for $k \geqslant 3$, is defined on the standard generators $g_1, g_2, \ldots, g_{k-1}$ of the braid group B_k on k-strands by the formulas:

$$\beta_q(g_1) = \begin{pmatrix} -q & 0 \\ -1 & 1 \end{pmatrix} \oplus \mathbf{1}_{k-3},$$

$$\beta_q(g_j) = \mathbf{1}_{j-2} \oplus \begin{pmatrix} 1 & -q & 0 \\ 0 & -q & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus \mathbf{1}_{k-j-2}, \quad \text{for } 2 \leqslant j \leqslant k-2,$$

$$\beta_q(g_{k-1}) = \mathbf{1}_{k-3} \oplus \begin{pmatrix} 1 & -q \\ 0 & -q \end{pmatrix}.$$

By taking $q \in \mathbb{C}^*$ we obtain a representation $\beta_k(q)$ with values in $GL(k-1, \mathbb{C})$. The representations $\beta_k(q)$ are irreducible unless q is a non-trivial kth root of unity, in which case it has a $(k-2)$-dimensional irreducible summand denoted by $\hat{\beta}_k(q)$. Following Formanek (see [17]) we call a complex representation of B_k of *Burau type* if it is isomorphic to the tensor product of $\beta_k(q)$ (or $\hat{\beta}_k(q)$) with some 1-dimensional representation. The latter are all of the form $\chi(y)$, where $\chi(y)(g_j) = y \in \mathbb{C}^*$, for $j \leqslant k-1$.

Lemma 3.1. *The representation $\rho_{p,\zeta}|_{B_{g-1}}$ on $W_{0,g}$ is of Burau type.*

Proof. This is known to be true for $g = 4, 5$ (see e.g. [20, 23]). By induction on g one shows that $\dim W_{0,g} = g-2$. Explicit computations as those in [20] (for even p) show that the elements $\rho_{p,\zeta}|_{B_{g-1}}(g_i)$ have only two non-trivial eigenvalues. Up to rescaling the images of g_i (i.e. twisting by a 1-dimensional representation), the two non-trivial eigenvalues are 1 and $-\zeta^8$. Also the image of g_i is a pseudo-reflection.

Formanek proved in [17, Theorem 10, 22] that irreducible representations of B_{g-1} of dimension at most $g-2$ are either 1-dimensional or of Burau type, if $g \geqslant 8$ or $g \geqslant 6$ and the image of g_i is a pseudo-reflection. When $p \geqslant 2g-1$ is prime, $\hat{\beta}_k(q)$ (with $q^{g-1} = 1$) cannot be a summand of $\rho_{p,\zeta}|_{B_{g-1}}$, because one eigenvalue of g_i is not a $(g-1)$th root of unity.

Notice that $\rho_{p,\zeta}$, and hence $\rho_{p,\zeta}|_{B_{g-1}}$, is semi-simple because it is Galois conjugate to the unitary representation ρ_{p,A_p}.

If $\rho_{p,\zeta}|_{B_{g-1}}$ were not irreducible then it would split as a direct sum of 1-dimensional representations. This is a contradiction, as the restriction of $\rho_{p,\zeta}|_{B_{g-1}}$ to $B_3 \subset B_{g-1}$ is of Burau type.

Therefore, up to twisting by some $\chi(y)$ (the explicit value of y is not needed) $\rho_{p,\zeta}|_{B_{g-1}}$ is equivalent to the Burau representation β_{q_p} of B_{g-1} at the root of unity q_p, where q_p is given by $q_p = \zeta^8$, for odd p. □

Now, for $k \geqslant 3$ the Burau representation $\beta_k(q)$ of B_k has an invariant Hermitian form defined by Squier in [43]. Squier's original Hermitian form is degenerate when q is a root of unity of order $n \leqslant k$. A slightly modified version H_q^S of this form can be found in [40, 48], where it is shown that it is non-degenerate unless q is a kth root of unity. Since a Burau-type representation is irreducible it admits a unique invariant Hermitian form, up to a real scalar.

In particular, the restriction of H_ζ to the space of conformal blocks is a real multiple of $H^S_{\zeta^8}$. This also follows from stronger results from [18, 33] concerning the density of images of Burau-type representation.

The signature of the form H_q^S is given in [40, Corollary 3.2]. Squier's form is definite (either positive or negative) if and only if $\arg(q_p) \in (-\frac{2\pi}{g-1}, \frac{2\pi}{g-1})$ (see also [1, Lemma 9]). If we set $\zeta = \exp(\frac{(2k+1)\pi i}{p})$, then it suffices to restrict to those integral $k \in \{0, 1, \ldots, \frac{p-3}{2}\}$. This condition on $\arg(q_p)$ amounts to counting all such integers k for which in addition

$$2s\pi - \frac{2\pi}{g-1} \leqslant \frac{4(2k+1)\pi}{p} \leqslant 2s\pi \frac{2\pi}{g-1}, \quad \text{where } s \in \{0, 1, 2, 3, 4\}.$$

The number of such integers is at most $\frac{p}{g-1} - \frac{3}{2}$.

In [21] the first author proved that the factors of $\mathbb{G}_{p,g}$, for $p \equiv -1 \pmod 4$ are in one–one correspondence with the $\frac{p-1}{2}$ primitive $2p$th roots of unity up to conjugacy. If we discard the compact ones we derive that the number of non-compact factors in $\mathbb{G}_{p,g}$ is at least $\frac{g-3}{2(g-1)}p + \frac{3}{2}$. □

Remark 3.1. It seems that there are precisely two conjugate values for which H_ζ is positive when $p \geqslant 5$ is odd prime and four values (obtained by conjugacy or changing the sign) when p is twice an odd prime, respectively, unless H_ζ is totally positive. A similar statement might hold for all (not necessarily prime) odd large enough p.

Remark 3.2. Similar estimates hold true for $g \in \{2, 3\}$, by using the homomorphisms $PB_3 \to M_2$ and $PB_4 \to M_3$ from [23]. We skip the details.

3.3. Estimates for the rank of $\mathbb{G}_{p,g}(\mathbb{R})$

Proposition 3.2. *For $g \geqslant 2$, prime $p \geqslant 7$ and $p \equiv -1 \pmod 4$ the real rank of $\mathbb{G}_{p,g}(\mathbb{R})$ is at least 2. Furthermore, for $g \geqslant 4$ and odd $p \geqslant 5$ each simple non-compact factor of $\mathbb{G}_{p,g}(\mathbb{R})$ has rank at least 2. Moreover, the real rank of $\mathbb{G}_{p,g}(\mathbb{R})$ is at least $(\lceil \frac{g-3}{2(g-1)}p + \frac{3}{2}\rceil)(\frac{p-1}{2})^{g-3}$, for $g \geqslant 4$, $p \geqslant 2g+1$ and $p \equiv -1 \pmod 4$.*

Proof. Let $W_g^{\pm}(\zeta)$ be a maximal positive/negative subspace of the space W_g of conformal blocks in genus g, for the Hermitian form H_ζ. Consider a separating curve γ on the closed orientable surface Σ_g whose complementary subsurfaces have genus $g-1$ and 1 respectively. If we label γ by 0 then the spaces of conformal blocks associated to these two subsurfaces are isometrically identified with the spaces of conformal blocks of the closed surfaces obtained by capping off the boundary components. Therefore we have natural isometric embeddings $W_{g-1} \otimes W_1 \hookrightarrow W_g$. It is well known that $W_1 = W_1^+(\zeta)$ is positive for any ζ. Therefore we obtain the following isometric embeddings: $W_{g-1}^+(\zeta) \otimes W_1 \hookrightarrow W_g^+(\zeta)$ and $W_{g-1}^-(\zeta) \otimes W_1 \hookrightarrow W_g^-(\zeta)$. In particular, we have for odd p

$$\dim W_g^+(\zeta) \geqslant (\dim W_1)^g = \left(\frac{p-1}{2}\right)^g.$$

Lemma 3.2. *If ζ is such that $W_3^+(\zeta) = W_3$, then $W_g^+(\zeta) = W_g$, i.e. the simple factor associated to ζ is compact.*

