J. Appl. Prob. **55,** 94–111 (2018)
*doi:*10.1017/jpr.2018.8

ON A JUMP-TELEGRAPH PROCESS DRIVEN BY AN ALTERNATING FRACTIONAL POISSON PROCESS

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Abstract

The basic jump-telegraph process with exponentially distributed interarrival times deserves interest in various applied fields such as financial modelling and queueing theory. Aiming to propose a more general setting, we analyse such a stochastic process when the interarrival times separating consecutive velocity changes (and jumps) have generalized Mittag-Leffler distributions, and constitute the random times of a fractional alternating Poisson process. By means of renewal theory-based issues we obtain the forward and backward transition densities of the motion in series form, and prove their uniform convergence. Specific attention is then given to the case of jumps with constant size, for which we also obtain the mean of the process. Finally, we investigate the first-passage time of the process through a constant positive boundary, providing its formal distribution and suitable lower bounds.

Keywords: Finite velocity; random motion; generalized Mittag-Leffler distribution; jump process; first-passage time

2010 Mathematics Subject Classification: Primary 60K15
Secondary 60G22; 60J75

1. Introduction

The (integrated) telegraph process is a continuous-time stochastic process that describes a random motion on the real line. The motion has finite (constant) velocity and its direction is reversed at every event of a homogeneous Poisson process. The transition density of the telegraph process satisfies a second-order (hyperbolic) telegraph equation (see the seminal articles by Goldstein [15] and Kac [19]). Under suitable conditions, the aforementioned equation tends asymptotically to the heat diffusion equation. In other words, the transition density of the telegraph process tends to the transition density of the one-dimensional Brownian motion, the former being more general but more difficult to deal with than the latter. Various extensions of the telegraph process have been proposed in the literature towards motions characterized by two or more velocities, or by random velocities, or with velocity changes governed by an alternating renewal process. The telegraph process and its generalizations have been widely applied in biomathematics and in queueing theory (see [7, Section 1], [14], and [31]).

Motions with deterministic or random jumps along the alternating direction at each velocity reversal have been studied in detail (see [9] and [12], also with a special focus to some general

Received 17 March 2017; revision received 2 November 2017.
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rescaling properties [22]). Damped versions of the telegraph process have been considered in [35] and [8] in the presence and in the absence of jumps, respectively. A jump-telegraph process is interesting for the purposes of financial modelling. For the sake of brevity, we only mention two works: Ratanov [36], in which the author proposed a new generalisation of the jump-telegraph process with variable velocities and jumps, and then applied this construction to market modelling; and Kolesnik and Ratanov (see [21] and the references therein), in which the authors presented a thorough investigation on the telegraph process and its applications to option pricing. Estimation procedures for the standard and geometric telegraph process (see [5], [6], and [17]), and for a Brownian motion governed by a telegraph process [34] have been recently provided under the hypothesis of discrete-time sampling.

In the last couple of decades a number of works have appeared in which the authors analysed processes governed by (space)-time fractional telegraph equations, obtained by replacing the ordinary derivatives in the telegraph equation by suitable fractional derivatives (see [26] and [29]). The key features of the resulting processes include long-range memory, path-dependence, non-Markovian properties, and anomalous diffusion behaviour. Masoliver [24] justified on physical grounds the fractional telegraph equation. Special forms of fractional telegraph equations with rational order were studied in [1]. Another approach was adopted in [2], in which the authors proposed a finite-velocity planar random motion whose changes of direction occur at times spaced by a fractional Poisson process. In general, fractional calculus is useful in computing probability distribution functions with fat tails. In the recent past, pure-jump fractional processes have attracted great attention. Just to mention a few examples, Beghin and Orsingher [3] illustrated various results on the fractional Poisson process and also focused on certain higher-order extensions, whilst a fractional counting process with multiple jumps was studied in [11]. See also [33] for a generalization of the space-fractional Poisson process. Birth, birth–death, and death processes have been investigated in [27], [28], and [30] respectively.

