

Genericity in topological dynamics

MICHAEL HOCHMAN[†]

*Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram,
The Hebrew University of Jerusalem, Jerusalem 91904, Israel
(e-mail: mhochman@math.huji.ac.il)*

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Abstract. We study genericity of dynamical properties in the space of homeomorphisms of the Cantor set and in the space of subshifts of a suitably large shift space. These rather different settings are related by a Glasner–King type correspondence: genericity in one is equivalent to genericity in the other. By applying symbolic techniques in the shift-space model we derive new results about genericity of dynamical properties for transitive and totally transitive homeomorphisms of the Cantor set. We show that the isomorphism class of the universal odometer is generic in the space of transitive systems. On the other hand, the space of totally transitive systems displays much more varied dynamics. In particular, we show that in this space the isomorphism class of every Cantor system without periodic points is dense and the following properties are generic: minimality, zero entropy, disjointness from a fixed totally transitive system, weak mixing, strong mixing and minimal self joinings. The latter two stand in striking contrast to the situation in the measure-preserving category. We also prove a correspondence between genericity of dynamical properties in the measure-preserving category and genericity of systems supporting an invariant measure with the same property.

1. Introduction

1.1. *Genericity.* A question of some interest is: what does a ‘typical’ dynamical system look like? To make this question precise one first fixes a parametrization of dynamical systems with an appropriate complete metric, and states that a property is *generic* if the set of systems satisfying the property is residual (contains a dense G_δ), and is *exotic* if it is meagre (its complement is residual).

This question goes back to the early twentieth century, at least as far as the work of Oxtoby and Ulam [19], who showed that in dimension two and up, a generic volume-preserving homeomorphism of a cube (and other ‘nice’ manifolds) is ergodic. A few years later, Halmos [13–15] showed that in the space of automorphisms of a Lebesgue space, a generic automorphism is ergodic and weak mixing. In contrast, Rohlin [21] showed that

[†] Current e-mail address: hochman@math.princeton.edu

the strongly mixing automorphisms are exotic (this established the existence of weakly but not strongly mixing systems before any explicit examples were available). Since then, the status of many other properties in this category have been established. A parallel program has been carried out for volume preserving homeomorphisms of manifolds, and a certain unification has been achieved between these two categories [5, 7]. Similar questions have been studied in the smooth category, where there are many open questions.

In the present paper we will be concerned with the category of topological dynamics, where there has recently been renewed interest in questions of genericity [2, 3, 6, 12]. By a *topological dynamical system* we mean a pair (X, φ) where X is a compact metric space and $\varphi : X \rightarrow X$ is a homeomorphism. Our results concern two parametrizations of dynamical systems which are, somewhat surprisingly, closely related: the space of subsystems of a suitably large shift space, which is universal, and the group of homeomorphisms of the Cantor set. We will mainly work in the first of these, where symbolic techniques can be applied, but we begin the discussion with the second, the better known of the two, where some interesting new phenomena have recently come to light.

1.2. *New results for homeomorphisms of the Cantor set.* Much work on genericity has focused on *Cantor systems*, that is, systems whose phase space is the Cantor set K . Their classical parametrization is the space of homeomorphisms of K , denoted

$$\mathcal{H} = \text{Homeo}(K)$$

and topologized by the complete metric

$$d(\varphi, \psi) = \max_{x \in K} (d(\varphi(x), \psi(x)) + d(\varphi^{-1}(x), \psi^{-1}(x)))$$

where on the right-hand side $d(\cdot, \cdot)$ is some fixed metric for K (the same topology is induced by the usual metric of uniform convergence, but note that that metric is not complete). We identify each $\varphi \in \mathcal{H}$ with the Cantor dynamical system (K, φ) .

A striking and quite unexpected result about \mathcal{H} was obtained recently by Kechris and Rosendal [16], who showed that there is a single Cantor system whose isomorphism class is a residual set in \mathcal{H} . Hence, generically, there is only one Cantor system. While Kechris and Rosendal's argument is non-constructive, Akin *et al* [2] describe this system explicitly and it turns out its dynamics are quite degenerate and completely understood. In particular, any question about genericity of a property in \mathcal{H} is reduced to the question of whether this generic system satisfies the property.

For this reason we restrict attention to more dynamically interesting subspaces of \mathcal{H} . One candidate is the space of transitive systems. It turns out that here the periodic phenomena are dominant, and again there is a single typical system.

THEOREM 1.1. *The space of transitive systems in $\text{Homeo}(K)$ is Polish, and the isomorphism class of the universal odometer is generic there.*

The universal odometer is the unique Cantor system which factors onto every finite cycle, and whose points are separated by these factors. See §5.4 for details.

Much more interesting is the situation in the space of totally transitive systems, which may be viewed as analogs of the aperiodic ergodic systems. The absence of global periodicity leaves room for much more complex dynamics.

THEOREM 1.2. *The space of totally transitive systems in $\text{Homeo}(K)$ is Polish. Within this space, the following hold.*

- (1) *For a fixed Cantor system without periodic points, the systems isomorphic to it are dense.*
- (2) *The zero-entropy, minimal and uniquely ergodic systems are generic.*
- (3) *For a fixed totally transitive system, the systems disjoint from it are generic.*
- (4) *The weakly mixing systems are a dense G_δ .*
- (5) *Strong mixing is generic.*
- (6) *Minimal self-joinings (and hence semi-simplicity and primeness) are generic.*

It is interesting to compare these results with the situation in the measure-preserving category, e.g. with the space of automorphisms of a Lebesgue space. There are several points of similarity. (1) is an analog of Halmos’s classical theorem [15] that the isomorphism class of any aperiodic ergodic system is dense. The statement about entropy in (2) is analogous to the situation in the measure-preserving category. Minimality and unique ergodicity may be viewed as ‘irreducibility’ conditions, and are somewhat analogous to ergodicity, which is likewise generic. The measure-theoretic analog of (4) is due to Halmos [15]. (3) is an analog of a theorem of del Junco’s [8], stating that generically an automorphism is disjoint from a fixed ergodic system. Note that in (3) the fixed system need not be a Cantor system but can be any totally transitive system.

On the other hand, there are some striking differences. (5) stands in contrast to the classical fact, due to Rohlin [21], that measure-theoretic strong mixing is exotic. (6) should be compared to the fact, noted by del Junco [8], that the measure preserving systems with minimal self joinings are exotic. In particular, we note that primeness (i.e. having only trivial factors) is generic in our setting but was recently proved to be exotic in the measure-preserving case (Ageev [1]).

It should be noted that Theorem 1.2 isn’t a result of the existence of a single generic system, as in the space of transitive systems. This possibility is ruled out by part (3) of the theorem, since a generic system would have to be disjoint from itself and hence consist of one point, which is impossible for a Cantor system.

1.3. The shift-space model and correspondence theorems. Our motivation for this work was the question of genericity of general topological systems, and in this regard \mathcal{H} is not a particularly good setting to investigate, since it represents only Cantor systems. In general, the typical dynamics in spaces $\text{Homeo}(X)$ can depend on X in surprising and non-trivial ways. An example of this, due to Glasner and Weiss [12], is that zero entropy is generic for homeomorphisms of the Cantor set, but infinite entropy is generic for homeomorphisms of the Hilbert cube. These differences are one reason it is desirable to have a universal model for dynamical systems.

Let $Q = [0, 1]^{\mathbb{N}_0}$ denote the Hilbert cube and consider the space $Q^{\mathbb{Z}}$ of bi-infinite sequences $x = (\dots, x(-1), x(0), x(1), \dots)$ over Q , which is compact and metrizable in the product topology. Let σ be the shift homeomorphism of $Q^{\mathbb{Z}}$, i.e. $(\sigma x)(i) = x(i + 1)$. Let

$$S = \{X \subseteq Q^{\mathbb{Z}} : X \text{ is closed, non-empty and } \sigma\text{-invariant}\}$$

be the space of subsystems of $Q^{\mathbb{Z}}$. In the Hausdorff metric \mathcal{S} is compact. We associate each $X \in \mathcal{S}$ with the dynamical system $(X, \sigma|_X)$, making \mathcal{S} into a parametrization of dynamical systems.

This parametrization is universal, that is, \mathcal{S} contains members of every isomorphism class of dynamical systems (see §2.5 below). However, it turns out that for purposes of studying genericity, \mathcal{S} and \mathcal{H} are essentially the same. In order to state this precisely we make the following definition. A dynamical property \mathbf{P} is a set of isomorphism classes of topological dynamical systems. We abuse notation and write $(X, T) \in \mathbf{P}$ to indicate that the isomorphism class of (X, T) is in \mathbf{P} . If \mathcal{U} is some parametrization of dynamical systems, we write $\mathcal{U}_{\mathbf{P}} \subseteq \mathcal{U}$ for the set of systems in \mathcal{U} satisfying \mathbf{P} .

THEOREM 1.3. (Correspondence theorem) *Let \mathbf{P} be a dynamical property. Suppose that:*

- (a) $\mathcal{H}_{\mathbf{P}}$ is a G_{δ} in \mathcal{H} ;
- (b) $\mathcal{S}_{\mathbf{P}}$ is a G_{δ} in \mathcal{S} ; and
- (c) $\mathcal{S}_{\mathbf{P}}$ contains a dense set of Cantor systems.

Let $\mathbf{Q} \subseteq \mathbf{P}$ be a dynamical property. Then:

- (1) \mathbf{Q} is generic in $\mathcal{H}_{\mathbf{P}}$ if and only if \mathbf{Q} is generic in $\mathcal{S}_{\mathbf{P}}$; and
- (2) if \mathbf{Q} contains only Cantor systems, then \mathbf{Q} is dense in $\mathcal{H}_{\mathbf{P}}$ if and only if \mathbf{Q} is dense in $\mathcal{S}_{\mathbf{P}}$.

Conditions (a) and (b) are necessary for the statement of (1), since in order to speak of genericity in $\mathcal{S}_{\mathbf{P}}$, $\mathcal{H}_{\mathbf{P}}$ these spaces must be Polish, and a classical theorem of Alexandroff states that this is equivalent to their being G_{δ} 's (see §2.7). Condition (c) cannot be omitted either. For example, let \mathbf{P} be the property of being connected, and let \mathbf{Q} be the property of consisting of one point. Then $\mathcal{H}_{\mathbf{P}} = \mathcal{H}_{\mathbf{Q}} = \emptyset$ and so density and genericity are satisfied trivially, whereas $\mathcal{S}_{\mathbf{Q}}$ is not dense in $\mathcal{S}_{\mathbf{P}}$.

Theorem 1.3 is the analog of a similar result in ergodic theory, where there are again two classical models of measure-preserving systems. One is the automorphism group \mathcal{A} of a Lebesgue space, given the so-called coarse topology, which is analogous to \mathcal{H} (the phase space is fixed and the map varies). The other is the space \mathcal{M} of shift-invariant Borel probability measures on $Q^{\mathbb{Z}}$, with the weak-* topology, which is analogous to \mathcal{S} (we fix the map and vary the subsystem/measure). See §2.8 for more details. Both these spaces are universal, and it has been shown by Glasner and King [11] and also by Rudolph [22] that a property is generic in one if and only if it is generic in the other. It is worth mentioning that in ergodic theory there is a third universal parametrization of dynamical systems, namely the space of transformations which are orbit equivalent to a fixed ergodic transformation. Rudolph [22] has examined this space and shown that genericity of dynamical properties there is equivalent to genericity in \mathcal{A} and \mathcal{M} . It remains to be explored to what extent there is a topological analog of this.

Besides its intrinsic interest, our main use of the correspondence theorem is in the proof of Theorems 1.1 and 1.2. We will verify the hypotheses of the correspondence theorem for the properties \mathbf{P} of being a Cantor system, being transitive and being totally transitive systems. Using symbolic techniques we will then be able to prove results on the genericity of various properties in \mathcal{S} , and transfer them to \mathcal{H} . We can also go the other way, and deduce from Kechris and Rosendal's results that in \mathcal{S} there is a single generic system.

An interesting correspondence also exists between genericity in the measure-preserving category and the topological one. For a dynamical property \mathbf{P} in the measure-preserving category, we write $\mathcal{M}_{\mathbf{P}} \subseteq \mathcal{M}$ for the set of invariant Borel probability measures μ on $Q^{\mathbb{Z}}$ such that the measure-preserving system $(Q^{\mathbb{Z}}, \sigma, \mu)$ is represented in \mathbf{P} . We have the following correspondence principle.

THEOREM 1.4. *Let \mathbf{P} be a dynamical property in the measure-preserving category and suppose that $\mathcal{M}_{\mathbf{P}}$ is a dense G_{δ} in \mathcal{M} . Let $\tilde{\mathbf{P}}$ be the property (in the topological category) of supporting an invariant measure from \mathbf{P} . Then $\tilde{\mathbf{P}}$ is generic in the space of totally transitive systems in \mathcal{S} and \mathcal{H} .*

This theorem is similar to results of Alpern [4], who showed that under some conditions a G_{δ} property which is generic for automorphisms of a Lebesgue space is also generic for volume-preserving homeomorphisms of a manifold. The hypothesis in Theorem 1.4, that $\mathcal{M}_{\mathbf{P}}$ is a G_{δ} , is unfortunate (note that such an hypothesis is present also in Alpern's work). It seems possible that the correspondence holds more generally, but we do not know how to prove it.

Finally, an application of Glasner and King's techniques [11] give a zero-one law for 'well-behaved' dynamical properties. Recall that a subset A of a Polish space is Baire measurable if it can be written as $A = U \Delta M$, where U is open and M is first category.

THEOREM 1.5. *Let \mathbf{P} be a topological dynamical property such that $\mathcal{H}_{\mathbf{P}}$ is Baire measurable. Then either \mathbf{P} is generic in the space of totally transitive systems in \mathcal{H} and \mathcal{S} , or it is exotic there.*

Although our use of \mathcal{S} is mainly for the study of Cantor systems, we note again that it is a universal model, and thus contains many other interesting subspaces, e.g. the space of all connected systems. There are many interesting questions here which we have not touched on. We discuss some of them in §10.

1.4. Organization. The rest of this paper is organized as follows. In the next section we give the basic definitions and notation. Section 3 is fairly independent and is devoted to the proof of the correspondence between \mathcal{H} and \mathcal{S} and the zero-one law. Sections 4–9 contain the remaining results. Section 4 contains some further definitions and symbolic machinery, §5 deals with transitive systems and proves Theorem 1.1, §6 deals with totally transitive systems and proves parts (1) and (2) of Theorem 1.2 as well as Theorem 1.4. Section 7 proves part (3) of Theorem 1.2. Section 8 discusses parts (4) and (5) of that theorem, and part (6) is proved in §9. Finally, in §10 we outline some extensions and open questions.

1.5. Notation. For the readers convenience, we conclude the introduction with a summary of our main notational and typographical conventions. Further definitions and notation appear in §§2 and 4.

K	The Cantor set
Q	The Hilbert cube, $[0, 1]^{\aleph_0}$
X, Y, Z, \dots	The phase space of dynamical systems
x, y, z, \dots	Points in X, Y, Z, \dots etc.; see §4.2 regarding subscripts
$x(i)$	For $x \in Q^{\mathbb{Z}}$, the i th coordinate of x
$S, T, f, g, \varphi, \psi, \dots$	Functions (usually continuous)
σ	The shift map on $Q^{\mathbb{Z}}$
$[(X, T)]$	Isomorphism class of (X, T)
$\mathbf{P}, \mathbf{Q}, \dots$	Dynamical properties, i.e. sets of isomorphism classes
$\mathcal{H}, \mathcal{H}_{\mathbf{P}}$	The space of homeomorphisms of the Cantor set [satisfying \mathbf{P}]
$\mathcal{S}, \mathcal{S}_{\mathbf{P}}$	The space of shift-invariant subsystems of $Q^{\mathbb{Z}}$ [satisfying \mathbf{P}]
$\mathcal{A}, \mathcal{A}_{\mathbf{P}}$	The automorphism group of a Lebesgue space [satisfying \mathbf{P}]
$\mathcal{M}, \mathcal{M}_{\mathbf{P}}$	The shift-invariant probability measures on $Q^{\mathbb{Z}}$ [satisfying \mathbf{P}]
d, d_{∞}	Metrics, vary with context; see §§2.2, 2.3, 4.1, 4.3
π_0	Projection onto the zeroth coordinate of $Q^{\mathbb{Z}}$
f_T^*	Orbit image map of f ; see §2.5
θ	Pseudo-orbit-to-orbit map; see §4.2
τ_Y	Symbolic retract ‘towards’ Y ; see §4.3

2. Definitions

In this section we present some definitions and notation. For convenience, we repeat here some of the definitions given in the introduction. We warn the reader in advance that the letter d will simultaneously denote metrics on many spaces; the metric depends on the context.

2.1. Dynamical systems. A *topological dynamical system* is a pair (X, T) where X is a non-empty compact metric space and T a homeomorphism of X . The space X is called the *phase space*. We often write only X or T in place of (X, T) . We will sometimes restrict ourselves to the case where X does not have isolated points; in this case X is said to be *perfect*.

A system (Y, S) is a *factor* of (X, T) if there is a continuous onto map $\varphi : X \rightarrow Y$ such that $\varphi T = S\varphi$. Such a φ is called a *homomorphism* or *factor map* from X to Y . If φ is a homeomorphism it is called an *isomorphism*, and (X, T) , (Y, S) are said to be isomorphic.

A subset $X_0 \subseteq X$ of a dynamical system (X, T) is invariant if $TX_0 = T^{-1}X_0 = X_0$. A closed, non-empty invariant subset X_0 of X defines a dynamical system by restricting T to X_0 and is called a *subsystem* of (X, T) .

2.2. Homeomorphisms of the Cantor set. We denote by K the Cantor set. For some fixed metric d on K we define

$$d(\varphi, \psi) = \max_{x \in K} (d(\varphi(x), \psi(x)) + d(\varphi^{-1}(x), \psi^{-1}(x)))$$

for φ, ψ homeomorphisms of K . We denote by

$$\mathcal{H} = \text{Homeo}(K)$$

the space of homeomorphisms with the metric defined above, which is complete.

2.3. *The Hausdorff metric.* Let X be a compact metric space with metric d ; $B_r(x)$ is the open ball of radius r around x as usual. The space of non-empty, closed subsets of X is denoted by 2^X , and the *Hausdorff metric* is defined on 2^X by

$$d(Y_0, Y_1) = \inf\{\varepsilon \mid \text{for } i = 0, 1 \text{ and } y \in Y_i \text{ there is } y' \in Y_{1-i} \text{ with } d(y, y') < \varepsilon\}$$

for non-empty, closed subsets $Y_0, Y_1 \subseteq X$. With the Hausdorff metric, 2^X is compact. This topology on 2^X can be characterized as follows. If $Y_n, Y \subseteq X$ are closed sets, then $Y_n \rightarrow Y$ if and only if for any sequence (y_n) with $y_n \in Y_n$ the set of accumulation points of (y_n) is contained in Y , and every point in Y arises in this way.