Proof. For a p-admissible coloring X of the trivalent graph G with g loops we denote by the same letter X the corresponding vector of the basis of W_g defined in § 3.1. For a vertex v we denote by a_v, b_v, c_v the colors of the three edges incident to v and for any edge e we denote by c_e the color of the edge e, as prescribed by X. The Hermitian norm of such a vector X was computed in [7, 4.11], as follows:

$$H_\zeta(X, X) = \eta^{g-1} \prod_{v \in V(G)} \langle a_v, b_v, c_v \rangle \cdot \prod_{e \in E(G)} \langle c_e \rangle^{-1},$$

where η is a constant independent of the genus, $V(G)$ denotes the set of vertexes and $E(G)$ the set of edges of the graph G. The precise values of the symbols η, $\langle a, b, c\rangle$ and

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$\langle a \rangle$ in terms of quantum numbers are given in [7] but they will not be explicitly needed in what follows. We only need to know that all of them are real numbers.

Observe also that the positivity of the Hermitian form in genus 3 implies the positivity for genus 2, as well. Now, there are two graphs with two loops and without leaves (degree-one vertexes), the theta graph and the graph made of two loops joined by a segment. The above formula for a vector corresponding to a coloring of the theta graph shows that:

$$\eta\langle a\rangle\langle b\rangle\langle c\rangle > 0,$$

for any p-admissible triple a, b, c at a vertex. Therefore all symbols $\langle a \rangle$ have the same sign as η. Using the other graph with two loops, we find that

$$\langle a, a, b\rangle\langle c, c, b\rangle > 0,$$

for every p-admissible coloring for which the symbols above are defined. Thus the sign of $\langle a, a, b\rangle$ is $\epsilon_b \in \{-1, +1\}$ and it only depends on b. Consider next a graph made of three loops joined together by means of a tree with one vertex and three edges, each edge having its endpoint on one loop. Take an arbitrary p-admissible triple of colors a, b, c for the three edges of the tree and color the loops in a p-admissible way. This is always possible, no matter how we chose the p-admissible triple a, b, c. The formula above implies that:

$$\langle a, b, c\rangle\epsilon_a\epsilon_b\epsilon_c > 0.$$

But now it is immediate that for any vector X corresponding to a colored trivalent graph without leaves with $g \geqslant 2$ loops we have $H_\zeta(X, X) > 0$. This implies that the Hermitian form on every space of conformal blocks associated to a closed orientable surface is positive definite. □

It follows that either $W_g^+(\zeta) = W_g$ is positive or else

$$\dim W_g^-(\zeta) \geqslant (\dim W_1)^{g-3} \dim W_3^-(\zeta) \geqslant \left(\frac{p-1}{2}\right)^{g-3}.$$

The two formulas above show that the rank of each simple non-compact factor of $\mathbb{G}_{p,g}$ is at least $(\frac{p-1}{2})^{g-3}$.

On the other hand, if p is odd and $(p, g) \neq (2, 5)$ then, by direct calculation one obtains that the Hermitian form associated to the 1-holed torus with the boundary circle colored by 2 is not totally positive. The argument above implies that the real rank of $\mathbb{G}_{p,g}(\mathbb{R})$ is at least 2. A similar statement is valid for even $p \geqslant 14$. □

Remark 3.3. When $g = 2$ and $p = 7$ the group $\mathbb{G}_{p,g}(\mathbb{R})$ is the product of two pseudo-unitary groups $SU(11, 3) \times SU(10, 4)$. When $g = 3$ and $p = 7$ the group $\mathbb{G}_{p,g}(\mathbb{R})$ is the product of two pseudo-unitary groups $SU(58, 40) \times SU(44, 54)$.

3.4. Proofs of Propositions 1.1 and 1.2

Proof of Proposition 1.1. Let $\Sigma_{g,n}$ denote the compact orientable surface of genus g with n boundary components and $M_{g,n}$ the mapping class group of $\Sigma_{g,n}$. Then Σ_g decomposes into $g+[\frac{g-2}{2}]$ pieces with disjoint interiors among which are g subsurfaces $\Sigma_{1,1}$, $[\frac{g-2}{2}]$ subsurfaces $\Sigma_{0,4}$, and $g-2[\frac{g}{2}]\in\{0,1\}$ pieces homeomorphic to $\Sigma_{0,3}$.

If $p\notin\{2,3,4,6,8,12\}$, $g\geqslant 2$ and $(p,g)\neq(10,2)$, then every subgroup of the form $\rho_p(M_{1,1})$ or $\rho_p(M_{0,4})$ associated to a subsurface $\Sigma_{1,1}$ or $\Sigma_{0,4}$ of Σ_g contains a free non-abelian group $\mathbb{F}_2$ on two generators (see [22, 23]). In particular, we find that $\mathbb{F}_2^{g+[\frac{g-2}{2}]}\subset\rho_p(M_g)$. Recall that vcd is increasing with respect to the inclusion of groups (see [9, Ch. VIII, 11, Ex. 1, Proposition 2.4]). Thus

$$\mathrm{vcd}(\widetilde{\rho}_p(\widetilde{M}_g))\geqslant \mathrm{vcd}(\mathbb{F}_2^{g+[\frac{g-2}{2}]})\geqslant \mathrm{vcd}(\mathbb{Z}^{g+[\frac{g-2}{2}]})=g+\left[\frac{g-2}{2}\right].$$

Notice that we also have, by the same argument, $\mathrm{vcd}(\rho_p(Mg))\geqslant g+[\frac{g-2}{2}]$. Observe that torsion-free nilpotent subgroups of $\widetilde{\rho}_p(\widetilde{M}_g)$ are abelian, because $\mathbb{G}_{p,g}(\mathbb{Z})$ contains no non-trivial unipotents, so that they cannot be used to get better lower bounds. □

Remark 3.4. When $p\equiv -1 \pmod 4$, $\mathrm{vcd}(\mathbb{G}_{p,g}(\mathbb{Z}))$ is the dimension of the corresponding non-compact symmetric space, since lattices are cocompact. If $\widetilde{\rho}_p(\widetilde{M}_g)$ were of infinite index in $\mathbb{G}_{p,g}(\mathbb{Z})$ then its top dimensional cohomology would vanish (see [9, VIII, Proposition 8.1]). Therefore $\widetilde{\rho}_p(\widetilde{M}_g)$ has finite index in $\mathbb{G}_{p,g}(\mathbb{Z})$ if and only if $\mathrm{vcd}(\widetilde{\rho}_p(\widetilde{M}_g))=\mathrm{vcd}(\mathbb{G}_{p,g}(\mathbb{Z}))$. Compare also with [44], where the author proved that passing to an infinite index subgroup of a Poincaré duality group strictly decreases the cohomological dimension.

