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Existence of nodal solutions for quasilinear elliptic problems in $\mathbb{R}^N$

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We prove the existence of one positive, one negative and one sign-changing solution of a p-Laplacian equation on $\mathbb{R}^N$ with a p-superlinear subcritical term. Sign-changing solutions of quasilinear elliptic equations set on the whole of $\mathbb{R}^N$ have scarcely been investigated in the literature. Our assumptions here are similar to those previously used by some authors in bounded domains, and our proof uses fairly elementary critical point theory, based on constraint minimization on the nodal Nehari set. The lack of compactness due to the unbounded domain is overcome by working in a suitable weighted Sobolev space.

Keywords: quasilinear elliptic equations; unbounded domain; sign-changing solutions; Nehari manifold; weighted Sobolev spaces

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1. Introduction

In this paper we study sign-changing solutions of the quasilinear elliptic equation

$$-\Delta_p u = \lambda A(x)|u|^{p-2}u + g(x,u) \quad \text{on } \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < N$, $N \geqslant 1$, and λ is a real parameter. We suppose that the coefficient $A\colon \mathbb{R}^N \to \mathbb{R}$ satisfies the following condition:

(A_1) A is measurable with $A > 0$ almost everywhere (a.e.), $A \in L^\infty(\mathbb{R}^N)$ and $A^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Note that the last condition in (A_1) holds if, for example, $\operatorname{ess\,inf}_\Omega A > 0$ for any bounded open set $\Omega \subset \mathbb{R}^N$. An additional condition relating A and λ, formulated in § 1.1, will also play a crucial role in our analysis.

We suppose that the function g satisfies the following assumptions, where $p^* = Np/(N-p)$:

(g_1) $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g(x,\cdot) \in C^1(\mathbb{R})$ for almost every $x \in \mathbb{R}$, and there exist $q \in (p, p^*)$ and $B \in L^{p^*/(p^*-q)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $B > 0$ a.e., such that

$$|g_s(x,s)| \leqslant B(x)|s|^{q-2} \quad \text{for almost every } x \in \mathbb{R}^N,\ s \in \mathbb{R};$$

(g_2) there exist $\theta > p$ and $R > 0$ such that

$$g(x,s)s \geqslant \theta G(x,s) > 0 \quad \text{for almost every } x \in \mathbb{R}^N,\ |s| \geqslant R,$$

where $G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t$;

(g_3) the mapping $s \mapsto g(x,s)/|s|^{p-1}$ is strictly increasing in $s \in \mathbb{R} \setminus \{0\}$ for almost every $x \in \mathbb{R}^N$.

Assumptions (g_1) and (g_2) are often referred to by saying that g is *subcritical* and *p-superlinear*,[1] respectively. The hypothesis $B \in L^\infty(\mathbb{R}^N)$ will be convenient to establish compactness of a weighted Sobolev embedding, which plays a important role in the paper. However, it can be relaxed to a local integrability assumption, namely, $B \in L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for some large enough s (see remark A.1 for a more a precise statement). Our assumptions on B imply that (1.1) is a compact perturbation (in a sense that will be made precise in lemma 2.5) of the p-linear eigenvalue problem

$$-\Delta_p u = \lambda A(x)|u|^{p-2}u \quad \text{on } \mathbb{R}^N. \tag{1.2}$$

The existence of eigenvalues and eigenfunctions for this problem has been discussed by Allegretto and Huang [1], even in the case of an indefinite weight A (i.e. when A does not have a constant sign), as long as A is positive on a set of positive measure and satisfies an integrability condition at infinity. Bifurcation of solutions of (1.1) from the principal eigenvalue of (1.2) has been studied by Drábek and Huang [10], who obtained global continua of positive and negative solutions of (1.1). Bifurcation from higher eigenvalues, which would provide sign-changing solutions, is a difficult problem. In fact, sign-changing solutions for quasilinear equations in the whole of $\mathbb{R}^N$ have scarcely been investigated; see [8], however, where a problem with cylindrical symmetry is considered, and [13], which deals with a p-asymptotically linear problem.

On the other hand, there is a fair amount of literature on solutions of quasilinear elliptic equations in bounded domains. In the radial case, the Dirichlet problem in a ball has been solved by Del Pino and Manásevich [9] by the bifurcation approach, yielding infinitely many nodal[2] solutions. In the non-radial setting, an important contribution (probably the most general so far) is due to Bartsch *et al.* [2], who proved existence and multiplicity of nodal solutions for p-Laplacian Dirichlet problems in smooth bounded domains. Their results are obtained by critical-point theory in Banach spaces, making clever use of a suitable pseudo-gradient flow. In fact, in $\mathbb{R}^N$, using similar arguments, a sign-changing solution of an auxiliary p-superlinear

[1] The term '$(p-1)$-superhomogeneous' is sometimes used instead of p-superlinear.

[2] Throughout the paper we will use the terms 'nodal' and 'sign-changing' interchangeably.

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problem (denoted $(P)_\mu$) is obtained in the course of the proof in [13]. However, equation $(P)_\mu$ considered there has a different structure from ours, mainly due to the p-asymptotically linear nature of the original problem.

It is worth remarking that the hypotheses (H_0)–(H_3) that were used to prove the existence of a nodal solution in [2] are structurally similar to ours. However, in this work we use a more elementary method, based on constraint minimization on the 'nodal Nehari set'. This approach originated in a series of works on the Dirichlet problem for semilinear equations, a review of which can be found in [3]. It has also been used more recently in the context of a prescribed mean-curvature problem in [5], from which some of our arguments are inspired. Note that the Nehari approach strongly relies on the monotonicity assumption (g_3), which is not needed in [2]. We will focus here on the case of a positive coefficient A and, under the hypotheses (A_1), (g_1)–(g_3), we will give a sufficient condition relating λ and A for the existence of at least one positive, one negative, and one nodal solution of (1.1). This condition (assumption (A,λ), below) should be compared with hypothesis (H_3) in [2], involving the first eigenvalue of the p-linear problem. We will also deduce corresponding existence results for the problem

$$-\Delta_p u = \tilde{A}(x)|u|^{p-2}u + g(x,u) \quad \text{on } \mathbb{R}^N, \tag{1.3}$$

under appropriate conditions on the coefficient $\tilde{A} \in L^\infty(\mathbb{R}^N)$.

Our approach is based on a variational formulation of (1.1) in a weighted Sobolev space. More precisely, let us define the norm $\|\cdot\|_{W_A}$ on $C_0^\infty(\mathbb{R}^N)$ by

$$\|u\|_{W_A} = \left(\int_{\mathbb{R}^N} |\nabla u|^p + A(x)|u|^p \,\mathrm{d}x \right)^{1/p}. \tag{1.4}$$

We will work in the space $W_A(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{W_A}}$. Some elementary properties of $W_A(\mathbb{R}^N)$ will be established in § 2. In particular, $W_A(\mathbb{R}^N)$ is a separable and uniformly convex (hence, reflexive) Banach space. Since $A \in L^\infty(\mathbb{R}^N)$, $W_A(\mathbb{R}^N) \supseteq W^{1,p}(\mathbb{R}^N)$, with equality if A is bounded away from zero. We will often merely write W_A for $W_A(\mathbb{R}^N)$, although some properties of $W_A(\Omega)$ will be given in § 2 for more general open subsets $\Omega \subset \mathbb{R}^N$.

With G defined as in (g_2), we introduce the functional $S_\lambda \colon W_A \to \mathbb{R}$,

$$S_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p - \lambda A(x)|u|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} G(x,u) \,\mathrm{d}x. \tag{1.5}$$

Due to (g_2), S_λ is not coercive, so one cannot apply the direct method of the calculus of variations to find critical points of S_λ. We will see that constraint minimization on Nehari-type sets provides an efficient alternative to obtain critical points and to discuss their nodal properties.

1.1. Main results

We now formulate the assumption relating A and λ that we shall use to prove our results. For A satisfying (A_1), let

$$\lambda_A = \inf_{u \in W_A \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x}{\int_{\mathbb{R}^N} A(x)|u|^p \,\mathrm{d}x} \geqslant 0. \tag{1.6}$$

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Provided that A and λ satisfy

(A,λ) $\lambda < \lambda_A$,

it will be shown in § 2 that

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^p - \lambda A(x)|u|^p \,\mathrm{d}x \right)^{1/p} \tag{1.7}$$

defines a (quasi-)norm[3] on W_A that is equivalent to $\|\cdot\|_{W_A}$. This enables one to rewrite $S_\lambda\colon W_A \to \mathbb{R}$ as

$$S_\lambda(u) = \frac{1}{p}\|u\|_\lambda^p - \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x, \tag{1.8}$$

which is a convenient way to exhibit the main properties of the p-homogeneous part of S_λ with respect to the variational procedure. Note that if $\lambda < 0$, then (A,λ) is always satisfied, and $\|\cdot\|_\lambda$ is a norm that is equivalent to $\|\cdot\|_{W_A}$ on W_A. The more interesting case of positive λ requires $\lambda_A > 0$. As will be shown in § 3, a sufficient condition for this to hold is

(A_2) $A \in L^{N/p}(\mathbb{R}^N)$.

