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On the ergodicity of hyperbolic Sinaĭ–Ruelle–Bowen measures II: the low-dimensional case

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Abstract. In this paper, we consider diffeomorphisms on a closed manifold M preserving a hyperbolic Sinaĭ–Ruelle–Bowen probability measure μ having intersections for almost every pair of stable and unstable manifolds. In this context, we show the ergodicity of μ when the dimension of M is at most three. If μ is smooth, then it is ergodic when the dimension of M is at most four. As a byproduct of our arguments, we obtain sufficient (topological) conditions which guarantee that there exists at most one hyperbolic ergodic Sinaĭ–Ruelle–Bowen probability measure. Even in higher dimensional cases, we show that every transitive *topological* Anosov diffeomorphism admits at most one hyperbolic Sinaĭ–Ruelle–Bowen probability measure.

1. *Introduction*

1.1. *Sinaĭ–Ruelle–Bowen measures.* Broadly speaking, Sinaĭ–Ruelle–Bowen (SRB) measures are the invariant measures most compatible with the smooth volume when we investigate the ergodic theory of dissipative systems. These measures were introduced by pioneering works of Sinaĭ [**30**] and Bowen and Ruelle [**8**] on the ergodic theory of Anosov or Axiom A systems. (Note that Anosov systems satisfy Axiom A.) Their main result can be stated in short that Axiom A attractors do support a unique invariant probability measure having 'nice' ergodic properties. This measure is called SRB. See §2.5 for the definition.

Subsequently, such measures have been shown to exist for dynamical systems beyond uniform hyperbolicity, for instance, certain partially hyperbolic attractors (see Pesin and Sinaĭ [**23**] and Alves, Bonatti and Viana [**1**, **7**]) and Hénon attractors (see Benedicks and

Young [**5**]). Note that, in fact, some of them concern u-Gibbs measures rather than SRB (see [**4**, **6**] for more details). These results also give the finitude (including uniqueness) of SRB measures. We should note here that the topology of the basins of SRB measures could be much more complicated. For instance, due to the example constructed by Kan [**17**], one can show that there exists a topologically transitive diffeomorphism on $\mathbb{T}^3$ admitting two hyperbolic SRB measures such that both basins are dense. Contrary to such phenomena, Rodriguez Hertz *et al* [**26**] show that there exists at most one SRB measure for topologically transitive surface diffeomorphisms. In this paper, we show that there exists at most one hyperbolic SRB measure for some diffeomorphisms of dimension greater than two (see Corollary 1.8 and also [**13**]).

This paper constitutes a direct continuation of [**13**].

1.2. *Statement of results.* Let M be a compact smooth Riemannian manifold with a norm $\|\cdot\|$, $f : M \to M$ a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of M preserving a Borel probability measure μ and with $Df : TM \to TM$ being the derivative of f. We let d be the distance on M induced by the Riemannian metric.

For $x \in M$ and $v \in T_xM$, we define the *Lyapunov exponent of* v *at* x by

$$\chi(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|D_x f^n(v)\|.$$

The measure μ is said to be *hyperbolic* if $\chi(x, v) \neq 0$ for μ-almost every x and every $v \in T_xM$ and $\min_{v\in T_xM} \chi(x, v) < 0 < \max_{v\in T_xM} \chi(x, v)$ for μ-almost every x. For $x \in M$, we set

$$\mathcal{W}^u(x) = \Big\{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0\Big\},$$
$$\mathcal{W}^s(x) = \Big\{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{n}(x), f^{n}(y)) < 0\Big\}.$$

If μ is hyperbolic, then $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are, in fact, injectively immersed manifolds with $T_x\mathcal{W}^u(x) \oplus T_x\mathcal{W}^s(x) = T_xM$ for μ-almost every x (see [**4**]). These are called the *unstable* and *stable manifolds* at x, respectively. Clearly, they are f-invariant, that is, $f(W^\tau(x)) = W^\tau(f(x))$ for $\tau = u, s$.

In [**13**], we introduce two notions of accessibility on μ, as follows. We say that μ satisfies:

- the *transversally almost accessibility property* (TAAP for short) if, for $\mu\otimes\mu$-almost every pair $(x, y) \in M \times M$, there exist integers $p, q \in \mathbb{Z}$ and a point $z \in \mathcal{W}^u(f^p(x)) \cap \mathcal{W}^s(f^q(y))$ such that
$$T_z\mathcal{W}^u(f^p(x)) + T_z\mathcal{W}^s(f^q(y)) = T_zM$$
or there exist integers $j, k \in \mathbb{Z}$ and a point $w \in \mathcal{W}^s(f^j(x)) \cap \mathcal{W}^u(f^k(y))$ such that
$$T_w\mathcal{W}^s(f^j(x)) + T_w\mathcal{W}^u(f^k(y)) = T_wM; \text{ and}$$
- the *almost accessibility property* (AAP for short) if, for $\mu\otimes\mu$-almost every pair $(x, y) \in M \times M$, there exist integers $p, q \in \mathbb{Z}$ satisfying either
$$\mathcal{W}^u(f^p(x)) \cap \mathcal{W}^s(f^q(y)) \neq \emptyset \tag{1.1}$$

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or

$$\mathcal{W}^s(f^p(x)) \cap \mathcal{W}^u(f^q(y)) \neq \emptyset. \tag{1.2}$$

Clearly, TAAP implies AAP.

In [**13**], the authors showed that for Sinaĭ–Ruelle–Bowen (SRB) measure-preserving diffeomorphisms, the AAP implies the ergodicity when $\dim \mathcal{W}^u$ is constant almost everywhere. In this paper, we show, without the assumption that $\dim \mathcal{W}^u$ is constant, that the AAP implies the ergodicity when the ambient manifold is of low dimension.

THEOREM 1.1. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M preserving a hyperbolic SRB probability measure μ. If μ satisfies the AAP and* $\dim M \leq 3$*, then μ is ergodic.*

For smooth measure-preserving (i.e. measures which are equivalent to the Riemannian volume) diffeomorphisms, the AAP implies the ergodicity even for four-dimensional manifolds.

THEOREM 1.2. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth Riemannian manifold M preserving a hyperbolic smooth measure μ. If μ satisfies the AAP and* $\dim M \leq 4$*, then μ is ergodic.*

Next, we give several topologically sufficient conditions for the AAP, and hence for the ergodicity of SRB (or smooth) measure. Let $\mathrm{Per}(f)$ be the set of all periodic points of f. For $p \in \mathrm{Per}(f)$, set

$$W^u(p) = \{x \in M : d(f^{-n}(x), f^{-n}(p)) \to 0 \ (n \to \infty)\},$$
$$W^s(p) = \{x \in M : d(f^{n}(x), f^{n}(p)) \to 0 \ (n \to \infty)\}$$

and define

$$\mathrm{Per}^*(f) = \{p \in \mathrm{Per}(f) : W^u(p) \setminus \{p\} \neq \emptyset,\ W^s(p) \setminus \{p\} \neq \emptyset\}.$$

THEOREM 1.3. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M of* $\dim M \leq 3$ *preserving a hyperbolic SRB probability measure μ. Assume that f satisfies the following condition.*

(P) *For every $p, q \in \mathrm{Per}^*(f)$ there exist $j, k \in \mathbb{Z}$ such that*

$$W^u(p) \cap W^s(f^j(q)) \neq \emptyset \quad \textit{and} \quad W^s(p) \cap W^u(f^k(q)) \neq \emptyset.$$

Then μ satisfies the AAP.

Note here that the intersection in condition (P) is not supposed to be transverse.

THEOREM 1.4. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M of* $\dim M \leq 3$ *preserving a hyperbolic SRB probability measure μ. If f has the specification property, then μ satisfies the AAP.*

As an immediate consequence of Theorems 1.1, 1.3 and 1.4, we obtain the following corollary.

COROLLARY 1.5. *Let f be as in Theorems 1.3 or 1.4. Then it is ergodic.*

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Here the specification property is stronger than topological transitivity (see §4 (Remark 4.4) for the definition). When M is a surface, it is shown by Rodriguez Hertz *et al* [**26**] that topological transitivity implies the ergodicity of hyperbolic SRB measures (see also [**13**]). When the dimension of the manifold M is greater than or equal to three, the ergodicity of hyperbolic SRB measures would not follow from the topological transitivity, in general. In fact, there exists a topologically transitive diffeomorphism of $\mathbb{T}^3$ preserving a hyperbolic SRB probability measure which is not ergodic (see [**6**, **26**]).

Example 1.6. Katok [**18**] constructed an area-preserving diffeomorphism k of the disk with non-zero Lyapunov exponents almost everywhere such that the map is an identity and has derivatives which are zero on the boundary of the disk. Take another disk and consider the identity map on the disk. By gluing the two disks along the boundary circle, we obtain a sphere and a diffeomorphism f (of class C^∞) which preserves a smooth measure.

For such an f, $\mathrm{Per}^*(f) = \mathrm{Per}^*(k)$. By the construction of k, we see that $\mathrm{Per}^*(k)$ fulfil condition (P). Therefore, by Theorem 1.3, there exists at most one hyperbolic ergodic SRB probability measure for f and even for diffeomorphisms that are topologically conjugate to f, since condition (P) is preserved by the conjugacy.

Even in high-dimensional cases, we can obtain the ergodicity of hyperbolic SRB probability measures as follows.

THEOREM 1.7. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M preserving a hyperbolic SRB probability measure μ. If f is a transitive topological Anosov diffeomorphism (that is, it is topologically conjugate to a topologically transitive Anosov diffeomorphism), then μ satisfies the AAP, and hence it is ergodic.*

All of the results related to the ergodicity show that there is at most one hyperbolic SRB probability measure since almost every ergodic component of a hyperbolic SRB measure is hyperbolic SRB. In particular, there is the following corollary.

COROLLARY 1.8. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M. Then it admits at most one hyperbolic SRB probability measure if either:*

(1) $\dim M \leq 3$ *and f satisfies the condition* (P)*;*

(2) $\dim M \leq 3$ *and f satisfies the specification property; or*

(3) *f is a transitive topological Anosov diffeomorphism.*

Remark 1.9. We should stress that Corollary 1.8 does not assert the existence of an SRB measure. In fact, even for case (3), there are both existence and non-existence results (see, for example, [**11**] and [**16**]).