Proof of Proposition 1.2. Since $\rho_p(M_g)$ is of finite index in $\rho_p(\widetilde{M}_g^u)$, the map $\rho_p^*: H^2(\rho_p(M_g),\mathbb{R})\to H^2(M_g,\mathbb{R})$ factors through $\rho_p^*: H^2(\rho_p(\widetilde{M}_g{}^u),\mathbb{R})\to H^2(\widetilde{M}_g{}^u,\mathbb{R})$, but the group $\widetilde{M}_g^u$ has no non-split extensions, so this last cohomology group is trivial. □

4. Dupont–Guichardet–Wigner quasi-homomorphisms on mapping class groups

4.1. Quasi-homomorphisms on $\widetilde{M}_g$

Guichardet–Wigner [31] and Dupont [16] introduced explicit bounded continuous cocycles $c_{SU(m,n)}$, whose classes generate $H_b^2(SU(m,n);\mathbb{R})\cong\mathbb{R}$ and could be interpreted in terms of the symplectic area of triangles. Let K be the maximal compact subgroup $S(U(m)\times U(n))$, A the group of unitary diagonal matrixes with real entries and N the group of unitary unipotent matrixes in $SU(m,n)$. Corresponding to the Iwasawa decomposition $SU(m,n)=KAN$, we denote by $x=k(x)a(x)n(x)$ the Iwasawa decomposition of the element $x\in SU(m,n)$. The construction due to Guichardet and Wigner in [31, Theorem 1] is as follows.

Proposition 4.1. *Let $\mathfrak{k}$ be the Lie algebra of the compact group K and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $SU(m,n)$. Consider a smooth function $v: SU(m,n)\to\mathbb{C}^*$ satisfying the following conditions:*

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1. *the restriction of* v *to the maximal compact* K *is a non-trivial morphism of* K *into* $U(1) \subset \mathbb{C}^*$*;*
2. *the restriction of* v *to* $\exp \mathfrak{p}$ *is strictly positive and* K*-invariant;*
3. $v(k \cdot \exp p) = v(k)v(\exp p)$*, for any* $k \in K$ *and* $p \in \mathfrak{p}$*.*

Then there exists a unique smooth 2-cocycle $c_v : SU(m,n) \times SU(m,n) \to \mathbb{R}$ *such that*

$$\exp(2\pi\sqrt{-1}c_v(g_1, g_2)) = \arg(v(g_1g_2)^{-1} \cdot v(g_1) \cdot v(g_2)), \quad \text{and} \quad c_v(1,1) = 0.$$

Moreover, the class of c_v *generates the Borel cohomology group* $H^2(SU(m,n), \mathbb{R})$*.*

An example is the function $v_0 : K \to U(1)$ given by $v_0(x) = \det(x_+)$, where $x = \begin{pmatrix} x_+ & 0 \\ 0 & x_- \end{pmatrix} \in S(U(m) \times U(n))$ and x_+ is the $U(m)$ component of x. Setting $v_0(\exp p) = 1$, and $v_0(k \cdot \exp p) = v_0(k)v_0(\exp p)$, extends v_0 to a function on all of $SU(m,n)$ with values in $U(1)$ that satisfies the conditions stated in Proposition 4.1. We therefore have the associated continuous bounded cocycle denoted by $c_{SU(m,n)}$. We will later normalize the cocycle $c_{SU(m,n)}$ to a cocycle whose class is the generator of the image of $H^2(SU(m,n), \mathbb{Z})$ in $H^2(SU(m,n), \mathbb{R})$. We also consider the unique continuous lift $\Phi : \widetilde{SU(m,n)} \to \mathbb{R}$ of v_0 to the universal covering, which is determined by the condition $\Phi(1) = 0$.

Let G be a topological group. The ordinary cohomology group $H^2(G, \mathbb{R})$ is usually an extremely large group; for instance, for non-compact Lie groups its dimension is typically uncountable (see [41]). This is not anymore the case for the continuous cohomology of Lie groups and in particular for their *bounded* cohomology group $H_b^2(G; \mathbb{R})$. There is a canonical comparison map $H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R})$ whose kernel is described by quasi-homomorphisms: a map $\varphi : G \to \mathbb{R}$ is a *quasi-homomorphism* if $\sup_{a,b \in G} |\partial\varphi(a,b)| < \infty$, where $\partial\varphi(a,b) = \varphi(ab) - \varphi(a) - \varphi(b)$ is the boundary 2-cocycle. The quasi-homomorphism φ is *homogeneous* if $\varphi(a^n) = n\varphi(a)$, for every $a \in G$ and $n \in \mathbb{Z}$. Let us denote the vector space of quasi-homomorphisms by $QH(G)$ and its quotient by the subspace generated by the bounded functions and the group homomorphisms by $\widetilde{QH}(G)$. It is known that there is an exact sequence:

$$0 \to \widetilde{QH}(G) \to H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R}).$$

Bestvina and Fujiwara proved in [6] that $\widetilde{QH}(M_g)$, and hence $\widetilde{QH}(\widetilde{M}_g)$ has uncountably many generators.

Let $g \geqslant 3$, $p \geqslant 5$ be a prime number and $SU(m,n)$ be the non-compact simple factor of $\mathbb{G}_{p,g}(\mathbb{R})$ corresponding to the primitive $2p$th root of unity ζ. Since the universal extension $\widetilde{M}_g{}^{\mathrm{u}}$ is perfect and has no non-trivial extensions, we have an isomorphism $\widetilde{QH}(\widetilde{M}_g{}^{\mathrm{u}}) \simeq H_b^2(\widetilde{M}_g{}^{\mathrm{u}}, \mathbb{R})$. As $\widetilde{M}_g{}^{\mathrm{u}}$ is of finite index in $\widetilde{M}_g$, we also have $\widetilde{QH}(\widetilde{M}_g) \simeq H_b^2(\widetilde{M}_g, \mathbb{R})$. Thus, there exists a quasi-homomorphism $L_\zeta : \widetilde{M}_g \to \mathbb{R}$, unique up to a bounded quantity, verifying

$$\partial L_\zeta = \widetilde{\rho}_{p,\zeta}^*(c_{SU(m,n)}).$$

Let $\overline{L}_\zeta$ denote the unique homogeneous quasi-homomorphism in the class of L_ζ. To give an explicit formula for the quasi-homomorphism $\overline{L}_\zeta : \widetilde{M}_g \to \mathbb{R}$, we have to introduce the Dupont–Guichardet–Wigner quasi-homomorphism Φ on the universal covering $\widetilde{SU(m,n)}$ of $SU(m,n)$.

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Definition 4.1. A Dupont–Guichardet–Wigner quasi-homomorphism $\Phi : \widetilde{SU(m,n)} \to \mathbb{Q}$ is a quasi-homomorphism satisfying:

$$\Phi(\widetilde{x}\widetilde{y}) - \Phi(\widetilde{x}) - \Phi(\widetilde{y}) = c_{SU(m,n)}(x, y)$$

for all $x, y \in SU(m,n)$ and their arbitrary lifts $\widetilde{x}, \widetilde{y} \in \widetilde{SU(m,n)}$.

The quasi-homomorphism is *normalized* if

$$\Phi(Tz) = \Phi(z) + 1, \quad \text{for } z \in \widetilde{SU(m,n)},$$

where T denotes the generator of $\ker(\widetilde{SU(m,n)} \to SU(m,n))$. All Dupont–Guichardet–Wigner quasi-homomorphisms are at bounded distance from each other and the unique homogeneous normalized Dupont–Guichardet–Wigner quasi-homomorphism is given by $\overline{\Phi}(z) = \lim_{n\to\infty} \Phi(z^n)/n$. In fact, it was noticed by Barge and Ghys in [5, Remarque fondamentale 2] that there is a unique homogeneous normalized quasi-homomorphism on any central extension of a uniformly perfect group, in particular on $\widetilde{SU(m,n)}$. The homogeneous quasi-homomorphism associated to a continuous quasi-homomorphism is also continuous, by the result of Shtern (see [42, Proposition 1]; thus $\overline{\Phi}$ is continuous.

Barge and Ghys gave a formula for the homogeneous symplectic quasi-homomorphism in [5, Theorem 2.10]. In the remaining part, we need the following extension to the pseudo-unitary case.