In the light of the previous investigations, and aiming to construct a more general model that takes into account both the occurrence of jumps and the fractional nature, in this paper we propose and study a one-dimensional jump-telegraph process with deterministic jumps occurring at velocity changes, and with intertimes governed by a fractional alternating counting process studied in detail in [10]. We obtain the probability law of the new process, which is given in a series form involving the generalized Mittag-Leffler function, also known as the Prabhakar function, and defined as

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k z^k}{k!\,\Gamma(\alpha\,k+\beta)},\qquad z\in\mathbb{C}, \tag{1.1}$$

with $\alpha,\beta,\gamma\in\mathbb{C}$, and $\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma)\}>0$, and where $(\gamma)_k=\Gamma(\gamma+k)/\Gamma(\gamma)$, $k\in\mathbb{N}$, is the Pochhammer symbol. We also discuss the uniform convergence of the distribution.

We devote special attention to the case of jumps having constant size. It turns out that the structure of the solution (see Proposition 3.1) is quite similar to that obtained in [9], even though the shape of the relevant density is qualitatively different, due to the special form of the underlying generalized Mittag-Leffler distributions. We also obtain the mean of the process in the special case of identically distributed upward and downward intertimes, and compare it to the means of other fractional processes. We stress that the mean results in the sum of two terms. The first term is linear in time and refers to the alternating component of the motion. The second term is a power of time, where the exponent (ν) is the 'tail' index of the underlying generalized Mittag-Leffler distribution, and is related to the jump component of the motion.

An interesting but difficult problem is the determination of the first-passage time distribution through a constant boundary. This problem for an asymmetric telegraph process was treated in [23], where the distributions of the first-passage times were described by using the Laplace transforms and their inversions. We provide a formal expression of the first-passage time distribution (in series form) by conditioning on the number of jumps. The formal expression is finally used to provide suitable lower bounds.

We point out that our proposed model (as well as the model studied by Di Crescenzo and Martinucci [9]) cannot be used in financial modelling. Indeed, the process is not a martingale under the assumption that the particle jumps upwards after forward motion (respectively, downwards after backward motion). For market modelling another assumption is more adequate, i.e. when the jump directions are opposite to the current tendency. The proposed model can be more effective in other fields, such as queueing, to describe alternating fluid workload with batch arrivals and services.

2. Probability law of the jump-telegraph process

Let $\{(X_t, V_t),\ t \geq 0\}$ denote a jump-telegraph process, where X_t and V_t represent, respectively, the position and the velocity of a particle running on the real line. The motion is performed starting at the origin at time 0, with two alternating constant velocities, say $-v, c$. The initial velocity can be either $-v$ or c. The velocities change at random times, which are the epochs of an alternating counting process $\{N_t,\ t \geq 0\}$. A jump occurs at each velocity change, the displacement of the jump being $\alpha_k > 0$ (upward jump) or $-\beta_k < 0$ (downward jump) if it follows the kth period of forward or backward motion, respectively.

Formally, the process is described by the following stochastic equations, for $t > 0$ and $V_0 \in \{-v, c\}$:

$$X_t = \int_0^t V_s \, \mathrm{d}s + \sum_{k=1}^{N_t} w_k, \tag{2.1}$$

$$V_t = \frac{c-v}{2} + \left[V_0 - \left(\frac{c-v}{2}\right)\right](-1)^{N_t},$$

where

$$w_k = \frac{\alpha_k - \beta_k}{2} - \operatorname{sgn}(V_0)\frac{\alpha_k + \beta_k}{2}(-1)^k. \tag{2.2}$$

We assume that $\{U_k\}_{k\in\mathbb{N}}$ and $\{D_k\}_{k\in\mathbb{N}}$ are independent sequences of independent copies of the nonnegative random variables U and D, which describe the duration of the kth random period in which the motion proceeds forward or backward, respectively. Note that the interarrival random times of the alternating counting process $\{N_t,\ t \geq 0\}$ are $U_1, D_1, U_2, D_2, \ldots$ (respectively, $D_1, U_1, D_2, U_2, \ldots$) when the initial velocity is positive (respectively, negative). For instance, the generic structure of a sample path of X_t is shown in Figure 1.