1.3. *The Hopf argument beyond Anosov systems.* Our approach is based upon developing the Hopf argument into the ergodic theory of non-uniformly hyperbolic systems. The Hopf argument, introduced by Hopf [**15**] and improved by Anosov and Sinaĭ [**2**, **3**], is a simple but strong method in the ergodic theory of Anosov systems. This method gives us a fairly geometric argument for ergodicity problems since it is, in part, based upon the geometry of mutually transverse invariant foliations, called the strong stable and strong unstable foliations. Although these foliations are not smooth in general, they possess the crucial absolute continuity. Subsequently, such foliations are shown to exist for a broad class of dynamical systems, one partially hyperbolic by Brin and Pesin [**9**] and by Hirsh, Pugh and Shub [**14**] and the other non-uniformly hyperbolic by Pesin [**22**]. Nevertheless, the Hopf argument does not extend naively to these dynamical systems.

It is conjectured by Pugh and Shub [**24**] that smooth measure-preserving partially hyperbolic diffeomorphisms with the (essential) accessibility property are ergodic. (In fact, it is a part of their main conjecture: stable ergodicity is C^r $(r > 1)$ dense among the smooth measure-preserving C^r partially hyperbolic diffeomorphisms.) Here a partially hyperbolic diffeomorphism is called accessible if any two points can be connected by a path which consists of pieces of smooth curves lying on strong stable and strong unstable manifolds. It is almost needless to say that the main difficulty emerged from the non-conformal dynamics along the central direction. This conjecture is proved by Rodriguez Hertz, Rodriguez Hertz and Ures [**28**] in the case where the center bundle is one-dimensional or under an additional assumption called center-bunching by Burns and Wilkinson [**10**] (see [**12**, **27**] for details).

In the case where the diffeomorphism is non-uniformly hyperbolic, the main issue is ensuring the transverse intersection between the stable and unstable manifolds. Notice that these two manifolds, $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$, intersect transversely at almost every x. However, *a priori*, there could be no points in $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ at which the intersection is transverse even though both x and y are typical (that is, Lyapunov regular points; see §2.1). In this paper, we show that the set of points $(x, y) \in M \times M$ for which the set $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ contains some transverse intersection points (or is empty) could have, *a posteriori*, full measure (see Proposition 3.5 for precise details). Consequently, under the assumption of the AAP, we could implement the Hopf argument, as in the case of Anosov diffeomorphisms, to show the ergodicity of SRB or smooth probability measures for non-uniformly hyperbolic diffeomorphisms (Theorems 1.1 and 1.2).

1.4. *Organization of this paper.* The rest of this paper is as follows. In §2, we recall some preliminary material from the non-uniformly hyperbolic theory and fix notation. In §3, we investigate the Hopf argument and the absolute continuity property of laminations in the non-uniformly hyperbolic dynamics and prove Theorems 1.1 and 1.2. We prove Theorems 1.3 and 1.4 in §4. Theorem 1.7 is proved in §5.

2. *Preliminaries*

2.1. *Lyapunov exponents.* Let μ be a Borel probability measure that is invariant under f. A point $x \in M$ is said to be *Lyapunov regular* if there exist real numbers $\chi_1(x) > \chi_2(x) > \cdots > \chi_{r(x)}(x)$ and a Df-invariant decomposition $T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that, for each $i = 1, 2, \ldots, r(x)$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_i(x) \quad (v \in E_i(x) \setminus \{0\})$$

exists and

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |\det(D_x f^n)| = \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x).$$

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We denote by Γ the set of Lyapunov regular points. By the Oseledec multiplicative ergodic theorem, Γ has full μ-measure. The numbers $\chi_i(x)$ are called the *Lyapunov exponents* of f at the point x. The functions $x \mapsto \chi_i(x)$, $r(x)$ and $\dim E_i(x)$ are Borel measurable and f-invariant. If the invariant measure is supposed to be ergodic, then they are constant functions and we denote the constants by χ_i and $\dim E_i$ $(i = 1, 2, \ldots, r)$.

Note that the measure μ is hyperbolic if none of the Lyapunov exponents for μ vanish and there exist Lyapunov exponents with different signs for μ-almost everywhere. In what follows, we always assume that μ is hyperbolic, and we will denote $u(x) = \max\{i : \chi_i(x) > 0\}$ and $s(x) = \min\{i : \chi_i(x) < 0\}$ for μ-almost every $x \in M$. Then the associated decomposition is represented by $T_xM = E^u(x) \oplus E^s(x)$ and $E^u(x) = T_x\mathcal{W}^u(x)$ and $E^s(x) = T_x\mathcal{W}^s(x)$, where $E^u(x) = \bigoplus_{i=1}^{u(x)} E_i(x)$, $E^s(x) = \bigoplus_{i=s(x)}^{r(x)} E_i(x)$ for μ-almost every $x \in M$. Note that $s(x) = u(x) + 1$ (see [**4**]).

2.2. *Lyapunov charts.* Let ν be an ergodic hyperbolic Borel probability measure that is invariant under f. Denote by $\chi_u = \chi_u^\nu$ the smallest positive Lyapunov exponents and by $\chi_s = \chi_s^\nu$ the largest negative Lyapunov exponent. Fix sufficiently small $\varepsilon = \varepsilon^\nu > 0$ with $\varepsilon < \min\{\chi^u, |\chi^s|\}$.

It follows from [**22**] and [**4**, Theorem 5.4.6] that there exist $\Gamma^\nu \subset \Gamma$ with $\nu(\Gamma^\nu) = 1$ and a measurable function $C_\varepsilon : \Gamma^\nu \to [1, \infty)$ such that for, every $x \in \Gamma^\nu$:

- for $n \geq 0$,

$$\|D_x f^{-n}(v)\| \leq C_\varepsilon(x) e^{-(\chi_u - \varepsilon)n} \|v\| \quad (v \in E^u(x)),$$
$$\|D_x f^{n}(v)\| \leq C_\varepsilon(x) e^{(\chi_s + \varepsilon)n} \|v\| \quad (v \in E^s(x));$$

- $|\sin \angle(E^u(x), E^s(x))| \geq C_\varepsilon(x)^{-1}$; and
- $C_\varepsilon(f^{\pm 1}(x)) \leq e^\varepsilon C_\varepsilon(x)$.

For each $l \in \mathbb{N}$, we let

$$\Lambda_\varepsilon^\nu = \bigcup_{l \geq 1} \Lambda_{l,\varepsilon}^\nu, \qquad \Lambda_{l,\varepsilon}^\nu = \{x \in \Gamma^\nu : C_\varepsilon(x) \leq l\}.$$

For notational simplicity, we may write Λ_l^ν or Λ_l instead of $\Lambda_{l,\varepsilon}^\nu$ and Λ^ν or Λ instead of Λ_ε^ν if no confusion may arise. Obviously, $\Lambda_{l_1} \subset \Lambda_{l_2}$ if $l_1 \leq l_2$, and $\nu(\Lambda) = 1$.

Let $d^u = \dim E^u$ and $d^s = \dim E^s$. For notational simplicity, we write $\mathbb{R}^u = \mathbb{R}^{d^u}$ and $\mathbb{R}^s = \mathbb{R}^{d^s}$. For $v = (v_1, v_2) \in \mathbb{R}^{\dim M} = \mathbb{R}^u \times \mathbb{R}^s$ we define a norm as

$$|v| = \max\{|v_1|_u, |v_2|_s\},$$

where $|\cdot|_u$ and $|\cdot|_s$ denote the Euclidean norm on $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively. For $\rho > 0$, we write

$$\mathbb{D}(\rho) = \{v \in \mathbb{R}^{\dim M} : |v| \leq \rho\},$$
$$\mathbb{D}^u(\rho) = \{v \in \mathbb{R}^u : |v|_u \leq \rho\},$$
$$\mathbb{D}^s(\rho) = \{v \in \mathbb{R}^s : |v|_s \leq \rho\}.$$

For each $l \in \mathbb{N}$, by [**19**, **22**], there exist a constant $q_l \in (0, 1]$ and a family of embeddings $\{\Psi_x : \mathbb{D}(q_l) \to M\}_{x \in \Lambda_l}$ such that the following hold.

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(1) $\Psi_x(0) = x$. $D_0\Psi_x(\mathbb{R}^u) = E^u(x)$ and $D_0\Psi_x(\mathbb{R}^s) = E^s(x)$.

(2) Let $f_x = \Psi_{f(x)}^{-1} \circ f \circ \Psi_x$, if it makes sense. Then, for each $i = 1, 2, \ldots, r$,

$$|D_0 f_x(v)| \geq e^{\chi_u - \varepsilon}|v| \quad (v \in \mathbb{R}^u),$$
$$|D_0 f_x(v)| \leq e^{\chi_s + \varepsilon}|v| \quad (v \in \mathbb{R}^s).$$

(3) $\mathrm{Lip}(f_x - D_0 f_x) \leq \varepsilon$, where $\mathrm{Lip}(g)$ denotes the Lipschitz constant of a map g.

(4) there exists a constant $K > 1$ such that for every $v, w \in \mathbb{D}(q_l)$,

$$K^{-1}\, d(\Psi_x(v), \Psi_x(w)) \leq |v - w| \leq q_l^{-1}\, d(\Psi_x(v), \Psi_x(w)).$$

Here we identify $\mathbb{R}^\tau$ $(\tau = u, s)$ with a subspace of $T\mathbb{R}^{\dim M}$.

In what follows, for $x \in \Lambda_l$ and $\delta \in (0, 1]$, we will write

$$R(x; \delta) = \Psi_x(\mathbb{D}(\delta q_l))$$

and call it a *rectangle*. In the case where the precise choice of the scaling parameter δ of the rectangle does not matter, we simply write it as $R(x)$.