Note that if (X, T) is a dynamical system then T induces a homeomorphism \tilde{T} of 2^X , and the subsystems are precisely the fixed points of \tilde{T} . In particular, the space of subsystems is closed in the Hausdorff metric.

2.4. *The space of subshifts.* Let Q denote the Hilbert cube (the product of $[0, 1]$ with itself countably many times) with a fixed metric d . Let $Q^{\mathbb{Z}}$ be the space of bi-infinite sequences $(\dots, x(-1), x(0), x(1), \dots)$ with $x(i) \in Q$, along with the product topology, which is also compact and metrizable. To be concrete, we define a compatible metric d on $Q^{\mathbb{Z}}$ by

$$d(y, y') = \inf\{\varepsilon > 0 : d(y(i), y'(i)) < \varepsilon \text{ for all } |i| \leq 1/\varepsilon\}$$

for $y, y' \in Q^{\mathbb{Z}}$. Thus y, y' are close in $Q^{\mathbb{Z}}$ if their coordinates agree well on a large block of indices around zero. Note that we have arranged things so that if $d(y, y') < \varepsilon$ then $d(y(0), y'(0)) < \varepsilon$.

The *shift* σ on $Q^{\mathbb{Z}}$ is the homeomorphism defined by

$$\sigma(x)(n) = x(n + 1).$$

The dynamical system $(Q^{\mathbb{Z}}, \sigma)$ is called the *full shift* on Q .

The space of closed subsystems of $(Q^{\mathbb{Z}}, \sigma)$ is

$$\mathcal{S} = \{X \subseteq Q^{\mathbb{Z}} : X \neq \emptyset \text{ is closed and } \sigma\text{-invariant}\}.$$

As noted above, \mathcal{S} is closed and compact in the Hausdorff metric.

We denote by π_0 the projection $Q^{\mathbb{Z}} \rightarrow Q$ onto the zeroth coordinate, i.e. $\pi_0(x) = x(0)$. For $X \in \mathcal{S}$ the set $\pi_0(X)$ is called the *cross-section* of X .

2.5. *Embedding dynamical systems in shift spaces.* Let (X, T) be a dynamical system and $x \in X$. The (full) *orbit* of x is the set $\{T^k x\}_{k \in \mathbb{Z}} = \{\dots, T^{-1}x, x, Tx, T^2x, \dots\}$. Let $f : X \rightarrow Q$ be a continuous map. f defines an *orbit picture* of (X, T) in $Q^{\mathbb{Z}}$ by assigning to $x \in X$ the sequence of images under f of its orbit:

$$f_T^*(x) = (f(T^k x))_{k \in \mathbb{Z}}.$$

One verifies that $f_T^*(Tx) = \sigma f_T^*(x)$, so f_T^* is a factor map from (X, T) onto its image. If f is an embedding of X in Q then $(f_T^*(X), \sigma)$ is isomorphic to (X, T) as a dynamical system via the isomorphism f_T^* .

Since Q has the property that any compact metric space can be embedded in it, the previous discussion implies that any dynamical system (X, T) can be embedded as a subsystem of $(Q^{\mathbb{Z}}, \sigma)$; thus \mathcal{S} contains representatives of every isomorphism class of dynamical systems.

2.6. Dynamical properties. A dynamical property \mathbf{P} is a family of isomorphism classes of dynamical systems. For a system (X, T) for its isomorphism class, although we shall write $(X, T) \in \mathbf{P}$ instead of $[(X, T)] \in \mathbf{P}$ (or just $X \in \mathbf{P}$ or $T \in \mathbf{P}$ when T or X are understood). When considering a space of dynamical systems such as \mathcal{H} or \mathcal{S} we identify \mathbf{P} with the subset of the space made up of those systems which have the property \mathbf{P} , and introduce the notation

$$\begin{aligned}\mathcal{H}_{\mathbf{P}} &= \{\varphi \in \mathcal{H} : (K, \varphi) \in \mathbf{P}\}, \\ \mathcal{S}_{\mathbf{P}} &= \{X \in \mathcal{S} : (X, \sigma|_X) \in \mathbf{P}\}.\end{aligned}$$

When we attribute topological properties such as openness or denseness to \mathbf{P} , we are actually referring to the sets $\mathcal{S}_{\mathbf{P}}$ or $\mathcal{H}_{\mathbf{P}}$. In particular we say \mathbf{P} is dense, generic or exotic in \mathcal{S}, \mathcal{H} if $\mathcal{S}_{\mathbf{P}}, \mathcal{H}_{\mathbf{P}}$ are respectively dense, residual or meagre.

2.7. Baire category. We briefly review the basic facts of Baire's category theory that we will use. A good reference for this is Oxtoby's book [20].

A topological space X is *Polish* if there is a complete separable metric d on X which induces the topology. A countable intersection of open sets is called a G_{δ} set. A set containing a dense G_{δ} is *residual*; the complement of a residual set is *meagre* or *first category* (any set which isn't first category is second category, but this doesn't imply residuality). Baire's theorem states that in a Polish space the intersection of countably many dense open sets is dense; thus the intersection of countably many residual sets is residual. The family of residual sets forms a σ -filter on X ; in this sense, residual sets are the topological analog of sets of full measure.

Alexandroff's classical theorem characterizes the Polish subsets of a Polish space.

THEOREM 2.1. (Alexandroff [20, Theorems 12.1 and 12.3]) *For a Polish space X , a set $A \subseteq X$ is Polish if and only if it is a G_{δ} .*

2.8. Measure-preserving systems. A measure-preserving system is a quadruple (X, \mathcal{F}, μ, T) where (X, \mathcal{F}, μ) is a standard probability space, $T : X \rightarrow X$ is bi-measurable (i.e. T and T^{-1} are measurable) and T preserves μ , i.e. $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$. Often X will be a topological space, and then we always assume that \mathcal{F} is the completion of the Borel σ -algebra with respect to a regular measure μ . Two measure-preserving systems (X, \mathcal{F}, μ, T) and $(Y, \mathcal{F}', \nu, S)$ are isomorphic if there is a measure-preserving invertible map $\pi : X \rightarrow Y$ such that $\pi T = S\pi$.

Let λ denote Lebesgue measure on $[0, 1]$ and \mathcal{L} the Lebesgue sets. Let

$$\mathcal{A} = \text{Aut}([0, 1], \mathcal{L}, \lambda)$$

denote the set of measure-preserving automorphisms of $([0, 1], \mathcal{L}, \lambda)$. Each $\varphi \in \mathcal{A}$ may be identified with the measure-preserving system $([0, 1], \mathcal{L}, \lambda, \varphi)$. This space is universal for non-atomic measure-preserving systems. Define a topology on \mathcal{A} by $\varphi_n \rightarrow \varphi$ if $\varphi_n(A) \rightarrow \varphi(A)$ for all $A \in \mathcal{L}$. This topology is Polish.

A Borel probability measure μ on $Q^{\mathbb{Z}}$ is invariant under the shift if $\mu(\sigma^{-1}(A)) = \mu(A)$ for every Borel set $A \subseteq Q^{\mathbb{Z}}$. Denote the space of shift-invariant measures by \mathcal{M} . We may associate to $\mu \in \mathcal{M}$ the measure-preserving system $(Q^{\mathbb{Z}}, \mathcal{F}, \sigma, \mu)$. Identifying measures with positive linear functionals on $C(Q^{\mathbb{Z}})$ and using the Riesz representation theorem, we may equip \mathcal{M} with the weak-* topology, which turns it into a compact metrizable space. This space is universal for measure-preserving systems.

3. The correspondence theorem and zero-one laws

3.1. *Formulation and proof outline.* Our correspondence theorem is modeled after Glasner and King’s result [11], which states that a dynamical property in the measure-preserving category is generic in the automorphism group \mathcal{A} of a Lebesgue space if and only if it is generic in the space \mathcal{M} of shift invariant measures on $Q^{\mathbb{Z}}$ (the topologies on these spaces were defined in the previous section). In our setting, the exact analogy of this would be that a dynamical property \mathbf{P} of topological systems is generic in \mathcal{H} if and only if it is generic in \mathcal{S} . Although this is true it is of limited interest, since as we have already mentioned, generically there is only one system in \mathcal{H} up to isomorphism, so genericity of \mathbf{P} reduces to the question of whether this generic system satisfies \mathbf{P} or not. The version of the correspondence described below is relative, asserting that if \mathbf{P} is a dynamical property satisfying certain conditions and $\mathbf{Q} \subseteq \mathbf{P}$, then \mathbf{Q} is generic in $\mathcal{H}_{\mathbf{P}}$ if and only if \mathbf{Q} is generic in $\mathcal{S}_{\mathbf{P}}$. We will later apply this to the case where \mathbf{P} is the class of transitive or totally transitive systems.

For convenience, we recall the formulation of Theorem 1.3.

THEOREM. (Correspondence theorem) *Let \mathbf{P} be a dynamical property. Suppose that:*

- (a) $\mathcal{H}_{\mathbf{P}}$ is a G_{δ} in \mathcal{H} ;
- (b) $\mathcal{S}_{\mathbf{P}}$ is a G_{δ} in \mathcal{S} ; and
- (c) $\mathcal{S}_{\mathbf{P}}$ contains a dense set of Cantor systems.

Let $\mathbf{Q} \subseteq \mathbf{P}$ be a dynamical property. Then:

- (1) \mathbf{Q} is generic in $\mathcal{H}_{\mathbf{P}}$ if and only if \mathbf{Q} is generic in $\mathcal{S}_{\mathbf{P}}$; and
- (2) if \mathbf{Q} contains only Cantor systems, then \mathbf{Q} is dense in $\mathcal{H}_{\mathbf{P}}$ if and only if \mathbf{Q} is dense in $\mathcal{S}_{\mathbf{P}}$.

Proof. The proof proceeds as follows. We first construct a Polish space \mathcal{E} and a map $\beta : \mathcal{H}_{\mathbf{P}} \times \mathcal{E} \rightarrow \mathcal{S}_{\mathbf{P}}$ such that:

- for all $\varphi \in \mathcal{H}_{\mathbf{P}}$ and $f \in \mathcal{E}$, the system $(\beta(\varphi, f), \sigma)$ is isomorphic to the system (K, φ) ;
- β is a topological embedding; and
- the image of β is dense and furthermore if $X \in \mathcal{S}_{\mathbf{P}}$ is a Cantor system then $X = \lim X_n$ for a sequence of X_n which are isomorphic to X and contained in the image of β .

Before undertaking the construction of \mathcal{E} and β , which is somewhat involved, let us show how their existence proves the theorem. Denote β 's image by $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{P}}$. We first claim that \mathcal{B} is a dense G_δ in $\mathcal{S}_{\mathbf{P}}$. Indeed, it is dense by assumption, and since $\mathcal{H}_{\mathbf{P}} \times \mathcal{E}$ is Polish and β a topological embedding, \mathcal{B} is Polish and by Alexandroff's theorem is a G_δ in $\mathcal{S}_{\mathbf{P}}$.

Denote $\mathcal{B}_{\mathbf{Q}} = \mathcal{B} \cap \mathcal{S}_{\mathbf{Q}}$ (this may not be a G_δ). We claim that $\beta^{-1}(\mathcal{B}_{\mathbf{Q}}) = \mathcal{H}_{\mathbf{Q}} \times \mathcal{E}$. Indeed, this is because

$$\begin{aligned} \beta^{-1}(\mathcal{B}_{\mathbf{Q}}) &= \{(\varphi, f) \in \mathcal{H}_{\mathbf{P}} \times \mathcal{E} : (\beta(\varphi, f), \sigma) \in \mathbf{Q}\} \\ &= \{(\varphi, f) \in \mathcal{H}_{\mathbf{P}} \times \mathcal{E} : (K, \varphi) \in \mathbf{Q}\} \\ &= \mathcal{H}_{\mathbf{Q}} \times \mathcal{E}. \end{aligned}$$

In summary, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{H}_{\mathbf{Q}} \times \mathcal{E} & \xrightarrow{\beta} & \mathcal{B}_{\mathbf{Q}} & \subset & \mathcal{S}_{\mathbf{Q}} \\ \cap & & \cap & & \cap \\ \mathcal{H}_{\mathbf{P}} \times \mathcal{E} & \xrightarrow{\beta} & \mathcal{B} & \subset & \mathcal{S}_{\mathbf{P}}. \end{array}$$

We can now prove (1). Since \mathcal{B} is residual in $\mathcal{S}_{\mathbf{P}}$, we see that $\mathcal{S}_{\mathbf{Q}}$ is residual in $\mathcal{S}_{\mathbf{P}}$ if and only if $\mathcal{B}_{\mathbf{Q}}$ is residual in \mathcal{B} , and since β is a homeomorphism this is equivalent to $\beta^{-1}(\mathcal{B}_{\mathbf{Q}}) = \mathcal{H}_{\mathbf{Q}} \times \mathcal{E}$ being residual in $\mathcal{H}_{\mathbf{P}} \times \mathcal{E}$. But the latter happens if and only if $\mathcal{H}_{\mathbf{Q}}$ is residual in $\mathcal{H}_{\mathbf{P}}$. This completes the proof.

The proof of (2) is very similar. First note that $\mathcal{H}_{\mathbf{Q}}$ is dense in $\mathcal{H}_{\mathbf{P}}$ if and only if $\mathcal{H}_{\mathbf{Q}} \times \mathcal{E}$ is dense in $\mathcal{H}_{\mathbf{P}} \times \mathcal{E}$, which happens if and only if $\beta(\mathcal{H}_{\mathbf{Q}} \times \mathcal{E}) = \mathcal{B}_{\mathbf{Q}}$ is dense in \mathcal{B} . Since \mathcal{B} is dense in $\mathcal{S}_{\mathbf{P}}$ this is equivalent to $\mathcal{B}_{\mathbf{Q}}$ being dense in $\mathcal{S}_{\mathbf{P}}$.

From this, one direction of (2) is immediate: if $\mathcal{H}_{\mathbf{Q}}$ is dense in $\mathcal{H}_{\mathbf{P}}$ then $\mathcal{B}_{\mathbf{Q}}$ is in $\mathcal{S}_{\mathbf{P}}$ and consequently $\mathcal{S}_{\mathbf{Q}}$ is also. Conversely, the remaining hypothesis about β shows that $\mathcal{B}_{\mathbf{Q}}$ is dense in $\mathcal{S}_{\mathbf{Q}}$, since if $X \in \mathcal{S}_{\mathbf{Q}}$ then $X = \lim X_n$ with $X_n \in \mathcal{B}_{\mathbf{Q}}$; so if $\mathcal{S}_{\mathbf{Q}}$ is dense in $\mathcal{S}_{\mathbf{P}}$, so is $\mathcal{B}_{\mathbf{Q}}$.

This completes the proof, assuming the existence of \mathcal{E} and β . □

3.2. *Construction of \mathcal{E} and β .* We turn to the details of the construction of \mathcal{E} and β . Write \mathcal{D} for the space of all topological embeddings of K in Q . This is a subspace of the space $\mathcal{C} = C(K, Q)$ of all continuous maps from K to Q , which carries the usual metric

$$d_\infty(f, g) = \sup_{x \in K} d(f(x), g(x))$$

for $f, g \in C(K, Q)$ (the symbol d_∞ now represents metrics on both \mathcal{H} and \mathcal{C} , which is intended will be clear from the context). This metric is complete on \mathcal{C} , and one may verify that \mathcal{D} is a G_δ subset of \mathcal{C} so it is a Polish space.

Our next lemma is a sharpening of a classical theorem of Kuratowski, stating that the Cantor sets constitute a dense G_δ subset of 2^Q .

PROPOSITION 3.1. *There exists a G_δ subset $\mathcal{E} \subseteq \mathcal{D}$ such that the map $f \mapsto f(K)$ from \mathcal{E} to 2^Q is an embedding, and its image is dense.*

Proof. We construct a sequence $N_i \in \mathbb{N}$ and a family $(U_\alpha)_{\alpha \in A}$ of open sets in Q whose index set A consists of finite words

$$\alpha_1 \dots \alpha_k \in \{1, 2, \dots, N_1\} \times \{1, 2, \dots, N_2\} \times \dots \times \{1, 2, \dots, N_k\}.$$

Write $\alpha_1 \dots \alpha_n < \alpha'_1 \dots \alpha'_m$ if $n \geq m$ and $\alpha_1 \dots \alpha_m = \alpha'_1 \dots \alpha'_m$. Let \emptyset denote the empty word, and write A_k for all words of length k in A . We construct the U_α so that they satisfy the following conditions.

- (1) For each k the family $\{U_\alpha\}_{\alpha \in A_k}$ is pairwise disjoint, and if $\alpha < \alpha'$ then $U_\alpha \subseteq U_{\alpha'}$ (so the partially ordered set $(\{U_\alpha\}, \subseteq)$ is isomorphic to $(A, <)$).
- (2) For each k , the union $\bigcup_{\alpha \in A_k} U_\alpha$ is an open dense set in Q .
- (3) We have $\text{diam } U_\alpha \leq 1/k$ for $\alpha \in A_k$; this is the condition that will determine the value of the N_i 's.

Such a family can easily be constructed by recursion on k by dividing each U_α , $\alpha \in A_k$ into finitely many small disjoint open sets plus a meagre remainder, obtaining the family $\{U_\beta\}_{\beta \in A_{k+1}}$. We omit the details. Let

$$D = \bigcap_{k=1}^{\infty} \bigcup_{\alpha \in A_k} U_\alpha$$

which is dense G_δ in Q . Define

$$\mathcal{K}_k = \left\{ C \subseteq \bigcup_{\alpha \in A_k} U_\alpha : C \text{ is a Cantor set} \right\}.$$

This set is open in 2^Q . Finally, let

$$\mathcal{K} = \bigcap_{k=1}^{\infty} \mathcal{K}_k = \{C \subseteq D : C \text{ is a Cantor set}\}.$$

\mathcal{K} is a G_δ subset of 2^Q ; we next verify that it is dense. Let $A \subseteq Q$ and $\varepsilon > 0$, and let $A_\varepsilon = \{x \in Q; d(x, A) < \varepsilon\}$ and let $B = A_\varepsilon \cap D$. Since D is a dense G_δ in Q and A_ε is open we see that B is a G_δ without isolated points and is dense in A_ε . In particular, $d(\overline{B}, A) \leq \varepsilon$. Since B is relatively compact we can choose a finite set $x_1, \dots, x_N \in B$ which is ε -dense in B . Since B is a G_δ subset of a Polish space we may find Cantor subsets $C_1, \dots, C_N \subseteq B$ with C_i contained in the ball $B_\varepsilon(x_i)$. Let $C = \bigcup_{i=1}^N C_i$. Then $C \subseteq D$ is a Cantor set, so $C \in \mathcal{K}$, and $d(C, A) < 3\varepsilon$. This establishes that \mathcal{K} is a dense G_δ in 2^Q .