In a previous paper [9], the case of Erlang-distributed random periods U and D and deterministic jumps was studied in detail. As a novelty, in the present paper we investigate the case when the random times U and D separating consecutive velocity changes (and jumps) follow a Mittag-Leffler distribution with parameters (λ, ν) and (μ, ν), respectively. With reference to the function (1.1) introduced in Section 1, for parameters $\lambda, \mu > 0$, and $0 < \nu < 1$, we consider the probability density functions (PDFs)

$$f_U(t) = \lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu), \qquad f_D(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu), \qquad t > 0, \tag{2.3}$$

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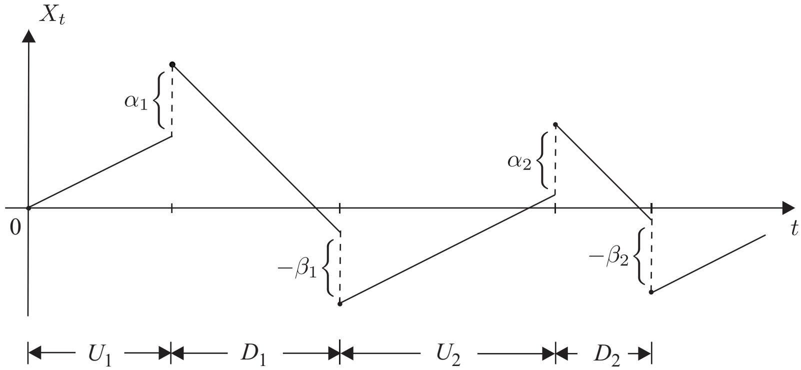

FIGURE 1: A sample path of X_t.

and the complementary cumulative distribution functions

$$\overline{F}_U(t) = \mathbb{P}(U > t) = E_{\nu,1}(-\lambda t^\nu), \qquad \overline{F}_D(t) = \mathbb{P}(D > t) = E_{\nu,1}(-\mu t^\nu), \qquad t > 0. \tag{2.4}$$

Note that, from (1.1), we have $E_{\alpha,\beta}(z) = E^1_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} z^k/\Gamma(\alpha k + \beta)$. We recall that, for $\nu = 1$, (2.3) and (2.4) lead to exponential distributions. Furthermore, for $k \in \mathbb{N}$, the PDFs of

$$U^{(k)} = U_1 + U_2 + \cdots + U_k \quad \text{and} \quad D^{(k)} = D_1 + D_2 + \cdots + D_k, \tag{2.5}$$

are, respectively, (see Equation (2.19) of [3]),

$$f_U^{(k)}(t) = \lambda^k t^{\nu k-1} E^k_{\nu,\nu k}(-\lambda t^\nu) \quad \text{and} \quad f_D^{(k)}(t) = \mu^k t^{\nu k-1} E^k_{\nu,\nu k}(-\mu t^\nu), \qquad t > 0, \tag{2.6}$$

where $E^k_{\nu,\nu k}(\cdot)$ was defined in (1.1). Note that the distributions given in (2.6) can be viewed as generalized Mittag-Leffler distributions (see [18]); these are also called positive Linnik distributions (see [4] and [32]). Moreover, such distributions are involved in the analysis of the fractional Poisson process and its extensions (see [3] and [11]).

It is worth pointing out that under assumptions (2.3) and (2.4), the process $\{N_t,\ t \geq 0\}$ constitutes a fractional alternating Poisson process previously investigated in [10].