It is well known that there exist families $\{\mathcal{W}^\tau_{\mathrm{loc}}(x)\}$ $(\tau = u, s)$ of smooth disks passing through $x \in \Lambda_l$ and positive numbers c_l, r_l, δ_l, A_l and B_l such that:

(i) $f^n(\Lambda_l) \subset \Lambda_q$ for some positive integer $q = q(l, n)$;

(ii) $\mathcal{W}^u(x) = \bigcup_{n=0}^\infty f^n(\mathcal{W}^u_{\mathrm{loc}}(f^{-n}(x)))$ and $\mathcal{W}^s(x) = \bigcup_{n=0}^\infty f^{-n}(\mathcal{W}^s_{\mathrm{loc}}(f^n(x)))$ for $x \in \Gamma$;

(iii) for each $x \in \Lambda_l$, the disk $\mathcal{W}^\tau_{\mathrm{loc}}(x)$ contains the closed ball centered at x of radius δ_l with respect to the induced distance d^τ_x on $\mathcal{W}^\tau(x)$;

(iv) for each $x \in \Lambda_l$, $y \in \Lambda_l \cap B(x, rc_l)$ and $r \in (0, r_l]$, $\mathcal{W}^\tau_{\mathrm{loc}}(y) \cap B(x, r)$ is connected, and the map

$$\Lambda_l \cap B(x, r_l c_l) \ni y \mapsto \mathcal{W}^\tau_{\mathrm{loc}}(y) \cap B(x, r_l)$$

is continuous with respect to the Hausdorff metric on the space of all closed subsets of $B(x, r_l)$;

(v) if $y \in \mathcal{W}^u_{\mathrm{loc}}(x)$ and $x \in \Lambda_l$, then, for every $n \geq 0$,

$$d^u_{f^{-n}(x)}(f^{-n}(y), f^{-n}(x)) \leq A_l e^{-nB_l} d^u_x(y, x),$$

and if $y \in \mathcal{W}^s_{\mathrm{loc}}(x)$ and $x \in \Lambda_l$, then, for every $n \geq 0$,

$$d^s_{f^n(x)}(f^n(y), f^n(x)) \leq A_l e^{-nB_l} d^s_x(y, x)$$

(see, for example, [**21**]). Here we denote the ball in M centered at x of radius r by $B(x, r)$.

2.3. *Admissible manifolds.* We recall the notion of admissible manifolds (see [**19**]). Let $x \in \Lambda_l$, $\delta \in (0, 1]$ and $\gamma \in (0, 1)$. A d^u-manifold $\mathbb{V} \subset \mathbb{D}(\delta q_l)$ is called a *γ-admissible unstable manifold near zero* if $\mathbb{V} = \mathrm{graph}\ \psi^u = \{(v, \psi^u(v)) : v \in \mathbb{D}^u(\delta q_l)\}$, where $\psi^u : \mathbb{D}^u(\delta q_l) \to \mathbb{D}^s(\delta q_l)$ is a C^1 map such that $|\psi^u(0)| \leq q_\varepsilon(x)/10$ and $|D\psi^u| \leq \gamma$. A d^s-manifold $\mathbb{V} \subset \mathbb{D}(\delta q_l)$ called a *γ-admissible stable manifold near zero* is defined similarly by reversing the roles of stable and unstable directions.

A d^u-manifold $V \subset M$ is called a *γ-admissible unstable manifold near x* if $V = \Psi_x(\mathbb{V})$ for some γ-admissible unstable manifold $\mathbb{V}$ near zero. A d^s-manifold $V \subset M$ called a *γ-admissible stable manifold near x* is defined similarly. Denote the family of all γ-admissible unstable manifolds near x by $\mathcal{U}_{\delta,\gamma}(x)$ and the family of all γ-admissible stable manifolds near x by $\mathcal{S}_{\delta,\gamma}(x)$.

LEMMA 2.1. [**19**, Proposition 2.5 and Corollary 2.2] *Let $x \in \Lambda_l$. Then any admissible unstable manifold near x intersects any admissible stable manifold near x transversely and in exactly one point.*

Moreover, for $\delta > \delta' > 0$ and $\gamma' > \gamma > 0$, there is a constant $\kappa = \kappa(\Lambda_l) > 0$ such that if $y \in \Lambda_l$ with $d(x, y) < \kappa$ and $V \in \mathcal{U}_{\delta,\gamma}(x)$, then

$$V \cap R(y; \delta') \in \mathcal{U}_{\delta',\gamma'}(y).$$

2.4. *Absolute continuity.* In what follows, given $x \in \Lambda_l$ and $r \in (0, c_l r_l]$, we set

$$\mathcal{V}^s(y) = \mathcal{W}^s_{\mathrm{loc}}(y) \cap B(x, r_l)$$

for $y \in \Lambda_l \cap B(x, r)$, where c_l and r_l come from §2.2. Set

$$Q^s(x) = \bigcup_{y \in \Lambda_l \cap B(x,r)} \mathcal{V}^s(y). \tag{2.1}$$

Note that if $\mathcal{V}^s(y) \cap \mathcal{V}^s(y') \neq \emptyset$, then $\mathcal{V}^s(y) = \mathcal{V}^s(y')$. Thus, for $z \in Q^s(x)$, we will write

$$\mathcal{V}^s(z) = \mathcal{V}^s(y), \tag{2.2}$$

where $y \in \Lambda_l \cap B(x, r)$ with $z \in \mathcal{V}^s(y)$ (if no ambiguity arises).

Let

$$\mathcal{L}^s(x) = \{\mathcal{V}^s(y) : y \in \Lambda_l \cap B(x, r)\}$$

and let W_1, W_2 be two transversals to the family $\mathcal{L}^s(x)$. Using (2.2), we can write the family $\mathcal{L}^s(x)$ in the form

$$\mathcal{L}^s(x) = \{\mathcal{V}^s(z) : z \in Q^s(x)\}.$$

Then the homeomorphism, called a *holonomy map*, is defined as

$$h : Q^s(x) \cap W_1 \to Q^s(x) \cap W_2 \tag{2.3}$$

by setting

$$h(z) = \mathcal{V}^s(z) \cap W_2$$

for $z \in Q^s(x) \cap W_1$.

The holonomy map h is said to be *absolutely continuous* if it transforms Lebesgue zero sets of W_1 to Lebesgue zero sets of W_2: that is, if the inherited volume m_2 on W_2 is absolutely continuous with respect to $h_* m_1$.

LEMMA 2.2. [**22**], [**4**, §8.6.1] *The holonomy map h associated to the stable lamination $\mathcal{W}^s$ is absolutely continuous and the Jacobian $dm_2/d(h_* m_1)$ of the holonomy map is bounded from above and bounded away from zero.*

2.5. *SRB measures.* Let $\mathcal{B}$ be the Borel σ-algebra of M completed with respect to μ and ξ a partition of M. Denote by $\mathcal{B}_\xi$ the sub σ-algebra of $\mathcal{B}$ whose elements are unions of elements of ξ. A countable system $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{B}$ is said to be a *basis* of ξ if, for any two distinct elements C_1, C_2 of ξ, there exists A_{i_0} such that, up to sets of measure zero, either $C_1 \subset A_{i_0}$ and $C_2 \not\subset A_{i_0}$ or $C_1 \not\subset A_{i_0}$ and $C_2 \subset A_{i_0}$. A partition with a basis is said to be *measurable*.

We denote by $C_\xi(x)$ the element of ξ containing $x \in M$. For a measurable partition ξ of M, there exists a *canonical system of conditional measures*: for μ-almost every $x \in M$, there is a probability measure μ_x^ξ defined on $C_\xi(x)$ such that the function $x \mapsto \mu_x^\xi(A)$ is $\mathcal{B}_\xi$-measurable and $\mu(A) = \int \mu_x^\xi(A)\, d\mu(x)$ for every $A \in \mathcal{B}$ (see [**29**] for more details).

Let $\mathcal{W}^u = \{\mathcal{W}^u(x) : x \in \Lambda\}$ be the unstable lamination and let ξ^u a measurable partition of M. We say that ξ^u is *subordinate* to the $\mathcal{W}^u$-lamination if, for μ-almost every $x \in M$, $C_{\xi^u}(x) \subset \mathcal{W}^u(x)$ and $C_{\xi^u}(x)$ contains an open neighborhood of x in $\mathcal{W}^u(x)$.

The measure μ is said to be *absolutely continuous with respect to the* $\mathcal{W}^u$*-lamination* or *Sinaĭ–Ruelle–Bowen* (*SRB*) if, for every measurable partition ξ^u subordinate to the $\mathcal{W}^u$-lamination, $\mu_x^{\xi^u}$ is absolutely continuous with respect to m_x^u for μ-almost every $x \in M$, where m_x^u denotes the inherited volume on $\mathcal{W}^u(x)$. It is known that the densities $d\mu_x^{\xi^u}/dm_x^u$ are strictly positive along $\mathcal{W}^u(x)$ for μ-almost every $x \in M$.

LEMMA 2.3. [**20**, **22**] *Let μ be a hyperbolic SRB probability measure invariant under f. Then there exist finite or countably many invariant sets $\Lambda_0, \Lambda_1, \ldots$ such that:*

(1) $\bigcup_{i\geq 0} \Lambda_i = M$ *and* $\Lambda_i \cap \Lambda_j = \emptyset$ *whenever* $i \neq j$;

(2) $\mu(\Lambda_0) = 0$, *and* $\mu(\Lambda_i) > 0$ *for each* $i \geq 1$;

(3) $f|\Lambda_i$ *is ergodic with respect to* $\nu_i(\cdot) = \mu(\cdot \cap \Lambda_i)/\mu(\Lambda_i)$ *for each* $i \geq 1$; *and*

(4) *there is a sequence of numbers* n_i $(i \geq 1)$ *and there are measurable sets* Λ_i^k $(k = 1, 2, \ldots, n_i)$ *such that:*
   - (a) $\bigcup_{k=1}^{n_i} \Lambda_i^k = \Lambda_i$ *and* $\Lambda_i^{k_1} \cap \Lambda_j^{k_2} = \emptyset$ *whenever* $k_1 \neq k_2$;
   - (b) $f(\Lambda_i^k) = \Lambda_i^{k+1}$ $(k = 1, \ldots, n_i - 1)$, $f(\Lambda_i^{n_i}) = \Lambda_i^1$;
   - (c) $(f^{n_i}|\Lambda_i^k, n_i\nu_i|\Lambda_i^k)$ *is isomorphic to a Bernoulli automorphism.*

Let $\mathcal{M}_f$ be the set of all f-invariant Borel probability measures on M and let $\mu \in \mathcal{M}_f$ be a hyperbolic SRB measure. Using the notation as in Lemma 2.3, we define $\mathcal{E}(\mu) = \{\nu_i\}_{i\geq 1}$. Then there exist positive numbers $\{a_\nu\}_{\nu \in \mathcal{E}(\mu)}$ with $\sum_{\nu \in \mathcal{E}(\mu)} a_\nu = 1$ such that $\mu(A) = \sum_{\nu \in \mathcal{E}(\mu)} a_\nu \nu(A)$ for any Borel set A.