Proposition 4.2. *The homogeneous quasi-homomorphism $\overline{L}_\zeta$ is given by the formula:*

$$\overline{L}_\zeta(x) = \overline{\Phi}(\widehat{\rho}_{p,\zeta}(x)),$$

where $\widehat{\rho}_{p,\zeta} : \widetilde{M}_g{}^{\mathrm{u}} \to \widetilde{SU(m,n)}$ is the unique lift of $\widetilde{\rho}_{p,\zeta}(x)$ to $\widetilde{SU(m,n)}$. Moreover, we have:

$$\overline{L}_\zeta(x) \equiv \frac{1}{2\pi}\left(\sum_{\lambda\in S(\widetilde{\rho}_{p,\zeta}(x))} n^+(\lambda)\arg(\lambda)\right) \in \mathbb{R}/\mathbb{Z},$$

where $S(u)$ is the set of eigenvalues of u and $n^+(\lambda)$ is the positive multiplicity of λ (see § 4.4 for details).

4.2. Non-triviality of the quasi-homomorphism space

Proposition 4.3. *If $s^*_{p,g} > r_{p,g}$ then $\widetilde{QH}(\widetilde{\rho}_p(\widetilde{M}_g{}^{\mathrm{u}}))$ cannot be trivial.*

Proof. Denote by $i_{p,\zeta} : \widetilde{\rho}_{p,\zeta}(\widetilde{M}_g{}^{\mathrm{u}}) \to PU(m,n)$ the obvious inclusion.

In [10, Theorem 1.3], Burger and Iozzi proved that for any discrete group Γ, two Zariski dense representations $\rho : \Gamma \to SU(m,n)$, with $1 \leqslant m < n$, are non-conjugate if and only if the corresponding cohomology classes $\rho^*(c_{SU(m,n)}) \in H^2_b(\Gamma; \mathbb{R})$ are distinct. Moreover, if distinct, then these classes are $\mathbb{Q}$-linearly independent.

Following [21], when ζ runs over the non-compact primitive roots of non-zero signature and positive imaginary part the bounded classes $i^*_{p,\zeta}(c_{SU(m,n)}) \in H^2_b(\widetilde{\rho}_p(\widetilde{M}_g{}^{\mathrm{u}}), \mathbb{R})$ are linearly independent over $\mathbb{Q}$. If $\widetilde{QH}(\widetilde{\rho}_p(\widetilde{M}_g{}^{\mathrm{u}}))$ were trivial, then the cohomology classes

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$i^*_{p,\zeta}(c_{SU(m,n)}) \in H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ would also be independent over $\mathbb{Q}$. But these are integral classes, i.e. they lie in the image of $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{Z})$, because they are pull-backs of integral classes from $H^2(\mathbb{G}_{p,g}(\mathbb{R}), \mathbb{R})$. Therefore, they would be linearly independent over $\mathbb{R}$. In other words, we would produce $s^*_{p,g}$ linearly independent classes living within the vector space $\mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M_g}^{\mathrm{u}}}$, which is of dimension at most $r_{p,g}$. This contradiction proves the claim. □

Remark 4.1. If $\widetilde{QH}(\rho_p(M_g))$ were infinite-dimensional, then $\rho_p(M_g)$ would not be boundedly generated.

4.3. Proof of Proposition 4.2

We reduce the problem to the computations made earlier by Barge and Ghys in [5] for the symplectic case. Given an integer $n \geqslant 1$, let $Sp(2n, \mathbb{R})$ denote the real symplectic group of $2n \times 2n$ matrixes. There are two natural homomorphisms $i : SU(m,n) \hookrightarrow Sp(2(m+n), \mathbb{R})$ and $j : Sp(2n, \mathbb{R}) \hookrightarrow SU(n,n)$, and these lift uniquely to continuous group homomorphisms $\widetilde{SU}(m,n) \hookrightarrow \widetilde{Sp}(2(m+n), \mathbb{R})$ and $\widetilde{Sp}(2n, \mathbb{R}) \hookrightarrow \widetilde{SU}(n,n)$. Let us set in this section $\Phi_{SU(m,n)}$ for the homogeneous quasi-homomorphism $\overline{\Phi}$ and $\Phi_{Sp(2n,\mathbb{R})}$ for its symplectic cousin. Standard arguments show the following proposition.

Proposition 4.4. 1. *The unitary homogeneous quasi-homomorphism $\Phi_{SU(n,n)}$ restricts along the embedding $Sp(2n, \mathbb{R}) \hookrightarrow SU(n,n)$ to the symplectic homogeneous quasi-homomorphism $\Phi_{Sp(2n,\mathbb{R})}$.*

2. *The symplectic homogeneous quasi-homomorphism $\Phi_{Sp(2(m+n),\mathbb{R})}$ restricts along the embedding $SU(m,n) \hookrightarrow Sp(2(m+n), \mathbb{R})$ to $2\Phi_{SU(m,n)}$, if $mn \neq 0$.*

Remark 4.2. It was already noticed in [25, § 4] that the restriction of $\Phi_{Sp(2(m+n),\mathbb{R})}$ to $SU(m+n) \hookrightarrow Sp(2(m+n), \mathbb{R})$ is trivial, as this subgroup is simply connected. The fact that the restriction of the Maslov class on $SU(m,n)$ is non-trivial was also stated in [25, Corollary 4.4].

Then from [5, Theorem 2.10] we deduce the following proposition.

Proposition 4.5. *The homogeneous Dupont–Guichardet–Wigner quasi-homomorphism $\overline{\Phi} : \widetilde{SU(m,n)} \to \mathbb{R}$ is the unique continuous lift of the map $\overline{\phi} : SU(m,n) \to \mathbb{R}/\mathbb{Z}$ sending 1 to 0, defined when g is semi-simple by the formula:*

$$\overline{\phi}(g) = \frac{1}{2\pi} \left(\sum_{\lambda \in S(g)} n^+(\lambda) \arg(\lambda) \right) \in \mathbb{R}/\mathbb{Z},$$

where $S(g)$ is the set of eigenvalues of g and $n^+(\lambda)$ their positive multiplicity.

We postpone the discussion and the definition of positive multiplicity to § 4.5.

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End of the proof of Proposition 4.2. Proposition 4.5 shows that $\overline{\Phi}$ is uniquely determined as a continuous lift of $\overline{\phi}$ and the formula follows because $\widetilde{\rho}_{p,\zeta}(\widetilde{M_g}) \subset SU(m,n)$ consists only of semi-simple elements. □

The independence of $\overline{L}_\zeta, \overline{\Phi}$ on the chosen bounded cocycle $c_{SU(m,n)}$ is a consequence of the fact that $SU(m,n)$ is uniformly perfect.

Although the fact that all simple Lie groups are uniformly perfect seems to be folklore, we did not find it explicitly in the literature. For all semi-simple Lie groups whose maximal compact is semi-simple any element is the product of two commutators (see [13]). However, this does not apply precisely to $SU(m,n)$. One also knows that there are elements that are not commutators (from [45]). An explicit bound for the number of reflections needed to write any element in $U(m,n)$ as a product was given in [14] and the number of commutators could be deduced from it. Using a similar reasoning one shows the following proposition.

Proposition 4.6. *The group $SU(m,n)$ is uniformly perfect, more precisely: any element is a product of at most $14(m+n)$ commutators.*

4.4. Useful properties of Dupont–Guichardet–Wigner cocycles

Notice that the reduction mod $\mathbb{Z}$ of $\overline{\Phi}$ descends to a map $\overline{\phi} : SU(m,n) \to \mathbb{R}/\mathbb{Z}$, given by $\overline{\phi}(x) = \overline{\Phi}(\widetilde{x})$, where $\widetilde{x}$ is an arbitrary lift of x. The quasi-homomorphism is easy to compute on lifts of Borel subgroups of $SU(m,n)$ such as AN. Recall that all Borel subgroups of $SU(m,n)$ are conjugate. The subgroup AN is simply connected, and contains the identity matrix; therefore its preimage $\widetilde{AN}$ is a disjoint union of (simply) connected components, each one homeomorphic to AN and canonically indexed by an element of $\mathbb{Z} = \ker(\widetilde{SU}(m,n) \to SU(m,n))$.