Let us now introduce the following forward and backward transition PDFs for $x \in \mathbb{R}$, $t > 0$, and $y \in \{-v, c\}$:

$$f(x, t \mid y)\, \mathrm{d}x = \mathbb{P}[X_t \in \mathrm{d}x,\ V_t = c \mid X_0 = 0,\ V_0 = y],$$
$$b(x, t \mid y)\, \mathrm{d}x = \mathbb{P}[X_t \in \mathrm{d}x,\ V_t = -v \mid X_0 = 0,\ V_0 = y].$$

Clearly, the probability law of (X_t, V_t) has an absolutely continuous component

$$p(x, t \mid y) = f(x, t \mid y) + b(x, t \mid y), \tag{2.7}$$

and a discrete component

$$\mathbb{P}[X_t = yt,\ V_t = y \mid X_0 = 0,\ V_0 = y].$$

In order to provide the formal expression of the above functions, we denote by

$$\alpha^{(k)} = \alpha_1 + \alpha_2 + \cdots + \alpha_k \quad (\beta^{(k)} = \beta_1 + \beta_2 + \cdots + \beta_k)$$

the total amplitude of the first k upward (downward) jumps. Moreover, we set

$$I_{j,k}(x,t) := \begin{cases} 1 & \text{if } -vt+\alpha^{(j)}-\beta^{(k)} < x < ct+\alpha^{(j)}-\beta^{(k)}, \\ 0 & \text{otherwise,} \end{cases} \qquad x\in\mathbb{R},\ t>0,$$

and

$$\tau_* = \frac{vt+x}{c+v}, \qquad \theta_k = \frac{\alpha^{(k)}-\beta^{(k)}}{c+v}, \qquad \eta_k = \frac{\alpha^{(k+1)}-\beta^{(k)}}{c+v}. \tag{2.8}$$

Theorem 2.1. *The probability law of* (X_t, V_t), $t>0$, *conditional on a positive initial velocity is given by the discrete component*

$$\mathbb{P}[X_t = ct, V_t = c \mid X_0 = 0, V_0 = c] = E_{\nu,1}(-\lambda t^{\nu}), \tag{2.9}$$

and by the absolutely continuous component for the forward and backward transition PDFs for $x\in\mathbb{R}$*:*

$$f(x,t\mid c) = \sum_{k=1}^{+\infty}\bigg\{\frac{I_{k,k}(x,t)}{c+v}\mu^k(t-\tau_*+\theta_k)^{\nu k-1}E^k_{\nu,\nu k}(-\mu(t-\tau_*+\theta_k)^{\nu}) \\ \times\lambda^k(\tau_*-\theta_k)^{\nu k}E^{k+1}_{\nu,\nu k+1}(-\lambda(\tau_*-\theta_k)^{\nu})\bigg\}, \tag{2.10}$$

$$b(x,t\mid c) = \sum_{k=0}^{+\infty}\bigg\{\frac{I_{k+1,k}(x,t)}{c+v}\lambda^{k+1}(\tau_*-\eta_k)^{\nu(k+1)-1}E^{k+1}_{\nu,\nu(k+1)}(-\lambda(\tau_*-\eta_k)^{\nu}) \\ \times\mu^k(t-\tau_*+\eta_k)^{\nu k}E^{k+1}_{\nu,\nu k+1}(-\mu(t-\tau_*+\eta_k)^{\nu})\bigg\}, \tag{2.11}$$

where the function $E^{k+1}_{\nu,\nu k+1}(\cdot)$ *was defined in (1.1) and* $\tau_* = \tau_*(x,t)$ *in (2.8).*