3. *Proofs of Theorems 1.1 and 1.2*

3.1. *AAP and TAAP.*

Recall that two submanifolds V and W of M are called *transverse* at z provided that either $z \notin V \cap W$ or $T_zV + T_zW = T_zM$. We denote this by $V \pitchfork_z W$.

For $p, q \in \mathbb{Z}$ and $(x, y) \in \Gamma^\nu \times \Gamma^\varrho$, define

$$I_{p,q}(x, y) = \mathcal{W}^u(f^p(x)) \cap \mathcal{W}^s(f^q(y))$$

and its subset

$$T_{p,q}(x, y) = \{z \in I_{p,q}(x, y) : \mathcal{W}^u(f^p(x)) \pitchfork_z \mathcal{W}^s(f^q(y))\}.$$

Here we remark that if $x \neq y$, then $I_{p,q}(x, y) \neq I_{q,p}(y, x)$ and hence $T_{p,q}(x, y) \neq T_{q,p}(y, x)$, in general.

Put

$$\mathcal{A}^{\nu,\varrho} = \bigcup_{p,q\in\mathbb{Z}} \{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : I_{p,q}(x, y) \neq \emptyset\}$$

and

$$\mathcal{T}_0^{\nu,\varrho} = \bigcup_{p,q\in\mathbb{Z}} \{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : T_{p,q}(x, y) \neq \emptyset\}.$$

Note again that given ν and ϱ ($\nu \neq \varrho$), the set $\mathcal{A}^{\varrho,\nu}$ differs from $\mathcal{A}^{\nu,\varrho}$. We know that $\mathcal{T}_0^{\nu,\varrho} \subset \mathcal{A}^{\nu,\varrho}$ and that both sets are $f^n \times f^m$-invariant for every $n, m \in \mathbb{Z}$. Thus, for every $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$, one can check by (3) and (4) of Lemma 2.3 that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 0$ or 1 and that $\nu \otimes \varrho(\mathcal{T}_0^{\nu,\varrho}) = 0$ or 1. Moreover, we note that μ satisfies the AAP if and only if $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$ or $\varrho \otimes \nu(\mathcal{A}^{\varrho,\nu}) = 1$ for every $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ and that it satisfies the TAAP if and only if $\nu \otimes \varrho(\mathcal{T}_0^{\nu,\varrho}) = 1$ or $\varrho \otimes \nu(\mathcal{T}_0^{\varrho,\nu}) = 1$ for every $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$.

LEMMA 3.1. [**13**, Lemma 3.2 and Remark 3.3] *If $\nu, \varrho \in \mathcal{E}(\mu)$ are such that $\nu \otimes \varrho(\mathcal{T}_0^{\nu,\varrho}) = 1$ or $\varrho \otimes \nu(\mathcal{T}_0^{\varrho,\nu}) = 1$, then $\nu = \varrho$.*

Let us define

$$\mathcal{T}_1^{\nu,\varrho} = \bigcap_{p,q \in \mathbb{Z}} \{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : T_{p,q}(x, y) = I_{p,q}(x, y)\}.$$

Then $\mathcal{T}_1^{\nu,\varrho}$ is $f^n \times f^m$-invariant for every $n, m \in \mathbb{Z}$.

For $\nu \in \mathcal{E}(\mu)$, the dimension of the unstable manifold is constant ν-almost everywhere. We denote the constant by $d^u(\nu)$ and set $d^s(\nu) = \dim M - d^u(\nu)$. Note that $d^s(\nu) = \dim \mathcal{W}^s(x)$ for ν-almost every x since ν is hyperbolic.

3.2. *The case of $d^u(\nu) \leq d^u(\varrho)$.*

PROPOSITION 3.2. *Let μ be a hyperbolic SRB Borel probability measure. Then $\nu \otimes \varrho(\mathcal{T}_1^{\nu,\varrho}) = 1$ for every pair $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ such that $d^u(\nu) \leq d^u(\varrho)$.*

Proof. This is proved in [**13**, Proposition 4.1] for every pair $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ such that $d^u(\nu) = d^u(\varrho)$.

Next, for every pair $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ such that $d^u(\nu) < d^u(\varrho)$, we see that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 0$ and hence $\nu \otimes \varrho(\mathcal{T}_1^{\nu,\varrho}) = 1$, trivially. Indeed, suppose that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$. Then there exists X_ν of full ν-measure such that, for every $x \in X_\nu$, it holds that $\varrho(\mathcal{A}_x^{\nu,\varrho}) = 1$, where we set $\mathcal{A}_x^{\nu,\varrho} = \{y \in \Gamma^\varrho : (x, y) \in \mathcal{A}^{\nu,\varrho}\}$ for $x \in X_\nu$. Fix $x \in X_\nu$ in the following. Since ϱ is an SRB measure,

$$\varrho(\mathcal{A}_x^{\nu,\varrho}) = \int \varrho_y^{\xi^u}(\mathcal{A}_x^{\nu,\varrho})\, d\varrho(y),$$

where ξ^u is a measurable partition subordinate to the $\mathcal{W}^u$-lamination. It follows that $\varrho_y^{\xi^u}(\mathcal{A}_x^{\nu,\varrho}) = 1$ for ϱ-almost every $y \in \mathcal{A}_x^{\nu,\varrho}$, and thereby $m_y^u(\mathcal{A}_x^{\nu,\varrho}) > 0$. However, it holds that $m_y^u(\{b \in \Gamma^\varrho : I_{p,q}(x, b) \neq \emptyset\}) = 0$ for every $p, q \in \mathbb{Z}$, since $d^u(\nu) < d^u(\varrho)$. This can be proved by the same argument as in [**13**, §4.1]. For the sake of completeness, we give a sketch of the proof. Fix any $p, q \in \mathbb{Z}$, take a $d^u(\varrho)$-dimensional disk $V^u(z)$ for every $z \in I_{p,q}(x, b)$ at which $V^u(z)$ intersects transversely with $\mathcal{W}^s(b)$ and consider a projection π_z^s along the stable manifolds (at Γ^ϱ) from $B^u(z, r) \subset \mathcal{W}^u(x)$ onto $V^u(z)$. Since $d^u(\nu) < d^u(\varrho)$, we see that there exists a constant $C > 0$ such that, for any $\gamma > 0$, there exists $r = r(z) > 0$ such that $m_z^u(\pi_z^s(B^u(z, r) \cap Q^s(b))) \leq C\gamma$ (see (2.1) for the definition of $Q^s(b)$). By using the Besicovitch covering lemma for a cover $\{B^u(z, r) : z \in I_{p,q}(x, b), b \in \Gamma^\varrho \cap \mathcal{W}^u(y)\}$, we obtain that $m_y^u(\{b \in \Gamma^\varrho : I_{p,q}(x, b) \neq \emptyset\}) = 0$ (see [**13**, Lemmas 4.1 and 4.3]).

Therefore $m^u_y(\mathcal{A}^{\nu,\varrho}_x) = m^u_y(\bigcup_{p,q\in\mathbb{Z}}\{b \in \Gamma^\varrho : I_{p,q}(x, b) \neq \emptyset\}) = 0$. This is a contradiction and $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$ is impossible. □

When $\mathcal{E}(\mu) \ni \nu \mapsto d^u(\nu)$ is constant, we have the following theorem.

THEOREM 3.3. [**13**, Theorem 1.2] *Let μ be a hyperbolic SRB Borel probability measure. If μ satisfies the AAP and the dimension of the unstable manifold is constant μ-almost everywhere, then it is ergodic.*

3.3. *Tangential intersection.* We first recall the notion of tangency. Let V and W be submanifolds of M. If the transverse property fails for V and W, that is, if there exists $z \in V \cap W$ such that

$$T_z V + T_z W \subsetneq T_z M,$$

then we say that z is a *point of tangential intersection* of V and W or that V and W have a *tangency* at z.

LEMMA 3.4. *Let μ be a hyperbolic SRB Borel probability measure. Then for $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$,*

$$\nu \otimes \varrho(\{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : m^u_{f^p(x)}(I_{p,q}(x, y) \setminus T_{p,q}(x, y)) > 0\}) = 0.$$

Proof. For notational simplicity, set

$$Z = \{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : m^u_{f^p(x)}(I_{p,q}(x, y) \setminus T_{p,q}(x, y)) > 0\}.$$

Suppose, on the contrary, that $\nu \otimes \varrho(Z) > 0$. Then there exists a positive ν-measure set $X \subset \Gamma^\nu$ such that, for every $x \in X$,

$$\varrho(\{y \in \Gamma^\varrho : (x, y) \in Z\}) > 0.$$

Note that, for every $x \in X$ and $y \in Z_x = \{y \in \Gamma^\varrho : (x, y) \in Z\}$, $m^u_{f^p(x)}(I_{p,q}(x, y) \setminus T_{p,q}(x, y)) > 0$ and $T_z\mathcal{W}^u(f^p(x)) + T_z\mathcal{W}^s(f^q(y)) \subsetneq T_z M$ for every $z \in I_{p,q}(x, y) \setminus T_{p,q}(x, y)$.

By the absolute continuity of an unstable lamination, there exists $X_0 \subset X$ with $\nu(X \setminus X_0) = 0$ such that, for every $x \in X_0$, $m^u_{f^p(x)}(\mathcal{W}^u(f^p(x)) \setminus \Lambda^\nu) = 0$. Note that $T_z\mathcal{W}^u(z) \oplus T_z\mathcal{W}^s(z) = T_z M$ for every $z \in \Lambda^\nu$, since μ is hyperbolic, and that $m^u_{f^p(x)}((I_{p,q}(x, y) \setminus T_{p,q}(x, y)) \cap \Lambda^\nu) > 0$ for every $x \in X_0$ and $y \in Z_x$.