For each $C \in \mathcal{K}$ we define a homeomorphism $\psi_C : C \rightarrow K$ as follows. Assume without loss of generality that $K \subseteq [0, 1]$ is the standard middle-third realization of the Cantor set. Let k be the first index such that for some $\alpha = \alpha_1 \dots \alpha_k \in A_k$ we have $C \subseteq U_\alpha$ and there are at least two distinct indices $1 \leq i, j \leq N_{k+1}$ such that $C \cap U_{\alpha r_i} \neq \emptyset$ and $C \cap U_{\alpha r_j} \neq \emptyset$. Let r_1, \dots, r_n be those indices such that $C \cap U_{\alpha r_i} \neq \emptyset$; by our assumption $n \geq 2$. Divide K into n closed and open sets W_1, \dots, W_n of diameter $< 1/n$ in a manner depending only on k and r_1, \dots, r_n but not on C , and as a first approximation prescribe ψ_C maps $C \cap U_{\alpha r_i}$ to W_i . Now continue inductively to define a homeomorphism from $C \cap U_{\alpha r_i}$ onto W_i . The map ψ_C is defined in the limit and is a homeomorphism from C to K .

We claim that $C \mapsto \psi_C^{-1}$ is an embedding of \mathcal{K} in \mathcal{D} . The inverse is continuous, since if $\psi_{C_n}^{-1} \rightarrow \psi_C^{-1}$ in d_∞ then clearly $C_n \rightarrow C$ in the Hausdorff metric.

To see that $C \mapsto \psi_C^{-1}$ is continuous, let $C', C'' \in \mathcal{K}$ be close enough so that the first k steps of the construction of $\psi_{C'}, \psi_{C''}$ agree. In particular, there will be a pairwise disjoint partition of $C' \cup C''$ by closed and open sets V_1, \dots, V_n , each of diameter less than $1/k$, and a partition W_1, \dots, W_n of K into closed and open sets of diameter less than $1/2^k$, such that both $\psi_{C'}$ maps $V_i \cap C'$ onto W_i and similarly $\psi_{C''}$ maps $V_i \cap C''$ onto W_i . Thus $d(\psi_{C'}^{-1}(z), \psi_{C''}^{-1}(z)) < 1/k$ for every $z \in K$, so $d_\infty(\psi_{C'}, \psi_{C''}) < 1/k$.

Thus the map $C \rightarrow \psi_C^{-1}$ is an embedding of \mathcal{K}_∞ in \mathcal{D} . Refer to its image as \mathcal{E} . Since \mathcal{K} is a Polish space so is \mathcal{E} . This completes the proof. □

Given an embedding $f \in \mathcal{E}$ of K into Q and a homeomorphism φ of K , we can embed the system (K, φ) in $Q^{\mathbb{Z}}$ via the orbit picture map f_φ^* (§2.5). We define $\beta : \mathcal{H}_P \times \mathcal{E} \rightarrow \mathcal{S}$ in this way: $\beta(\varphi, f) = f_\varphi^*(K)$. Then $(\beta(\varphi, f), \sigma)$ is a system isomorphic to (K, f) .

LEMMA 3.2. *The map $\beta : \mathcal{H} \times \mathcal{E} \rightarrow \mathcal{S}$, which sends $(\varphi, f) \in \mathcal{H} \times \mathcal{E}$ to the orbit picture $f_\varphi^*(K)$ of the systems (K, φ) , is a topological embedding.*

Proof. Verification. □

It remains to show that, if the Cantor systems are dense in \mathcal{S}_P , then the image of $\mathcal{H}_P \times \mathcal{E}$ under β is dense in \mathcal{S}_P .

Recall that $\pi_0 : Q^{\mathbb{Z}} \rightarrow Q$ is the projection onto the zeroth coordinate. π_0 is a continuous open and closed map. A system $X \in \mathcal{S}$ is a *graph* if there is a homeomorphism $\varphi : \pi_0(X) \rightarrow \pi_0(X)$ such that X is the orbit picture of the system $(\pi_0(X), \varphi)$ under the inclusion map $i : \pi_0(X) \hookrightarrow Q$. Denote

$$\mathcal{S}_{Gr} = \{X \in \mathcal{S} : X \text{ is a graph}\}.$$

We remark without proof that \mathcal{S}_{Gr} is a dense G_δ subset of \mathcal{S} (but note that being a graph is not an isomorphism invariant, but rather a property of the embedding of the system in \mathcal{S} , so it is not a dynamical property in our sense).

PROPOSITION 3.3. *Let $\mathcal{K}_0 \subseteq \mathcal{K}$ be a dense set of Cantor sets in Q . Let $X \in \mathcal{S}$ be a Cantor system. Then $X = \lim X_n$ for a sequence $X_n \in \mathcal{S}_{Gr}$ with $\pi_0(X_n) \in \mathcal{K}_0$ and $(X_n, \sigma) \cong (X, \sigma)$.*

Proof. Fix $\varepsilon > 0$ and find $C \in \mathcal{K}_0$ such that $d(C, \pi_0(X)) < \varepsilon$ in 2^Q ; this can be done because \mathcal{K}_0 is dense in 2^Q . Since X is zero-dimensional and $\pi_0 : Q^{\mathbb{Z}} \rightarrow Q$ is open and closed, $\pi_0(X)$ is zero-dimensional. Since $d(C, \pi_0(X)) < \varepsilon$ and X is a Cantor set, there exists a homeomorphism $\alpha : X \rightarrow C$ such that

$$d(x(0), \alpha x) < \varepsilon$$

for all $x \in X$. Define $T : C \rightarrow C$ by $T = \alpha \sigma \alpha^{-1}$, so $(C, T) \cong (X, \sigma)$. Let $i : C \rightarrow Q$ be the inclusion map and define

$$Y = i_T^*(C)$$

the orbit picture of the system (C, T) in $Q^{\mathbb{Z}}$. By definition $Y \in \mathcal{S}_{\mathbf{Gr}}$, and $(Y, \sigma) \cong (X, \sigma)$ via the homeomorphism $j = i_T^* \circ \alpha : X \rightarrow Y$. We have

$$(jx)(0) = (i^* \alpha x)(0) = \alpha x$$

so

$$d(x(0), (jx)(0)) = d(x(0), \alpha x) < \varepsilon$$

and more generally, since φ commutes with σ we have

$$d(x(k), (jx)(k)) = d((\sigma^k x)(0), (j\sigma^k x)(0)) < \varepsilon$$

for all $x \in X$ and $k \in \mathbb{Z}$. Consequently, $d(X, Y) = d(X, jX) < \varepsilon$ in \mathcal{S} . □

COROLLARY 3.4. *Under the hypotheses of the correspondence theorem and with \mathcal{E} as in Proposition 3.1 and β as in Lemma 3.2, the image of $\mathcal{H}_{\mathbf{P}} \times \mathcal{E}$ under β is dense in $\mathcal{S}_{\mathbf{P}}$.*

Proof. Write \mathcal{B} for the image of $\mathcal{H}_{\mathbf{P}} \times \mathcal{E}$ under β and let $\mathcal{K}_0 = \{f(K) : f \in \mathcal{E}\} \subseteq \mathcal{K}$. By assumption there is a dense set \mathcal{K}' of Cantor systems in $\mathcal{S}_{\mathbf{P}}$, so it suffices to show that \mathcal{B} is dense in \mathcal{K}' . If $X \in \mathcal{K}'$ is a Cantor system, then by the lemma above $X = \lim X_n$ for $X_n \in \mathcal{S}_{\mathbf{Gr}}$, $\pi_0(X_n) \in \mathcal{K}_0$ and $(X, \sigma) \cong (X_n, \sigma)$. The last fact implies that $(X_n, \sigma) \in \mathcal{S}_{\mathbf{P}}$. Let $\psi_n : \pi_0(X_n) \rightarrow \pi_0(X_n)$ be a homeomorphism such that X_n is the orbit picture of ψ_n , and let $f \in \mathcal{E}$ such that $f : K \rightarrow \pi_0(X_n)$ is a homeomorphism. Define $\varphi_n \in \mathcal{H}$ by $\varphi_n = f^{-1} \psi_n f$. Since $(K, \varphi_n) \cong (X_n, \sigma)$ we have $\varphi_n \in \mathcal{H}_{\mathbf{P}}$, and $\beta(\varphi_n, f) = X_n \in \mathcal{B}$. □

This completes the proof of the correspondence theorem.

Let us show that the hypotheses of the correspondence theorem are satisfied for the class \mathbf{P} of all systems, i.e. that the genericity status of a dynamical property \mathbf{Q} is the same in \mathcal{H} and \mathcal{S} . Conditions (a) and (b) are satisfied trivially; (c) follows from the following proposition.

PROPOSITION 3.5. *The Cantor systems are dense in \mathcal{S} .*

Proof. Let $X \in \mathcal{S}$ and $\varepsilon > 0$, and write $D = \pi_0(X)$. Let $C \in 2^Q$ be a Cantor set with $d(C, D) < \varepsilon$. Let

$$X' = \{x' \in C^{\mathbb{Z}} : \exists x \in X \text{ such that } d(x(i), x'(i)) \leq \varepsilon \text{ for all } i \in \mathbb{Z}\}.$$

This set is seen to be non-empty, closed and shift-invariant, so $X' \in \mathcal{S}$. Clearly $d(X, X') < \varepsilon$. We also have $X' \subseteq C^{\mathbb{Z}}$ so X' is zero-dimensional, and it is easily seen to have no isolated points, so X' is a Cantor system. □

COROLLARY 3.6. *The Kechris–Rosendal system is generic in \mathcal{S} .*

3.3. Zero-one laws. We turn now to the zero-one laws. Since there are sets which are neither meagre nor residual, one would expect that there are dynamical properties which are neither generic nor exotic in \mathcal{S} . In [11] Glasner and King proved a zero-one law asserting that in the automorphism group \mathcal{A} of a Lebesgue space every Baire measurable dynamical property is either generic or exotic. We show next that a similar situation holds for certain subspaces of \mathcal{H} .

A subset A of a complete metric space X is *Baire measurable* if it belongs to the completion of the Borel σ -algebra of X with respect to the σ -ideal of meagre sets. Equivalently, it is Baire measurable if it can be written as $U \Delta M$ where U is open and M meagre. We follow the convention from [11] and use the above terminology instead of the usual ‘sets with the property of Baire’. All Borel sets, and more generally all analytic sets, are Baire measurable. For more information see Oxtoby [20].

THEOREM 3.7. *Let \mathbf{P} be a dynamical property such that $\mathcal{H}_{\mathbf{P}}$ is a G_{δ} and there is a Cantor system (C, φ) with isomorphism class dense in $\mathcal{H}_{\mathbf{P}}$. Then for every dynamical property $\mathbf{Q} \subseteq \mathbf{P}$ with $\mathcal{H}_{\mathbf{Q}}$ a Baire set, either \mathbf{Q} is generic in $\mathcal{H}_{\mathbf{P}}$ or it is exotic there.*

This is an immediate corollary of the following.

THEOREM 3.8. (Glasner and King [11]) *Suppose a group Γ acts by homeomorphisms on a Polish space X . If the action is transitive (i.e. there is a $x \in X$ such that $\overline{\Gamma x} = X$) then every Baire measurable subset of X which is invariant under the action of Γ is either of meagre or residual.*

The proof is not complicated and can be found in [11].

Proof of Theorem 3.7. Let \mathcal{H} act on itself by conjugation, and note that both $\mathcal{H}_{\mathbf{P}}$ and $\mathcal{H}_{\mathbf{Q}}$ are invariant under this action. Furthermore, by assumption $\mathcal{H}_{\mathbf{P}}$ is Polish, the \mathcal{H} -orbit of φ is dense in $\mathcal{H}_{\mathbf{P}}$ and $\mathcal{H}_{\mathbf{Q}}$ is a \mathcal{H} -invariant Baire measurable set. By Theorem 3.8 it is therefore either residual or meagre. \square

Theorem 1.5 now follows from part 1 of Theorem 1.2 and the theorem above.

4. Symbolic approximation

4.1. *Symbolic systems.* Recall that $\pi_0 : Q^{\mathbb{Z}} \rightarrow Q$ is the projection onto the zeroth coordinate. A system $X \in \mathcal{S}$ is *symbolic* if its cross-section $\pi_0(X)$ is finite. In this case $\pi_0(X)$ is also called the *alphabet* of X .

Write Q^* for the set of all finite words over Q . We denote words by the letters a, b, c or u, v, w . The i th letter of a word a is $a(i)$. If $a = a(1)a(2) \dots a(k)$ then k is the length of a and is denoted by $\ell(a)$. The concatenation of words $a, b \in Q^*$ is written ab . For $a, b \in Q^*$ we say that a is a subword of b at index $i \leq \ell(b) - \ell(a)$ if $a(j) = b(i + j)$ for $j = 1, \dots, \ell(a)$. The index i is called the *alignment* of a in b . If such an i exists, we say that a appears in b , or is a subword of b .

We measure the distance between two words a, b with the same length (finite or infinite) by

$$d_{\infty}(a, b) = \sup_i d(a(i), b(i)).$$

(The symbol d_{∞} was defined in §3; the new meaning can be distinguished from the old one by its context.) For $\varepsilon > 0$ we say the the word a is an ε -subword in b if there is a word a' appearing in b with $d_{\infty}(a, a') < \varepsilon$. Such a subword a' of b is called an ε -appearance of a in b .

Note that $x, y \in Q^{\mathbb{Z}}$ are ε -close, i.e. $d(x, y) < \varepsilon$ with respect to the metric on $Q^{\mathbb{Z}}$ introduced in §2, if and only if for $n = \lfloor 1/\varepsilon \rfloor$ the words $a = x(-n) \dots x(n)$ and $b = y(-n) \dots y(n)$ satisfy $d_{\infty}(a, b) < \varepsilon$.

Every $X \in \mathcal{S}$ is a set of bi-infinite sequences, which we think of as bi-infinite words over Q . We say that a finite word $a = a(1) \dots a(n)$ appears (or ε -appears) in X if there is an $x \in X$ such that a appears (or ε -appears) in x . By shift invariance of X , if a is a subword (ε -subword) of X then there are appearances (ε -appearances) of a in X with every alignment.

Using this terminology, for $X, Y \in \mathcal{S}$ we have $d(X, Y) \leq \varepsilon$ if and only if whenever a is a subword of X and $\ell(a) \leq 1 + 2/\varepsilon$ then a is an ε -subword of Y , and similarly with the roles of X, Y reversed.

4.2. *Approximation by symbolic systems.* A finite set of finite words $L \subseteq Q^*$ is called a *Language*. A system $X \in \mathcal{S}$ is said to be *constructed from L* if every word in X is a bi-infinite concatenation of words from L and every word from L appears in X .

Given a system $X \in \mathcal{S}$ our first goal is to find workable conditions under which a system Y constructed from a language L is close to X in \mathcal{S} . The basic idea will be to work with languages L which are made up of words which appear (or ε -appear) in X . Thus a concatenation of words from L will look locally like a subword (or ε -subword) of X , provided we take care not to splice together subwords of X which don't 'fit'. We will also want L to be large enough so that it contain words representing all subwords of X , up to some degree of accuracy.

The following notion is central to making the above precise. Let (X, T) be a dynamical system and $\varepsilon > 0$. A finite, infinite or bi-infinite sequence $\bar{x} = (x_i)$ is called an ε -pseudo-orbit if $d(x_{i+1}, Tx_i) < \varepsilon$ for all i . A finite ε -pseudo-orbit x_1, \dots, x_N is an ε -pseudo-period if in addition $d(Tx_N, x_1) < \varepsilon$.

For $X \in \mathcal{S}$ and a pseudo-orbit $\bar{x} = (x_i)$ in X , note that each x_i is itself a bi-infinite sequence over Q , $x_i = (\dots, x_i(-1), x_i(0), x_i(1) \dots)$. We adopt the convention that sequences of points in $Q^{\mathbb{Z}}$ (finite, one-sided infinite or bi-infinite) are always written using the bar notation \bar{x} . The member points of such a sequence are written as x_i , with added superscripts if necessary; thus $x_i(j) \in Q$ is the j th coordinate of the i th point $x_i \in Q^{\mathbb{Z}}$ of the sequence \bar{x} .

Let $X \in \mathcal{S}$. Given a bi-infinite sequence $\bar{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ of points in X we define $\theta(\bar{x}) \in Q^{\mathbb{Z}}$ by $\theta(\bar{x})(i) = x_i(0)$, so

$$\theta(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, x_{-1}(0), x_0(0), x_1(0), \dots).$$

We define θ similarly on finite sequences of points from X , so $\theta(x_1, \dots, x_N)$ is a finite word a of length N with $a(i) = x_i(0)$.

LEMMA 4.1. *If $\bar{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ is a bi-infinite ε -pseudo-orbit in $(Q^{\mathbb{Z}}, \sigma)$ and $\varepsilon < 1$, then $d(\theta(\bar{x}), x_0) \leq \sqrt{\varepsilon}$.*

Proof. Suppose $k \leq 1/\sqrt{\varepsilon}$. It suffices to show that $d(x_0(i), \theta(\bar{x})(i)) < \sqrt{\varepsilon}$ for $|i| \leq k$. But

$$d(x_0(i), \theta(\bar{x})(i)) = d(x_0(i), x_i(0)) \leq \sum_{j=0}^{i-1} d(x_j(i-j), x_{j+1}(i-(j+1)))$$

since $d(\sigma x_j, x_{j+1}) < \varepsilon$, for every $m \leq 1/\varepsilon$ it holds that $d(\sigma x_j(m), x_{j+1}(m)) = d(x_j(m+1), x_{j+1}(m)) < \varepsilon$ and since $k \leq 1/\sqrt{\varepsilon}$ this is certainly true for $m = i-j$ as j ranges from 0 to i . Therefore each summand is less than ε so

$$d(x_0(i), \theta(\bar{x})(i)) < i\varepsilon \leq \sqrt{\varepsilon}$$

as claimed. □

Let $X \in \mathcal{S}$ and for $i = 1, \dots, M$ suppose that $\bar{x}^i = x_1^i, \dots, x_{N(i)}^i$ are ε -pseudo-orbits with $\varepsilon < 1$. Let $a^i = \theta(\bar{x}^i)$ and $L = \{a^1, \dots, a^M\}$. Suppose $Y \neq \emptyset$ is a system constructed from L and that $a^i a^j$ appears in Y only if $d(\sigma(x_{N(i)}^i), x_1^j) < \varepsilon$. Every $y \in Y$ can be written as a concatenation of the form $y = \dots a^{k(-1)} a^{k(0)} a^{k(1)} \dots$, and then $y = \theta(\bar{y})$ where \bar{y} is a bi-infinite sequence of points from X given by the concatenation $\bar{y} = \dots \bar{x}^{k(-1)} \bar{x}^{k(0)} \bar{x}^{k(1)} \dots$ aligned in the obvious way. By the assumption that $a^i a^j$ appears in Y only if $d(\sigma(x_{N(i)}^i), x_1^j) < \varepsilon$, \bar{y} is an ε -pseudo-orbit. The lemma now implies that every $y \in Y$ is $\sqrt{\varepsilon}$ -close to one of the x_j^i , so every $y \in Y$ is $\sqrt{\varepsilon}$ -close to a point in X .