Lemma 4.1. *The quasi-homomorphism $\overline{\Phi}$ is locally constant on $\widetilde{AN}$. More precisely, $\overline{\Phi}$ takes the value d on the sheet of $\widetilde{AN}$ indexed by d. Consequently, if B is an arbitrary Borel subgroup of $SU(m,n)$ and $\widetilde{B}$ denotes its preimage in $\widetilde{SU}(m,n)$, then $\overline{\Phi}$ takes integer values on $\widetilde{B}$.*

Proof. By construction, the function v_0 is constant with value 1 on AN; therefore its continuous lift Φ takes integral values on $\widetilde{AN}$, and as it is continuous, these values are given by the integer indexing the connected component. Moreover, if $g \in \widetilde{AN}$ belongs to the component indexed say by d, then for any $n \in \mathbb{Z}$ the element g^n belongs to the component indexed by nd. Therefore we have:

$$\overline{\Phi}(g) = \lim_{n\to\infty} \frac{1}{n}\Phi(g^n) = \lim_{n\to\infty} \frac{1}{n} dn = d.$$

If B is an arbitrary Borel subgroup, then there is an element $g \in SU(m,n)$ such that $gBg^{-1} \subset AN$. As a consequence, if we denote by $\widetilde{g}$ a preimage of g in $\widetilde{SU}(m,n)$, conjugation by $\widetilde{g}$ embeds $\widetilde{B}$ into $\widetilde{AN}$. As $\overline{\Phi}$ is invariant under conjugation, the result follows. □

Proposition 4.7. *The homogeneous normalized quasi-homomorphism on $\widetilde{SU}(m,n)$ is the unique continuous normalized lift of the map $\overline{\phi}\circ e : SU(m,n)\to\mathbb{R}/\mathbb{Z}$, where $g = e(g)h(g)u(g)$ is the Jordan decomposition of $g\in SU(m,n)$. Recall that $e(g)$ is the elliptic part, $h(g)$ the hyperbolic part and $u(g)$ the unipotent part of g.*

Proof. Let g be an arbitrary element in $SU(m,n)$ and $\widetilde{g}\in\widetilde{SU}(m,n)$ one of its lifts. Choose also a lift $\widetilde{e}(g)$ of $e(g)$. Since $e(g)$ commutes with g we have that $\overline{\Phi}(\widetilde{e}(g)^{-1}\widetilde{g}) = \overline{\Phi}(\widetilde{e}(g)^{-1}) + \overline{\Phi}(\widetilde{g})$. By construction, $\widetilde{e}(g)^{-1}\widetilde{g} = h(g)u(g)$ and since $h(g)$ is conjugate to some element in A and $u(g)$ to some element in N, $h(g)u(g)$ belongs to some Borel subgroup of $SU(m,n)$. By Lemma 4.1, this implies that $\overline{\Phi}(\widetilde{e}(g)^{-1}\widetilde{g})\in\mathbb{Z}$ or equivalently:

$$\begin{aligned}\overline{\Phi}(\widetilde{g}) &= -\overline{\Phi}(\widetilde{e}(g)^{-1}) \bmod \mathbb{Z}\\ &= \overline{\Phi}(\widetilde{e}(g)) \bmod \mathbb{Z}\\ &= \overline{\phi}(e(g)) \text{ by definition of } \overline{\phi}.\end{aligned}$$

The second equality comes from the fact that, as $\overline{\Phi}$ is homogeneous and normalized, for any $h\in\widetilde{SU}(m,n)$, $\overline{\Phi}(h^{-1}) = -\overline{\Phi}(h)$. □

4.5. Positive eigenvalues of pseudo-unitary operators

Consider a pseudo-unitary operator $g\in SU(m,n)$. Let $H : V\times V\to\mathbb{C}$ be the indefinite Hermitian form defining the group $SU(m,n)$, where $\dim_{\mathbb{C}} V = m+n$. We will assume henceforth that $1\leqslant m\leqslant n$.

The spectrum $S(g)$ of g is symmetric with respect to the unit circle; namely if $\lambda\in S(g)$ then $\overline{\lambda}^{-1}\in S(g)$ (see [29, Ch. 10, § 5]). For a given $\lambda\in S(g)$ we consider the root space $V_\lambda(g) = \ker(g-\lambda I)^{m+n}\subset V$. We have then $V = \bigoplus_{\lambda\in S(g)} V_\lambda(g)$. Moreover, each $V_\lambda(g)$ splits as $V_\lambda(g) = \bigoplus_i V_{\lambda,i}(g)$, where each subspace $V_{\lambda,i}(g)$ corresponds to a Jordan block with diagonal λ in the Jordan decomposition of g. The number of such subspaces $V_{\lambda,i}(g)$ (i.e. Jordan blocks) is the geometric multiplicity of λ, namely $\dim\ker(g-\lambda I)$. The collection of dimensions $\dim V_{\lambda,i}$ is the collection of partial multiplicities of λ. Furthermore, the collection of partial multiplicities of $\lambda\in S(g)$ agrees with the one for $\overline{\lambda}^{-1}$.

We will use the canonical form of pseudo-unitary operators from [30, Theorem 5.15.1]. We will only need a weaker form and state it in the simplest form, though the statement in [30] is more precise.

Proposition 4.8. *Let $g\in SU(m,n)$ have the set of Jordan blocks $J_1, J_2, \dots, J_{a+2b}$ (where $a+2b\leqslant m+n$) and corresponding eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_{a+2b}$, not necessarily distinct. We suppose that $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_a| = 1$, $|\lambda_{a+2i-1}|>1$ and $\lambda_{a+2i-1} = \overline{\lambda}_{a+2i}^{-1}$, for $1\leqslant i\leqslant b$. Then there exists a non-singular matrix C such that the following two conditions hold simultaneously:*

$$C^{-1}gC = \bigoplus_{i=1}^{m^+(g)} \lambda_{j_i}K_{j_i} \bigoplus_{i=1}^{m^-(g)} \lambda_{s_i}K_{s_i} \bigoplus_{1\leqslant i\leqslant b}\begin{pmatrix}\lambda_{a+2i-1}K_{a+2i-1} & 0\\ 0 & \overline{\lambda}_{a+2i-1}^{-1}K_{a+2i}\end{pmatrix},$$

$$C^*HC = \bigoplus_{i=1}^{m^+(g)} P_{j_i} \bigoplus_{i=1}^{m^-(g)} -P_{s_i} \bigoplus_{1\leqslant i\leqslant b} \begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix},$$

where we have the following:

1. *The blocks K_j are unipotent upper triangular matrixes (also called Toeplitz blocks), for all $j \leqslant a+2b$.*
2. *Each matrix P_j is a permutation matrix of the form* $\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ *having the size of the Jordan block J_j, for all $j \leqslant a+2b$.*
3. *The two sets $\{j_1, j_2, \ldots, j_{m_+(g)}\}$ and $\{s_1, s_2, \ldots, s_{m_-(g)}\}$ form a partition of $\{1, 2, \ldots, a\}$, so that $m_+(g)+m_-(g) = a$. The sign characteristic $\varepsilon_i \in \{\pm 1\}$ for $1 \leqslant i \leqslant a$ is given by $\varepsilon_i = 1$ iff $i \in \{j_1, j_2, \ldots, j_{m_+(g)}\}$.*
4. *The canonical form is unique, up to a permutation of equal Toeplitz blocks respecting the sign characteristic.*

When g is semi-simple the canonical form is simpler, as follows.