Take $x \in X_0$ and $y \in Z_x$. Since $\mathcal{W}^u(f^p(x)) = \mathcal{W}^u(z)$ and $\mathcal{W}^s(f^q(y)) = \mathcal{W}^s(z)$ for every $z \in (I_{p,q}(x, y) \setminus T_{p,q}(x, y)) \cap \Lambda^\nu$, $T_z\mathcal{W}^u(f^p(x)) \oplus T_z\mathcal{W}^s(f^q(y)) = T_z M$. We arrive at a contradiction. □

3.4. *The case of $d^u(\varrho) = 1$.* The following is a counterpart of Proposition 3.2. Then, using this result, we show Theorems 1.1 and 1.2 in §3.5.

PROPOSITION 3.5. *Let μ be a hyperbolic SRB Borel probability measure. Then $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \setminus \mathcal{T}^{\nu,\varrho}_0) = 0$ for every pair $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ such that $d^u(\varrho) = 1$.*

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Proof. Note, first, that if $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 0$, then $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \setminus \mathcal{T}_0^{\nu,\varrho}) = 0$, trivially. Thus it is enough to show that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \setminus \mathcal{T}_0^{\nu,\varrho}) = 0$ for (ν, ϱ) satisfying $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$. Suppose, on the contrary, that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \setminus \mathcal{T}_0^{\nu,\varrho}) = 1$ while $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$. Then there exist $p, q \in \mathbb{Z}$ such that $\nu \otimes \varrho(\{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : I_{p,q}(x, y) \neq \emptyset, T_{p,q}(x, y) = \emptyset\}) > 0$. We divide the rest of the proof into two parts.

Step 1. Let

$$A = \left\{(x, y) \in \Gamma^\nu \times \Gamma^\varrho : \begin{matrix} I_{0,0}(x, y) \neq \emptyset, \quad T_{0,0}(x, y) = \emptyset, \\ m_x^u(I_{0,0}(x, y) \setminus T_{0,0}(x, y)) = 0. \end{matrix}\right\}$$

Then $a = \nu \otimes \varrho(A) > 0$ by Lemma 3.4 and the $f^n \times f^m$-invariance of A. For any $\kappa > 0$, there exists $X(\kappa) \subset \Gamma^\nu$ with $\nu(X(\kappa)) > 0$ such that, for any $x \in X(\kappa)$,

$$\varrho(\{y \in \Gamma^\varrho : I_{0,0}(x, y) \cap B^s(y, \kappa) \neq \emptyset,\ (x, y) \in A\}) > a/2. \tag{3.1}$$

Indeed, by §2.2-(v) and the Fubini theorem, there exist $N = N(\kappa) \in \mathbb{N}$ and $\tilde{X} = \tilde{X}(\kappa) \subset \Gamma^\nu$ with $\nu(\tilde{X}) > 0$ such that, for any $n \geq N$ and $x \in \tilde{X}$,

$$\varrho(\{y \in \Gamma^\varrho : I_{n,n}(x, y) \cap B^s(f^n(y), \kappa) \neq \emptyset,\ (f^n(x), f^n(y)) \in A\}) > a/2.$$

Put $X(\kappa) = f^n(\tilde{X})$, which is the desired set.

Take $l > 1$ so large that $\varrho(\Lambda_l^\varrho) > 1 - (a/3)$ and fix $x \in X(\kappa_l)$, where $\kappa_l = \min\{q_l, r_l\}/2$ (see §2.2 for q_l and r_l). Consider

$$Y_l = Y_l(x) = \{y \in \Lambda_l^\varrho : I_{0,0}(x, y) \cap B^s(y, \kappa_l) \neq \emptyset,\ (x, y) \in A\}.$$

By (3.1), we may assume that $\varrho(Y_l) > 0$.

Let ξ^u be a measurable partition subordinate to the $\mathcal{W}^u$-lamination. Then

$$\int \varrho_y^{\xi^u}(Y_l)\, d\varrho(y) = \varrho(Y_l).$$

It follows that $\varrho_y^{\xi^u}(Y_l) > 0$ for ϱ-almost every $y \in Y_l$, which means that $m_y^u(Y_l) > 0$ for ϱ-almost every $y \in Y_l$, since ϱ is an SRB measure. Define

$$Y_l^u = \left\{y \in Y_l : \lim_{r \to 0} \frac{m_y^u(B^u(y, r) \cap Y_l)}{m_y^u(B^u(y, r))} = 1\right\}. \tag{3.2}$$

By the Lebesgue density lemma, $\varrho(Y_l^u) = \varrho(Y_l) > 0$.

Step 2. Take $y_1 \in Y_l^u$. Recall that $R(y_1; 1) = \Psi_{y_1}(\mathbb{D}(q_l))$, where $\Psi_{y_1} : \mathbb{D}(q_l) \to M$ is as in §2.2. Below, for notational simplicity, we will put $R(y_1) = R(y_1; 1)$.

Let $Q^s(y_1)$ be as in (2.1). By §2.2-(iv), for $y \in \Lambda_l \cap B(y_1, r_l c_l)$, there exists a C^1 function $\psi_y^s : \mathbb{D}^s(q_l) \to \mathbb{D}^u(q_l)$ such that

$$\Psi_{y_1}^{-1}(\mathcal{V}^s(y)) = \{(\psi_y^s(v), v);\ v \in \mathbb{D}^s(q_l)\} \subset \mathbb{R}^u \times \mathbb{R}^s$$

and the map $y \mapsto \psi_y^s$ is continuous with respect to the C^1 topology.

Let $z \in I_{0,0}(x, y_1) \cap B^s(y_1, \kappa_l)$. Then:

- $T_z\mathcal{W}^u(x) \subset T_z\mathcal{W}^s(y_1)$; and
- $m_x^u(\mathcal{W}^u(x) \cap \mathcal{W}^s(y_1)) = 0$,

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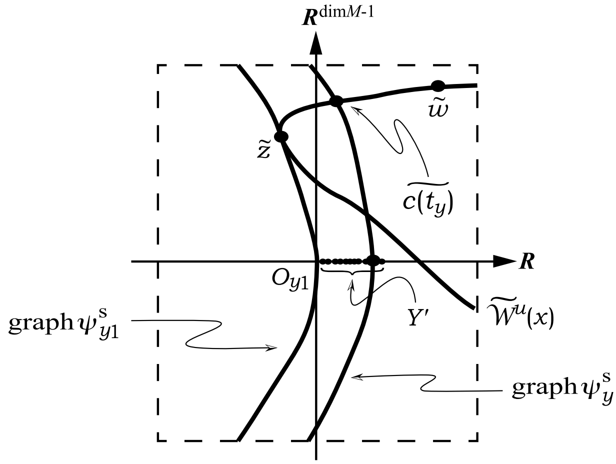

FIGURE 1. For the proof of Proposition 3.5. The rectangle denotes the Lyapunov chart $R(y_1) \subset \mathbb{R}^u \times \mathbb{R}^s = \mathbb{R} \times \mathbb{R}^{\dim M-1}$ at $y_1 \in Y^u_l$. We write $\tilde{a} = \Psi^{-1}_{y_1}(a)$ for $a \in R(y_1)$ and $\widetilde{\mathcal{W}}^u(x) = \Psi^{-1}_{y_1}(\mathcal{W}^u(x) \cap R(y_1))$ in this picture. Here $z \in I_{0,0}(x, y_1)$ is a point of tangential intersection of $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y_1)$.

since $d^s(\varrho) = \dim M - 1$ and $(x, y_1) \in A$. Thus there exist a point $w \in \mathcal{W}^u(x) \setminus \mathcal{W}^s(y_1)$ and a C^1 curve $c : [0, 1] \to \mathcal{W}^u(x)$ such that

$$c(0) = z, \quad c(1) = w \text{ and } c(t) \in R(y_1) \text{ for every } t \in [0, 1]. \tag{3.3}$$

Since $z \in B^s(y_1, \kappa_l) \subset \mathcal{V}^s(y_1)$, $\tilde{z}_1 = \psi^s_{y_1}(\tilde{z}_2)$, where $\Psi^{-1}_{y_1}(z) = (\tilde{z}_1, \tilde{z}_2) \in \mathbb{D}(q_l) \subset \mathbb{R}^u \times \mathbb{R}^s$. Without loss of generality, we may assume that

$$\tilde{w}_1 > \psi^s_{y_1}(\tilde{w}_2),$$

where $\Psi^{-1}_{y_1}(w) = (\tilde{w}_1, \tilde{w}_2) \in \mathbb{D}(q_l)$.

Since $y_1 \in Y^u_l$, we can take a sufficiently small $r > 0$ and a subset $Y' \subset Y_l \cap B^u(y_1, r)$ with positive m^u-measure such that

$$\tilde{w}_1 > \psi^s_y(\tilde{w}_2) \quad \text{and} \quad \tilde{z}_1 < \psi^s_y(\tilde{z}_2)$$

for every $y \in Y'$. By (3.3) and the intermediate value theorem, there exists $0 < t_y < 1$ such that

$$\widetilde{c(t_y)}_1 = \psi^s_y(\widetilde{c(t_y)}_2),$$

where $\Psi^{-1}_{y_1}(c(t_y)) = (\widetilde{c(t_y)}_1, \widetilde{c(t_y)}_2)$. This means that $c(t_y) \in \mathcal{W}^u(x) \cap \mathcal{W}^s(y)$.

Since the dimension of the curve c is equal to $d^u(\varrho)(= 1)$, by the same argument as in the proof of [**13**, Proposition 4.1], the curve c has a transverse intersection point at $c(t_y)$ for m^u-almost every $y \in Y'$ (see Figure 1). This implies that

$$T_{c(t_y)}\mathcal{W}^u(x) + T_{c(t_y)}\mathcal{W}^s(y) = T_{c(t_y)}M$$

for m^u-almost every $y \in Y'$, which contradicts the fact that $T_{0,0}(x, y) = \emptyset$ for every $y \in Y_l = Y_l(x)$. □

3.5. *Proofs.* Here we prove Theorems 1.1 and 1.2. In both of the proofs, we consider $\mathcal{A}^{\nu,\varrho}$, $\mathcal{T}_1^{\nu,\varrho}$ and $\mathcal{T}_0^{\nu,\varrho}$ corresponding to possible cases of pairs $(d^u(\nu), d^u(\varrho))$. Recall here that these sets are not symmetric with respect to ν and ϱ. This causes the asymmetry in the argument below.