Now suppose in addition that the union of the collection $\{x_j^i : 1 \leq i \leq M, 1 \leq j \leq N(i)\} \subseteq X$ of points making up the pseudo-orbits $\bar{x}^1, \dots, \bar{x}^M$ is ε -dense in X , that is, for every $x \in X$ we have $d(x, x_j^i) < \varepsilon$ for some i, j . Recall that we are assuming that Y was constructed from L , so every a^i appears in Y . We claim that this ensures that $d(x, Y) < 2\sqrt{\varepsilon}$ for every $x \in X$. Indeed, let $x \in X$ be arbitrary. By assumption there are i, k such that $d(y, x_i^k) < \varepsilon$. Therefore there is a $y \in Y$ and an ε -pseudo-orbit $\bar{y} = (\dots, x_0, x_1, \dots)$ with $x_0 = x_i^k$ and $y = \theta(\bar{y})$ follows from the denseness of the x_j^i . By the previous lemma, $d(\theta(\bar{y}), x_i^k) = d(\theta(\bar{y}), x_0) \leq \sqrt{\varepsilon}$ so

$$d(x, y) \leq d(x, x_i^k) + d(x_i^k, y) = d(x, x_i^k) + d(\theta(\bar{y}), x_i^k) \leq \varepsilon + \sqrt{\varepsilon} \leq 2\sqrt{\varepsilon}.$$

We have proved the following.

PROPOSITION 4.2. *Let $X \in \mathcal{S}$ and suppose that for $i = 1, \dots, M$ we are given ε -pseudo-orbits $\bar{x}^i = x_1^i, \dots, x_{N(i)}^i$ in X such that their union is ε -dense in X . Write $a^i = \theta(\bar{x}^i)$ and $L = \{a^1, \dots, a^M\}$ and let Y be any system constructed from L under the restriction that $a^i a^j$ appears in Y only if $d(\sigma(x_{N(i)}^i), x_1^j) < \varepsilon$. Then $d(X, Y) < 2\sqrt{\varepsilon}$.*

A special and very useful case of this is the following.

COROLLARY 4.3. *Let $X \in \mathcal{S}$ and suppose that for $i = 1, \dots, M$ we are given ε -pseudo-periods $\bar{x}^i = x_1^i, \dots, x_{N(i)}^i$ in X with the same initial point, i.e. $x_1^1 = x_1^2 = \dots = x_1^M$. Assume that the union of the pseudo-periods is ε -dense in X . Write $a^i = \theta(\bar{x}^i)$ and $L = \{a^1, \dots, a^M\}$ and let Y be any system constructed from L . Then $d(X, Y) < 2\sqrt{\varepsilon}$.*

4.3. *Projection into symbolic systems.* So far we have seen how to construct systems Y close to X , provided we have a good language to work with. We now would like this Y to have the property that any system Z sufficiently close to Y inherits some of Y 's structure. This is our next task.

Let $Y \in \mathcal{S}$ and $a \in Q$. If there is in $\pi_0(Y)$ a unique point closest to a , denote it by $\tau_Y(a)$. This defines a partial map $\tau_Y : Q \rightarrow \pi_0(Y) \subseteq Q$, which is defined on an open subset of Q and is continuous there. Extend this to a partial map $\tau_Y : Q^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$ by $\tau_Y(z)(i) = \tau_Y(z(i))$. When τ_Y is defined on a point $z \in Q^{\mathbb{Z}}$ it is defined also on $\sigma^k z$ for all k and commutes with σ in the sense that $\sigma(\tau_Y z) = \tau_Y(\sigma z)$; also, τ_Y is continuous where it is defined (though its domain may not be open). In particular, if $Z \in \mathcal{S}$ is such that τ_Y is defined on every $z \in Z$, we see that τ_Y is a factor map from Z onto some subsystem of Y , and that if Y is symbolic then $\tau_Y(Z) \subseteq \pi_0(Y)^{\mathbb{Z}}$, and so $\tau_Y(Z)$ is symbolic over the same alphabet as Y .

LEMMA 4.4. *If $Y \in \mathcal{S}$ is symbolic then τ_Y is defined on every $Z \in \mathcal{S}$ sufficiently close to Y .*

Proof. Let $Y, Z \in \mathcal{S}$ with Y symbolic. Define

$$\rho_Y = \min\{d(a, b) : a, b \in \pi_0(Y) \text{ and } a \neq b\}$$

where $\pi_0(Y)$ is finite, hence the minimum in the definition of ρ_Y exists. Suppose that $d(Z, Y) < \rho_Y/2$. If $z \in Z$ then there is a $y \in Y$ with $d(z, y) < \rho_Y/2$, so $d(z(0), y(0)) < \frac{1}{2}\rho_Y$. Thus clearly τ_Y is defined on $z(0)$. Since $z \in Z$ was arbitrary and using the shift invariance of Z we conclude that τ_Y is defined on every $z \in Z$. □

Definition 4.5. We say that a finite language $L \subseteq Q^*$ has the *unique parsing property with window size N* if every word $a \in Q^*$ of length at least N has at most one parsing $a = uv_1 \dots v_m w$ such that (a) $v_1, \dots, v_m \in L$, and (b) $u' u, w w' \in L$ for some $u', w' \in Q^*$.

If L is a language such that distinct words in L contain distinct letters, then L has the unique parsing property. Since Q has no isolated points, we can always perturb the letters of words of a finite alphabet L by an arbitrarily small amount to make them distinct, and achieve unique parsing in this way.

Another way to get unique parsing is to add prefixes. Suppose, for instance, that u, v are two words. Let w be a third word and n such that $w^n = w \dots w$ (n times) does not appear as a subword of u, v . Then setting $u' = w^n u$ and $v' = w^n v$, we see that $L = \{u', v'\}$ has the unique parsing property with window size $N = n + \max\{\ell(u'), \ell(v')\}$.

There are many other ways to get unique parsing. In the sequel we will not spell out the details of this.

Suppose $Y \in \mathcal{S}$ is a symbolic system constructed from a language L which has the unique parsing property with window size N . Let Z be close enough to Y so that for every $z \in Z$ there is a $y \in Y$ such that $d(z(i), y(i)) < \rho_Y/2$ for every $|i| \leq N$, with ρ_Y as in the proof of the previous lemma. It follows that for every $z \in Z$ the central $(2N + 1)$ -long subword of $\tau_Y(z)$ equals $y(-N), \dots, y(N)$ for some $y \in Y$. Given $z \in Z$ let $a = \tau_Y(z)$ and $a_i = a(i - N) \dots a(i + N)$. Each a_i has a unique parsing as in Definition 4.5, and by uniqueness the parsing of the word $a(i - N + 1), \dots, a(i + N)$ induced by the parsings

of a_i and of a_{i+1} must agree. This means that we can merge all the parsings of the a_i 's and obtain a parsing of a into words from L . This proves the following.

LEMMA 4.6. *Let Y be constructed from a language L with the unique parsing property. If Z is close enough to Y (in a manner depending only on L), then τ_Y is defined on Z and $\tau_Y(Z)$ is also constructed from L .*

There is one last important property of τ_Y that we will use. For $x, y \in Q^{\mathbb{Z}}$ let

$$d_{\infty}(x, y) = \sup_{k \in \mathbb{Z}} d(x(k), y(k))$$

denote the uniform distance between x and y .

LEMMA 4.7. *Let $Y \in \mathcal{S}$ be a symbolic system and $\varepsilon > 0$. Then for every system Z sufficiently close to Y , the projection $\tau_Y : Z \rightarrow Y$ is defined and displaces points by at most ε , that is, $d_{\infty}(z, \tau_Y(z)) < \varepsilon$ for all $z \in Z$.*

Proof. It is easy to check that if $d(Y, Z) < \varepsilon$ then $d(y(0), \pi_0(Z)) < \varepsilon$ for every $y \in Y$, implying $d(z(0), \tau_Y(z)(0)) < \varepsilon$. Of course there is nothing special about the index zero, so assuming that τ_Y is defined on Z (which will be true for Z close enough to Y) we have $d_{\infty}(z, \tau_Y(z)) < \varepsilon$ for all $z \in Z$. □

5. The space of transitive systems

In this section we prove Theorem 1.1. We begin with some definitions.

5.1. *Transitivity.* A dynamical system (X, T) is *bi-transitive* if there is a point $x \in X$ whose full orbit $\{T^n x\}_{n \in \mathbb{Z}}$ is dense in X ; such a point is called a *bi-transitive point*. The system is *forward transitive* if there is a point $x \in X$ whose forward orbit $\{T^n x\}_{n \in \mathbb{N}}$ is dense; such a point is called a *forward transitive point*. A point $x \in X$ is *forward recurrent* if there is a sequence $n(k) \in \mathbb{N}$ with $n(k) \rightarrow \infty$ and $T^{n(k)} x \rightarrow x$.

For brevity, we say that a system is *transitive* if it is forward transitive, and denote this class of systems by \mathbf{T} .

The notions of bi-transitivity and forward-transitivity are distinct in general, but for systems without isolated points (and in particular the Cantor set), they coincide. The following sequence of lemmas establishes this, along with some other well-known facts we will use. For completeness we provide proofs, or see e.g. [24].

LEMMA 5.1. *Let (X, T) be forward transitive. Then every forward transitive point is forward recurrent, and either X is perfect or else X is finite and T is a cyclical permutation of X .*

Proof. Let $x \in X$ be forward transitive. If $T^n x = x$ for some $n > 0$ then the forward orbit of x , and hence X , is $\{x, Tx, \dots, T^{n-1}x\}$, and T permutes this set cyclically. We also see that $T^{nk} x \rightarrow x$ as $k \rightarrow \infty$, so x is forward recurrent. Otherwise $T^n x \neq x$ for all $n \geq 1$, so $T^n x \neq T^{-1}x$ for all $n \geq 0$. Since $T^{-1}x$ is in the closure of $\{T^n x\}_{n \geq 0}$ there is a sequence $n_k \rightarrow \infty$ with $T^{n_k} x \rightarrow T^{-1}x$, so $T^{n_k+1} x \rightarrow x$, showing that x is forward recurrent. Since $T^{n_k+1} x \neq x$, this shows that x is not isolated, and, since T is a homeomorphism, neither are any of the points $T^n x$. Since $\{T^n x\}_{n \in \mathbb{N}}$ is dense, X has no isolated points. □

LEMMA 5.2. *A dynamical system (X, T) is transitive if and only if for every non-empty open set $U, V \subseteq X$ there is an $n > 0$ with $U \cap T^{-n}V \neq \emptyset$.*

Proof. Suppose x is forward transitive. We may assume X is perfect, since otherwise X consists of a single orbit and the conclusion holds trivially. Hence U, V are infinite, so the forward orbit $\{T^n x\}_{n \in \mathbb{N}}$ visits each of them infinitely often (otherwise $U \setminus \{T^n x\}_{n=1}^\infty$ would be a non-empty open set not visited by x 's forward orbit, and similarly for V). We can therefore find integers $0 < i < j$ with $T^i x \in V$ and $T^j x \in U$. Hence $T^j x \in U \cap T^{i-j}V$, so $U \cap T^{i-j}V \neq \emptyset$.

Conversely, let $\{U_i\}_{i \in I}$ be a countable basis for X , and $U_i \neq \emptyset$. It is clear that the set of forward transitive points of X is precisely $\bigcap_{i \in I} \bigcup_{n \in \mathbb{N}} T^{-n}U_i$. By assumption $\bigcup_{n \in \mathbb{N}} T^{-n}U_i$ is dense in X for all $i \in I$. Hence $\bigcap_{i \in I} \bigcup_{n \in \mathbb{N}} T^{-n}U_i$ is a dense G_δ , and in particular non-empty. □

LEMMA 5.3. *If X has no isolated points, then (X, T) is bi-transitive if and only if it is forward transitive.*

Proof. One direction is trivial. In the other, suppose then that (X, T) is transitive and X is perfect. Let $U, V \neq \emptyset$ be open sets. We must show that $U \cap T^{-n}V \neq \emptyset$ for some $n > 0$. Since X is perfect U, V are infinite. The orbit of x must visit U infinitely many times, since otherwise $U \setminus \{T^n x; n \in \mathbb{Z}\}$ is open, non-empty and not visited by the orbit of x . The same holds for V . Therefore there are integers $i < j$ such that $T^i x \in V$ and $T^j x \in U$; so $T^j x \in U \cap T^{-n}V \neq \emptyset$ for $n = j - i > 0$. □

Since the Cantor set has no isolated points, the spaces of bi-transitive and of forward transitive systems in $\mathcal{H} = \text{Homeo}(K)$ coincide. This equality does not hold in \mathcal{S} . To see this, let $0, 1 \in \mathbb{Q}$ denote distinct elements and let $x = (\dots, 0, 0, 0, 1, 1, 1, \dots) \in \mathbb{Q}^{\mathbb{Z}}$ and X be the closure of the full orbit of x . Then X consists of the fixed points $(\dots, 0, 0, 0, \dots)$ and $(\dots, 1, 1, 1, \dots)$ and a single orbit spiralling between them, all of whose points are bi-transitive but not forward transitive.

We note without proof that the space of transitive systems in \mathcal{S} , which is a Polish subset, generically contains just one system, and this system is not forward transitive. We henceforth concentrate on the space of forward transitive systems.

5.2. *The spaces of transitive systems are G_δ 's.* In order to apply the correspondence theorem to the spaces of forward-transitive systems we must first establish that these spaces are G_δ 's. Recall that the property of being (forward) transitive is denoted by \mathbf{T} .

PROPOSITION 5.4. *\mathcal{H}_T is a G_δ subset of \mathcal{H} .*

Proof. Let $\{U_i\}_{i \in I}$ be a countable basis for the Cantor set, with $U_i \neq \emptyset$. For each $i, j \in I$ and $n \in \mathbb{N}$, the set of $\varphi \in \text{Homeo}(K)$ such that

$$\varphi^{-n}(U_i) \cap U_j \neq \emptyset$$

is an open set in \mathcal{H} , and thus for each i, j the set of $\varphi \in \mathcal{H}$ so that $\varphi^{-n}(U_i) \cap U_j \neq \emptyset$ for some $n \in \mathbb{N}$ is open. Intersecting over $i, j \in I$ we obtain a G_δ , and this is precisely the set of transitive homeomorphisms by Lemma 5.2. □

The proof for \mathcal{S} is similar.

PROPOSITION 5.5. \mathcal{S}_T is a G_δ subset of \mathcal{S} .

Proof. Let $\{U_i\}_{i \in I}$ be a countable basis for the topology of $Q^{\mathbb{Z}}$. A system $X \in \mathcal{S}$ is transitive if and only if for every $i, j \in I$

$$X \cap U_i \neq \emptyset \quad \text{and} \quad X \cap U_j \neq \emptyset \Rightarrow \exists k > 0 : X \cap U_i \cap \sigma^{-k}U_j \neq \emptyset. \quad (5.1)$$

Thus it suffices to show that for fixed $i, j \in I$, the systems that have this property with respect to U_i, U_j are a G_δ set.

Fix $i, j \in I$. The set of systems which fail to intersect one of U_i or U_j is closed in the Hausdorff metric, and hence is a G_δ in \mathcal{S} .

Let $\mathcal{U} \subseteq \mathcal{S}$ denote the class of system which intersect both U_i and U_j ; by the above this set is open in \mathcal{S} . For each $k > 0$, the set \mathcal{U}_k of systems $X \in \mathcal{U}$ such that $X \cap U_i \cap \sigma^{-k}U_j \neq \emptyset$ is open. The set of systems which satisfy condition (5.1) is $\bigcup_{k>0} \mathcal{U}_k$, which is open (and of course a G_δ).

We have shown that for fixed $i, j \in I$, the set of systems in \mathcal{S} which satisfy condition (5.1) above consists of the union of two G_δ sets, and hence is itself a G_δ (in general, the union of finitely many G_δ 's is a G_δ). □

We remark that the spaces \mathcal{H}_T and \mathcal{S}_T are not closed. To see that \mathcal{S}_T is not closed, let $0, 1 \in Q$ be distinct points, let $0_n, 1_n$ be sequences of n repetitions of 0, 1 respectively, and let x_n be the periodic sequence $x_n = \dots 0_n 1_n 0_n 1_n \dots$. The orbit closure X_n of x_n is transitive (it consists of a single periodic orbit) but $X_n \rightarrow X$ where X is the simple system generated by the points $(\dots, 0, 0, 1, 1, \dots)$ and $(\dots, 1, 1, 0, 0, \dots)$ and this systems is not transitive. One can easily construct examples of this sort where the X_n are infinite. A similar construction may be carried out in \mathcal{H} (see [6]).

5.3. *Finite cycles.* The simplest transitive systems are those which are a cyclical permutation of a finite set. We call such systems *finite cycles*. A finite cycle with n points is isomorphic to the finite group $\mathbb{Z}/n\mathbb{Z}$ with the map $i \mapsto i + 1 \pmod n$. We denote this system by C_n and use the same symbol to denote the group $\mathbb{Z}/n\mathbb{Z}$.

Note that a system $Y \in \mathcal{S}$ is a finite cycle if and only if it consists of the translates of a single periodic Q -sequence.

It is well known that in the space \mathcal{M} of invariant probability measures on $Q^{\mathbb{Z}}$, the measures supported on periodic orbits are dense. Below, we prove a topological analog of this. We require some preparation.