Proof. Since $\mathbb{P}[X_t = ct, V_t = c \mid X_0 = 0, V_0 = c] = \overline{F}_U(t)$, (2.9) follows immediately from (2.4). In Theorem 2.1 of [9], the following general expressions were proved for $t>0$ and $x\in\mathbb{R}$:

$$f(x,t\mid c) = \sum_{k=1}^{+\infty}\frac{I_{k,k}(x,t)}{c+v}f_D^{(k)}(t-\tau_*+\theta_k)\int_{t-\tau_*+\theta_k}^{t}f_U^{(k)}(s-t+\tau_*-\theta_k)\overline{F}_U(t-s)\,\mathrm{d}s, \tag{2.12}$$

$$b(x,t\mid c) = \frac{I_{1,0}(x,t)}{c+v}f_U(\tau_*-\eta_0)\overline{F}_D(t-\tau_*+\eta_0) \\ +\sum_{k=1}^{+\infty}\frac{I_{k+1,k}(x,t)}{c+v}f_U^{(k+1)}(\tau_*-\eta_k)\int_{\tau_*-\eta_k}^{t}f_D^{(k)}(s-\tau_*+\eta_k)\overline{F}_D(t-s)\,\mathrm{d}s, \tag{2.13}$$

where τ_*, θ_k, and η_k were defined in (2.8). Therefore, densities (2.10) and (2.11) can be obtained from (2.3), (2.4), and (2.6) and noting that the integrals on the right-hand side of (2.12) and (2.13) can be computed by means of (see Theorem 2 of [20])

$$\int_0^x (x-t)^{\beta-1}E^{\gamma}_{\alpha,\beta}[a(x-t)^{\alpha}]t^{\nu-1}E^{\sigma}_{\alpha,\nu}(at^{\alpha})\,\mathrm{d}t = x^{\beta+\nu-1}E^{\gamma+\sigma}_{\alpha,\beta+\nu}(ax^{\alpha})$$

for $\alpha,\beta,\gamma,a,\nu,\sigma\in\mathbb{C}$ ($\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\nu)>0$). □

Theorem 2.2. *The series on the right-hand sides of (2.10) and (2.11) are uniformly convergent for $x \in \mathbb{R}$ and for fixed $t > 0$.*

Proof. Set, for $x \in \mathbb{R}$, $k \in \mathbb{N}$, and $t > 0$,

$$
\begin{aligned}
f_k(x, t \mid c) &:= \mathbb{P}[X_t \in \mathrm{d}x,\ V_t = c,\ N_t = k \mid X_0 = 0,\ V_0 = c] \\
&= \frac{I_{k,k}(x,t)}{c+v} \mu^k (t - \tau_* + \theta_k)^{\nu k - 1} E^k_{\nu,\nu k}(-\mu(t - \tau_* + \theta_k)^\nu) \\
&\quad \times \lambda^k (\tau_* - \theta_k)^{\nu k} E^{k+1}_{\nu,\nu k+1}(-\lambda(\tau_* - \theta_k)^\nu).
\end{aligned}
$$

The forward transition PDF (2.10) can thus be split as

$$
f(x, t \mid c) = \sum_{k=1}^{k^*-1} f_k(x, t \mid c) + \sum_{k=k^*}^{+\infty} f_k(x, t \mid c),
$$

where $k^* \in \mathbb{N}$ is determined by the Archimedean property of the real numbers so that $\nu k^* > 1$. In general, for all $k \in \mathbb{N}$,

$$
f_k(x, t \mid c) \le \frac{I_{k,k}(x,t)}{c+v} \mu^k (t - \tau_* + \theta_k)^{\nu k - 1} E^k_{\nu,\nu k}(-\mu(t - \tau_* + \theta_k)^\nu).
$$

In fact, the generalized Mittag-Leffler function $E^{k+1}_{\nu,\nu k+1}(-\lambda t^\nu)$, $k \ge 0$, suitably normalized by the factor $(\lambda t^\nu)^k$, represents a proper probability distribution (see [3]). If $k > k^*$, for fixed $t > 0$, the function $((ct - x)/(c + v) + \theta_k)^{\nu k - 1}$ is monotonically decreasing in $x \in (-vt + \alpha^{(k)} - \beta^{(k)}, ct + \alpha^{(k)} - \beta^{(k)})$. Consequently, we have

$$
\left(\frac{ct - x}{c + v} + \theta_k\right)^{\nu k - 1} \le \left(\frac{ct - x}{c + v} + \theta_k\right)^{\nu k - 1}\Bigg|_{x = -vt + \alpha^{(k)} - \beta^{(k)}} = t^{\nu k - 1}.
$$