Proof of Theorem 1.1. Let μ be a hyperbolic SRB measure satisfying the AAP. Then every pair of ergodic components $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ satisfies that either $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$ or $\varrho \otimes \nu(\mathcal{A}^{\varrho,\nu}) = 1$. First, we consider the case $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$.

When the manifold M is two dimensional, there is only one case for the pair, $(d^u(\nu), d^u(\varrho)) = (1, 1)$. Thus, by Theorem 3.3, μ is ergodic.

When the manifold M is three dimensional, there are four cases for the pair of $(d^u(\nu), d^u(\varrho)) = (i, j)$, where $i, j \in \{1, 2\}$. By Proposition 3.2, $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \cap \mathcal{T}_1^{\nu,\varrho}) = 1$ when $(d^u(\nu), d^u(\varrho)) = (1, 1)$, $(1, 2)$ and $(2, 2)$. By applying Proposition 3.5 to the remaining case $(d^u(\nu), d^u(\varrho)) = (2, 1)$, $\nu \otimes \varrho(\mathcal{T}_0^{\nu,\varrho}) = 1$. Therefore it follows from Lemma 3.1 that $\nu = \varrho$ in every case, which means that μ itself is ergodic.

The proof of the other case $\varrho \otimes \nu(\mathcal{A}^{\varrho,\nu}) = 1$ is similar and is omitted. □

Proof of Theorem 1.2. Let μ be a hyperbolic smooth measure satisfying the AAP. Then every pair of ergodic components $(\nu, \varrho) \in \mathcal{E}(\mu) \times \mathcal{E}(\mu)$ satisfies that either $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$ or $\varrho \otimes \nu(\mathcal{A}^{\varrho,\nu}) = 1$. First, we consider the case $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$. By Theorem 1.1, it is enough to consider the case where the dimension of M is four.

Since the manifold M is four dimensional, there are nine cases for the pair of $(d^u(\nu), d^u(\varrho)) = (i, j)$, where $i, j \in \{1, 2, 3\}$. By Proposition 3.2, $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \cap \mathcal{T}_1^{\nu,\varrho}) = 1$ when $(d^u(\nu), d^u(\varrho)) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$, $(2, 3)$, $(3, 3)$. By Proposition 3.5, $\nu \otimes \varrho(\mathcal{T}_0^{\nu,\varrho}) = 1$ when $(d^u(\nu), d^u(\varrho)) = (2, 1)$, $(3, 1)$. Henceforth, we consider the remaining case $(d^u(\nu), d^u(\varrho)) = (3, 2)$. Since both ν and ρ are absolutely continuous with respect to the volume, they are absolutely continuous with respect to the stable lamination (see [**20**, Corollaire 5.6]). Since $(d^s(\varrho), d^s(\nu)) = (2, 1)$, by applying Proposition 3.5, $\nu \otimes \varrho(\mathcal{T}_0^{\nu,\varrho}) = 1$. Therefore it follows from Lemma 3.1 that $\nu = \varrho$ in every case, which means that μ itself is ergodic.

The proof of the other case $\varrho \otimes \nu(\mathcal{A}^{\varrho,\nu}) = 1$ is similar and is omitted. □

4. *Proofs of Theorems 1.3 and 1.4*

In what follows, we give the proof of Theorem 1.3 only. The proof of Theorem 1.4 is the same as that of Theorem 1.3, except for one part. See Remark 4.4 about the modification for proving Theorem 1.4.

4.1. *Set-up.* Let ν and ϱ be any ergodic components of μ. By ergodicity, there exist positive integers $d^u(\nu)$ and $d^u(\varrho)$ such that $\dim \mathcal{W}^u(x) = d^u(\nu)$ for ν-almost every $x \in M$ and $\dim \mathcal{W}^u(x) = d^u(\varrho)$ for ϱ-almost every $x \in M$, respectively. In the following sections, we consider possible cases of pairs $(d^u(\nu), d^u(\varrho))$, respectively, and show, in each case, that there exist subsets A_ν and A_ϱ with $\nu(A_\nu) > 0$ and $\varrho(A_\varrho) > 0$ such that $A_\nu \times A_\varrho \subset \mathcal{A}^{\nu,\varrho}$ or $A_\varrho \times A_\nu \subset \mathcal{A}^{\varrho,\nu}$. This will imply that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho}) = 1$ or $\varrho \otimes \nu(\mathcal{A}^{\varrho,\nu}) = 1$, that is, that μ satisfies the AAP.

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4.1.1. *An auxiliary lemma.* To begin with, fix the notation. All through this subsection 4.1.1, dim M, $d^u(\nu)$ and $d^u(\varrho)$ are arbitrary. Choose $l > 1$ so large that $\nu(\Lambda_l) > 0$ and take $x_0 \in \mathrm{supp}(\nu|\Lambda_l)$. Here we denote by $\mathrm{supp}(\nu|\Lambda_l)$ the support of $\nu|\Lambda_l$. Thus $\nu(B(x_0, r) \cap \Lambda_l) > 0$ for a given small number $r > 0$. Denote by $B(\nu)$ the ergodic basin of ν defined as

$$B(\nu) = \Big\{x \in M : \lim_{n\to\pm\infty} \delta_n(x) = \nu\Big\},$$

where $\delta_n(x) = \sum_{i=0}^{n-1} \delta(f^{-i}(x))/n$ and $\delta(y)$ denotes the Dirac measure at $y \in M$. Since $\nu(B(\nu)) = 1$ by the Birkhoff ergodic theorem, $\nu(B(x_0, r) \cap \Lambda_l \cap B(\nu)) = \nu(B(x_0, r) \cap \Lambda_l) > 0$. Thus by the Poincaré recurrence theorem, there exists a subset $A_\nu \subset B(x_0, r) \cap \Lambda_l \cap B(\nu)$ with $\nu(A_\nu) = \nu(B(x_0, r) \cap \Lambda_l \cap B(\nu))$ such that every $x \in A_\nu$ returns infinitely often to A_ν under both forward and backward iteration of f. That is, for every $x \in A_\nu$ there exists an increasing sequence $n_k = n_k(x)$ that tends to $\pm\infty$ as $k \to \pm\infty$ such that $f^{n_k}(x) \in A_\nu$ for all $k \in \mathbb{Z}$.

Similarly, choose $l > 1$ so large that $\varrho(\Lambda_l) > 0$ and take $y_0 \in \mathrm{supp}(\varrho|\Lambda_l)$. Then there is a positive ϱ-measure set $A_\varrho \subset B(y_0, r) \cap \Lambda_l \cap B(\varrho)$ such that, for every $y \in A_\varrho$, there exists an increasing sequence $m_k = m_k(y)$ that tends to $\pm\infty$ as $k \to \pm\infty$ such that $f^{m_k}(y) \in A_\varrho$ for all $k \in \mathbb{Z}$.

For $x \in B(x_0, r) \cap \Lambda_l$, set $\mathcal{V}^\tau(x)$ as the component of $\mathcal{W}^\tau(x) \cap R(x_0)$ that contains x: that is,

$$\mathcal{V}^\tau(x) = \mathcal{V}^\tau_\nu(x) = C(\mathcal{W}^\tau(x) \cap R(x_0) : x),$$

where $\tau = s, u$. Similarly, for $y \in B(y_0, r) \cap \Lambda_l$ and $\tau = s, u$, set

$$\mathcal{V}^\tau(y) = \mathcal{V}^\tau_\varrho(y) = C(\mathcal{W}^\tau(y) \cap R(y_0) : y).$$

LEMMA 4.1. *For every $x \in A_\nu$, there exist $x_1, x_2 \in \{f^n(x)\}_{n\in\mathbb{Z}}$ such that $\mathcal{V}^u(x)$, $\mathcal{V}^u(x_1)$ and $\mathcal{V}^u(x_2)$ are mutually disjoint.*

Proof. Take $x \in A_\nu$ and an increasing sequence $n_k = n_k(x) \in \mathbb{Z}$ such that $f^{n_k}(x) \in A_\nu$, chosen by the Poincaré recurrence theorem, as above.

First, we show that there exists $k \in \mathbb{Z}$ such that $f^{n_k}(x) \in A_\nu \setminus \mathcal{V}^u(x)$. Suppose that $f^{n_k}(x) \in \mathcal{V}^u(x)$ for every $k \in \mathbb{Z}$. By the property (v) in §2.2, we see that

$$f^{-n_k}(\mathcal{V}^u(x)) \subsetneq \mathcal{V}^u(f^{-n_k}(x)) = \mathcal{V}^u(x)$$

and that $f^{-n_k}|\mathcal{V}^u(x) : \mathcal{V}^u(x) \to \mathcal{V}^u(x)$ is a contraction for large $n_k \gg 1$. Fix such an $n_k \in \mathbb{N}$. Then there there exists (unique) $x_* \in \mathcal{V}^u(x)$ such that:

- $f^{-n_k}(x_*) = x_*$; and
- for every $y \in \mathcal{V}^u(x)$, $f^{-n_k m}(y) \to x_*$ as $m \to \infty$.

Therefore

$$\delta_{n_k m}(x) \to \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta(f^{-j}(x_*))$$

weakly as $m \to \infty$. On the other hand, $\delta_{n_k m}(x) \to \nu$ weakly as $m \to \infty$, since $x \in A_\nu$. This gives a contradiction to the fact that ν is SRB. Therefore there exists $n_k \in \mathbb{Z}$ such that $f^{n_k}(x) \in A_\nu \setminus \mathcal{V}^u(x)$.

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By applying the same argument, we see that there exists $n_j \in \mathbb{Z}$ such that $f^{n_j+n_k}(x) \in A_\nu \setminus (\mathcal{V}^u(x) \cup \mathcal{V}^u(f^{n_k}(x)))$.