LEMMA 5.6. *If (X, T) is forward transitive then it contains an ε -dense ε -pseudo-period.*

Proof. Let x be a forward transitive point. Then $\{x, Tx, \dots, T^m x\}$ is ε -dense in X for some m . Choose $n > m$ with $d(T^n x, x) < \varepsilon$; then $x, Tx, \dots, T^{n-1}x$ is an ε -dense ε -pseudo-period. □

The following lemma implies that approximation in \mathcal{S} by periodic systems, which *a priori* is a property of the embedding of the system in $Q^{\mathbb{Z}}$, is really an intrinsic property of the system. Note that the condition in the lemma does not depend on the metric.

LEMMA 5.7. *Let $X \in \mathcal{S}_T$. Then X is the limit of finite cycles if and only if for every $\varepsilon > 0$ there is an ε -dense ε -pseudo-period in X .*

Proof. If $\bar{x} = x_1, \dots, x_n$ is an ε -dense ε -pseudo-period in X let $a = \theta(\bar{x})$ and let $X_\varepsilon \in \mathcal{S}$ be the cycle consisting of the orbit of the infinite concatenation a with itself. Then by Corollary 4.3 we have $d(X, X_\varepsilon) < 2\sqrt{\varepsilon}$, so X is the limit of the finite cycles system X_ε .

Conversely, suppose X_n are finite cycles and $X_n \rightarrow X$. Suppose X_n is the orbit of the periodic point x_n of period $N(n)$. For each $k = 0, \dots, N(n) - 1$ let $x'_{n,k} \in X$ be one of the points in X closest to $\sigma^k x_n$. One verifies that $(x_{n,k})_{k=0}^{N(n)-1}$ is an 2ε -dense $\delta(\varepsilon)$ -pseudo-period in X for some $\delta(\varepsilon) \rightarrow 0$ with ε . □

PROPOSITION 5.8. *The finite cycles are dense in \mathcal{S}_T .*

Proof. Suppose X is transitive and let $\varepsilon > 0$. Take a recurrent transitive point $x \in X$, which is forward recurrent by Lemma 5.1. We can therefore choose n so that $x, \sigma x, \dots, \sigma^n x$ is an ε -dense ε -pseudo-period. Apply the previous lemma. □

5.4. *Odometers.* Given a factor map $f : (X, T) \rightarrow (Y, S)$, the sets $f^{-1}(y)$ for $y \in Y$ are called the *fibers* (of f), and induce a partition of X into closed, pairwise disjoint sets. If Y is a finite cycle then the fibers are also open sets.

The next simplest transitive systems after the finite cycles are those systems that are determined by factors onto finite cycles. A system (X, T) is an *odometer* (or *adding machine*) if, for every $\varepsilon > 0$, there is a factor map π from X onto a finite cycle (Y, S) with fibers of diameter $< \varepsilon$.

Equivalently, for every $\varepsilon > 0$ there is a partition of X into closed, pairwise disjoint sets of diameter $< \varepsilon$, which are permuted cyclically by T .

In the next few lemmas we establish some well-known properties of odometers and define the universal odometer.

LEMMA 5.9. *Odometers are zero-dimensional and forward transitive.*

Proof. The first statement is clear from the definition, since fibers of factor maps to cycles form a closed and open basis for the topology.

If (X, T) is an odometer, we show that every point is forward transitive. Fix $x \in X$ and let $\emptyset \neq U \subset X$ be an open and closed set. Choose a factor onto a finite cycle with fibers of diameter less than $\max\{d(x, y) : x \in U, y \in X \setminus U\}$. Then each fiber is contained in U or $X \setminus U$ and in particular there are fibers contained in U . Since $T^n x$ visits this fiber for some $n > 0$, for this n we have $T^n x \in U$. Since the closed and open sets are a basis for the topology on X , the forward orbit of x is dense. □

It follows from Lemma 5.1 that an odometer is either a finite cycle or a Cantor system.

LEMMA 5.10. *Let (X, T) be a dynamical system and let $f : X \rightarrow C_m$ and $g : X \rightarrow C_n$ be factor maps. Then there is a factor map $h : X \rightarrow C_k$ such that the maps f, g factor through h , i.e. the fibers of h refine the fibers of f and of g , and m, n divide k .*

Proof. Let X_0, \dots, X_{m-1} and Y_0, \dots, Y_{n-1} be the fibers of f, g respectively. Consider the partition

$$\{X_i \cap Y_j : 0 \leq i < m, 0 \leq j < n\}.$$

Let k be the number of these atoms. The atoms are open and closed and T acts on them by permutation. By transitivity, this permutation is cyclic; otherwise we could choose two atoms whose T -orbits are disjoint. Let h be the projection to the quotient space of this partition with the quotient action. Clearly f, g factor through h . Since T is a homeomorphism and maps fibers to fibers, the number of fibers of h in each fiber of f is constant, so $m|k$, and similarly $n|k$. \square

One way to construct an odometer is as follows. Fix a sequence $(k(n))_{n \in \mathbb{N}}$ of positive integers such that $n(k)|n(k+1)$. Let $\pi_{k-1} : C_{n(k)} \rightarrow C_{n(k-1)}$ be the factor map given by $\pi_{k-1}(i) = i \bmod n(k-1)$. We obtain a system of factor maps

$$\dots \xrightarrow{\pi_{k+1}} C_{n(k+1)} \xrightarrow{\pi_k} C_{n(k)} \xrightarrow{\pi_{k-1}} C_{n(k-1)} \xrightarrow{\pi_{k-2}} \dots \xrightarrow{\pi_1} C_{n(1)}.$$

The inverse limit of this system is an odometer (X, T) . (Note that the π_k 's are group homomorphisms, so the inverse limit also has a group structure; the map T of X is a translation on this group by the element which is the limit of $1 \in C_{n(k)}$.)

Let (X^*, T^*) denote the odometer constructed as above from the sequence $n(k) = k!$. We call this the *universal odometer*. This system may be characterized as follows.

LEMMA 5.11. *Up to isomorphism, (X^*, T^*) is the unique odometer which factors onto every finite cycle.*

Proof. Fix $m \in \mathbb{N}$. By construction, (X^*, T^*) factors onto $C_m!$ and this system factors onto C_m by reduction modulo m .

Conversely, suppose that (X, T) is an odometer which factors onto every finite cycle. We define a sequence $n(k)$ and factor maps $f_k : X \rightarrow C_{n(k)}$ by induction, as follows. Set $n(1) = 1$ and f_1 maps X to a point. Suppose we are given $f_k : X \rightarrow C_{n(k)}$. Since X factors onto every C_N it factors onto $C_{n(k)!}$, and by Lemma 5.10 we can find $n(k+1)$ and a factor $f_{k+1} : X \rightarrow C_{n(k+1)}$ such that $n(k)!|n(k+1)$, the fibers of f_{k+1} are of diameter $< 1/k$ and they refine the fibers of f_k . Let ρ_k be the factor map from $C_{n(k+1)} \rightarrow C_{n(k)}$ so that $f_k = \rho_k \circ f_{k+1}$. It now follows that (X, T) is the inverse limit of the diagram

$$\dots \xrightarrow{\rho_{k+1}} C_{n(k)} \xrightarrow{\rho_k} C_{n(k-1)} \xrightarrow{\rho_{k-1}} C_{n(k-2)} \xrightarrow{\rho_{k-2}} \dots \xrightarrow{\rho_1} C_{n(1)}.$$

Since $n(k)|n(k)!|n(k+1)$, we may interpolate $C_{n(k)!}$ between $C_{n(k+1)}$ and $C_{n(k)}$. We obtain

$$\dots \rightarrow C_{n(k)!} \rightarrow C_{n(k)} \rightarrow C_{n(k-1)!} \rightarrow C_{n(k-1)} \rightarrow \dots \rightarrow C_{n(1)}.$$

It follows that (X, T) is the inverse limit of

$$\dots \rightarrow C_{n(k)!} \rightarrow C_{n(k-1)!} \rightarrow \dots \rightarrow C_{n(1)!}.$$

We may now interpolate all the other $C_m!$'s for $m = 1, 2, 3, \dots$ into this sequence, and this gives us the sequence which defines (X^*, T^*) ; so $X \cong X^*$. \square

Similar arguments show that every odometer can be obtained as the inverse limit of $C_{n(k)}$'s for some sequence $n(k)$, and that every odometer is a factor of (X^*, T^*) . This justifies the claim of universality. We will not use these facts, and omit the proofs.

5.5. *The universal odometer is generic in $\mathcal{S}_{\mathbf{T}}$.* We can now bring everything together and show that the universal odometer is generic in $\mathcal{S}_{\mathbf{T}}$.

LEMMA 5.12. *Suppose that $Y \in \mathcal{S}$ factors onto a finite cycle of period n with fibers of diameter $< \varepsilon$. Then the same is true of every $X \in \mathcal{S}$ sufficiently close to Y .*

Proof. Suppose Y_0, \dots, Y_{n-1} is a partition of $Y \in \mathcal{S}$ into closed sets of diameter $< \varepsilon$. Let $\delta = \min_{i \neq j} d(Y_i, Y_j)$. Since the shift σ on $Q^{\mathbb{Z}}$ is uniformly continuous there is an η so that if $d(x', x'') \leq \eta$ then $d(\sigma x', \sigma x'') \leq \delta/3$. We may assume that $\eta < \delta$ and $2\eta + \text{diam } Y_i < \varepsilon$ for $i = 0, \dots, n - 1$. Let $X \in \mathcal{S}$ be a system with $d(X, Y) < \eta$. Let

$$X_i = \{x \in X : d(x, Y_i) < \eta\}.$$

This is a closed partition of X into sets of diameter $< 2\eta + \max_i \text{diam } Y_i < \varepsilon$. For every point $x \in X_i$ there is a $y \in Y_i$ with $d(x, y) < \eta$, so $d(\sigma x, \sigma y) < \delta/3$. Since $\sigma y \in Y_{i+1}$ and $\eta < \delta/3$ we conclude that $d(\sigma x, Y_j) > \eta$ for $j \neq i$, so $\sigma x \in X_{i+1}$. Thus σ permutes the X_i cyclically. This completes the proof. \square

LEMMA 5.13. *For every n , the finite cycles of period divisible by n are dense in $\mathcal{S}_{\mathbf{T}}$.*

Proof. Fix $n \in \mathbb{N}$ and let $Y \in \mathcal{S}$ be a finite cycle of period k . Then Y consists of the orbit of a point $y \in Q^{\mathbb{Z}}$ of period k , that is, $y(i + k) = y(i)$ for all $i \in \mathbb{Z}$ and k is the least positive integer with this property. Since Q has no isolated points we can perturb the coordinates of y by a small amount in a manner which has period kn . We obtain a point y' , which can be made arbitrarily close to y in d_{∞} , with period kn . The orbit of this point is a finite cycle Y' of period kn , and $Y' \rightarrow Y$ as $y' \rightarrow y$.

We have shown that the cycles of period divisible by n are dense among the finite cycles, and since by Proposition 5.8 the latter are dense in \mathcal{S} , the proof is complete. \square

THEOREM 5.14. *The isomorphism class of (X^*, T^*) is a dense G_{δ} set in $\mathcal{S}_{\mathbf{T}}$.*

Proof. Let $\mathcal{U}_n \subseteq \mathcal{S}$ be the set of transitive systems which factor onto a finite cycle of period divisible by n with fibers of diameter $< 1/n$. By Lemma 5.12 this set is open in \mathcal{S} , and hence in $\mathcal{S}_{\mathbf{T}}$, and by Lemma 5.13 this set is dense. By Lemma 5.11, $\bigcap_{n=1}^{\infty} \mathcal{U}_n$ is the isomorphism class of (X^*, T^*) . \square

In particular, this establishes that there is a dense set of Cantor systems in $\mathcal{S}_{\mathbf{T}}$. Together with Propositions 5.4 and 5.5 we have verified the hypotheses of the correspondence theorem, proving Theorem 1.1.

6. The space of totally transitive systems

As we have seen, among the transitive systems periodicity phenomena are quite dominant. These are absent from the class of totally transitive systems. This class, which contains a rich variety of dynamics and a few surprises, will occupy us for the remainder of this paper.

6.1. *Total transitivity.* A system (X, T) is *totally transitive* if T^n is transitive for every $n > 0$. The class of totally transitive systems is denoted by \mathbf{T}^* .

We remark that in this definition it is not important whether we interpret transitivity as forward- or bi-transitivity; the result is the same. Indeed, if T^n is forward transitive for each n , then it is bi-transitive for every n . Conversely, suppose (X, T) is a system with T^n bi-transitive for each n and suppose $x \in X$ is an isolated point. Since $\{x\}$ is open it must be a bi-transitive point for each T^n . In particular there is a $k \in \mathbb{Z}$ with $T^{2k}x = Tx$. But this is impossible because T is injective. Therefore X was perfect, so bi-transitivity of T^n implies forward transitivity of T^n by Lemma 5.3.

We now say a few words about the relation between \mathcal{S}_T and \mathcal{S}_{T^*} . The totally transitive systems are not dense in \mathcal{S}_T since the set of systems in \mathcal{S}_T with periodic factors is open (Lemma 5.12) and dense (in particular, non-empty), and a totally transitive system cannot have a periodic factor. Note, however, that total transitivity is not equivalent to the non-existence of periodic factors.

The closure $\overline{\mathcal{S}_{T^*}}$ is not contained in \mathcal{S}_T . For instance, let $0, 1 \in Q$ be distinct points and let X_n be the system over the alphabet $\{0, 1\}$ in which every maximal block of consecutive 0's or consecutive 1's has length at least n . Then X_n are seen to be totally transitive, but $\lim X_n$ exists and is the system generated by the points $(\dots, 0, 0, 1, 1, \dots)$ and $(\dots, 1, 1, 0, 0, \dots)$. This system is not transitive.

\mathcal{S}_{T^*} is not even relatively closed in \mathcal{S}_T . Consider for example the systems $X_n \subseteq \{0, 1\}^{\mathbb{Z}}$ which contain all sequences of 0's and 1's such that if two occurrences of 1's occur at distance $k \leq n$ then k is even. These systems are totally transitive and $X_n \rightarrow X \subseteq \{0, 1\}^{\mathbb{Z}}$, where X contains all sequences of 0's and 1's in which the distance between every two 1's is even. X is not totally transitive.

The following is needed in order to apply the correspondence theorem to \mathbf{T}^* .

PROPOSITION 6.1. \mathcal{S}_{T^*} is a G_δ in \mathcal{S} and \mathcal{H}_{T^*} is a G_δ in \mathcal{H} .

Proof. For each r one can imitate the proof that \mathcal{S}_T is a G_δ and deduce that the space

$$\mathcal{T}_r = \{X \in \mathcal{S} : (X, \sigma^r) \text{ is transitive}\}$$

is a G_δ . By definition, $\mathcal{S}_{T^*} = \bigcap_r \mathcal{T}_r$. The proof in \mathcal{H} is similar. □

We conclude with an important combinatorial property of totally transitive systems, upon which we will rely heavily in the sequel.

PROPOSITION 6.2. Let $(X, T) \in \mathbf{T}^*$. Then for every $\varepsilon > 0$ there exist two relatively prime numbers n, m and two ε -dense ε -pseudo-periods x_1, \dots, x_n , and x'_1, \dots, x'_m in X with the same starting point, $x_1 = x'_1$.

Proof. Suppose (X, T) is totally transitive, let $\varepsilon > 0$, and consider the set I of integers n such that there exists an ε -dense ε -pseudo-period x_1, x_2, \dots, x_n in X . Let r be the least common divisor of I . If $r = 1$ the proof is complete. Otherwise $r \neq 1$ and $r|n$ for every $n \in I$. Fix a transitive point $x_1 \in X$ and for $i = 0, \dots, r - 1$ let $A_i \subseteq X$ consist of the points y such that there is an ε -pseudo-orbit x_1, x_2, \dots, x_n with $n \equiv i \pmod{r}$ and $x_n = y$.

The A_i are clearly open, and their union is all of X because x_1 is a transitive point.

The A_i are also disjoint: if $x' \in A_i \cap A_j$ and $i < j$ then we can construct an ε -pseudo-period starting at x_1 of length $j - i \pmod r$ by choosing ε -pseudo-orbits $x_1, x'_2, \dots, x'_m = x', x'_{m+1}, \dots, x'_M = x_1$ with $m = i \pmod r$ and $x'_1, \dots, x'_n = x', x'_{n+1}, \dots, x'_N$ with $n = j \pmod r$, and forming the ε -pseudo-period $x_1, x'_2, \dots, x'_m, x''_{n+1}, \dots, x''_N$. This contradicts the definition of r .

Since $TA_i = A_{i+1 \pmod r}$ the projection taking $x \in X$ to the unique A_i to which it belongs is a factor map from X to the periodic system $(\{A_0, \dots, A_{r-1}\}, T)$. This contradicts the total transitivity of X . □

6.2. *The Rohlin property.* There is a classical theorem of Rohlin in ergodic theory, stating that the isomorphism class of every aperiodic ergodic system is dense in the automorphism group of a Lebesgue space (aperiodicity means that the set of periodic points has measure zero). We next discuss the topological analog of this, i.e. when the isomorphism class of a topological system (X, T) is dense in $\mathcal{S}_{\mathbf{T}^*}$.

There are two obvious obstructions to this. One is the existence of periodic points. Suppose that (X, T) has a periodic point x of period k and that $X_n \rightarrow Y$ with $(X_n, \sigma) \cong (X, T)$. Let $x_n \in X_n$ have period k . If y is any accumulation point of x_n then $y \in Y$, and $\sigma^k y = y$. In particular, the closure of systems with periodic points cannot contain systems without periodic points, such as infinite minimal systems.

Another obstruction is topological. Clearly, if X is connected and Y disconnected, then one cannot approximate Y in the Hausdorff metric by homeomorphic images of X .

It turns out that these are essentially the only obstructions. We will show that the isomorphism class of any zero-dimensional system without periodic points is dense in $\mathcal{S}_{\mathbf{T}^*}$. In the proof we use the following lemma, which is a variation of a lemma of Krieger [17].