We shall see in lemma 2.4 that $S_\lambda \in C^1(W_A,\mathbb{R})$ provided that A satisfies (A_1). Our main results concern weak solutions of problems (1.1) and (1.3). A function $u \in W_A$ is called a *solution* of (1.1) if and only if $S'_\lambda(u) = 0$, that is, if and only if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} A(x)|u|^{p-2}u\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,u)v\,\mathrm{d}x = 0 \quad \forall\varphi\in W_A, \tag{1.9}$$

with a similar definition for solutions of (1.3). The existence of weak solutions will be proved by a variational approach in § 3, under hypotheses (A_1), (A,λ) and (g_1)–(g_3). In addition to ensuring that $\lambda_A > 0$, assumption (A_2) is also needed to obtain extra regularity of the solutions. Our main result is the following theorem.

Theorem 1.1. *Suppose that hypotheses (A_1), (A_2), (A,λ) and (g_1)–(g_3) are satisfied. There then exist three solutions $u_1,u_2,u_3 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ of (1.1), with $u_1 > 0$, $u_2 < 0$ and $u_3^\pm \not\equiv 0$. Furthermore, u_3 has exactly two nodal domains.*

The definition of the class $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ is recalled in § 4. As usual, we denote by $u^\pm$ the positive and negative parts of u. More precisely, we use here the convention $u^\pm(x) := \pm\max\{\pm u(x),0\}$, so that $u = u^+ + u^-$. For a continuous function $u\colon \mathbb{R}^N \to \mathbb{R}$, a *nodal domain* is a connected component of $\mathbb{R}^N \setminus u^{-1}(\{0\})$.

It is also possible to formulate a version of our results from which the parameter λ is absent, which pertains to problem (1.3).

Corollary 1.2. *Suppose that (g_1)–(g_3) are satisfied. If either*

(i) *$\tilde{A}$ satisfies (A_1), (A_2) and $\lambda_{\tilde{A}} > 1$,*

(ii) *or $-\tilde{A}$ satisfies (A_1) and (A_2),*

(iii) *or $\tilde{A} = 0$ a.e.,*

[3] See lemma 2.6 and remark 2.7.

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then (1.3) has three solutions in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, the solutions have the same nodal structure as in theorem 1.1, i.e. one is positive, one is negative and one is sign-changing with two nodal domains.

Proof. If case (i) holds, then the result follows from theorem 1.1 by letting $A = \tilde{A}$ and $\lambda = 1 < \lambda_{\tilde{A}}$. Case (ii) follows by letting $A = -\tilde{A}$ and $\lambda = -1$, while for (iii) one can take any A satisfying (A_1)–(A_2), and $\lambda = 0$. □

The rest of the paper is organized as follows. In § 2 we first establish some properties of the spaces $W_A(\Omega)$, in particular, a compact embedding that plays a central role in the proof of our main result. The existence of three critical points is proved in § 3 by minimization on Nehari-type sets. The regularity of solutions is discussed in § 4, where the proof of theorem 1.1 is then completed.

We shall use the letter C (possibly with an index) to denote various positive constants, the exact values of which are not relevant to the analysis.

2. Preliminaries

We start this section with some preliminary results about the functional setting, which will be useful to our existence theory. In particular, we establish embedding properties of the space $W_A(\Omega)$, which will play an important role in our analysis. For an arbitrary open set $\Omega \subset \mathbb{R}^N$, let

$$W_A(\Omega) = \{u|_\Omega : u \in W_A(\mathbb{R}^N)\},$$

where $W_A(\mathbb{R}^N)$ was defined in the introduction. Given a positive measurable function $B : \Omega \to \mathbb{R}_+$ and $q \geqslant 1$, we define the weighted Lebesgue space

$$L^q_B(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^q B(x)\,\mathrm{d}x < \infty \right\}$$

endowed with its natural norm $\|u\|_{L^q_B(\Omega)} = (\int_\Omega |u|^q B(x)\,\mathrm{d}x)^{1/q}$. When there is no risk of confusion, we shall merely write W_A and L^q_B for $W_A(\Omega)$ and $L^q_B(\Omega)$. We may also use the shorthand notation $\|\cdot\|_r \equiv \|\cdot\|_{L^r}$ for the usual Lebesgue norms.

Under appropriate assumptions, we will show that $W_A(\Omega)$ is continuously and compactly embedded into $L^q_B(\Omega)$. Our proof will rely on the classic theory of Sobolev spaces, as presented, for example, in [6, chapitre IX].[4] But let us first state the following elementary properties.

Proposition 2.1. *Let A satisfy (A_1) and $B : \Omega \to \mathbb{R}$ be measurable with $B > 0$ almost everywhere.*

(i) *$W_A(\Omega)$ is a separable, uniformly convex (hence, reflexive) Banach space satisfying $W^{1,p}(\Omega) \subseteq W_A(\Omega)$, with equality if* $\operatorname{ess\,inf}_{\mathbb{R}^N} A > 0$.

(ii) *For any $1 < q < \infty$, $L^q_B(\Omega)$ is a separable reflexive Banach space.*

[4]Note that an English translation of Brezis's book is also available [7]. However, in the original (French) version [6], the compactness result we shall use is formulated in a way that is slightly better suited to our proof of proposition 2.2, so we will refer to [6] throughout.

Proof. The fact that $W_A(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$ if $\operatorname{ess\,inf}_{\mathbb{R}^N} A > 0$ follows from the definition of $W_A(\mathbb{R}^N)$. The remaining statements can be found in [4] and [11]. □

Proposition 2.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is either $\mathbb{R}^N$ or a bounded open set with C^1 boundary. Let $q \in (p, p^*)$ and let $B \in L^{p^*/(p^*-q)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We then have $W_A(\Omega) \subset L^q_B(\Omega)$ and there is a constant $C > 0$ such that*

$$\|u\|_{L^q_B(\Omega)} \leqslant C\|u\|_{W_A(\Omega)}, \quad u \in W_A(\Omega). \tag{2.1}$$

Furthermore, the embedding is compact.

Proof. To prove (i), let us first consider the case in which $\Omega = \mathbb{R}^N$. Notice that $W^{1,p}(\mathbb{R}^N)$ is a dense subspace of $W_A(\mathbb{R}^N)$. We shall thus start by proving (2.1) for $u \in W^{1,p}(\mathbb{R}^N)$, and then argue by density. For $u \in W^{1,p}(\mathbb{R}^N)$, it follows from Hölder's inequality and the Sobolev embedding theorem that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^q B(x)\,\mathrm{d}x \leqslant \|B\|_{p^*/(p^*-q)}\|u\|^q_{p^*} &\leqslant \|B\|_{p^*/(p^*-q)} C\|\nabla u\|^q_p \\ &\leqslant C\|B\|_{p^*/(p^*-q)}\|u\|^q_{W_A},\end{aligned}$$

and so $\|u\|_{L^q_B} \leqslant C\|u\|_{W_A}$ for all $u \in W^{1,p}(\mathbb{R}^N)$. Now, for any $u \in W_A(\mathbb{R}^N)$, there is a sequence $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ such that $u_n \to u$ in $W_A(\mathbb{R}^N)$. But for each n there holds $\|u_n\|_{L^q_B} \leqslant C\|u_n\|_{W_A}$, so passing to the limit using Fatou's lemma yields (2.1). The case in which Ω is a bounded domain with smooth boundary follows from the case $\Omega = \mathbb{R}^N$ by adapting the extension theorem [6, théorème IX.7] to the present context. The compactness of the embedding is proved in appendix A.1. □

Remark 2.3. Observe that, by density, the classic Sobolev inequality, $\|u\|_{p^*} \leqslant C\|\nabla u\|_p$ for $u \in W^{1,p}(\Omega)$, extends to $u \in W_A(\Omega)$, so that

$$\|u\|_{L^{p^*}(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega)} \leqslant C\|u\|_{W_A(\Omega)}, \quad u \in W_A(\Omega). \tag{2.2}$$

We are now in a position to prove that the functional $S_\lambda \colon W_A(\mathbb{R}^N) \to \mathbb{R}$ defined in (1.5) is of class C^1. We shall denote by $\langle \cdot, \cdot \rangle \colon W_A^* \times W_A \to \mathbb{R}$ the duality pairing between W_A and its topological dual W_A^*.