Corollary 4.1. *Let $g \in SU(m, n)$ be a semi-simple element with eigenvalues λ_i, $1 \leqslant i \leqslant m+n$. Let us denote by $\lambda_\alpha, \overline{\lambda}_\alpha^{-1}$, with $\alpha \in N(g) \subset \{1, 2, \ldots, m+n\}$ those eigenvalues of modulus different from 1, where $|\lambda_\alpha| > 1$. Then there exists a non-singular matrix C such that the following two conditions hold simultaneously:*

$$C^{-1}gC = \bigoplus_{i=1}^{m^+(g)} (\lambda_{j_i}) \oplus \bigoplus_{i=1}^{m^-(g)} (\lambda_{s_i}) \oplus \bigoplus_{\alpha\in N(g)} \begin{pmatrix} \lambda_\alpha & 0 \\ 0 & \overline{\lambda}_\alpha^{-1} \end{pmatrix},$$

$$C^*HC = \bigoplus_{i=1}^{m^+(g)} (+1) \oplus \bigoplus_{i=1}^{m^-(g)} (-1) \oplus \bigoplus_{\alpha\in N(g)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here the sets of indices $\{j_1, j_2, \ldots, j_{m_+(g)}\}$, $\{s_1, s_2, \ldots, s_{m_+(g)}\}$ and $N(g)$ form a partition of $\{1, 2, \ldots, m+n\}$. The canonical form is unique up to a permutation preserving the eigenvalues and the sign characteristic.

Proof. This result seems to have been stated explicitly first by Krein (see [32]) for the symplectic group and by Yakubovich in the present setting (see [49, p. 124]). □

Definition 4.2. Let g be a semi-simple element of $SU(m, n)$. The eigenvalues λ_i of g, for $i \in \{j_1, j_2, \ldots, j_{m_+(g)}\}$, i.e. those for which $\varepsilon_i = +1$, will be called *positive* (after Gelfand and Lidskii, Krein and Yakubovich) and their *positive multiplicity* n_i^+ is the multiplicity among positive eigenvalues. By convention, the eigenvalues λ_α with $|\lambda_\alpha| > 1$ are said to be *positive* and their positive multiplicity coincides with the usual multiplicity. The

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remaining eigenvalues will be called *negative* eigenvalues of g. We will also denote by $n^+(\lambda)$ the positive multiplicity of the eigenvalue λ (which is 0 for negative ones) of the semi-simple g.

The positivity seems more subtle when g is not semi-simple. In fact, the signature of each block $\varepsilon_j P_j$ equals 0 when its dimension n_j is even and ε_j, when its dimension n_j is odd, respectively. Further, the signature of $\begin{pmatrix} 0 & P_{a+2i-1} \\ P_{a+2i} & 0 \end{pmatrix}$ is always 0. Thus, every eigenvalue involved in a Jordan block is positive with a positive multiplicity equal to approximatively half of its partial multiplicity.

Lemma 4.2. *Let $g \in SU(m,n)$. Then in a suitable basis of V we can write simultaneously:*

$$e(g) = \bigoplus_{i=1}^{a} \operatorname{diag}(\lambda_i) \bigoplus_{1\leqslant i \leqslant b} \begin{pmatrix} \operatorname{diag}\left(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}\right) & 0 \\ 0 & \operatorname{diag}\left(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}\right) \end{pmatrix},$$

$$H = \bigoplus_{i=1}^{a} \varepsilon_i X_i \bigoplus_{1\leqslant i \leqslant b} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $\operatorname{diag}(\lambda_i)$ *is a diagonal matrix of the size equal to the partial multiplicity n_i of λ_i and X_i is the diagonal matrix of the same size with entries ± 1 of signature $\frac{1}{2}(1-(-1)^{n_i})$.*

Proof. The proof is clear from the previous discussion. □

Furthermore, the elliptic element $e(g)$ is conjugate to some element $\begin{pmatrix} e(g)_+ & 0 \\ 0 & e(g)_- \end{pmatrix}$ of $S(U(m)\times U(n))$, where $e(g)_+ \in U(m)$ corresponds to a maximal invariant positive subspace of V for the Hermitian form H. The previous lemma gives an explicit formula for $e(g)_+$ in the form:

$$e(g)_+ = \bigoplus_{i=1}^{a} \operatorname{diag}_+(\lambda_i) \bigoplus_{1\leqslant i \leqslant b} \operatorname{diag}\left(\frac{\lambda_{a+2i-1}}{|\lambda_{a+2i-1}|}\right),$$

where $\operatorname{diag}_+(\lambda_i)$ is a diagonal matrix of the size equal to its partial positive multiplicity, defined as: $n_i^+ = \begin{cases} \frac{n_i}{2}, & \text{even } n_i \\ \frac{n_i+\varepsilon_i}{2}, & \text{odd } n_i \end{cases}$.

An immediate consequence is that

$$\det(e(g)_+) = \exp\left(2\pi\sqrt{-1}\left(\sum_{i=1}^{a} n_i^+ \arg(\lambda_i) + \sum_{i=1}^{b} n_{a+2i-1}\arg(\lambda_{a+2i-1})\right)\right).$$

When g is already semi-simple, this formula simplifies to

$$\det(e(g)_+) = \exp\left(2\pi\sqrt{-1}\left(\sum_{\lambda\in S(g)} n^+(\lambda)\arg(\lambda)\right)\right).$$

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We formulate the result obtained so far in the following.

Lemma 4.3. *For $g \in SU(m,n)$ we have*

$$\overline{\phi}(g) = \frac{1}{2\pi}\left(\sum_{i=1}^{a} n_i^+ \arg(\lambda_i) + \sum_{i=1}^{b} n_{a+2i-1}\arg(\lambda_{a+2i-1})\right) \in \mathbb{R}/\mathbb{Z}.$$

5. Evaluation of quasi-homomorphisms

5.1. Arithmetic properties of dimensions of conformal blocks

The aim of this section is to provide ground for the explicit computations of values of quasi-homomorphisms in § 5.2. Our results here are far from being complete and might only be seen as quantitative evidence in favor of various non-degeneracy conditions of arithmetic nature.

5.1.1. Dimensions. The first step is an apparently unnoticed congruence satisfied by the dimensions $N(g,p)$ of the space of conformal blocks arising in the TQFT $\mathcal{V}_p$. Before proceeding, we need to introduce some notation.

We denote by $\theta(p)$ the order of the root of unity $\zeta_{2p}^{-12-p(p+1)}$, where ζ_{2p} is a primitive $2p$th root of unity. Specifically, we have the following.

Lemma 5.1.