LEMMA 6.3. *Let (X, T) be a zero-dimensional system without periodic points. Then for every k there is a closed and open subset $U \subseteq X$ such that $U, TU, \dots, T^k U$ are pairwise disjoint, and $X = \bigcup_{i=-k}^k T^i U$.*

Proof. Since no point in X is periodic, for $x \in X$ the points $x, Tx, \dots, T^k x$ are distinct, so we may choose a closed and open set E_x containing x and $E_x, TE_x, \dots, T^k E_x$ are pairwise disjoint. The collection $\{E_x\}_{x \in X}$ covers X ; choose a finite subcover E_1, \dots, E_m . Set $E'_1 = E_1$ and

$$E'_{i+1} = E'_i \cup \left(E_{i+1} \setminus \bigcup_{j=-k}^k E'_j \right).$$

One now checks that $U = E'_m$ satisfies our requirements. □

THEOREM 6.4. *Let (X, T) be a zero-dimensional system without periodic points. Then the set of systems isomorphic to X is dense in the totally transitive systems, i.e. $\mathcal{S}_{\mathbf{T}^*} \subseteq \overline{\mathcal{S}}_{[(X, T)]}$ in \mathcal{S} . In particular, if (X, T) is totally transitive then its isomorphism class is dense in $\mathcal{S}_{\mathbf{T}^*}$.*

Proof. Let $Y \in \mathcal{S}_{\mathbf{T}^*}$ be totally transitive and $\varepsilon > 0$; we will find a system $X' \in \mathcal{S}$ isomorphic to (X, T) and with $d(X', Y) < \varepsilon$.

The proof has two steps. First, use Lemma 6.3 to get a symbolic factor of X which is within ε of Y , and then modify this map to make it an embedding of X into $Q^{\mathbb{Z}}$ without moving the image more than ε .

The proof of the first step follows. Select two ε -dense ε -pseudo-periods in Y with relatively prime lengths m, n and the common starting point: $\bar{y}' = y'_1, \dots, y'_m$ and $\bar{y}'' = y''_1, \dots, y''_n$. Let $a' = \theta(\bar{y}')$ and $a'' = \theta(\bar{y}'')$ (recall the notation of §4).

There exists an integer k_0 such that every integer $k > k_0$ can be written as $k = rm + sn$ for integers $r, s \in \mathbb{N}$. For each $k > k_0$ we can therefore fix a word a_k of length k consisting of concatenations of a', a'' .

Using Lemma 6.3, select a closed and open set $U \subseteq X$ such that if $x \in U$ then $T^i x \notin U$ for $i = 1, \dots, k_0$, but $\bigcup_{i=-k_0}^{k_0} T^i U = X$; hence for every $x \in X$ we have $T^i x \in U$ for some $0 \leq i \leq 2k_0$ and $T^j x \in U$ for some $-2k_0 \leq j < 0$.

We now define a continuous map $f : X \rightarrow Q^{\mathbb{Z}}$ using visits to U as ‘markers’. For $x \in X$, let $I \subseteq \mathbb{Z}$ be the set of times $i \in \mathbb{Z}$ such that $T^i x \in U$. By choice of U the set I is bounded neither above nor below, and the gap between consecutive times in I is at most $M = 2k_0$. By choice of U we also know that if $i, j \in I$ and $i \neq j$ then $|i - j| > k_0$. Fix an ordering $I = \{\dots < i(-1) < i(0) < i(1) < i(2) < \dots\}$ and form the word $f(x) \in Q^{\mathbb{Z}}$ such that at index $i(m)$ we see the word $a_{i(m+1)-i(m)}$. The map $x \mapsto f(x)$ obviously satisfies $f(Tx) = \sigma f(x)$, and f is continuous since the symbol $f(x)(0)$ is determined by the values of the minimal $i, j \in \mathbb{N}$ such that $T^i x \in U$ and $T^{-j} x \in U$. These vary continuously in x because U is open and closed. We see that f is a factor map onto its image. The image $f(X)$ is constructed from the language $\{a_k\}_{k \geq k_0}$, so it is constructed from the language $\{a', a''\}$, which by the choice of a', a'' and Corollary 4.3 gives $d(f(X), Y) \leq 2\sqrt{\varepsilon}$.

In general $f(X) \neq X$ because f need not be injective. The second step is to perturb f to make it so. Let

$$Q_0 = \{a'(i), a''(j) : i = 1, \dots, m, j = 1, \dots, n\} \subseteq Q$$

be the list of symbols appearing in a', a'' . For each $z \in Q_0$ let $X_z \subseteq Q$ be a homeomorphic copy of X such that $d(z, x) < \varepsilon$ for every $x \in X_z$, and such that the X_z ’s are pairwise disjoint. This can be done because Q_0 is finite and Q is perfect. For each $z \in Q_0$ let $g_z : X \rightarrow X_z$ be a fixed homeomorphism.

Define a map $h : X \rightarrow Q^{\mathbb{Z}}$ by

$$h(x)(i) = g_{f(x)(i)}(\sigma^i x).$$

h is clearly continuous, and commutes with the shift since

$$h(\sigma x)(i) = g_{f(\sigma x)(i)}(\sigma^i(\sigma x)) = g_{f(x)(i+1)}(\sigma^{i+1} x) = h(x)(i + 1).$$

The map h is an injection, since if $y \in h(X)$ then $y(0) \in X_z$ for some $z \in Q_0$, implying that $g_z^{-1}(y)$ is the unique pre-image of y under h . Hence h is an isomorphism onto its image.

Finally, since $d(z, g_z(x)) < \varepsilon$ for every $x \in X$ and $z \in Q_0$, we see that $d_\infty(f(x), h(x)) < \varepsilon$. Therefore $d(f(X), h(X)) < \varepsilon$, so

$$d(h(X), Y) \leq d(h(X), f(X)) + d(f(X), Y) \leq \varepsilon + 2\sqrt{\varepsilon}.$$

So $h(X)$ is the desired system. □

COROLLARY 6.5. *The correspondence theorem and the zero-one law hold in \mathbf{T}^* .*

Proof. There exist totally transitive Cantor systems without periodic points: any minimal totally transitive Cantor system, for instance. We conclude that in $\mathcal{S}_{\mathbf{T}^*}$ there is a dense set of Cantor systems. Together with Proposition 6.1, this completes the hypotheses of the correspondence theorem for $\mathcal{S}_{\mathbf{T}^*}$ and $\mathcal{H}_{\mathbf{T}^*}$. \square

COROLLARY 6.6. *The isomorphism class of any Cantor system without periodic points is dense in $\mathcal{H}_{\mathbf{T}^*}$. In particular the zero-one law (Theorem 1.5) holds.*

Proof. Let (X, T) be a totally transitive Cantor system without periodic points. Set $\mathbf{Q} = [(X, T)]$ and apply part (2) of the correspondence theorem to conclude that \mathbf{Q} is dense in $\mathcal{H}_{\mathbf{T}^*}$. The zero-one law follows from Theorem 1.5. \square

6.3. *Minimality, unique ergodicity and entropy.* A dynamical system (X, T) is minimal if it has no non-trivial subsystems, or equivalently, if every point is (forward) transitive. Minimality can also be characterized by the property that for every non-empty open set $U \subseteq X$ there is an N such that for every $x \in X$ we have $T^i x \in U$ for some $i \in \{1, \dots, N\}$. For a proof of this, see [24, Theorem 5.1]. The class of minimal systems is denoted \mathbf{Min} .

A system (X, T) is *uniquely ergodic* if there is a unique T -invariant Borel probability measure on X . The class of uniquely ergodic systems is denoted \mathbf{U} . Unique ergodicity can be characterized by the property that for every continuous function $f \in C(X)$ and every $\varepsilon > 0$ there is an N such that for every $x, y \in X$,

$$\left| \frac{1}{N} \sum_{i=1}^N f(T^i x) - \frac{1}{N} \sum_{i=1}^N f(T^i y) \right| < \varepsilon.$$

This follows from [24, Theorem 6.19].

PROPOSITION 6.7. *\mathbf{Min} and \mathbf{U} are dense G_δ subsets of $\mathcal{S}_{\mathbf{T}^*}$.*

Proof. Density of both classes follows from Theorem 6.4.

Let $\{U_i\}_{i=1}^\infty$ be a countable basis for the topology of \mathcal{S} . As in the proof of Proposition 5.5, for each i and N the set of $X \in \mathcal{S}$ such that

$$\mathcal{U}_{i,N} = \left\{ X \in \mathcal{S} : X \cap U_i \neq \emptyset \text{ and } X \subseteq \bigcup_{k=0}^N \sigma^k U_i \right\}$$

is open in \mathcal{S} , as is the union $\mathcal{U}_{i,N}$ over N . Since the set

$$\mathcal{V}_i = \{X \in \mathcal{S} : X \cap U_i = \emptyset\}$$

is closed it is a G_δ . Therefore

$$\mathcal{W}_i = \mathcal{V}_i \cap \bigcup_{N=1}^\infty \mathcal{U}_{i,N} = \left\{ X \in \mathcal{S} : X \cap U_i \neq \emptyset \Rightarrow X \subseteq \bigcup_{k=0}^N \sigma^k U_i \right\}$$

is a G_δ . Now $\mathcal{S}_{\mathbf{Min}} = \bigcap_{i=1}^\infty \mathcal{W}_i$ is a G_δ .

A similar argument shows that \mathcal{S}_U is a G_δ . Briefly, for each i and n , N one shows that the set

$$\mathcal{U}_{i,n,N} = \left\{ X \in \mathcal{S} : \left| \frac{1}{N} \sum_{i=1}^N f_i(\sigma^i x) - \frac{1}{N} \sum_{i=1}^N f_i(\sigma^i y) \right| < \frac{1}{n} \text{ for all } x, y \in X \right\}$$

is open. Then $\mathcal{S}_U = \bigcap_{i,n} \bigcup_N \mathcal{U}_{i,n,N}$ is a G_δ .

The proofs in \mathcal{H} are similar and slightly easier, so we omit them. □

Our next result requires familiarity with Bowen’s definition of entropy [24].

PROPOSITION 6.8. *The class of zero-entropy systems is a dense G_δ in $\mathcal{S}_{\mathbf{T}^*}$.*

Proof. Density is clear from Theorem 6.4, since there exists minimal totally transitive Cantor systems with zero entropy.

To see that the class of zero-entropy systems is a G_δ , let $\mathcal{U}_n \subseteq \mathcal{S}$ be the set of systems X such that for some k the number of $1/n$ -separated words of length $k > n$ in X is less than $2^{k/n}$. One verifies that the \mathcal{U}_n are open, and their intersection consists exactly of the zero-entropy systems. □

6.4. *Connections with the space of invariant measures.* In this section we will prove a partial correspondence theorem relating the genericity of a dynamical property \mathbf{P} in the measure-theoretic category to the genericity in the class of totally transitive topological systems which support an invariant measure in \mathbf{P} .

Recall that \mathcal{M} is the space of shift-invariant Borel probability measures on $Q^{\mathbb{Z}}$ with the weak- $*$ topology.

THEOREM 6.9. *Let \mathbf{P} be a dynamical property in the measure theoretic category and suppose that $\mathcal{M}_{\mathbf{P}}$ is a dense G_δ in \mathcal{M} . Let $\tilde{\mathbf{P}}$ be the class of topological dynamical systems which support a measure from \mathbf{P} . Then $\tilde{\mathbf{P}}$ is generic in $\mathcal{S}_{\mathbf{T}^*}$.*

Proof. For a system $X \in \mathbf{Min} \cap \mathbf{U}$, denote the unique invariant measure by μ_X . Let $m : \mathcal{S}_{\mathbf{Min} \cap \mathbf{U}} \rightarrow \mathcal{M}$ be the map $X \mapsto \mu_X$. One may verify that this map is continuous (but note that its image is meagre in \mathcal{M} , since generically a measure in \mathcal{M} has global support). Thus $m^{-1}(\mathcal{M}_{\mathbf{P}}) \subseteq \mathcal{S}_{\mathbf{Min} \cap \mathbf{U}}$ is a G_δ in $\mathbf{Min} \cap \mathbf{U}$ and hence in \mathbf{T}^* ; and $m^{-1}(\mathcal{M}_{\mathbf{P}}) \subseteq \mathcal{S}_{\tilde{\mathbf{P}}}$.

Since zero entropy and weak mixing (in the ergodic sense) are generic in \mathcal{M} and $\mathcal{M}_{\mathbf{P}}$ is generic in \mathcal{M} , there is a weak-mixing zero-entropy system $(Y, S, \nu) \in \mathcal{M}_{\mathbf{P}}$. By Krieger’s generator theorem, there is a minimal uniquely ergodic symbolic system on two symbols (Z, T) such that $(Z, T, \mu_Z) \cong (Y, S, \nu)$ in the measure-theoretic category. Thus there is a system $X \in \mathcal{S}_{\mathbf{Min} \cap \mathbf{U}}$ with $(X, \sigma) \cong (Z, T)$ in the topological category. Since (X, σ, μ_X) is measure-theoretically weak mixing, (X, σ) is totally transitive. Applying Theorem 6.4 we see that the isomorphism class of (X, σ) is dense in $\mathcal{S}_{\mathbf{Min} \cap \mathbf{U} \cap \mathbf{T}^*}$. But $X \in m^{-1}(\mathcal{M}_{\mathbf{P}})$, and so is every system isomorphic to X . Thus $m^{-1}(\mathcal{M}_{\mathbf{P}})$ is dense in $\mathbf{Min} \cap \mathbf{U} \cap \mathbf{T}^*$, so $\tilde{\mathbf{P}}$ is generic there. □

COROLLARY 6.10. *The systems supporting a global weak mixing rigid invariant measure are generic in \mathbf{T}^* , and in particular systems supporting a strong mixing measure are exotic there.*

We note that the relation between *topological* weak and strong mixing is somewhat different; see §8 below.

For the proof of Theorem 6.9 it was necessary to assume that \mathbf{P} is a dense G_δ in \mathcal{M} . It would be nice to weaken this assumption to the weaker one that \mathbf{P} is generic in \mathcal{M} . The problem is that in this case $\mathcal{M}_{\mathbf{P}}$ does indeed contain a dense G_δ subset $\mathcal{M}_{\mathbf{P}}^* \subseteq \mathcal{M}_{\mathbf{P}}$, but this G_δ set may not be saturated with respect to the isomorphism relation, and in the above proof we cannot conclude that the system (X, σ) is in $m^{-1}(\mathcal{M}_{\mathbf{P}}^*)$. We do not know if the theorem holds under weaker hypotheses.

7. *Disjointness*

For dynamical systems $(X, T), (Y, S)$ let $T \times S$ denote the homeomorphism of $X \times Y$ given by $(T \times S)(x, y) = (Tx, Sy)$.

Let $(X, T), (Y, S)$ be bi-transitive dynamical systems. A *joining* of $(X, T), (Y, S)$ is a bi-transitive subsystem $Z \subseteq X \times Y$ whose projection to the first coordinate is onto X and to the second coordinate is onto Y . Systems $(X, T), (Y, S)$ are *disjoint* if their only joining is the product system $(X \times Y, T \times S)$.

The notion of disjointness, along with an analogous notion in the measure-preserving category, was introduced by Furstenberg in [9] as a tool in the classification of dynamical systems and has proved a very successful one. In [8], del Junco showed that for any automorphism T of a Lebesgue space the set of automorphisms measure-theoretically disjoint from T is residual in the coarse topology on the space of automorphisms. We prove the following analog of this in the following theorem.

THEOREM 7.1. *Let $X \in \mathbf{T}^*$. Then a generic system in $S_{\mathbf{T}^*}$ is disjoint from X .*

For the proof we will need a few simple facts, which we provide for completeness.

LEMMA 7.2. *In order for bi-transitive systems $(X, T), (Y, S)$ to be disjoint it suffices that for every two bi-transitive points $x \in X$ and $y \in Y$, the point (x, y) is a bi-transitive point for $X \times Y$.*

Proof. Suppose this holds and $Z \subseteq X \times Y$ is a joining; then it has a bi-transitive point $(x, y) \in Z$. Since the closure of the full orbit of x is the projection of the closure of the full orbit of (x, y) in Z , and Z is a joining, x is a bi-transitive point for X . Similarly, y is bi-transitive for Y , and by our assumption, $Z = X \times Y$. □

LEMMA 7.3. *If (X, T) is totally transitive and if $x \in X$ is a bi-transitive point, then x is bi-transitive for (X, T^n) .*

Proof. Fix n and let X_0 be the closure of $\{T^{kn}x\}_{k \in \mathbb{Z}}$. Then $Y = X_0 \cup TX_0 \cup \dots \cup T^{n-1}X_0$ is closed and contains the two-sided orbit of x so $Y = X$. By Baire's theorem one of the translates T^iX_0 must have a non-empty interior so this is true of X_0 . Since $T^nX_0 = X_0$ and in particular T^n preserves the interior of X_0 we conclude from the transitivity of (X, T^n) that the interior of X_0 is dense in X so $X_0 = X$; as desired. □

LEMMA 7.4. *Every totally transitive system is disjoint from every finite cycle.*

Proof. Let (X, T) be totally transitive, and (Y, S) a finite cycle with period k . Let $x \in X$ be a transitive point and fix $y \in Y$. We have $(T \times S)^k(x, y) = (T^k(x), y)$, so that the orbit closure of (x, y) under the map $(T \times S)^k$ is $X \times \{y\}$. But then the orbit closure of (x, y) under $T \times S$ is the union $\bigcup_{i=0}^{k-1} X \times \{S^i y\} = X \times Y$. This implies disjointness. \square

A similar argument shows that every odometer is disjoint from every totally transitive system.

Proof of Theorem 7.1. Let us say that $Y \in \mathcal{S}_{\mathbf{T}^*}$ is ε -disjoint from X if, for every bi-transitive point $x \in X$ and every bi-transitive point $y \in Y$, the full orbit closure of (x, y) in $X \times Y$ is ε -dense in $X \times Y$ with respect to the metric

$$d((x', y'), (x'', y'')) = \max\{d(x', x''), d(y', y'')\}$$

on $X \times Y$. If a system Y is ε -disjoint from X for every $\varepsilon > 0$ then it is disjoint from X .

Let $U_r \subseteq \mathcal{S}_{\mathbf{T}^*}$ be the set of systems $1/r$ -disjoint from X ; we will complete the proof by showing that U_r contains an open dense set.

Fix r and $Y \in \mathcal{S}_{\mathbf{T}^*}$. It suffices to show that for every $\varepsilon > 0$ there exists a system $Z \in \mathcal{S}_{\mathbf{T}^*}$ with $d(Y, Z) < \varepsilon$ such that some neighborhood of Z in $\mathcal{S}_{\mathbf{T}^*}$ is contained in U_r . Fix $\varepsilon > 0$ and a positive δ to be determined later. Choose two δ -dense δ -pseudo-periods $\bar{a} = a_1, \dots, a_m$ and $\bar{b} = b_1, \dots, b_n$ in Y with m, n relatively prime and $a_1 = b_1$. Let $a = \theta(\bar{a}), b = \theta(\bar{b})$, and assume that $\{a, b\}$ has the unique parsing property; see remark after Definition 4.5.