Lemma 2.4. *Let A satisfy assumption (A_1). We then have $S_\lambda \in C^1(W_A(\mathbb{R}^N), \mathbb{R})$ and, for all $u, v \in W_A(\mathbb{R}^N)$,*

$$\langle S'_\lambda(u), v\rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda A(x)|u|^{p-2}uv\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,u)v\,\mathrm{d}x. \tag{2.3}$$

Proof. The proof of lemma 2.4 follows from the continuity of the embedding in proposition 2.2. To avoid disrupting the exposition with technicalities, we postpone it until appendix A.2. □

Lemma 2.5. *Under assumptions (g_1), (A_1) and (A_2), the following statements hold.*

(i) *The functionals $W_A(\mathbb{R}^N) \to \mathbb{R}$, $u \mapsto \int_{\mathbb{R}^N} g(x,u)u\,\mathrm{d}x$, $u \mapsto \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x$ are compact, in the sense that they map bounded sequences to relatively compact ones.*

(ii) *The functional $W_A(\mathbb{R}^N) \to \mathbb{R}$, $u \mapsto \int_{\mathbb{R}^N} A(x)|u|^p\,\mathrm{d}x$ is also compact.*

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Proof. (i) Consider a bounded sequence $(u_n) \subset W_A$. By proposition 2.2, (u_n) is bounded in $L_B^q(\mathbb{R}^N)$ and there exist a subsequence (still denoted by (u_n)) and an element $u \in L_B^q(\mathbb{R}^N)$ such that $u_n \to u$ in $L_B^q(\mathbb{R}^N)$. It follows from (g_1) that

$$\begin{aligned}|\Phi(u_n) - \Phi(u)| &\leqslant \int_{\mathbb{R}^N} |g(x,u_n)u_n - g(x,u)u| \,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|\,|u_n| \,\mathrm{d}x + \int_{\mathbb{R}^N} |g(x,u)|\,|u_n - u| \,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^N} B||u_n|^{q-2}u_n - |u|^{q-2}u|\,|u_n| \,\mathrm{d}x + \int_{\mathbb{R}^N} B|u|^{q-1}|u_n - u| \,\mathrm{d}x,\end{aligned}$$

where, by Hölder's inequality,

$$\begin{aligned}&\int_{\mathbb{R}^N} B||u_n|^{q-2}u_n - |u|^{q-2}u|\,|u_n| \,\mathrm{d}x \\ &\qquad\qquad \leqslant C\biggl(\int_{\mathbb{R}^N} B||u_n|^{q-2}u_n - |u|^{q-2}u|^{q/(q-1)} \,\mathrm{d}x\biggr)^{(q-1)/q}.\end{aligned}$$

Since $u_n \to u$ in $L_B^q(\mathbb{R}^N)$, we can suppose that $u_n \to u$ pointwise a.e. and that $B|u_n|^q \leqslant f$, uniformly in n, for some $f \in L^1(\mathbb{R}^N)$. It then follows by dominated convergence that the right-hand side of the above inequality goes to zero as $n \to \infty$. On the other hand, by Hölder's inequality,

$$\int_{\mathbb{R}^N} B|u|^{q-1}|u_n - u| \,\mathrm{d}x \leqslant \|u\|_{L_B^q}^{q-1} \|u_n - u\|_{L_B^q} \to 0 \quad \text{as } n \to \infty,$$

which concludes the proof. A similar argument shows that $u \mapsto \int_{\mathbb{R}^N} G(x,u)\,\mathrm{d}x$ is compact.

(ii) Consider again a bounded sequence $(u_n) \subset W_A$. By proposition 2.1(i), there exists a subsequence (still denoted by (u_n)) and an element $u \in W_A$ such that $u_n \rightharpoonup u$ weakly in W_A. By Hölder's inequality, we have, for any open set $\Omega \subset \mathbb{R}^N$,

$$\int_\Omega A(x)||u_n|^p - |u|^p| \,\mathrm{d}x \leqslant \|A\|_{L^r(\Omega)} \||u_n|^p - |u|^p\|_{L^s(\Omega)}$$

for some $r > N/p$ and $s < N/(N-p)$. Since $u \mapsto |u|^p$ is continuous from $L^{ps}(\Omega)$ to $L^s(\Omega)$, and, for Ω bounded, the embedding $W_A(\Omega) \subset L^{ps}(\Omega)$ is compact, it follows that

$$\int_\Omega A(x)||u_n|^p - |u|^p| \,\mathrm{d}x \to 0, \quad \Omega \text{ bounded}.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Omega} A(x)||u_n|^p - |u|^p| \,\mathrm{d}x &\leqslant \||u_n|^p - |u|^p\|_{L^{N/(N-p)}(\mathbb{R}^N\setminus\Omega)} \|A\|_{L^{N/p}(\mathbb{R}^N\setminus\Omega)} \\ &\leqslant (\|u_n\|_{Np/(N-p)}^p + \|u\|_{Np/(N-p)}^p) \|A\|_{L^{N/p}(\mathbb{R}^N\setminus\Omega)} \\ &\leqslant (\|\nabla u_n\|_p^p + \|\nabla u\|_p^p) \|A\|_{L^{N/p}(\mathbb{R}^N\setminus\Omega)} \\ &\leqslant C\|A\|_{L^{N/p}(\mathbb{R}^N\setminus\Omega)},\end{aligned}$$

which can be made arbitrarily small by choosing $|\Omega|$ large enough. □

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We conclude this section by showing that, under the hypothesis (A, λ), the (quasi-)norm $\|\cdot\|_\lambda$ defined in (1.7) is equivalent to $\|\cdot\|_{W_A}$.

LEMMA 2.6. *Let A and λ satisfy hypotheses (A_1) and (A, λ). There then exist constants $c_i = c_i(\lambda) > 0$, $i = 1, 2$, such that*

$$c_1\|u\|_{W_A} \leqslant \|u\|_\lambda \leqslant c_2\|u\|_{W_A}, \quad u \in W_A(\mathbb{R}^N). \tag{2.4}$$

Proof. The second inequality follows directly from the definition of $\|\cdot\|_\lambda$ in (1.7), with $c_2 = (\max\{1, |\lambda|\})^{1/p}$. It actually holds for any $\lambda \in \mathbb{R}$. Condition (A, λ) is required to prove the first inequality in (2.4), which we do now. Let $\varepsilon > 0$. By the definition of λ_A in (1.6), we have, for any $u \in W_A$,

$$\begin{aligned}
\|u\|_\lambda^p &= \varepsilon \int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x + (1-\varepsilon) \int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} A(x)|u|^p \, \mathrm{d}x \\
&\geqslant \varepsilon \int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x + (1-\varepsilon)\lambda_A \int_{\mathbb{R}^N} A(x)|u|^p \, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} A(x)|u|^p \, \mathrm{d}x \\
&= \varepsilon \int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x + [(1-\varepsilon)(\lambda_A - \lambda) - \varepsilon\lambda] \int_{\mathbb{R}^N} A(x)|u|^p \, \mathrm{d}x \\
&\geqslant \min\{\varepsilon, [(1-\varepsilon)(\lambda_A - \lambda) - \varepsilon\lambda]\}\|u\|_{W_A}^p.
\end{aligned}$$

Since $(1-\varepsilon)(\lambda_A - \lambda) - \varepsilon\lambda \to \lambda_A - \lambda > 0$ as $\varepsilon \to 0$, we can choose $\varepsilon > 0$ such that $c_1 := (\min\{\varepsilon, [(1-\varepsilon)(\lambda_A - \lambda) - \varepsilon\lambda]\})^{1/p}$ does the job. This concludes the proof. □

REMARK 2.7. It is worth noting that $\|\cdot\|_\lambda$ satisfies the usual properties of a norm except for the triangle inequality. However, it follows from (2.4) that

$$\|u + v\|_\lambda \leqslant c_2 c_1^{-1}(\|u\|_\lambda + \|v\|_\lambda).$$

Hence, $\|\cdot\|_\lambda$ is a norm or a quasi-norm, depending on whether $c_2 c_1^{-1}$ is smaller or larger than 1. In fact, it can be seen that $c_2 c_1^{-1} > 1$ if $\lambda > 0$, so $\|\cdot\|_\lambda$ is only a quasi-norm in this case.

3. Existence of weak solutions

We will prove the existence of at least one positive, one negative, and one sign-changing solution of (1.1) by constraint minimization of the functional S_λ defined in (1.5). Since it is easier to obtain solutions of a given sign, we will focus our attention on the existence of a sign-changing solution, and we will explain in the course of the proof how to modify it in order to get positive/negative solutions. The existence of a sign-changing solution is obtained by minimizing S_λ on the 'nodal Nehari set', which will be defined below.

We define the positive and negative parts $u^\pm$ of a function $u\colon \mathbb{R}^N \to \mathbb{R}$ by $u^\pm(x) := \pm\max\{\pm u(x), 0\}$, $x \in \mathbb{R}^N$, so that $u = u^+ + u^-$, with $\pm u^\pm \geqslant 0$. It follows from [15] that $u^\pm \in W_A(\mathbb{R}^N)$ whenever $u \in W_A(\mathbb{R}^N)$.