1. *If p is odd we have:*
$$\theta(p) = \begin{cases} p, & \text{if g.c.d.}(p,6) = 1 \\ \dfrac{p}{3}, & \text{if } p \equiv 0 \pmod 3. \end{cases}$$
2. *Assume p is even.*
   (a) *If $p = 12s$, $s \in \mathbb{Z}$*
$$\theta(p) = \begin{cases} 2s, & \text{if } s \equiv 0 \pmod 2 \\ s, & \text{if } s \equiv 1 \pmod 2. \end{cases}$$
   (b) *If $p = 4s$, $s \in \mathbb{Z}$, g.c.d.$(s,3) = 1$*
$$\theta(p) = \begin{cases} 2s, & \text{if } s \equiv 0 \pmod 2 \\ s, & \text{if } s \equiv 1 \pmod 2. \end{cases}$$
   (c) *If $p = 6s$, $s \in \mathbb{Z}$, g.c.d.$(s,2) = 1$ then $\theta(p) = 2s$.*
   (d) *If $p = 2s$, $s \in \mathbb{Z}$, g.c.d.$(s,6) = 1$ then $\theta(p) = 2s$.*

Proof. The proof is obtained by direct calculation. □

Proposition 5.1. *If $g \geqslant 3$ then*

$$N(g,p) \equiv 0 \pmod{\theta(p)}.$$

If $g = 2$ then

$$10N(g,p) \equiv 0 \pmod{\theta(p)}.$$

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Proof. The universal central extension $\widetilde{M_g}^u$ is a subgroup of the central extension $\widetilde{M_g}(12)$ arising in the TQFT representation, which has Euler class 12 (see [39]). It was already noticed in [15, 24] that the image $\widetilde{\rho}_p(\widetilde{M_g}^u)$ in the unitary group $U(N(g,p))$ is actually contained in the subgroup $SU(N(g,p))$ for $g \geqslant 3$. This is a consequence of the fact that $\widetilde{M_g}^u$ is perfect. The action of the central element of $\widetilde{M_g}^u$ is by means of the scalar matrix $\zeta_{2p}^{-12-p(p+1)}$ (see e.g. [39]). This matrix has therefore unit determinant and hence the first congruence follows. In the case $g = 2$ we have to use the fact that $H_1(\mathcal{M}_2) = \mathbb{Z}/10\mathbb{Z}$ and follow the same lines. □

We have also for small values of the genus g the following computations due to Zagier [50]:

$$N(g,2k) = \begin{cases} \frac{1}{6}(k^3-k), & \text{if } g = 2 \\ \frac{1}{180}(k^2(k^2-1)(k^2+11), & \text{if } g = 3 \\ \frac{1}{7560}(k^3(k^2-1)(2k^4+23k^2+191), & \text{if } g = 4 \end{cases}$$

and from [7]:

$$N(g,p) = \frac{1}{2^g} N(g,2p), \quad \text{if } p \text{ is odd.}$$

Notice that, with the notations from [50] we have $N(g,p) = \mathcal{D}(g,k)$, when $p = 2k$ and $N(g,p) = \frac{1}{2^g}\mathcal{D}(g,p)$ if p is odd. As an immediate corollary, we obtain the following.

Lemma 5.2.

1. *If $g = 3$ and $p = 4n+2$ or $p = 8n \pm 3$ then $N(3,p)$ is odd.*
2. *If $p = 5$ then $N(g,5)$ is odd iff the genus $g \not\equiv 1 \pmod 3$.*

Proof. Using the Verlinde formula (usually for even p) and the previous relation, we find that the dimension $N(g,5)$ is given by

$$N(g,5) = \left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}.$$

Thus $N(g,5)$ is determined by the following recurrence with the given initial conditions:

$$N(g+1,5) = 5N(g,5) - 5N(g-1,5), \quad N(1,5) = 2, N(2,5) = 5.$$

The mod 2 congruence follows by induction on g. □

Corollary 5.1. *The signature is non-zero (as needed in [10]) when $N(g,p)$ is odd, and thus for infinitely many values of g, p as in Lemma 5.2.*

5.1.2. Signatures. The Verlinde formula for the dimensions $N(g,p)$ admits refinements for the case of the signatures $\sigma(g,\zeta_{2p})$ of the Hermitian forms H_ζ in genus g. Here the root of unity ζ_{2p} is a primitive $2p$th root of unity. More details will appear in a forthcoming paper [12] devoted to this subject. The aim of this section is to gather evidence to back up the following.

Conjecture 5.1. *Let us consider ζ a primitive $2p$th root of unity, for prime $p \geqslant 5$ such that neither ζ nor $\overline{\zeta}$ is equal to A_p, for odd p and $\pm A_p$, for even p, respectively. Then for all g in some arithmetic progression, $\sigma(g, \zeta) \not\equiv 0 \pmod p$.*

We have the following general behavior.

Proposition 5.2 [12]. *For each p we have:*

$$\sigma(g, p, \zeta) = \sum_{i=1}^{[\frac{p-1}{2}]} \lambda_i(\zeta)^{g-1},$$

where $\lambda_i(\zeta)$ runs over the set of roots of some polynomials P_ζ with integer coefficients.

Remark 5.1. Observe that $N(1, p) = [\frac{p-1}{2}]$, which corresponds to the fact that $\sigma(g, p, \zeta) = N(1, p)$ for any ζ, because the genus one Hermitian form H_ζ is always positive, as the image of the quantum representations is always finite (see e.g. [27]).

In this section, we will denote by $\zeta_{2p} = \exp(\frac{\pi i}{p})$ the principal primitive root of unity. The other primitive roots of unity are of the form ζ_{2p}^k, with odd k. Moreover, it is enough to restrict to the case when $k \in \{1, 3, 5, \dots p-1\}$. Recall that $P_{\zeta_{2p}^{\frac{p-1}{2}}}$ for $p \equiv -1 \pmod 4$ and $P_{\zeta_{2p}^{\frac{p+1}{2}}}$ for $p \equiv 1 \pmod 4$, respectively, are the polynomials associated to the unitary TQFTs, thereby computing the dimensions of the space of conformal blocks according to the Verlinde formula. With the help of a computer program run by F. Costantino, one finds the following.

Example 5.1. 1. Let $p = 5$.

   (a) We have

   $$P_{\zeta_{10}} = x^2 - 3x + 3$$

   and the first terms of the sequence $\sigma(g, 5, \zeta_{10})$, $g \geqslant 1$ are

   $$2, 3, 3, 0, -9, -27, -54, -81, -81, 0, 243.$$

   (b) Further

   $$P_{\zeta_{10}^3} = x^2 - 5x + 5$$

   and the first terms of the sequence $\sigma(g, 5, \zeta_{10})$, $g \geqslant 1$ are the dimensions $N(g, 5)$:

   $$2, 5, 15, 50, 175, 625, 2250, 8125, 29\,375, 106\,250, 384\,375.$$

2. Let $p = 7$.

   (a) We have

   $$P_{\zeta_{14}} = x^3 - 8x^2 + 23x - 23$$

   and the first terms of the sequence $\sigma(g, 7, \zeta_{14})$, $g \geqslant 1$ are

   $$3, 8, 18, 29, 2, -237, -1275, -4703, -13\,750, -31\,156, -41\,167.$$

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(b) Also

$$P_{\zeta_{14}^3} = x^3 - 14x^2 + 49x - 49$$

and the first terms of the sequence $\sigma(g, 7, \zeta_{14}^3)$, $g \geqslant 1$ are given by the dimension $N(g, 7)$:

3, 14, 98, 833, 7546, 69 629, 645 869, 6000 099, 55 765 626, 518 361 494, 4818 550 093.

(c) Eventually, we have

$$P_{\zeta_{14}^5} = x^3 - 6x^2 + 23x - 23$$

and the first terms of the sequence $\sigma(g, 7, \zeta_{14}^5)$, $g \geqslant 1$ are

3, 6, −10, −129, −406, 301, 8177, 32 801, 15 658, −472 404, −2440 135.

3. Let $p = 9$.

(a) We have

$$P_{\zeta_{18}} = x^4 - 16x^3 + 97x^2 - 257x + 257$$

and the first terms of the sequence $\sigma(g, 9, \zeta_{18})$, $g \geqslant 1$ are

4, 16, 62, 211, 446, −1509, −29 113, −259 040, −1823 114, −11 137 172, −60 443 933.

(b) Further

$$P_{\zeta_{18}^5} = x^4 - 30x^3 + 243x^2 - 729x + 729$$

and the first terms of the sequence $\sigma(g, 9, \zeta_{18}^5)$, $g \geqslant 1$ are the dimensions $N(g, 9)$:

4, 30, 414, 7317, 137 862, 2637 765, 50 664 771, 974 133 540, 18 734 896 134, 360 344 121 174, 6930 952 607 259.