Fix a (forward) transitive point $x_0 \in X$ and let A be the periodic system whose single point up to translation is the bi-infinite concatenation of a 's; denote this point by a^* . Now X, A are disjoint, because X is totally transitive and A periodic, so the orbit of $(x_0, \sigma^i a^*)$ is dense in $X \times A$ for $i = 1, \dots, m$. There is therefore an integer k_a such that for each $i = 1, \dots, m$, the first k_a points on the orbit of $(x_0, \sigma^i a^*)$ are δ -dense in $X \times A$. Let b^*, B be defined in the same way; there is an integer k_b such that for every $j = 1, \dots, n$ the first k_b points on the orbit of $(x_0, \sigma^j b^*)$ are δ -dense in $X \times B$. Let $k' = \max\{k_a, k_b\}$.

Set $x_1 = T^{k'} x_0$; what we have so far is that for $i = 1, \dots, m$ the first k' points in the backward orbit of $(x_0, \sigma^i a^*)$ is δ -dense in $X \times A$ and similarly for $j = 1, \dots, n$ the first k' point on the backward orbit of $(x_0, \sigma^j b^*)$ are δ -dense in $X \times B$. Now x_1 is still a (forward) transitive point for X so by the same reasoning as in the previous paragraph there is a k'' such that the first k'' points on the orbits of $(x_1, T^i a^*), (x_1, \sigma^j b^*)$ are δ -dense in $X \times A, X \times B$ respectively, for every $i = 1, \dots, m$ and $j = 1, \dots, n$.

Let $k = \max\{k', k''\}$. The point x_1 has the property that for any $u \in A$ and any $v \in B$ the first k iterates of (x_1, u) and (x_1, v) in either direction are δ -dense in $X \times Y$. This remains true for any x close enough to x_1 ; let $V \subseteq X$ be a small neighborhood of x_1 so that every $x \in V$ has this property.

Choose two integers $M, N > 2k$ such that mM, nN are relatively prime and let $Z \in \mathbf{T}^*$ be any system constructed from $\{a^M, b^N\}$, where a^M is the M -fold concatenation of a and b^N the N -fold concatenation of b . By Corollary 4.3, for δ small enough we will have $d(Y, Z) < \varepsilon$; we claim that in addition Z is $1/2r$ -disjoint from X .

To see this, let $x \in X$ and $z \in Z$ be bi-transitive points. We must show that the full orbit of (x, z) is $1/2r$ -dense in $X \times Z$. For some $i, T^i x \in V$. Since it suffices to show that the full orbit of $(T^i x, \sigma^i z)$ is $1/2r$ -dense in $X \times Z$, we may assume that $x \in V$ to begin with.

Parse z into words of type a^M or b^N . The zeroth coordinate of z is in an a^M block or in a b^N block. In the first case since $M \geq 2k$ either the k -block starting at zero in z , or the k -block ending at 0 in z , looks like a concatenation of a 's. It follows that, for a small enough choice of δ , the first k points of either the forward or backward orbit of (x, z) is $1/2r$ -dense in $X \times Z$ (we assume that δ was chosen small enough to overcome any edge effects arising from the part of z outside the a^M block). The same argument applies in the case the 0th coordinate lies in a b^N block; thus Z is $1/2r$ -disjoint from X .

We will complete the proof by showing that any $W \in \mathcal{S}_{\mathbf{T}^*}$ close enough to Z in \mathcal{S} is $1/r$ -disjoint from X . Let $W \in \mathcal{S}_{\mathbf{T}^*}$ with $d(Z, W) < \delta$ and let $x \in X$ and $w \in W$ be bi-transitive points.

The system $W' = \tau_Z(W)$ is a totally transitive system (since it is a factor of the totally transitive system W) and assuming δ is small enough, W' is constructed from $\{a^M, b^N\}$ (see the discussion at the end of §4). Thus the argument for Z also applies to W' , so W' is $1/2r$ -disjoint from X . Since $w' = \tau_Z(w)$ is a bi-transitive point in W' , the full orbit of (x, w') in $X \times W'$ is $1/2r$ -dense in $X \times W'$.

Recalling the properties of the projection τ_Z from §4, we have that $d_\infty(w, \tau_Z(w)) < \delta$ and thus

$$d((T \times \sigma)^i(x, w), (T \times \sigma)^i(x, \tau_Z(w))) < \delta$$

for every $i \in \mathbb{Z}$, so we conclude that the full orbit of (x, w) is $(1/2r + \delta)$ -dense in $X \times W'$. Using the fact that

$$d(W, \tau_Z(W)) < \delta$$

we see that the full orbit of (x, w) is $(1/2r + 2\delta)$ -dense in $X \times W$. Thus W is $1/r$ -disjoint from X , and the proof is complete. □

THEOREM 7.5. *If $\mathcal{C} \subseteq \mathcal{S}_{\mathbf{T}^*}$ is compact (or σ -compact) then a generic system in $\mathcal{S}_{\mathbf{T}^*}$ is disjoint from every $X \in \mathcal{C}$.*

The proof is essentially the same as before. One must choose M, N in such a way that for some k and for every $X \in \mathcal{C}$ there is a transitive point $x_X \in X$ and a neighborhood $V_X \subseteq X$ of x_X such that for every $x \in V_X$ the first k iterates of (x, a^*) in $X \times A$ is δ -dense in $X \times A$, and similarly for (x, b^*) . This can be done because \mathcal{C} is compact.

8. Mixing

8.1. Weak mixing. A dynamical system (X, T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive. The class of weakly mixing systems is denoted by **WM**.

In the definition of weak mixing we can replace transitivity of $X \times X$ with bi-transitivity without changing the class **WM**, since the only way that $(X \times X, T \times T)$ could be bi-transitive but not transitive is if it has isolated points. Then X has isolated points and it is easy to check that in this case $(X \times X, T \times T)$ is not bi-transitive.

In particular, this implies that **WM** \subseteq **T**, since X is a factor of $X \times X$. The following is well known.

LEMMA 8.1. **WM** \subseteq **T**^{*}.

Proof. Suppose that $(X, T) \in \mathbf{WM}$ but X is not totally transitive. For some k there is a proper open subset $U \subseteq X$ such that $T^{-k}U \subseteq U$ and U is not dense in X . We may assume that $U \cap T^{-1}U = \emptyset$; for otherwise replace U with $U \cap T^{-1}U \cap \dots \cap T^{-i}U$ for the largest i for which it is not empty ($i \leq k - 2$ since otherwise we would get a T -invariant open subset of U , which contradicts transitivity). Let $V = \bigcup_{i=0}^{k-1} T^{-i}(U \times U)$. Then $T^{-1}V \subseteq V$. On the other hand V is not dense in $X \times X$ because it does not intersect $U \times T^{-1}U$. □

One consequence of this is that since $\mathcal{S}_{\mathbf{T}^*}$ is not dense in $\mathcal{S}_{\mathbf{T}}$, neither is $\mathcal{S}_{\mathbf{WM}}$.

THEOREM 8.2. \mathbf{WM} is a G_δ subset of \mathcal{S} and is dense in $\mathcal{S}_{\mathbf{T}^*}$.

Proof. The proof that \mathbf{WM} is a G_δ is similar to the proof that \mathbf{T} is a G_δ (Proposition 5.5). Density again follows from Theorem 6.4 and the existence of minimal weakly mixing Cantor systems. □

8.2. Strong mixing. A dynamical system (Y, T) is *strongly mixing* if, for every two open non-empty sets $U, V \subseteq Y$, there are only finitely many integers n for which $U \cap T^{-n}V = \emptyset$. The class of strong mixing systems is denoted by \mathbf{SM} .

LEMMA 8.3. A transitive system (Y, T) is strongly mixing if and only if for every non-empty open set $U \subseteq Y$ there are only finitely many n for which $U \cap T^{-n}U = \emptyset$.

Proof. One direction is trivial. For the other, suppose $U, V \subseteq Y$ are non-empty open sets. By transitivity we have $W = U \cap T^{-k}V \neq \emptyset$ for some $k \in \mathbb{N}$. By assumption, $W \cap T^{-n}W \neq \emptyset$ for all but finitely many $n \in \mathbb{Z}$. But $W \cap T^{-n}W \neq \emptyset$ implies $U \cap T^{-n+k}V \neq \emptyset$. □

In the group of automorphisms of a Lebesgue space the measure-theoretically weak mixing systems are generic while the measure-theoretically strong mixing systems are exotic. By analogy, one would expect that the strong mixing systems are exotic in $\mathcal{S}_{\mathbf{T}^*}$. Surprisingly, they are not.

THEOREM 8.4. \mathbf{SM} is generic in $\mathcal{S}_{\mathbf{T}^*}$.

We do not know whether $\mathcal{S}_{\mathbf{SM}}$ is a G_δ .

In order to prove this we will use the following approximation of strong mixing. Let $\varepsilon > 0$. We say that a system (X, T) is ε -strongly-mixing if there exists an integer N such that, for every $k > N$ and every $x \in X$, it holds that $B_\varepsilon(x) \cap T^{-k}(B_\varepsilon(x)) \neq \emptyset$.

Clearly (X, T) is strongly mixing if and only if it is $1/n$ -strongly-mixing for every $n \in \mathbb{N}$. Let

$$V_n = \{X \in \mathcal{S} : X \text{ is } 1/n\text{-strongly-mixing}\}.$$

In order to prove Theorem 8.4 it suffices to show that each V_n contains an open dense subset of $\mathcal{S}_{\mathbf{T}^*}$. This is a consequence of the following.

PROPOSITION 8.5. Let $\varepsilon > 0$ and $X \in \mathcal{S}_{\mathbf{T}^*}$. For every $\delta > 0$ there is a system $Y \in \mathcal{S}_{\mathbf{T}^*}$ with $d(X, Y) < \delta$ and an $r > 0$ such that any totally transitive system Z with $d(Y, Z) < r$ is ε -strongly-mixing.

We will break the proof into a sequence of lemmas. First, we establish a symbolic condition for ε -strongly-mixing.

LEMMA 8.6. *A system $X \in \mathcal{S}$ is ε -strongly-mixing if and only if there is a finite set $L \subseteq Q^*$ of words appearing in X and an integer N such that:*

- (1) *every subword of X of length $1 + 2/\varepsilon$ is an ε -subword of a member of L ; and*
- (2) *for every $a \in L$ and every $k > N$ there is some $x \in X$ in which a ε -appears at index zero and at index k .*

Proof. For $x \in X$, note that $B_\varepsilon(x) \cap \sigma^{-k}(B_\varepsilon(x)) \neq \emptyset$ if and only if there is some $x' \in X$ with $d(x, x') < \varepsilon$ and $d(x, \sigma^k x') < \varepsilon$. This in turn is equivalent to the fact that the subwords of x' of length $1 + 2/\varepsilon$ appearing at indices zero and k differ by at most ε from the central subword of x of the same length. The lemma now follows by compactness. \square

The main step in the proof of Proposition 8.5 is the construction of Y . Fix $X \in \mathcal{S}_{\mathbb{T}^*}$ and $\varepsilon, \delta > 0$. We will construct Y which is δ -close to X by symbolic approximation as described in Proposition 4.2. Our aim is to perform the construction in such a way that Y satisfies the hypothesis of Lemma 8.6, as does every system close enough to Y .

Fix $\lambda > 0$ to be determined later. Select two λ -pseudo-periods in Y , say $\bar{a} = a_1, \dots, a_N$ and $\bar{b} = b_1, \dots, b_M$, with $a_1 = b_1$ and lengths M, N relatively prime, such that:

- (a) the collection

$$\left\{ a_k : \frac{1}{\varepsilon} < k < N - \frac{1}{\varepsilon} \right\}$$

is λ -dense in Y ; and

- (b) for any sub-sequence \bar{u} of \bar{a} and any $0 \leq k < N$ there is an index i with $i = k \pmod N$ such that \bar{u} occurs in \bar{b} at i .

To construct \bar{a} , first select a transitive point a_1 and set $a_k = \sigma^k a_1$, then choose N so that (a) holds. Next, enumerate the possibilities for \bar{u}, k as in (b) and realize them in \bar{b} one at a time, using the total transitivity of Y to ensure that any possible alignment can be achieved given enough time. Extend \bar{b} as needed to ensure its length is prime relative to N .

Now \bar{b} is much longer than \bar{a} . Choose K so that the K -time concatenation $\bar{a}^K = \bar{a}\bar{a} \dots \bar{a}$ is much longer than \bar{b} , say $KN > 10M$, and such that KN and M are relatively prime. Write $\bar{c} = \bar{a}^K$; note that \bar{c} is also a λ -dense λ -pseudo-period starting at a_1 . We assume that the language $\{\bar{b}, \bar{c}\}$ has the unique parsing property; see Definition 4.5 and the remarks following it.

Set $a = \theta(\bar{a}), b = \theta(\bar{b}), c = \theta(\bar{c})$ (so c is the K -times concatenation of a). By Proposition 4.2, any symbolic system constructed from $\{b, c\}$ will be $2\sqrt{\lambda}$ -close to X . Let Y be such a totally transitive system, satisfying the additional constraint that b does not appear in it twice consecutively. Such a system exists since KN, M are relatively prime.

LEMMA 8.7. *If λ was chosen small enough (in a manner independent of Y) then Y is ε -strongly-mixing.*

Proof. Let L be the set of subwords of a of length $1 + 2/\varepsilon$. We will claim that the two conditions in Lemma 8.6 hold with respect to L .

Condition (1) follows from (a) assuming λ was chosen small enough; we omit the details.

We turn to (2). We claim that for every $w \in L$ and $n > KN$ there is a $y \in Y$ such that w appears in y at indices zero and n .

Let $n > KM$ and let $y \in Y$ be any transitive point for y . Since Y is totally transitive both b and c appear in y (otherwise y , and hence Y , would be periodic). For convenience, assume b appears at index zero. Consider the M -block in y at location n . If this block is made up entirely of concatenations of a 's then the proof is complete: w appears with period N in y between n and $n + M$, and if its alignment is $k \pmod N$ then we can find the occurrence of w in b with this alignment, say at index j in b , and then $j, j + n$ are the occurrences of w in y we were looking for.

Similarly, if the M -block starting in y at index $-n$ is made up entirely of concatenations of a the proof is complete.

If the argument above fails to produce the pair of w 's we wanted, then it must be because there is an occurrence of b at position $n + n'$ in y for some $-M \leq n' \leq M$ (or at $-n + n'$; this case is dealt with similarly). Let $u_1 u_2 \dots$ and $v_1 v_2 \dots$ be the unique parsing of y into $\{b, c\}$ words starting at index zero and $n + n'$ respectively, so $u_j, v_j \in \{b, c\}$ for all j . By assumption $u_1 = v_1 = b$. If $u_j = v_j$ for all j then the point y would be periodic, which is impossible since y generates Y and Y is totally transitive. So let j be the first index such that $u_j \neq v_j$. Then $u_{j-1} = v_{j-1}$ and they cannot be equal to b , because if both were equal to b then the next block in both cases would necessarily be c (recall that b 's do not occur consecutively in y) and we get $u_j = v_j$, a contradiction. So we have either $u_{j-1} u_j = c c = a^K a^K$ and $v_{j-1} v_j = c b = a^K b$, or else $u_{j-1} u_j = a^K b$ and $v_{j-1} v_j = a^K a^K$. In either case, since the beginning of u_j, v_j are exactly $n + n'$ apart and since the length of a^K is several times that of b , we are back in the situation from the previous paragraph and can find two occurrences of w separated by n , as desired. \square

To complete the proof of the proposition, we claim that if Z is totally transitive and is close enough to Y then Z enjoys the same kind of properties as Y . This can be put more precisely as follows.

LEMMA 8.8. *Let $Y \in \mathcal{S}_{\mathbf{T}^*}$ be constructed from $\{b, c\}$. For λ small enough (in a manner independent of Y), if Z is totally transitive and sufficiently close to Y , then Z is 10ϵ -strongly-mixing.*

Proof. For Z close enough to Y , the projection τ_Y is defined on Z . Furthermore, since $\{b, c\}$ has the unique parsing property when Z is close enough to Y , we have that for any $z \in Z$ its projection $\tau_Y(z)$ is a concatenation of b, c 's with no two b 's appearing consecutively. Also, since Y is non-trivial when Z is close enough to Y , we see that $\tau_Y(Z)$ is non-trivial. Therefore, $\tau_Y(Z)$ is a totally transitive system (since it is a factor of the totally transitive system Z) constructed from $\{b, c\}$ and satisfying the constraint that the word bb never appear, and hence by the previous lemma, $\tau_Y(Z)$ is ϵ -strongly-mixing.

Now use the fact that if Z is close enough to Y then $d_\infty(\tau_Y(z), z) < \epsilon$ for every $z \in Z$. Using Lemma 8.6 one sees that this implies that Z is 10ϵ -strongly mixing for all Z sufficiently close to Y . \square

This completes the proof of Proposition 8.5 and that strong mixing is generic in \mathcal{ST}^* . Theorem 1.1(5) now follows from the correspondence theorem.

9. *Minimal self-joinings*

In this section, by the *orbit* of a point x we mean the full orbit of x , i.e. the set $\{T^k x : k \in \mathbb{Z}\}$. For a system (X, T) we will denote the action of $T \times T$ on $X \times X$ simply by T , so $T(x', x'') = (Tx', Tx'')$.

A topological system (Y, S) has *minimal self-joinings* if it is infinite and for every $(x', x'') \in X \times X$, either x', x'' are on the same orbit (i.e. there is a $n \in \mathbb{Z}$ with $T^n x' = x''$) or else the orbit of (x', x'') is dense in $X \times X$. Such systems are also said to be *doubly minimal*.

There is a standard one-one correspondence between factors of (X, T) and closed invariant equivalence relations of $(X \times X, T)$. A factor Y of X given by a map $\varphi : X \rightarrow Y$ corresponds to the closed equivalence relation $\{(x', x'') \in X \times X : \varphi(x') = \varphi(x'')\}$.

When X has minimal self-joinings the only subsystems of $X \times X$ are the trivial subsystem $X \times X$ and the graphs

$$D_n = \{(x, T^n x) : x \in X\}$$

for $n \in \mathbb{N}$. Now, D_n is not an equivalence for $n \neq 0$. Otherwise for every $x \in X$, the fact that $(x, T^n x) \in D_n$ would imply (by symmetry of the equivalence relation D_n) that $(T^n x, x) \in D_n$ and thus by definition of D_n we get $T^{2n} x = x$. Since this holds for every $x \in X$ we deduce that every point of X has period $2n$, which is impossible because X has minimal self-joinings. (Proof: since X is infinite, we can choose two periodic points not on the same orbit. This pair violates the definition of double minimality.)