THEOREM 3.1. *Suppose that hypotheses (A_1), (A, λ) and (g_1)–(g_3) are satisfied. There then exist $u_1, u_2, u_3 \in W_A(\mathbb{R}^N)$ with $u_1 > 0$ a.e., $u_2 < 0$ a.e., and $u_3^\pm \not\equiv 0$ a.e., such that $S'_\lambda(u_i) = 0$, $i = 1, 2, 3$.*

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Remark 3.2. When $\lambda < 0$, condition (A, λ) of theorem 3.1 is trivially satisfied. Then A need only satisfy assumption (A_1) and the conclusion of theorem 3.1 holds. The choice $A \equiv 1$ is allowed in this case, yielding solutions in $W^{1,p}(\mathbb{R}^N)$. Observe that $\lambda_A = 0$ for $A \equiv 1$, reflecting the absence of Poincaré inequality on $\mathbb{R}^N$. In order to apply theorem 3.1 with $\lambda > 0$, we need conditions on A such that $\lambda_A > 0$.

Proposition 3.3. *If* A *satisfies* (A_1) *and* (A_2)*, then* $\lambda_A > 0$*. Moreover, there exists* $u^* \in W_A(\mathbb{R}^N)$ *such that*

$$\lambda_A = \frac{\int_{\mathbb{R}^N} |\nabla u^*|^p \,\mathrm{d}x}{\int_{\mathbb{R}^N} A(x)|u^*|^p \,\mathrm{d}x}.$$

Proof. By Hölder's inequality and remark 2.3, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} A(x)|u|^p \,\mathrm{d}x \leqslant C\|A\|_{N/p}\|\nabla u\|_p^p, \quad u \in W_A(\mathbb{R}^N).$$

Furthermore, variational arguments similar to the proof of [1, theorem 1] show that, when (A_1) and (A_2) hold, the infimum in (1.6) is actually achieved. □

We will now prove theorem 3.1. Without further mention, we shall suppose that the hypotheses of theorem 3.1 hold throughout the rest of this section. Before we proceed with the bulk of the proof, let us first derive some elementary consequences of hypotheses (g_1)–(g_3).

Lemma 3.4. *The function* g *has the following properties.*

(i) $|g(x,s)| \leqslant B(x)|s|^{q-1}$ *and* $|G(x,s)| \leqslant (1/q)B(x)|s|^q$ *for almost every* $x \in \mathbb{R}^N$ *and all* $s \in \mathbb{R}$*. In particular,*

$$g(x,s) = o(|s|^{p-1}) \quad \text{as } s \to 0, \text{ uniformly for almost every } x \in \mathbb{R}^N.$$

(ii) $$\lim_{|s|\to\infty} \frac{|g(x,s)|}{|s|^{p-1}} = \infty \quad \text{and} \quad \lim_{|s|\to\infty} \frac{|G(x,s)|}{|s|^p} = \infty \quad \text{for almost every } x \in \mathbb{R}^N.$$

(iii) *Letting* $h(x,s) = (1/p)g(x,s)s - G(x,s)$*, we have that*

$$sG_s(x,s) > 0 \quad \text{and} \quad sh_s(x,s) > 0 \quad \text{for almost every } x \in \mathbb{R}^N,\ s \neq 0.$$

Proof. Property (i) follows immediately from (g_1). To prove (ii), first note that, for $s > R$,

$$\frac{G'(x,s)}{G(x,s)} \geqslant \frac{\theta}{s} \quad \Longrightarrow \quad G(x,s) \geqslant R^\theta G(x,R)s^\theta = R^\theta G(x,R)|s|^\theta,$$

whereas, for $s < -R$,

$$\begin{aligned}
\frac{G'(x,s)}{G(x,s)} \leqslant \frac{\theta}{s} \quad &\Longrightarrow \quad \int_s^{-R} \frac{G'(x,t)}{G(x,t)} \,\mathrm{d}t \leqslant \theta \int_s^{-R} \frac{\mathrm{d}t}{t} \\
&\Longrightarrow \quad \ln \frac{G(x,-R)}{G(x,s)} \leqslant \ln \left(\frac{-s}{R}\right)^{-\theta} \\
&\Longrightarrow \quad G(x,s) \geqslant R^{-\theta}G(x,-R)(-s)^\theta = R^{-\theta}G(x,-R)|s|^\theta.
\end{aligned}$$

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Therefore, $G(x,s) \geqslant C(x)|s|^\theta$ for almost all $x \in \mathbb{R}^N$ and for all $|s| \geqslant R$, where $C(x) := \min\{R^\theta G(x,R), R^{-\theta} G(x,-R)\}$. Then (g_2) implies that

$$|g(x,s)| \geqslant \theta C(x)|s|^{\theta-1} \quad \text{for almost every } x \in \mathbb{R}^N,\ |s| \geqslant R,$$

with $C(x) := \min\{R^\theta G(x,R), R^{-\theta} G(x,-R)\}$, from which the limits in (ii) follow. Finally, (iii) follows from (g_2) and (g_3). □

Let us now describe the variational setting we shall use to obtain critical points of the functional S_λ defined in (1.5). By lemma 2.4, $S_\lambda \in C^1(W_A, \mathbb{R})$ and, recalling the definition of $\|\cdot\|_\lambda$ in (1.7), for all $\lambda < \lambda_A$ we define

$$J_\lambda(u) = \langle S'_\lambda(u), u\rangle = \|u\|_\lambda^p - \int_{\mathbb{R}^N} g(x,u)u \,\mathrm{d}x,$$
$$N_\lambda = \{u \in W_A \setminus \{0\} \colon J_\lambda(u) = 0\},$$
$$M_\lambda = \{u \in W_A \colon u^\pm \in N_\lambda\} \subset N_\lambda.$$

The sets N_λ and M_λ are known as the *Nehari manifold* and the *nodal Nehari set*, respectively. Clearly, N_λ contains all non-trivial solutions of (1.1) while M_λ contains all sign-changing solutions of (1.1).

For $u \in N_\lambda$, it follows from lemma 3.4(i), proposition 2.2 and lemma 2.6 that

$$\|u\|_\lambda^p = \int_{\mathbb{R}^N} g(x,u)u \,\mathrm{d}x \leqslant C \int_{\mathbb{R}^N} B(x)|u|^q \,\mathrm{d}x \leqslant C\|u\|_{W_A}^q \leqslant C_\lambda \|u\|_\lambda^q.$$

Hence, letting $\delta_\lambda := C_\lambda^{-1/(q-p)}$, we have

$$\|u\|_\lambda \geqslant \delta_\lambda > 0, \quad u \in N_\lambda. \tag{3.1}$$

Observing that

$$J_\lambda(u) = 0 \iff \|u\|_\lambda^p = \int_{\mathbb{R}^N} g(x,u)u \,\mathrm{d}x,$$

it follows from lemma 3.4(iii) that

$$S_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p} g(x,u)u - G(x,u) \,\mathrm{d}x = \int_{\mathbb{R}^N} h(x,u) \,\mathrm{d}x > 0, \quad u \in N_\lambda. \tag{3.2}$$

Therefore,[5]

$$m_\lambda := \inf_{M_\lambda} S_\lambda \geqslant \inf_{N_\lambda} S_\lambda \geqslant 0. \tag{3.3}$$

We will now show that the Nehari manifold is diffeomorphic to the unit sphere in $(W_A, \|\cdot\|_\lambda)$. Firstly, similar arguments to the proof of lemma 2.4 show that $J_\lambda \in C^1(W_A, \mathbb{R})$, and it follows from (g_3) that $\langle J'_\lambda(u), u\rangle < 0$ for all $u \in N_\lambda$. Therefore, by the submersion theorem, N_λ is a C^1 manifold of codimension 1 in W_A such that the tangent space $T_u N_\lambda$ is transversal to $\mathbb{R}_+ u$ for all $u \in N_\lambda$.

Lemma 3.5. *For any fixed $u \in W_A \setminus \{0\}$, there exists a unique $t = t_\lambda(u) > 0$ such that $t_\lambda(u)u \in N_\lambda$. Furthermore, the map $u \mapsto t_\lambda(u)u$ is a C^1 diffeomorphism*

[5]That $M_\lambda \neq \emptyset$ is easily seen from step 2 in the proof of proposition 3.6.