Set $x_1 = f^{n_k}(x)$ and $x_2 = f^{n_j+n_k}(x)$. Then $\mathcal{V}^u(x_1)$ and $\mathcal{V}^u(x_2)$ are well defined since both x_1 and x_2 belong to A_ν. It is clear that $\mathcal{V}^u(x)$, $\mathcal{V}^u(x_1)$ and $\mathcal{V}^u(x_2)$ are mutually disjoint. □

4.1.2. *The case of $d^u = 2$.* Suppose that $d^u(\nu) = 2$ in this subsection. By the C^1 continuity of the local unstable manifolds on Λ_l, every $\mathcal{V}^u(x)$ $(x \in A_\nu)$ is an admissible unstable two manifold near x_0. Since $\mathcal{V}^s(x_0)$ is an admissible stable one manifold near x_0, we see that every $\mathcal{V}^u(x)$ intersects $\mathcal{V}^s(x_0)$ transversely and in exactly one point, by Lemma 2.1. Therefore there is an ordering on $\{\mathcal{V}^u(x) : x \in A_\nu\}$ according to the canonical ordering on the one manifold $\mathcal{V}^s(x_0)$. In particular, we can specify the intermediate component among three elements from $\{\mathcal{V}^u(x) : x \in A_\nu\}$. More precisely, if $\mathcal{V}^u(x_i)$ $(x_i \in A_\nu, i = 1, 2, 3)$ are mutually disjoint and have the intersection point $\{p_i\} = \mathcal{V}^u(x_i) \cap \mathcal{V}^s(x_0)$ with $p_1 < p_2 < p_3$ according to the canonically given ordering $<$ on $\mathcal{V}^s(x_0)$, then $\mathcal{V}^u(x_2)$ is the intermediate one.

For an admissible two manifold $V = \Psi_{x_0}(\mathbb{V})$ such that $\mathbb{V} = \text{graph}\,\psi \subset \mathbb{R}^2 \times \mathbb{R}$, let

$$\mathbb{V}^+ = \{v = (v_1, v_2) \in \mathbb{D}(q_l) : v_2 > \psi(v_1)\},$$
$$\mathbb{V}^- = \{v = (v_1, v_2) \in \mathbb{D}(q_l) : v_2 < \psi(v_1)\},$$

where we write $v = (v_1, v_2)$ with respect to $\mathbb{D}^2 \times \mathbb{D}^1$.

LEMMA 4.2. *Suppose that $d^u(\nu) = 2$ and take $x \in A_\nu$. Then there exist $n \in \mathbb{Z}$ and a hyperbolic periodic point $z \in B(x_0, r)$ such that:*

- $\mathbb{V} = \Psi_{x_0}^{-1}(C(W^u(z) \cap R(x_0) : z))$ *is an admissible unstable two manifold; and*
- *for every connected subset $A \subset \mathbb{D}(q_l)$ so that $A \cap (\mathbb{V}^+ \cup \mathbb{V}^-) \neq \emptyset$ and $A \cap \mathbb{V} \neq \emptyset$, there exists $K > 0$ such that $f^{2kp}(\mathcal{W}^u(f^n(x))) \cap \Psi_{x_0}(A) \neq \emptyset$ for every $k \geq K$, where $p \in \mathbb{N}$ is the period of z.*

Proof. By Lemma 4.1, we can find two more local unstable manifolds $\mathcal{V}^u(x_1)$ and $\mathcal{V}^u(x_2)$ with $x_1, x_2 \in \{f^n(x)\}_{n\in\mathbb{Z}}$ such that $\mathcal{V}^u(x)$, $\mathcal{V}^u(x_1)$ and $\mathcal{V}^u(x_2)$ are mutually disjoint. Therefore we can specify the intermediate component among these three and denote it by $\mathcal{V}^u(\bar{x})$. (Hence $\bar{x}$ is the renamed point from one of x, x_1, x_2.) The others are denoted by $\mathcal{V}^u(x^+)$ and $\mathcal{V}^u(x^-)$, respectively.

Let $\varepsilon \in (0, \min_{\sigma\in\{+,-\}} d(\mathcal{V}^u(\bar{x}), \mathcal{V}^u(x^\sigma)))$. By the Katok closing lemma for non-uniformly hyperbolic systems [**19**, Main lemma and Theorem 4.1], there exists a hyperbolic periodic point $z \in B(\bar{x}, \varepsilon)$ of period $p \in \mathbb{N}$ such that the local stable and local unstable manifolds of z are admissible stable and admissible unstable manifolds near x_0. Denote these local manifolds by $V^\tau(z) = C(W^\tau(z) \cap R(x_0) : z)$ for $\tau = s, u$. Note that the component $V^u(z)$ is between $\mathcal{V}^u(x^+)$ and $\mathcal{V}^u(x^-)$. Note also that both $\mathcal{V}^u(x^+)$ and $\mathcal{V}^u(x^-)$ intersect $V^s(z)$ transversely (and in exactly one point, respectively), by Lemma 2.1. Denote them by z^+ and z^-, respectively.

Take any $A \subset \mathbb{D}(q_l)$ so that $A \cap (\mathbb{V}^+ \cup \mathbb{V}^-) \neq \emptyset$ and $A \cap \mathbb{V} \neq \emptyset$. Here we suppose that $A \cap \mathbb{V}^+ \neq \emptyset$ as the other case can be dealt with analogously. By the inclination lemma with respect to z, there is $K^+ \in \mathbb{N}$ such that

$$\Psi_{x_0}(A) \cap C(f^{2kp}(\mathcal{V}^u(x^+)) \cap R(x_0) : f^{2kp}(z^+)) \neq \emptyset \tag{4.1}$$

for every $k \geq K^+$. Since $x^+ \in \{f^n(x)\}_{n\in\mathbb{Z}}$, Lemma 4.2 is obtained. □

Remark 4.3. In fact, Lemma 4.2 is valid for the case $d^u = \dim M - 1$.

4.2. *The case of* $\dim M = 3$.

First, we consider the three-dimensional case.

4.2.1. *Case 1:* $(d^u(\nu), d^u(\varrho)) = (2, 1)$.

We divide the proof into three steps.

Step 1. Let A_ν be as above. Fix $x \in A_\nu$. Take a hyperbolic periodic point $z_\nu \in B(x_0, r)$ and an admissible unstable manifold $\mathbb{V}^u_\nu = \Psi^{-1}_{x_0}(V^u(z_\nu))$, as in Lemma 4.2.

Since $d^s(\varrho) = 2$, by Lemma 4.2 for $y \in A_\varrho$, we can take a hyperbolic periodic point $z_\varrho \in B(y_0, r)$ and an admissible stable manifold $\mathbb{V}^s_\varrho = \Psi^{-1}_{y_0}(V^s(z_\varrho))$ having the same (but 'dual') properties. Below we may assume that both z_ν and z_ϱ are fixed points.

Step 2. By condition (P), there exists $j \in \mathbb{Z}$ such that $W^u(z_\nu) \cap W^s(f^j(z_\varrho)) \neq \emptyset$. To avoid cumbersome notation, we consider $j = 0$.

Remark 4.4. Here we mention about the modification to prove Theorem 1.4. Recall first the notion of specification. We say that f satisfies the specification property if, for each $\delta > 0$, there is $N = N(\delta) \in \mathbb{N}$ such that, for any intervals $I_j \subset [a, b]$ $(j = 1, \ldots, n)$ of integers for some $a, b \in \mathbb{Z}$ with $d(I_j, I_k) \geq N(\delta)$ for $j \neq k$ and for $x_1, \ldots, x_n \in M$, there exists $p \in M$ such that $f^{b-a+N}(p) = p$ and $d(f^i(p), f^i(x_j)) < \delta$ for $i \in I_j$. Now, by the specification property, given $\delta > 0$, there exists $N = N(\delta) \in \mathbb{N}$ such that, for any $n \geq N$, there exists $p_n \in M$ such that $d(f^{-i}(p_n), z_\nu) < \delta$ and $d(f^{N+i}(p_n), z_\varrho) < \delta$ for every $i = 0, \ldots, n$. Since $\delta > 0$ can be chosen arbitrarily small, the limit point $p = \lim p_n$ belongs to both $W^u(z_\nu)$ and $W^s(z_\varrho)$. This means that $W^u(z_\nu) \cap W^s(z_\varrho) \neq \emptyset$. (This is the only place where we use the specification property to prove Theorem 1.4.)

There are two cases: $W^s(z_\varrho) \setminus \{z_\varrho\} \not\subset W^u(z_\nu)$ or $W^s(z_\varrho) \setminus \{z_\varrho\} \subset W^u(z_\nu)$. For the former case, there exists $q \in W^u(z_\nu) \cap W^s(z_\varrho) \cap R(x_0)$ such that the component

$$\mathbb{V}^s(q) = \mathbb{V}^s(z_\varrho : q) = \Psi^{-1}_{x_0}(C((W^s(z_\varrho) \setminus \{z_\varrho\}) \cap R(x_0) : q))$$

satisfies $\mathbb{V}^s(q) \cap (\mathbb{V}^u_\nu)^+ \neq \emptyset$ or $\mathbb{V}^s(q) \cap (\mathbb{V}^u_\nu)^- \neq \emptyset$. Therefore, by Lemma 4.2, there is $K \in \mathbb{N}$ such that $W^s(z_\varrho)$ intersects $f^{2k}(\mathcal{W}^u(x))$ for every $k \geq K$.

Next, for the latter case $W^s(z_\varrho) \setminus \{z_\varrho\} \subset W^u(z_\nu)$, we take another hyperbolic fixed point $w_\nu \in B(x_0, r) \setminus \{z_\nu\}$ by the Katok closing lemma. By repeating the argument in step 2, it should hold that $W^u(w_\nu) \cap W^s(z_\varrho) \neq \emptyset$. This is impossible since $W^s(z_\varrho) \setminus \{z_\varrho\} \subset W^u(z_\nu)$ and $W^u(w_\nu) \cap W^u(z_\nu) = \emptyset$.