(c) Eventually

$$P_{\zeta_{18}^7} = x^4 - 10x^3 + 101x^2 - 257x + 257$$

and the first terms of the sequence $\sigma(g, 9, \zeta_{18}^7)$, $g \geqslant 1$ are

4, 10, −102, −1259, −746, 90 915, 687 147, −2179 104, −67 636 010, −303 038 972, 3064 220 783.

Remark 5.2. We have $P_\zeta = P_{\bar{\zeta}}$. Moreover, for even p we also have $P_\zeta = P_{-\zeta}$.

Proposition 5.3. *Conjecture 5.1 is true for* $p \in \{5, 7, 9\}$.

Proof. We obtain from above that the sequence $\sigma(g, \zeta_{10}) \pmod 5$, $g \geqslant 1$ is periodic with period 24 and its terms read:

2, 3, 3, 0, 1, 3, 1, 4, 4, 0, 3, 4, 3, 2, 2, 0, 4, 2, 4, 1, 1, 1, 0, 2, 1, 2, 3, ...

Therefore $\sigma(g, \zeta_{10}) \equiv 0 \pmod 5$ if and only if $g \pmod{24} \in \{4, 10, 16, 22\}$.

Furthermore, for $p = 7$ the sequence $\sigma(g, \zeta_{14})$ (mod 7), $g \geqslant 1$ is periodic with period 12 and its first terms read:

$$3, 1, 4, 1, 2, 1, -1, 1, 5, 1, 0, 1, 3, 1, 4, \ldots.$$

Thus $\sigma(g, \zeta_{14}) \equiv 0 \pmod 7$ if and only if $g \equiv 11 \pmod{12}$.

The sequence $\sigma(g, \zeta_{14}^3)$ (mod 7), $g \geqslant 1$ is eventually periodic. One can check that $\sigma(g + 36, \zeta_{14}^3) \equiv \sigma(g, \zeta_{14}^3) \pmod 7$ for $g \geqslant 55$.

A more conceptual proof is as follows. It suffices to show that $P_\zeta(0)$ is invertible (mod p). The vector $v_g = (\sigma(h, \zeta)_{h \in \{g, g+1, \ldots, g+[\frac{p-1}{2}]-1\}})$ is obtained from v_1 by means of the formula

$$v_g = M_\zeta^g v_1,$$

where M_ζ is the companion matrix associated to P_ζ. Therefore $\det M_\zeta = P_\zeta(0)$. If the determinant is invertible mod p then the sequence of vectors $M_\zeta^g v_1$ cannot contain the null vector mod p. But this sequence is eventually periodic. Therefore for g in some arithmetic progression $\sigma(g, \zeta)$ is non-trivial mod p. Using the explicit values of P_ζ, one settles immediately the cases $p \in \{5, 7, 9\}$. □

5.2. Proof of Theorem 1.2

Recall from §4.1 that we have a homogeneous quasi-homomorphism $\overline{L}_\zeta : \widetilde{M}_g \to \mathbb{R}$ associated to a primitive $2p$th root of unity ζ. Consider the map

$$\bar{l}_\zeta = \overline{L}_\zeta|_{\ker \widetilde{\rho}_p} : \ker \widetilde{\rho}_p \to \mathbb{R}.$$

Lemma 5.3. *We have $\bar{l}_\zeta \in \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M}_g{}^{\mathrm{u}}}$, namely $\bar{l}_\zeta$ is a group homomorphism invariant by the conjugacy action of $\widetilde{M}_g{}^{\mathrm{u}}$.*

Proof. The boundary of $\overline{L}_\zeta$ is $\widetilde{\rho}_p^*(c_{SU(m,n)})$, which obviously vanishes on $\ker \widetilde{\rho}_p$, namely

$$\widetilde{\rho}_p^*(c_{SU(m,n)})(x, y) = 0, \quad \text{if either } x \text{ or } y \in \ker \widetilde{\rho}_p.$$

This implies that $\bar{l}_\zeta$ is a homomorphism.

Eventually recall that $\overline{L}_\zeta$ is a homogeneous quasi-homomorphism and thus it is a class function. This implies that $\bar{l}_\zeta$ is also a class function. □

Recall from the proof of Proposition 2.1 that there is an isomorphism

$$\iota : \mathrm{Hom}(\ker \widetilde{\rho}_p, \mathbb{R})^{\widetilde{M}_g{}^{\mathrm{u}}} \to H^2(\widetilde{\rho}_p(\widetilde{M}_g{}^{\mathrm{u}}), \mathbb{R}).$$

We want to show that $\bar{l}_\zeta \neq 0$ and consequently $\iota(\bar{l}_\zeta) \in H^2(\widetilde{\rho}_p(\widetilde{M}_g{}^{\mathrm{u}}), \mathbb{R})$ is not vanishing. Denote by $h_g^+(\zeta)$ the dimension of the maximal positive subspace of the Hermitian form H_ζ.

Proposition 5.4. *Suppose that $h_g^+(\zeta) \not\equiv 0 \pmod p$, $p \geqslant 5$ prime. Then $\iota(\bar{l}_\zeta) \neq 0 \in H^2(\widetilde{\rho}_p(\widetilde{M}_g{}^{\mathrm{u}}); \mathbb{R})$.*

Proof. Let c denote a generator of the center of $\widetilde{M_g}^{\mathrm{u}}$. We know that $\widetilde{\rho}_{p,\zeta}(c) = \zeta^{-12}$ (see [39]), when p is odd. The formula of Proposition 4.2 yields

$$\overline{L}_\zeta(c) \equiv -12h_g^+(\zeta)\arg(\zeta) \pmod{2\pi\mathbb{Z}}.$$

Now, if $\overline{L}_\zeta(c) \not\equiv 0 \in \mathbb{R}/2\pi\mathbb{Z}$, then $\overline{L}_\zeta(c) \neq 0$. This implies that $\overline{L}_\zeta(c^n) \neq 0$ for any $n \neq 0$. Recall that $c^p \in \ker \widetilde{\rho}_{p,\zeta}$. Thus $\bar{l}_\zeta(c^p) \neq 0$ so that $\bar{l}_\zeta$ is not identically zero, as claimed. □

Proposition 5.5. *If $p \in \{5, 7, 9\}$ then $\bar{l}_{\zeta_{2p}}$ is non-zero for infinitely many values of g in some arithmetic progression.*

Proof. According to Proposition 5.4, it suffices to show that $h^+(\zeta_{2p}) \not\equiv 0 \pmod p$. We proved in Proposition 5.1 that $N(g, p) \equiv 0 \pmod p$, so that this condition is equivalent to proving that $\sigma(g, \zeta_{2p}) \not\equiv 0 \pmod p$. But this last statement is part of Proposition 5.3. □

End of the proof of Theorem 1.2. From Proposition 5.5 and the proof of Proposition 2.1 we obtain that $\bar{l}_{\zeta_{2p}}$ is non-zero and hence a non-trivial class in $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$ for infinitely many values of g and $p \in \{5, 7, 9\}$.

Remark 5.3. The same method provides examples when $\overline{L}_\zeta(T_\gamma^p) \neq 0$, and hence slightly better lower bounds for the rank of $H^2(\widetilde{\rho}_p(\widetilde{M_g}^{\mathrm{u}}), \mathbb{R})$.

Remark 5.4. If we were able to show that there is at least one non-trivial quasi-homomorphism on $\rho_{p,\zeta}(M_g)$ then it would follow that this group cannot be an irreducible higher rank lattice in a semi-simple Lie group, according to the result of Burger and Monod from [11].

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