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from $\{u \in W_A \colon \|u\|_\lambda = 1\}$ *onto* N_λ*, with inverse* $u \mapsto u/\|u\|_\lambda$*. Moreover, for any* $u \in W_A \setminus \{0\}$*, we have*

$$t_\lambda(u) < 1 \quad \text{if } J_\lambda(u) < 0 \quad \text{and} \quad t_\lambda(u) > 1 \quad \text{if } J_\lambda(u) > 0. \tag{3.4}$$

Finally, $S_\lambda(tu)$ *is increasing for* $t \in (0, t_\lambda(u))$ *and decreasing for* $t \in (t_\lambda(u), \infty)$*, with*

$$S_\lambda(t_\lambda(u)u) = \max_{t>0} S_\lambda(tu) \quad \text{for all } u \in W_A \setminus \{0\}. \tag{3.5}$$

Proof. Define a C^1 function $\varphi\colon (0,\infty) \to \mathbb{R}$ by

$$\varphi_u(t) = \frac{1}{t^p} J_\lambda(tu) = \|u\|_\lambda^p - \int_{\mathbb{R}^N} \frac{g(x,tu)}{t^{p-1}} u \, \mathrm{d}x.$$

It follows from lemma 3.4(i) and (ii) that

$$\lim_{t\to 0^+} \varphi_u(t) = \|u\|_\lambda^p > 0, \qquad \lim_{t\to+\infty} \varphi_u(t) = -\infty.$$

Furthermore, by (g_3), $t \mapsto \varphi_u(t)$ is strictly decreasing on $(0,\infty)$, from which the existence and uniqueness of $t_\lambda(u)$ follow. The diffeomorphism statement is a consequence of the implicit function theorem and the transversality of $T_u N_\lambda$ and $\mathbb{R}_+ u$. Equation (3.4) follows easily from the properties of φ_u, while the behaviour of $S_\lambda(tu)$, $t > 0$, follows from the calculation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} S_\lambda(tu) &= \langle S_\lambda'(tu), u\rangle \\
&= \int_{\mathbb{R}^N} t^{p-1}(|\nabla u|^p - \lambda A(x)|u|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,tu)u\,\mathrm{d}x \\
&= t^{-1}\left(\|tu\|_\lambda^p - \int_{\mathbb{R}^N} g(x,tu)tu\,\mathrm{d}x\right) \\
&= t^{-1} J_\lambda(tu).
\end{aligned}$$

The lemma is proved. □

Proposition 3.6. *The infimum* m_λ *defined in (3.3) is achieved.*

Proof. In the course of this proof, we will take the liberty of passing to subsequences when necessary, without mentioning it explicitly. The proof proceeds in two steps.

Step 1 (boundedness of a minimizing sequence). Consider $(u_n) \subset M_\lambda$ such that $S_\lambda(u_n) \to m_\lambda$, and suppose by contradiction that $\|u_n\|_\lambda \to \infty$ as $n \to \infty$. Now let

$$v_n = p[m_\lambda + 1]^{1/p} \frac{u_n}{\|u_n\|_\lambda}.$$

Since the sequence (v_n) is bounded, we can suppose that there exists $v \in W_A$ such that $v_n \rightharpoonup v$ weakly in W_A. By lemma 2.5, we have

$$S(v_n) \to m_\lambda + 1 - \int_{\mathbb{R}^N} G(x,v)\,\mathrm{d}x.$$

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On the other hand, since $u_n \in N_\lambda$, it follows from (3.5) that $S(v_n) \leqslant S(u_n)$. We shall thus reach a contradiction by showing that $v \equiv 0$. If this is not the case, there exists a set $\Omega \subset \mathbb{R}^N$ with positive measure, and a number $\delta > 0$, such that $\operatorname{ess\,inf}_\Omega |v| \geqslant \delta$. Invoking proposition 2.2 and Egorov's theorem, we can suppose that

$$\operatorname*{ess\,inf}_{\Omega} |v_n| \geqslant \frac{\delta}{2} > 0, \quad n \geqslant n_0,$$

for some large enough $n_0 \in \mathbb{N}$. Since $G(x, 0) \equiv 0$ and the supports of v_n^+ and v_n^- are disjoint, it follows from lemma 3.4(iii) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{G(x, u_n)}{\|u_n\|_\lambda^p} \, \mathrm{d}x \\
&\quad \geqslant \int_{\Omega} \frac{G(x, (\|u_n\|_\lambda / p[m_\lambda + 1]^{1/p}) v_n)}{\|u_n\|_\lambda^p} \, \mathrm{d}x \\
&\quad = \int_{\Omega} \frac{G(x, (\|u_n\|_\lambda / p[m_\lambda + 1]^{1/p}) v_n^+) + G(x, (\|u_n\|_\lambda / p[m_\lambda + 1]^{1/p}) v_n^-)}{\|u_n\|_\lambda^p} \, \mathrm{d}x \\
&\quad \geqslant \int_{\Omega} \frac{G(x, (\|u_n\|_\lambda / p[m_\lambda + 1]^{1/p})(\delta/2)) + G(x, (\|u_n\|_\lambda / p[m_\lambda + 1]^{1/p})(-\delta/2))}{\|u_n\|_\lambda^p} \, \mathrm{d}x, \\
&\qquad\qquad n \geqslant n_0.
\end{aligned}$$

Then lemma 3.4(ii) yields

$$\int_{\mathbb{R}^N} \frac{G(x, u_n)}{\|u_n\|_\lambda^p} \, \mathrm{d}x \to \infty \quad \text{as } n \to \infty.$$

However, on the other hand,

$$\int_{\mathbb{R}^N} \frac{G(x, u_n)}{\|u_n\|_\lambda^p} \, \mathrm{d}x = \frac{(1/p)\|u_n\|_\lambda^p - S_\lambda(u_n)}{\|u_n\|_\lambda^p} \to \frac{1}{p} \quad \text{as } n \to \infty,$$

which gives the desired contradiction. Therefore, any minimizing sequence (u_n) is indeed bounded.

Step 2 (existence of a minimizer). Let $(u_n) \subset M_\lambda$ such that $S_\lambda(u_n) \to m_\lambda$. Since (u_n) is bounded in W_A, there exists $u \in W_A$ such that $u_n \rightharpoonup u$ and $u_n^\pm \rightharpoonup u^\pm$ weakly in W_A as $n \to \infty$. It immediately follows from the weak lower semi-continuity of $u \mapsto \|\nabla u\|_p^p$ and from lemma 2.5 that

$$S_\lambda(u) \leqslant \liminf_{n \to \infty} S_\lambda(u_n) = m_\lambda.$$

Hence, we need only prove that $u^\pm \in N_\lambda$. We first observe that $u^\pm \not\equiv 0$. Indeed, by lemma 2.5 and (3.1),

$$\int_{\mathbb{R}^N} g(x, u^\pm) u^\pm \, \mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{R}^N} g(x, u_n^\pm) u_n^\pm \, \mathrm{d}x = \lim_{n \to \infty} \|u_n^\pm\|_\lambda^p \geqslant \delta_\lambda^p > 0. \tag{3.6}$$

Invoking again the weak lower semi-continuity of $u \mapsto \|\nabla u\|_p^p$ and lemma 2.5, it follows from (3.6) that

$$J_\lambda(u^\pm) = \|u^\pm\|_\lambda^p - \lim_{n \to \infty} \|u_n^\pm\|_\lambda^p \leqslant 0.$$

Suppose by contradiction that

$$\|u^+\|_\lambda^p < \liminf_{n\to\infty} \|u_n^+\|_\lambda^p.$$

Then $t^+ := t_\lambda(u^+) < 1$ and $t^- := t_\lambda(u^-) \leqslant 1$, $t^+u^+ + t^-u^- \in M_\lambda \subset N_\lambda$, and so

$$S_\lambda(t^+u^+ + t^-u^-) = \int_{\mathbb{R}^N} h(x, t^+u^+ + t^-u^-)\,\mathrm{d}x$$

by (3.2). Since $h(x,0) \equiv 0$ and the supports of u^+ and u^- are disjoint, it follows from lemma 3.4(iii) that

$$\begin{aligned}
S_\lambda(t^+u^+ + t^-u^-) &= \int_{\mathbb{R}^N} h(x, t^+u^+ + t^-u^-)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} h(x, t^+u^+)\,\mathrm{d}x + \int_{\mathbb{R}^N} h(x, t^-u^-)\,\mathrm{d}x \\
&< \int_{\mathbb{R}^N} h(x, u^+)\,\mathrm{d}x + \int_{\mathbb{R}^N} h(x, u^-)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} h(x, u^+ + u^-)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} h(x, u)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} h(x, u_n)\,\mathrm{d}x \\
&= \lim_{n\to\infty} S_\lambda(u_n) = m_\lambda.
\end{aligned}$$

This contradiction concludes the proof. □

Remark 3.7. If $\lambda < 0$, then $\|\cdot\|_\lambda$ is a norm, $(W_A, \|\cdot\|_\lambda)$ is uniformly convex and the proof shows that $u_n \to u$ in W_A (up to a subsequence).

We are now in a position to complete the proof of theorem 3.1.

Proof of theorem 3.1. Proposition 3.6 yields an element $u_3 \in W_A$ that minimizes S_λ on the nodal Nehari set M_λ. To conclude the proof of theorem 3.1, we will now show that $S'_\lambda(u_3) = 0$. The existence of the critical points u_1 and u_2 (with $u_1 > 0$ a.e. and $u_2 < 0$ a.e.) follows similarly, by minimizing S_λ over $N_\lambda^\pm$ instead of M_λ, where

$$N_\lambda^\pm = \{u \in N_\lambda \colon u^\mp = 0\}.$$

Since M_λ is not a submanifold of W_A, we cannot use the Lagrange multiplier theorem to infer that the minimizer u_3 is indeed a critical point of S_λ. To overcome this difficulty, we appeal to a theorem of Miranda,[6] which was first established in [14]. For the reader's convenience, we recall here the two-dimensional version of this result. An elegant proof can be found in [18].