Moreover, we note that the function $E^k_{\nu,\nu k}(-\mu((ct - x)/(c + v) + \theta_k)^\nu)$ is monotonically increasing in $x \in (-vt + \alpha^{(k)} - \beta^{(k)}, ct + \alpha^{(k)} - \beta^{(k)})$ (see [25, Section 2.3]) so that

$$
E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct - x}{c + v} + \theta_k\right)^\nu\right) \le E^k_{\nu,\nu k}\left(-\mu\left(\frac{ct - x}{c + v} + \theta_k\right)^\nu\right)\Bigg|_{x = ct + \alpha^{(k)} - \beta^{(k)}} = \frac{1}{\Gamma(\nu k)}.
$$

The forward PDF thus satisfies the following relation:

$$
\begin{aligned}
f(x, t \mid c) &\le \sum_{k=1}^{k^*-1} f_k(x, t \mid c) + \sum_{k=k^*}^{+\infty} \frac{I_{k,k}(x,t)}{c+v} \mu^k \frac{t^{\nu k - 1}}{\Gamma(\nu k)} \\
&\le \frac{t^{-1}}{c+v} \sum_{k=k^*}^{+\infty} \frac{(\mu t^\nu)^k}{\Gamma(\nu k)} \\
&= \frac{t^{-1}}{c+v} \sum_{r=0}^{+\infty} \frac{(\mu t^\nu)^{r+k^*}}{\Gamma(\nu(r + k^*))} \\
&= \frac{t^{-1}}{c+v} (\mu t^\nu)^{k^*} E_{\nu,\nu k^*}(\mu t^\nu).
\end{aligned}
$$

Uniform convergence is then due to total convergence. □

Remark 2.1. Due to symmetry, if $V_0 = -v$, the probability law of (X_t, V_t) can be obtained from Theorem 2.1 by interchanging f with b, U with D, c with v, x with $-x$, and α_i with β_i for all $i \in \mathbb{N}$, thus yielding

$$f(x,t \mid -v) = \sum_{k=0}^{+\infty} \bigg\{ \frac{I_{k,k+1}(x,t)}{c+v} \mu^{k+1} (t - \tau_* - \tilde{\eta}_k)^{\nu(k+1)-1} \times E_{\nu,\nu(k+1)}^{k+1}(-\mu(t-\tau_*-\tilde{\eta}_k)^\nu) \lambda^k (\tau_* + \tilde{\eta}_k)^{\nu k} E_{\nu,\nu k+1}^{k+1}(-\lambda(\tau_* + \tilde{\eta}_k)^\nu) \bigg\}$$

and

$$b(x,t \mid -v) = \sum_{k=1}^{+\infty} \bigg\{ \frac{I_{k,k}(x,t)}{c+v} \lambda^k (\tau_* - \theta_k)^{\nu k-1} E_{\nu,\nu k}^{k}(-\lambda(\tau_* - \theta_k)^\nu) \times \mu^k (t - \tau_* + \theta_k)^{\nu k} E_{\nu,\nu k+1}^{k+1}(-\mu(t-\tau_*+\theta_k)^\nu) \bigg\},$$

where

$$\tilde{\eta}_k = \frac{\beta^{(k+1)} - \alpha^{(k)}}{c+v} = \frac{\beta_{k+1}}{c+v} - \theta_k.$$

Corollary 2.1. *If the initial velocity is random, i.e. V_0 is either c or $-v$ with equal probability, we obtain*

$$\mathbb{P}[X_t = ct \mid X_0 = 0] = \tfrac{1}{2} E_{\nu,1}(-\lambda t^\nu), \quad \mathbb{P}[X_t = -vt \mid X_0 = 0] = \tfrac{1}{2} E_{\nu,1}(-\mu t^\nu), \qquad V_0 = \begin{cases} c & \text{with probability } \frac{1}{2}, \\ -v & \text{with probability } \frac{1}{2}. \end{cases} \tag{2.14}$$