Step 3. Fix any $k \geq K$ and take $a \in W^s(z_\varrho) \cap \mathcal{W}^u(f^{2k}(x))$. For $n \in \mathbb{N}$, let

$$\mathbb{V}^u(f^n(a)) = \Psi^{-1}_{y_0}(C(\mathcal{W}^u(f^{2k+n}(x)) \cap R(y_0) : f^n(a))).$$

Then $\mathbb{V}^u(f^n(a)) \cap (\mathbb{V}^s_\varrho)^+ \neq \emptyset$ or $\mathbb{V}^u(f^n(a)) \cap (\mathbb{V}^s_\varrho)^- \neq \emptyset$ for sufficiently large $n \in \mathbb{N}$. Fix any such an $n \in \mathbb{N}$. By Lemma 4.2, there is $J \in \mathbb{N}$ such that $\mathcal{W}^u(f^{2k+n}(x))$ intersects $f^{-2j}(\mathcal{W}^s(y))$ for every $j \geq J$. Therefore we obtain that, for any $(x, y) \in A_\nu \times A_\varrho$, there exist $r, t \in \mathbb{Z}$ such that $\mathcal{W}^u(f^r(x)) \cap \mathcal{W}^s(f^t(y)) \neq \emptyset$.

4.2.2. *Case 2:* $(d^u(\nu), d^u(\varrho)) = (1, 2)$. Since $(d^s(\nu), d^s(\varrho)) = (2, 1)$, by replacing the role of stable and unstable manifolds in case 1, we obtain that, for any $(x, y) \in A_\nu \times A_\varrho$, there exist $r, t \in \mathbb{Z}$ such that $\mathcal{W}^s(f^r(x)) \cap \mathcal{W}^u(f^t(y)) \neq \emptyset$.

4.2.3. *Case 3:* $(d^u(\nu), d^u(\varrho)) = (2, 2)$. Let A_ν be as above. Fix $x \in A_\nu$. Take a hyperbolic periodic point z_ν and an admissible unstable manifold $\mathbb{V}^u_\nu = \Psi^{-1}_{x_0}(V^u(z_\nu))$, as in Lemma 4.2. Also by the Katok closing lemma, there exists a hyperbolic periodic point $z_\varrho \in B(y_0, r)$ such that the local stable and local unstable manifolds of z_ϱ are admissible stable and admissible unstable manifolds near y_0. Below we may assume that both z_ν and z_ϱ are fixed points. Then by exactly same argument as in step 2 of case 1, we find an intersection point between the unstable manifold $W^u(z_\nu)$ and the stable manifold $W^s(z_\varrho)$.

There are two cases: $W^s(z_\varrho) \setminus \{z_\varrho\} \not\subset W^u(z_\nu)$ or $W^s(z_\varrho) \setminus \{z_\varrho\} \subset W^u(z_\nu)$. For the former case, there exists $q \in W^u(z_\nu) \cap W^s(z_\varrho) \cap R(x_0)$ such that the component

$$\mathbb{V}^s(q) = \mathbb{V}^s(z_\varrho : q) = \Psi^{-1}_{x_0}(C((W^s(z_\varrho) \setminus \{z_\varrho\}) \cap R(x_0) : q))$$

satisfies $\mathbb{V}^s(q) \cap (\mathbb{V}^u_\nu)^+ \neq \emptyset$ or $\mathbb{V}^s(q) \cap (\mathbb{V}^u_\nu)^- \neq \emptyset$. Therefore, by Lemma 4.2, there exists $n \in \mathbb{Z}$ and $K \in \mathbb{N}$ such that $W^s(z_\varrho)$ intersects $f^{2k}(\mathcal{W}^u(f^n(x)))$ for every $k \geq K$. Furthermore, we have the following. Let $\{z\} = \mathcal{V}^u(f^n(x)) \cap V^s(z_\nu)$ and put

$$\mathbb{V}_k = \Psi^{-1}_{x_0}(C(f^{2k}(\mathcal{W}^u(f^n(x))) \cap R(x_0) : f^{2k}(z))).$$

Note that it is an admissible unstable manifold for every $k \in \mathbb{N}$ and that $\mathbb{V}^s(q) \cap \mathbb{V}_k \neq \emptyset$ for every $k \geq K$. Then, by (4.1), $\mathbb{V}^s(q) \cap \mathbb{V}^+_k \neq \emptyset$ and $\mathbb{V}^s(q) \cap \mathbb{V}^-_k \neq \emptyset$ for every $k \geq K + 1$.

Take $y \in A_\varrho$. Then, by the inclination lemma, for sufficiently large $m \gg 1$ there exists a C^1-curve $V'_m \subset f^m(\mathcal{W}^s(y)) \cap R(x_0)$ such that $\Psi^{-1}_{x_0}(V'_m)$ is C^1-close to $\mathbb{V}^s(q)$. Therefore $\Psi^{-1}_{x_0}(V'_m) \cap \mathbb{V}^+_k \neq \emptyset$ and $\Psi^{-1}_{x_0}(V'_m) \cap \mathbb{V}^-_k \neq \emptyset$ for every such $m \gg 1$. This implies that

$$\emptyset \neq V'_m \cap \Psi_{x_0}(\mathbb{V}_k) \subset \mathcal{W}^s(f^m(y)) \cap \mathcal{W}^u(f^{2k+n}(x)).$$

Thus we obtain that, for any $(x, y) \in A_\nu \times A_\varrho$, there exist $r, t \in \mathbb{Z}$ such that $\mathcal{W}^u(f^r(x)) \cap \mathcal{W}^s(f^t(y)) \neq \emptyset$.

Next, for the latter case, $W^s(z_\varrho) \setminus \{z_\varrho\} \subset W^u(z_\nu)$, we take another hyperbolic fixed point $w_\nu \in B(x_0, r) \setminus \{z_\nu\}$ by the Katok closing lemma. By repeating the argument in step 2, it should hold that $W^u(w_\nu) \cap W^s(z_\varrho) \neq \emptyset$. This is impossible since $W^s(z_\varrho) \setminus \{z_\varrho\} \subset W^u(z_\nu)$ and $W^u(w_\nu) \cap W^u(z_\nu) = \emptyset$.

4.2.4. *Case 4:* $(d^u(\nu), d^u(\varrho)) = (1, 1)$. Since $(d^s(\nu), d^s(\varrho)) = (2, 2)$, by replacing the role of stable and unstable manifolds in case 3, we obtain that, for any $(x, y) \in A_\nu \times A_\varrho$, there exist $r, t \in \mathbb{Z}$ such that $\mathcal{W}^s(f^r(x)) \cap \mathcal{W}^u(f^t(y)) \neq \emptyset$.

4.3. *The case of* dim $M = 2$. Next we consider the two-dimensional case. There is only one case for the pair, $(d^u(\nu), d^u(\varrho)) = (1, 1)$. For this case, the proof is analogous to the dim $M = 3$ case. Here note that Lemma 4.2 holds since $d^u(\nu) = 1 = \dim M - 1$ (see Remark 4.3).

5. *Proof of Theorem 1.7*

Suppose that $f : M \to M$ is topologically conjugate to a topologically transitive Anosov diffeomorphism $g : X \to X$ via a homeomorphism $h : M \to X$: that is, $g \circ h = h \circ f$. By the stable and unstable manifolds theorem for g, there exists a constant $\delta_g > 0$ such that if $d_X(x, y) \le \delta_g$, then $W^u_g(x) \cap W^s_g(y) \ne \emptyset$. Note that $W^s_g(y)$ passes through $W^u_g(x)$ at every point in $W^u_g(x) \cap W^s_g(y)$. Here and below, we refer the readers to the book [**25**] for the facts about Anosov diffeomorphisms.

Step 1. Let ν and ϱ be any ergodic components of $\mu \in \mathcal{M}_f$. Choose $l \ge 1$ so large that $\nu(\Lambda^\nu_l) > 0$ and $\varrho(\Lambda^\varrho_l) > 0$. Then take $x \in \Lambda^\nu_l$ and $y \in \Lambda^\varrho_l$ arbitrarily. We will find an intersection point between the unstable manifold $\mathcal{W}^u_f(f^p(x))$ and the stable manifold $\mathcal{W}^s_f(f^q(y))$ for some $p, q \in \mathbb{Z}$.

There exist $\varepsilon(x) > 0$ and $\varepsilon(y) > 0$ such that $B^u_X(h(x), \varepsilon(x)) \subset h(\mathcal{V}^u_f(x))$ and $B^s_X(h(y), \varepsilon(y)) \subset h(\mathcal{V}^s_f(y))$, respectively. Since g is topologically transitive Anosov, by the Smale–Bowen spectral decomposition theorem, there exists a periodic point $z \in X$ of g such that $W^s_g(z)$ intersects $B^u_X(h(x), \varepsilon(x))$ transversely and $W^u_g(z)$ intersects $B^s_X(h(y), \varepsilon(y))$ transversely. (Here we assume that z is a fixed point of g, without loss of generality.) Thus, by the inclination lemma, there exists $N = N(\delta_g) \in \mathbb{N}$ such that $g^n(h(\mathcal{V}^u_f(x)))$ intersects $g^{-n}(h(\mathcal{V}^s_f(y)))$ transversely for every $n \ge N$. Therefore $\mathcal{W}^u_f(f^n(x)) \cap \mathcal{W}^s_f(f^{-n}(y)) \ne \emptyset$.

Step 2. Note that the dimension of the unstable manifold $\mathcal{W}^u_f(x)$ is constant for μ-almost every $x \in M$, since f is conjugate to g. Indeed, for μ-almost every $x \in M$, $h(\mathcal{W}^u_f(x)) \subset W^u_g(h(x))$, $h(\mathcal{W}^s_f(x)) \subset W^s_g(h(x))$ and $\dim h(\mathcal{W}^u_f(x)) + \dim h(\mathcal{W}^s_f(x)) = \dim M = \dim X$. On the other hand, we see that both $\dim W^u_g(\cdot)$ and $\dim W^s_g(\cdot)$ are constant on X such that $\dim W^u_g(\cdot) + \dim W^s_g(\cdot) = \dim X$, as g is Anosov. It follows that $\dim h(\mathcal{W}^u_f(x))$ is constant for μ-almost every $x \in M$, and so is $\dim \mathcal{W}^u_f(x)$.

Thus, by Proposition 3.2, it follows that $\nu \otimes \varrho(\mathcal{A}^{\nu,\varrho} \cap \mathcal{T}^{\nu,\varrho}_1) = 1$, and hence $\nu = \varrho$ by Lemma 3.1. This means that μ is ergodic.

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