[6]This is essentially a version of Brouwer's fixed-point theorem.

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Lemma 3.8. *Let $L > 0$, let $R = (-L, L)^2 \subset \mathbb{R}^2$ and consider a continuous function $F = (F_1, F_2)\colon \bar{R} \to \mathbb{R}^2$ that satisfies $F(t, s) \neq 0$ for all $(s, t) \in \partial R$, and the following conditions on the boundary ∂R:*

$$F_1(-L, t) \geqslant 0, \qquad F_1(L, t) \leqslant 0, \qquad F_2(s, -L) \geqslant 0, \qquad F_2(s, L) \leqslant 0.$$

In other words, the vector field F, evaluated on the boundary ∂R, always points towards the interior of R. There then exists $(s_0, t_0) \in R$ such that $F(s_0, t_0) = 0$.

This lemma is applied in the following manner. Suppose by contradiction that $S'_\lambda(u_3) \neq 0$. Then there is a $\varphi \in W_A$ such that $\langle S'_\lambda(u_3), \varphi\rangle = -2$, and so, by continuity of S'_λ, there is an $\varepsilon > 0$ such that

$$\langle S'_\lambda(tu_3^+ + su_3^- + r\varphi), \varphi\rangle < -1 \tag{3.7}$$

for all $r \in (0, \varepsilon]$ and all $(s, t) \in \bar{R}$, where $R = (1-\varepsilon, 1+\varepsilon)^2 \subset \mathbb{R}^2$. Now consider a continuous function $\eta\colon \bar{R} \to [0, \varepsilon]$ such that $\eta(1, 1) = \varepsilon$, $\eta(\partial R) = 0$, and $\eta \neq 0$ on R. We define $F\colon \bar{R} \to \mathbb{R}^2$ by

$$F(s, t) = (J_\lambda((tu_3^+ + su_3^- + \eta(t, s)\varphi)^-), J_\lambda((tu_3^+ + su_3^- + \eta(t, s)\varphi)^+)).$$

First of all, it is clear that F is continuous. Next, for $s = 1-\varepsilon$ and $t \in [1-\varepsilon, 1+\varepsilon]$, using the function $\varphi_u(t)$ introduced in the proof of lemma 3.5 we have

$$\frac{F_1(1-\varepsilon, t)}{(1-\varepsilon)^p} = \frac{J_\lambda((1-\varepsilon)u_3^-)}{(1-\varepsilon)^p} = \varphi_{u_3^-}(1-\varepsilon) > \varphi_{u_3^-}(1) = 0$$

since $u_3^- \in N_\lambda$. Similar arguments show that $F_1(1+\varepsilon, t) < 0$ and $\mp F_2(s, 1\pm\varepsilon) > 0$. Hence, the hypotheses of lemma 3.8 are satisfied and there exists $(s_0, t_0) \in R$ such that $F(s_0, t_0) = 0$. Remarking that $t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi \neq 0$ by (3.7), it follows that $t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi \in M_\lambda$. We will reach a contradiction by showing that $S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi) < m_\lambda$. By (3.7), we have

$$\begin{aligned} S_\lambda(t_0u_3^+ &+ s_0u_3^- + \eta(t_0, s_0)\varphi) \\ &= S_\lambda(t_0u_3^+ + s_0u_3^-) + \int_0^{\eta(t_0, s_0)} \langle S'_\lambda(tu_3^+ + su_3^- + r\varphi), \varphi\rangle \,\mathrm{d}r \\ &< S_\lambda(t_0u_3^+ + s_0u_3^-) - \eta(t_0, s_0). \end{aligned}$$

If $(s_0, t_0) = (1, 1)$, then $S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi) < m_\lambda - \varepsilon$ and we are done. So suppose that $(s_0, t_0) \neq (1, 1)$. Since $u_3^\pm \in N_\lambda$, it follows from (3.5) that

$$\begin{aligned} S_\lambda(t_0u_3^+ + s_0u_3^- + \eta(t_0, s_0)\varphi) &< S_\lambda(t_0u_3^+ + s_0u_3^-) - \eta(t_0, s_0) \\ &= S_\lambda(t_0u_3^+) + S_\lambda(s_0u_3^-) - \eta(t_0, s_0) \\ &\leqslant S_\lambda(u_3^+) + S_\lambda(u_3^-) - \eta(t_0, s_0) \\ &= S_\lambda(u_3) - \eta(t_0, s_0) < m_\lambda, \end{aligned}$$

yielding the desired contradiction. This completes the proof of theorem 3.1. □

4. Regularity and conclusion of the proof

We will now prove a regularity result for the solutions given by theorem 3.1. We say that $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ if for any compact $K \subset \mathbb{R}^N$ there exists $\alpha = \alpha(K)$ such that $u \in C^{1,\alpha}(K)$. Once $C^{1,\alpha}$ regularity is proved, we will prove the remaining statements of theorem 1.1 about the nodal properties of the solutions.

PROPOSITION 4.1. *Suppose that the hypotheses* (A_1), (A_2) *and* (g_1) *hold. Then any weak solution* u *of (1.1) satisfies* $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. Assumption (A_1) implies that $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N) \subset L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N)$. It then follows from [16] that $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, provided that we know *a priori* that $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^N)$. For a fixed compact $K \subset \mathbb{R}^N$, we briefly explain how the results of [12] imply that $u \in L^{\infty}(K)$. Firstly, the integrability conditions on A and B given in hypotheses (A_1), (A_2) and (g_1) precisely ensure that assumptions (7.1) and (7.2) in [12, ch. 4, § 7] (together with conditions (1)–(3) there) hold. Thus, the idea is to apply theorem 7.1 of [12, ch. 4, § 7]. In the present context, the parameters m and q appearing there are given by $m = p$ and $q = p^*$. We need only explain here why the conclusion of this theorem still holds without the hypothesis that

$$\operatorname*{ess\,sup}_{\partial K} |u| < \infty. \tag{4.1}$$

This assumption is used in the proof of [12, ch. 4, theorem 7.1] in the two following instances. First, to derive the estimate (7.3), the test function $\eta(x) = \max\{u(x) - k, 0\}$ is used, where k is a positive parameter such that $k \geqslant \operatorname{ess\,sup}_{\partial K} |u|$. This restriction is due to the definition of a weak solution in [12, ch. 4], requiring that $\eta \in W^{1,p}_0(K)$. Our definition of a weak solution (see (1.9)) allows us to merely consider $\eta \in W^{1,p}(K)$, and we can derive the estimate (7.3) in the same way as in [12]. Finally, assumption (4.1) is used to conclude the proof of theorem 7.1 by invoking theorem 5.1 of [12, ch. 2]. It turns out that theorem 5.2 of [12, ch. 2] does the job as well, and does not require (4.1). This completes the proof. □

We can now finish the proof of theorem 1.1.

Proof of theorem 1.1. It only remains to show that $u_1 > 0$, $u_2 < 0$, and u_3 has exactly two nodal domains, as defined in § 1.1. The positivity of u_1 and the negativity of u_2 follow from the strong maximum principle (see, for example, [17, theorem 5]). Regarding u_3, we already know from the previous results that it has two nodal domains. Suppose by contradiction that u_3 has (at least) three distinct nodal domains $\Omega_i \subset \mathbb{R}^N$, $i = 1, 2, 3$, and define

$$v_i(x) = \begin{cases} u_3(x) & \text{if } x \in \Omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $v_i \in W_A$, $i = 1, 2, 3$, and without loss of generality we can suppose that $v_1 > 0$ and $v_2 < 0$. Since $S'_\lambda(u_3) = 0$, it follows that $v_i \in N_\lambda$, $i = 1, 2, 3$, $v_1 + v_2 \in M_\lambda$, and so (3.2) implies that

$$m_\lambda \leqslant S_\lambda(v_1 + v_2) < S_\lambda(u_3) = m_\lambda.$$

This contradiction concludes the proof. □

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Appendix A.

A.1. Compactness

Since we were not able to find the exact result we need in the literature, we now give a proof of the compactness of the embedding $W_A(\Omega) \subset L^r_B(\Omega)$ in proposition 2.2. We shall make extensive use of the classic Hölder and interpolation inequalities, which hold in the weighted spaces $L^q_B(\Omega)$, as in the usual case in which $B \equiv 1$.