Consequently, when X has minimal self-joinings, the only closed equivalence relations of $X \times X$ are the entire space $X \times X$, which corresponds to the factor map taking all of X to a single fixed point, or the diagonal $D_0 = \{(x, x) : x \in X\}$, which corresponds to the factor given by the identity map on X . Thus the only factors of a system with minimal self-joinings are the trivial factors. Systems with this property are called *prime*; the simplest example being periodic systems of prime period.

The notion of double minimality comes from a similar notion in the measure-preserving category, where examples were first constructed by Rudolph [23] and shown to be exotic by del Junco in [8]. However, it was recently shown by Ageev that primeness in the measure-theoretic category is generic [1].

Prime topological systems were first constructed by Furstenberg *et al* [10]. The first topological system with minimal self-joinings was constructed by King in [17]. Later Weiss showed in [25] that every ergodic system (Y, \mathcal{B}, μ, T) with zero entropy has a realization as an invariant measure on a topological system with minimal self-joinings. This implies that in some sense topological double minimality is a common phenomenon. Double minimality is also a common phenomenon in our setup.

THEOREM 9.1. *Double minimality is generic in \mathcal{ST}^* .*

We will use the following combinatorial fact which is a watered-down version of Lemma 2 from [25].

LEMMA 9.2. For any integer L and every large enough N there is a set $I \subseteq \{1, \dots, N\}$ such that, if $i \in I$, then $i + k \notin I$ for $1 \leq k \leq L$ and for every $N/10 \leq k \leq 9N/10$ it holds that $(I + k) \cap I \neq \emptyset$. Such a set I is called approximately random.

We also use the following simple observation.

LEMMA 9.3. Let $b = b(1) \dots b(M) \in Q^*$ and $c = c(1) \dots c(N) \in Q^*$ and assume $2M/5 < N < M/2$. Let $x, y \in Q^{\mathbb{Z}}$ be non-periodic points constructed from b, c such that c doesn't appear twice consecutively in x, y . Then either $x = \sigma^n y$ for some $n \in \mathbb{Z}$, or else there are occurrences u, v of b in x, y whose alignment differs by some k in the range $(2M/10, 8M/10)$.

Proof. Since x, y are constructed from L we may parse x, y as

$$\begin{aligned} x &= \dots u_{-1}u_0u_1u_2 \dots \\ y &= \dots v_{-1}v_0v_1v_2 \dots \end{aligned}$$

where $u_i, v_i \in \{b, c\}$ (note that the parsing may not be unique, so we choose one such parsing).

Suppose there is no pair of b 's in x, y whose alignment differs by an integer in the range $(2M/10, 8M/10)$. We will show that $x = \sigma^n y$ for some n .

We claim that if $u_i = c$ for some i then there is a unique j such that $v_j = c$ and u_i, v_j overlap. Indeed, there can obviously be at most one such j , since occurrences of c are separated by b 's and b is longer than c . Now if the statement were false then u_i overlaps with one or two occurrences of b in y . By the restriction that cc does not appear in x , we see that $u_{i-1}u_iu_{i+1} = bcb$, and one can check directly that no matter what configuration of b 's occurs in y opposite u_i , there is a b in y opposite one of the words u_{i-1} or u_{i+1} at a shift in the range $(4M/5, M/2)$, contrary to our assumption.

Similarly, if $v_i = c$ then there is a unique occurrence of c in the parsing of x overlapping v_i .

Now suppose $u_iu_{i+1} \dots u_{i+k} = cbb \dots bbc$. The distance between the end of u_i and the beginning of u_{i+k} is a multiple of M . From the above it is clear that there are occurrences v_j, v_{j+m} of c in y opposite u_i, u_{i+k} respectively. There can be no occurrence of c in y between v_j and v_{j+m} because this would imply an occurrence in x between u_i and u_{i+k} , contrary to our assumption. Thus $kM - 2N \leq mM \leq kM + 2N$. Since $N < M/2$, we must have $k = m$. In particular, the offset of v_j from u_i is equal to the offset of u_{i+k} from v_{j+k} and $u_iu_{i+1} \dots u_{i+k} = v_jv_{j+1} \dots v_{j+k}$.

Similarly, suppose $u_i = c$ and $u_k = b$ for all $k > i$. The same argument shows that there is a j such that $v_j = c$ overlaps u_i and $v_k = b$ for all $k > j$. A similar statement is true if $u_k = b$ for all $k < i$.

If $u_i = v_i = b$ for every i then clearly x, y are shifts of each other. Otherwise pick an occurrence of c in x , which we may assume is u_0 . Find the occurrence of c in y opposite it, which we assume is v_0 . Now look to the right; repeated application of the above shows that the right-infinite subword of x starting at u_0 equals the right-infinite subword of y starting at v_0 . A similar statement holds for left-infinite subwords, so y is a shift of x . □

The main step in the proof of Theorem 9.1 is the following construction.

LEMMA 9.4. *Let $X \in \mathcal{S}_T^*$. For every ε, δ there exists a totally transitive symbolic system $Y = Y(X, \varepsilon, \delta)$ with $d(X, Y) < \varepsilon$ and such that for every $y', y'' \in Y$ either $y' = \sigma^n y''$ for some $n \in \mathbb{Z}$ or else the orbit of $\{\sigma^n(y', y'') : n \in \mathbb{Z}\}$ is δ -dense in $Y \times Y$. Furthermore, if Z is close enough to Y then τ_Y takes Z into Y .*

Proof. Let X, ε, δ be given. Choose a small positive η and construct an η -dense η -pseudo-period $\bar{a} = a_1, \dots, a_R$ in X . Now form a very long η -pseudo-orbit $\bar{b} = b_1, \dots, b_M$ in X for which the set of indices $I \subseteq \{1, \dots, M\}$ at which \bar{a} occurs has the property that $I \cap (I + k) \neq \emptyset$ for every $M/10 \leq k \leq 9M/10$ (such a \bar{b} exists by total transitivity and Lemma 9.2). Let $1 \leq i, j \leq R$ and $M/5 \leq k \leq 4M/5$; then there are occurrences of a_i and a_j in \bar{b} whose alignment differs by k . To see this note that assuming $M > 10R$ we can find, by approximate randomness, two occurrences of \bar{a} in \bar{b} whose alignment differs by $k + i - j$.

The significance of the previous paragraph is that if we take two copies of \bar{b} and shift one of them right or left by a distance between $2M/10$ and $8M/10$, then every pair a_i, a_j occurs in the pair \bar{b} and $\sigma^k \bar{b}$. Since \bar{a} is η -dense, the collection of pairs (a_i, a_j) occurring in $(\bar{b}, \sigma^k \bar{b})$ is η -dense in $X \times X$.

Choose another η -pseudo-orbit $c = c_1, \dots, c_N$ with $c_1 = b_1$ and such that $4M/10 < N < M/2$ and M, N relatively prime (we can do this because we may assume M as large as necessary). Write b, c for the finite words $b = \theta(\bar{b})$ and $c = \theta(\bar{c})$. Let $Y \in \mathcal{S}$ be the system constructed from the language $\{b, bcb\}$ containing all infinite concatenations of b, c in which the word cc does not appear. This is a strongly mixing system since K, N are relatively prime (essentially a mixing subshift of finite type). We have by Proposition 4.2 that $d(X, Y) < 2\sqrt{\eta}$, which can be made $< \varepsilon$.

We may assume that $\{b, c\}$ has the unique parsing property (Definition 4.5) so $\{b, bcb\}$ do as well; this implies that $\tau_Y(Z) \subseteq Y$ for Z close enough to Y (Lemma 4.4 and what follows).

It remains to show that every $(y', y'') \in Y \times Y$ not on the same orbit have a δ -dense orbit under $Y \times Y$. Applying Lemma 9.3 we see if y', y'' are not on the same orbit then there are occurrences of b in y', y'' whose alignment difference is in the range $(2M/10, 8M/10)$. Now $y' = \theta(\bar{y}')$ and $y'' = \theta(\bar{y}'')$ for η -pseudo-orbits \bar{y}', \bar{y}'' made by concatenating \bar{b} 's and \bar{c} 's. Therefore, by our assumptions, every pair (a_i, a_j) occurs in the pair \bar{y}', \bar{y}'' . Since $\{(a_i, a_j)\}_{1 \leq i, j \leq R}$ is η -dense in $X \times X$ this means that the orbit of $(y', y'') = \theta(\bar{y}', \bar{y}'')$ is δ -dense in $Y \times Y$, assuming η small enough, as required. \square

Proof of Theorem 9.1. As in §4 let

$$\rho_Y = \min\{d(c', c'') : c', c'' \in \pi_0(Y), c' \neq c''\}$$

and recall that if Y is symbolic then $\rho_Y > 0$, and if $d(Z, Y) < \rho_Y$ then τ_Y is defined on Z (Lemma 4.4). Also write

$$\tilde{\rho}_Y(Z) = \max_{z \in Z} \min_{y \in Y} d(z, y).$$

For every Z close enough to Y we have $\tilde{\rho}_Y(Z) < \rho_Y/2$ and then τ_Y is defined on Z ; furthermore $d_\infty(z, \tau_Y(z)) < \tilde{\rho}_Y(Z)/2$ for all $z \in Z$.

For $X \in \mathcal{S}_{\mathbf{T}^*}$ and $\varepsilon, \delta > 0$ let $Y = Y(X, \varepsilon, \delta)$ be as in the lemma. Fix $n \in \mathbb{N}$ and choose $r = r(X, \varepsilon, \delta) < \delta$ in such a way that

$$Z \in B_r(Y) \Rightarrow \tilde{\rho}_Y(Z) < \min\left\{\frac{1}{4}\rho_Y, \frac{1}{n}\right\}.$$

This guarantees that τ_Y is defined on any $Z \in B_r(Y)$. We may assume also that the projection of $\tau_Y(Z)$ of Z is non-trivial, since we may assume X and hence Y are non-trivial.

Write $B(X, \varepsilon, \delta) = B_{r(X, \varepsilon, \delta)}(Y(X, \varepsilon, \delta))$. Set

$$U_n = \bigcup_{X \in \mathbf{T}^*} \bigcup_{\varepsilon > 0} B\left(X, \varepsilon, \frac{1}{n}\right)$$

and

$$D = \bigcap_{n \in \mathbb{N}} U_n.$$

D is a dense G_δ subset of $\mathcal{S}_{\mathbf{T}^*}$ (each U_n is open and it is dense in $\mathcal{S}_{\mathbf{T}^*}$ since $Y(X, \varepsilon, \delta) \in \mathcal{S}_{\mathbf{T}^*}$ for $X \in \mathcal{S}_{\mathbf{T}^*}$). We will show that if $Z \in D \cap \mathcal{S}_{\mathbf{T}^*} \cap \mathcal{S}_{\mathbf{Min}}$ then Z has minimal self-joinings; since \mathbf{Min} is a dense G_δ in $\mathcal{S}_{\mathbf{T}^*}$ this completes the proof.

Let $Z \in D \cap \mathcal{S}_{\mathbf{T}^*} \cap \mathcal{S}_{\mathbf{Min}}$ and for $n \in \mathbb{N}$ let $Y_n = Y(X_n, \varepsilon_n, 1/n)$, $r_n = r(X_n, \varepsilon_n, 1/n)$ such that $Z \in \bigcap B_{r_n}(Y_n)$. Fix n and $z', z'' \in Z$ and let $y' = \tau_{Y_n}(z')$ and $y'' = \tau_{Y_n}(z'')$. We distinguish the following two cases.

(1) If y', y'' are not on the same orbit in Y_n then the orbit of (y', y'') is $1/n$ dense in $Y_n \times Y_n$. For an arbitrary pair $(w', w'') \in Z \times Z$ there is some m such that

$$d(\sigma^m(y'), \tau_Y(w')) < \frac{1}{n} \quad \text{and} \quad d(\sigma^m(y''), \tau_Y(w'')) < \frac{1}{n}.$$

Now since

$$d_\infty(y', z') < \tilde{\rho}_{Y_n}(Z) \quad \text{and} \quad d_\infty(y'', z'') < \tilde{\rho}_{Y_n}(Z)$$

and since $\tilde{\rho}_{Y_n}(Z) < 1/n$ we can combine these inequalities and get

$$d(\sigma^m(y'), \sigma^m(z')) < \frac{2}{n} \quad \text{and} \quad d(\sigma^m(y''), \sigma^m(z'')) < \frac{2}{n}.$$

Similarly, we have

$$d(w', \tau_Y(w')) < \frac{1}{n} \quad \text{and} \quad d(w'', \tau_Y(w'')) < \frac{1}{n}$$

from which we conclude

$$d(\sigma^m(z'), w') < \frac{3}{n} \quad \text{and} \quad d(\sigma^m(z''), w'') < \frac{3}{n}.$$

Since w', w'' were arbitrary, the orbit of (z', z'') in $Z \times Z$ is $3/n$ dense in $Z \times Z$.

(2) Otherwise, suppose that y', y'' are on the same orbit, so $y' = \sigma^{k(n)}y''$ for some integer $k(n)$. Then since

$$d_\infty(y', z') \leq \tilde{\rho}_{Y_n}(Z) < \min\left\{\frac{1}{2}\rho_Y, \frac{1}{n}\right\}$$

and the same is true of y'', z'' we see that z' is $\min\{\rho_Y/2, 1/n\}$ -uniformly close to $\sigma^{k(n)}z''$.

If (1) holds for infinitely many n then the orbit of (z', z'') is dense in $Z \times Z$.

Otherwise (2) holds for infinitely many n . If there is some $m \in \mathbb{N}$ such that $k(n) = m$ for infinitely many n , we see that $z', \sigma^m(z'')$ are $1/n$ -uniformly close for infinitely many n , and so $z' = \sigma^m(z'')$, i.e. z', z'' are on the same orbit.

In the alternative case $k(n)$ would take on infinitely many values, and in particular at least to values. Suppose for simplicity that $k(1) \neq k(2)$ and that $\tilde{\rho}_{Y_1}(Z) \leq \tilde{\rho}_{Y_2}(Z)$. We have

$$d_\infty(z', \sigma^{k(1)}(z'')) < \frac{1}{2} \tilde{\rho}_{Y_1}(Z), \quad d_\infty(z', \sigma^{k(2)}(z'')) < \frac{1}{2} \tilde{\rho}_{Y_2}(Z)$$

and using the assumption $\tilde{\rho}_{Y_1}(Z) \leq \tilde{\rho}_{Y_2}(Z)$ we get

$$d_\infty(z'', \sigma^{k(2)-k(1)}(z'')) < \tilde{\rho}_{Y_2}(Z).$$

Therefore

$$\tau_{Y_2}(z'') = \sigma^{k(2)-k(1)}(\tau_{Y_2}(z'')).$$

Since Z is minimal the point z'' is a transitive point for Z , so $\tau_Y(z'')$ is transitive for Y , and we conclude that Y is periodic with period $k(2) - k(1)$. However, Z is totally transitive so cannot have periodic factors: a contradiction. \square

This completes the proof that the double minimality is generic in $\mathcal{S}_{\mathbf{T}^*}$. Theorem 1.1(6) now follows from the correspondence theorem.

10. Closing comments and open questions

We briefly mention some problems and extensions which arise in connection with this work.

Our original motivation for this work was our interest in \mathcal{S} as a universal space of topological systems. That it is equivalent to the homeomorphism group of K was a surprise and has proved useful in the study of the latter. However, there are many interesting subsets of \mathcal{S} which cannot be attacked by our methods.

A prime candidate is the space of connected systems in \mathcal{S} , which is universal for connected systems. This is a closed space in \mathcal{S} , hence Polish. What is generic there? More specifically, what is the relation between the transitive, totally transitive and rigid connected systems? What is generic in each? Is there a zero-one law?

Another interesting question is: what is the connection between the space of connected systems in \mathcal{S} and the homeomorphism groups of ‘nice’ manifolds?

Throughout this paper we have been working with subsystems of the shift space $Q^{\mathbb{Z}}$ where Q is the Hilbert cube. It may be verified that except for in the proof that \mathcal{S} is universal, the only property we have relied upon has been that Q has no isolated points. For any perfect compact metric space Δ our proofs work for the space $\mathcal{S}(\Delta)$ of closed shift-invariant subsystems of $\Delta^{\mathbb{Z}}$.

This is no longer true when Δ has isolated points. Most interesting is the symbolic case, e.g. $\Delta = \{0, 1\}$, so $\Delta^{\mathbb{Z}} = \{0, 1\}^{\mathbb{Z}}$ is the full shift on two symbols. Theorem 5.14 is no longer true; indeed the only symbolic odometers are finite cycles. Instead we have that the finite cycles form a dense G_δ (this also shows that the correspondence theorem is false here). Another theorem which requires modification is Theorem 6.4; to obtain an analog

of it, one must take entropy into account. Otherwise, however, Theorem 1.2 remains the same in this space.

A problem we have only partly settled is the relation between genericity in the space of measures and genericity of topological realizations of them. We have provided a partial answer to this in Theorem 6.9. It is possible that the hypothesis that the property be a G_δ is too strong. See the remarks after the proof of Theorem 6.9.

Finally, very little is known about genericity for actions of other groups. Here, part of our work carries over: the correspondence theorem and its proof remain valid for such actions. More precisely, consider a countable group Γ and the shift space Q^Γ along with the shift action of Γ on Q^Γ given by $(gx)(h) = x(hg)$ for $g, h \in \Gamma$. The correspondence theorem then relates genericity in the space \mathcal{S}_Γ of shift-invariant subsystems of Q^Γ to genericity in the space of representations by homeomorphisms of Γ on the Cantor set with an appropriate topology, and similarly for suitable subspaces.

However, Theorems 1.1 and 1.2 do not appear to have good analogs in this more general setting. The following example is instructive. Consider the case of the group $\Gamma = \mathbb{Z}$, and let $X \subseteq \{a, b\}^\mathbb{Z}$ be a shift of finite type, $a, b \in Q$. If Y is close enough to X in $\mathcal{S}_\mathbb{Z}$, then by Lemma 4.6 Y has a non-trivial factor in $\{a, b\}^\mathbb{Z}$ in which each sufficiently long word belongs to X ; this implies there is a non-trivial factor of Y embedded in X . If Y is prime, for instance, then Y embeds into X . In conjunction with our other results about the genericity of prime systems and the density of the isomorphism class of minimal totally transitive systems, this means that the subsystems of mixing SFTs are a very large family. The last observation is of course not new. The point is that when one goes to $\Gamma = \mathbb{Z}^2$ this argument cannot be true because there are minimal totally transitive \mathbb{Z}^2 shifts of finite type X (Mozes [18]). By the same reasoning as above, such a system has a neighborhood in $\mathcal{S}_{\mathbb{Z}^2}$ consisting entirely of extensions of this system, so the isomorphism class of any system not extending X does not have a dense isomorphism class. Compare this with Theorem 6.4.

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