Proof of proposition 2.2 (continued). We start by assuming that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary. We will first explain how the proof of the classic Rellich–Kondrachov theorem can be adapted to the present case. We follow the proof of Brezis [6, théorème IX.16], which is based on a criterion of strong compactness in L^p spaces [6, corollaire IV.26]. Note that [6, corollaire IV.26] is a consequence of the famous Riesz–Fréchet–Kolmogorov theorem [6, théorème IX.25]. It is easy to see that the proof of both [6, théorème IX.25] and [6, corollaire IV.26] remain virtually unchanged in the case of a weighted L^p space such as $L^r_B(\Omega)$. Therefore, we can merely follow the proof of [6, théorème IX.16] for the case in which $p < N$.

Letting $\mathcal{F}$ be the unit ball in $W_A(\Omega)$, this amounts to verifying assumptions (IV.23) and (IV.24) of [6, corollaire IV.26]. Following Brezis, for an open set $\omega \subset \Omega$ such that $\bar{\omega}$ is compact and $\bar{\omega} \subset \Omega$, we write $\omega \subset\subset \Omega$. For a given $h \in \mathbb{R}^N$, we also define the translation $\tau_h u$ of a function u by $\tau_h u(x) = u(x+h)$, $x \in \mathbb{R}^N$. Assumptions (IV.23) and (IV.24) of [6, corollaire IV.26] now read

$$\forall \varepsilon > 0 \quad \forall \omega \subset\subset \Omega, \quad \exists \delta \in (0, \operatorname{dist}(\omega, \Omega^{c})) \quad \text{such that (s.t.)}$$
$$\|\tau_h u - u\|_{L^q_B(\omega)} < \varepsilon \quad \forall h \in \mathbb{R}^N \text{ s.t. } |h| < \delta \quad \forall u \in \mathcal{F} \tag{A 1}$$

and

$$\forall \varepsilon > 0 \quad \exists \omega \subset\subset \Omega \quad \text{s.t.} \quad \|u\|_{L^q_B(\Omega \setminus \omega)} < \varepsilon \quad \forall u \in \mathcal{F}. \tag{A 2}$$

To prove (A 1), consider $\alpha \in (0,1]$ such that $1/q = \alpha/1 + (1-\alpha)/p^*$. By interpolation,

$$\|\tau_h u - u\|_{L^q_B(\omega)} \leqslant \|\tau_h u - u\|^{\alpha}_{L^1_B(\omega)} \|\tau_h u - u\|^{1-\alpha}_{L^{p^*}_B(\omega)}. \tag{A 3}$$

Now following the proof of [6, proposition IX.3], it is easily seen that

$$\|\tau_h u - u\|_{L^1_B(\omega)} \leqslant \|B\|_\infty \|\nabla u\|_{L^1(\Omega)} |h|. \tag{A 4}$$

Hence, recalling that Ω is bounded and using (2.2), we have

$$\begin{aligned} \|\tau_h u - u\|_{L^q_B(\omega)} &\leqslant \|B\|^{\alpha}_\infty \|\nabla u\|^{\alpha}_{L^1(\Omega)} |h|^{\alpha} (2\|u\|_{L^{p^*}_B(\Omega)})^{1-\alpha} \\ &\leqslant 2^{1-\alpha} \|B\|_\infty \|\nabla u\|^{\alpha}_{L^1(\Omega)} \|u\|^{1-\alpha}_{L^{p^*}(\Omega)} |h|^{\alpha} \\ &\leqslant C \|\nabla u\|^{\alpha}_{L^p(\Omega)} \|\nabla u\|^{1-\alpha}_{L^p(\Omega)} |h|^{\alpha} \\ &\leqslant C|h|^{\alpha} \quad (\text{since } \|\nabla u\|_{L^p(\Omega)} \leqslant 1 \text{ for } u \in \mathcal{F}), \end{aligned}$$

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which proves (A 1). On the other hand, for all $u \in \mathcal{F}$, it follows from Hölder's inequality and (2.2) that

$$\begin{aligned}\|u\|^q_{L^q_B(\Omega\setminus\omega)} = \int_{\Omega\setminus\omega} |u|^q B(x)\,dx &\leqslant \|B\|_\infty |\Omega\setminus\omega|^{(p^*-q)/p^*}\|u\|^q_{L^{p^*}_B(\Omega)} \\ &\leqslant C|\Omega\setminus\omega|^{(p^*-q)/p^*}\|u\|^q_{L^{p^*}(\Omega)} \\ &\leqslant C\|\nabla u\|^q_{L^p(\Omega)}|\Omega\setminus\omega|^{(p^*-q)/p^*} \\ &\leqslant C|\Omega\setminus\omega|^{(p^*-q)/p^*}.\end{aligned}$$

Therefore, $\|u\|_{L^q_B(\Omega\setminus\omega)} \leqslant C|\Omega\setminus\omega|^{1/q-1/p^*}$, which proves (A 2) and concludes the proof that the embedding is compact when Ω is a smooth bounded domain.

We now consider the case in which $\Omega = \mathbb{R}^N$. Let $(u_n) \subset W_A(\mathbb{R}^N)$ be a bounded sequence. We will show that (u_n) is relatively compact in $L^q_B(\mathbb{R}^N)$. Firstly, since $W_A(\mathbb{R}^N)$ is reflexive, we can suppose that $u_n \rightharpoonup u$ weakly in $W_A(\mathbb{R}^N)$ for some $u \in W_A(\mathbb{R}^N)$. Also, denoting by $B(0,R)$ the ball of radius R centred at $x=0$ in $\mathbb{R}^N$, $u_n|_{B(0,R)} \rightharpoonup u|_{B(0,R)}$ weakly in $W_A(B(0,R))$, and we already know that (up to a subsequence) $u_n|_{B(0,R)} \to u|_{B(0,R)}$ in $L^q_B(B(0,R))$. Furthermore, by Hölder's inequality and (2.2),

$$\begin{aligned}\int_{|x|\geqslant R}|u_n-u|^q B(x)\,dx &\leqslant C\int_{|x|\geqslant R}(|u_n|^q+|u|^q)B(x)\,dx \\ &\leqslant C\biggl(\int_{|x|\geqslant R}(|u_n|^q+|u|^q)^{p^*/q}\,dx\biggr)^{q/p^*}\|B\|_{L^{p^*/(p^*-q)}(|x|\geqslant R)} \\ &\leqslant C(\|u_n\|^q_{p^*}+\|u\|^q_{p^*})\|B\|_{L^{p^*/(p^*-q)}(|x|\geqslant R)} \\ &\leqslant C(\|\nabla u_n\|^q_p+\|\nabla u\|^q_p)\|B\|_{L^{p^*/(p^*-q)}(|x|\geqslant R)} \\ &\leqslant C\|B\|_{L^{p^*/(p^*-q)}(|x|\geqslant R)}. \end{aligned} \tag{A 5}$$

Since $B \in L^{p^*/(p^*-q)}(\mathbb{R}^N)$, the right-hand side of (A 5) can be made arbitrarily small by choosing $R>0$ large enough, uniformly in n, which concludes the proof. □

Remark A.1. Let us now explain how the hypothesis $B \in L^\infty(\mathbb{R}^N)$ in (g_1) can be relaxed to a local integrability condition in the above proof, and hence throughout the whole paper. Choosing $t \in (q,p^*)$ and $\alpha \in (0,1]$ such that $1/q = \alpha/1+(1-\alpha)/t$, we start by replacing the interpolation inequality (A 3) with

$$\|\tau_h u - u\|_{L^q_B(\omega)} \leqslant \|\tau_h u - u\|^\alpha_{L^1_B(\omega)}\|\tau_h u - u\|^{1-\alpha}_{L^t_B(\omega)}. \tag{A 6}$$

Then, instead of (A 4), the proof of [6, proposition IX.3] can be modified to show that

$$\|\tau_h u - u\|_{L^1_B(\omega)} \leqslant \|B\|_{L^{p/(p-1)}(\Omega)}\|\nabla u\|_{L^p(\Omega)}|h|. \tag{A 7}$$

On the other hand,

$$\begin{aligned}\|\tau_h u - u\|_{L^t_B(\omega)} \leqslant 2\|u\|_{L^t_B(\Omega)} &\leqslant 2\|B\|^{1/t}_{L^{p^*/(p^*-t)}(\Omega)}\|u\|_{L^{p^*}(\Omega)} \\ &\leqslant C\|B\|^{1/t}_{L^{p^*/(p^*-t)}(\Omega)}\|\nabla u\|_{L^p(\Omega)}.\end{aligned} \tag{A 8}$$

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That (A 1) holds now follows from (A 6)–(A 8), provided that[7]

$$B \in L^s_{\text{loc}}(\mathbb{R}^N), \quad \text{where } s = \max\left\{\frac{p}{p-1}, \frac{p^*}{p^*-t}\right\}. \tag{A 9}$$

Furthermore, we have

$$\begin{aligned}\|u\|^q_{L^q_B(\Omega\setminus\omega)} = \int_{\Omega\setminus\omega} |u|^q B(x)\,\mathrm{d}x &\leqslant \|B\|_{L^{p^*/(p^*-q)}(\Omega\setminus\omega)} \|u\|^q_{L^{p^*}(\Omega)} \\ &\leqslant C\|B\|_{L^{p^*/(p^*-q)}(\Omega\setminus\omega)} \|\nabla u\|^q_{L^p(\Omega)},\end{aligned}$$

and so (A 2) follows from the assumption that $B \in L^{p^*/(p^*-q)}(\mathbb{R}^N)$. This completes the proof of compactness of the embedding $W_A(\Omega) \subset L^q_B(\Omega)$ when Ω is a bounded domain with smooth boundary. The case in which $\Omega = \mathbb{R}^N$ then follows as before. Hence, we see that proposition 2.2 holds provided that $B \in L^{p^*/(p^*-q)}(\mathbb{R}^N)$ and B satisfies (A 9) for some $t \in (q, p^*)$.

A.2. Differentiability

This appendix is devoted to the proof of lemma 2.4. Before we proceed with the proof, let us first remark that, thanks to proposition 2.2,

$$u \in W_A(\mathbb{R}^N) \quad \Longrightarrow \quad u \in L^p_A(\mathbb{R}^N) \quad \text{and} \quad u \in L^q_B(\mathbb{R}^N),$$

where B and q were introduced in hypothesis (g_1). As in appendix A.1, we will take advantage of the Hölder inequality in the weighted Lebesgue spaces $L^p_A(\mathbb{R}^N)$ and $L^q_B(\mathbb{R}^N)$.

Proof of lemma 2.4. For given $u, v \in W_A(\mathbb{R}^N)$ we start by computing the Gâteaux derivative $\mathrm{D}\,S_\lambda(u)$ of S_λ in the direction v, and we show that it is equal to the right-hand side of (2.3). That is, we compute

$$\lim_{t\to 0} \frac{1}{t}[S_\lambda(u+tv) - S_\lambda(u)].$$

It follows from Hölder's inequality that $|\nabla u|^{p-2}\nabla u \cdot \nabla v \in L^1(\mathbb{R}^N)$. The derivation of the first term then follows in a standard manner, using the mean-value theorem and the dominated convergence theorem. Let us now consider the other two terms in more detail. For $t \neq 0$, and $x \in \mathbb{R}^N$, it follows from the mean-value theorem that there exists $s = s(t,x) \in [0,1]$ such that

$$|u+tv|^p - |u|^p = p|u+stv|^{p-2}(u+stv)tv,$$

and so

$$\frac{1}{t}\frac{1}{p}A(x)(|u+tv|^p - |u|^p) \to A(x)|u|^{p-2}uv \quad \text{as } t \to 0 \text{ for almost every } x \in \mathbb{R}^N.$$

Moreover,

$$\left|\frac{1}{t}\frac{1}{p}A(x)(|u+tv|^p - |u|^p)\right| \leqslant A(x)|u+stv|^{p-1}|v| \leqslant CA(x)(|u|^{p-1}|v| + |v|^p),$$

[7]Note that $\dfrac{p}{p-1} > \dfrac{p^*}{p^*-t} \iff t < \dfrac{N}{N-p}$.

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where $A|v|^p \in L^1(\mathbb{R}^N)$ since $v \in W_A$, and $A(x)|u|^{p-1}|v| \in L^1(\mathbb{R}^N)$ by Hölder's inequality in $L^p_A(\mathbb{R}^N)$. Hence, by dominated convergence,

$$\frac{1}{t}\frac{1}{p}\int_{\mathbb{R}^N} A(x)(|u+tv|^p - |u|^p)\,\mathrm{d}x \to \int_{\mathbb{R}^N} A(x)|u|^{p-2}uv\,\mathrm{d}x \quad \text{as } t \to 0.$$

To deal with the last term, we apply again the mean-value theorem, which yields a number $s = s(t,x) \in [0,1]$ such that

$$\frac{1}{t}(G(x,u+tv) - G(x,u)) = \frac{1}{t}g(x,u+stv)tv \to g(x,u)v \qquad \text{as } t \to 0, \text{ for almost every } x \in \mathbb{R}^N.$$

Also, by (g_1),

$$|g(x,u+stv)v| \leqslant B(x)|u+stv|^{q-1}|v| \leqslant CB(x)(|u|^{q-1}|v| + |v|^q) \in L^1(\mathbb{R}^N)$$

thanks to proposition 2.2 with $r = q$. It then follows by dominated convergence that

$$\frac{1}{t}\int_{\mathbb{R}^N}(G(x,u+tv) - G(x,u))\,\mathrm{d}x \to \int_{\mathbb{R}^N} g(x,u)v\,\mathrm{d}x \quad \text{as } t \to 0.$$

We have thus proved that the Gâteaux derivative $\mathrm{D}\,S_\lambda(u)v$ exists and is equal to the right-hand side of (2.3).

To complete the proof, we will now show that $\mathrm{D}\,S_\lambda(u) \in W_A^*$ for all $u \in W_A$, and that the mapping $u \mapsto \mathrm{D}\,S_\lambda(u)$ is continuous. Hölder's inequality yields

$$\begin{aligned}\left|\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}^N} |\nabla u|^{p-1}|\nabla v|\,\mathrm{d}x &\leqslant \||\nabla u|^{p-1}\|_{p/(p-1)}\|\nabla v\|_p \\ &= \|\nabla u\|_p^{p-1}\|\nabla v\|_p \\ &\leqslant \|u\|_{W_A}^{p-1}\|v\|_{W_A}, \\ \left|\int_{\mathbb{R}^N} A(x)|u|^{p-2}uv\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}^N} |u|^{p-1}|v|A(x)\,\mathrm{d}x &\leqslant \||u|^{p-1}\|_{L_A^{p/(p-1)}}\|v\|_{L_A^p} \\ &= \|u\|_{L_A^p}^{p-1}\|v\|_{L_A^p} \\ &\leqslant \|u\|_{W_A}^{p-1}\|v\|_{W_A},\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^N} g(x,u)v\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}^N} |g(x,u)||v|\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^N} |u|^{q-1}|v|B(x)\,\mathrm{d}x \\ &\leqslant \||u|^{q-1}\|_{L_B^{q/(q-1)}}\|v\|_{L_B^q} \\ &= \|u\|_{L_B^q}^{q-1}\|v\|_{L_B^q} \\ &\leqslant C\|u\|_{W_A}^{q-1}\|v\|_{W_A},\end{aligned}$$

where the last inequality follows from proposition 2.2. These estimates show that $\mathrm{D}\,S_\lambda(u) \in W_A^*$ for all $u \in W_A$.

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To prove that $u \mapsto \mathrm{D}\, S_\lambda(u)$ is continuous, consider $(u_n) \subset W_A$ such that $u_n \to u$ in W_A. We will show that

$$\|\mathrm{D}\, S_\lambda(u_n) - \mathrm{D}\, S_\lambda(u)\| = \sup_{v \in W_A \setminus \{0\}} \frac{|\langle \mathrm{D}\, S_\lambda(u_n) - \mathrm{D}\, S_\lambda(u), v\rangle|}{\|v\|_{W_A}} \to 0 \quad \text{as } n \to \infty. \tag{A 10}$$

We have

$$\begin{aligned} |\langle \mathrm{D}\, S_\lambda(u_n) - \mathrm{D}\, S_\lambda(u), v\rangle| \leqslant \int_{\mathbb{R}^N} &||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u|\, |\nabla v| \,\mathrm{d}x \\ &+ |\lambda| \int_{\mathbb{R}^N} A(x) ||u_n|^{p-2}u_n - |u|^{p-2}u|\, |v| \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|\, |v| \,\mathrm{d}x. \end{aligned}$$

Using Hölder's inequality in the same fashion as above, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} ||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u|\, |\nabla v| \,\mathrm{d}x& \\ \leqslant \||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\|_{p/(p-1)} \|v\|_{W_A},& \end{aligned} \tag{A 11}$$

$$\int_{\mathbb{R}^N} A(x) ||u_n|^{p-2}u_n - |u|^{p-2}u|\, |v| \,\mathrm{d}x \leqslant \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{L_A^{p/(p-1)}} \|v\|_{L_A^p} \tag{A 12}$$

and

$$\int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|\, |v| \,\mathrm{d}x \leqslant C \||u_n|^{q-2}u_n - |u|^{q-2}u\|_{L_B^{q/(q-1)}} \|v\|_{W_A}. \tag{A 13}$$

We now observe that, since $u_n \to u$ in W_A, we have (up to a subsequence) $u_n \to u$ and $\nabla u_n \to \nabla u$ pointwise a.e., $A|u_n|^p \leqslant f$ and $B|u_n|^q \leqslant g$ (by proposition 2.2) for some functions $f, g \in L^1(\mathbb{R}^N)$, uniformly in n. The limit in (A 10) then follows from estimates (A 11)–(A 13) by dominated convergence, up to a subsequence. Since the previous argument can be applied to any subsequence of (u_n), this completes the proof. □

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