Furthermore, from (2.7), the transition PDF of X_t is

$$\begin{aligned} p(x,t) &:= \mathbb{P}[X_t \in \mathrm{d}x \mid X_0 = 0] \\ &= \tfrac{1}{2} [p(x,t \mid c) + p(x,t \mid -v)] \\ &= \tfrac{1}{2}[f(x,t \mid c) + b(x,t \mid c) + f(x,t \mid -v) + b(x,t \mid -v)], \end{aligned} \tag{2.15}$$

where the forward and backward transition PDFs conditional on the initial velocity are given in Theorem 2.1 and Remark 2.1.

3. Constant jump sizes

We now focus on the special case when all the jumps have equal constant amplitude, say α. Hereafter, we obtain the explicit expression of the density $p(x,t)$ defined in (2.15).

Proposition 3.1. *Let $\alpha_k = \beta_k = \alpha > 0$ for all $k \in \mathbb{N}$, and let U and D be Mittag-Leffler distributed with parameters (λ, ν) and (μ, ν), respectively. Let $\mathbb{P}(V_0 = c) = \mathbb{P}(V_0 = -v) = \frac{1}{2}$. The probability law of X_t is characterized by the discrete component indicated in (2.14), and by the absolutely continuous component $p(x,t)$ specified hereafter.*

(i) *If $0 < t < \alpha/(c+v)$ then*

$$p(x,t) = \begin{cases} \varphi_{-1}(x,t), & -vt - \alpha < x < ct - \alpha, \\ \varphi_0(x,t), & -vt < x < ct, \\ \varphi_1(x,t), & -vt + \alpha < x < ct + \alpha, \\ 0 & \text{otherwise}. \end{cases}$$

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(ii) *If* $\alpha/(c+v) \le t < 2\alpha/(c+v)$ *then*

$$p(x,t) = \begin{cases} \varphi_{-1}(x,t), & -vt-\alpha < x < -vt, \\ \varphi_{-1}(x,t)+\varphi_0(x,t), & -vt < x < ct-\alpha, \\ \varphi_0(x,t), & ct-\alpha < x < -vt+\alpha, \\ \varphi_0(x,t)+\varphi_1(x,t), & -vt+\alpha < x < ct, \\ \varphi_1(x,t), & ct < x < ct+\alpha, \\ 0 & \textit{otherwise}. \end{cases}$$

(iii) *If* $t \ge 2\alpha/(c+v)$ *then*

$$p(x,t) = \begin{cases} \varphi_{-1}(x,t), & -vt-\alpha < x < -vt, \\ \varphi_{-1}(x,t)+\varphi_0(x,t), & -vt < x < -vt+\alpha, \\ \varphi_{-1}(x,t)+\varphi_0(x,t)+\varphi_1(x,t), & -vt+\alpha < x < ct-\alpha, \\ \varphi_0(x,t)+\varphi_1(x,t), & ct-\alpha < x < ct, \\ \varphi_1(x,t), & ct < x < ct+\alpha, \\ 0 & \textit{otherwise}, \end{cases}$$

with

$$\varphi_{-1}(x,t) = \frac{1}{2(c+v)} \sum_{k=0}^{+\infty} \Big\{ \mu^{k+1}(t-\tau_*-\eta_0)^{\nu(k+1)-1} E_{\nu,\nu(k+1)}^{k+1}(-\mu(t-\tau_*-\eta_0)^\nu) \times \lambda^k (\tau_*+\eta_0)^{\nu k} E_{\nu,\nu k+1}^{k+1}(-\lambda(\tau_*+\eta_0)^\nu) \Big\},$$

$$\varphi_0(x,t) = \frac{1}{2(c+v)} \sum_{k=0}^{+\infty} \Big\{ \mu^{k+1}(t-\tau_*)^{\nu(k+1)-1} E_{\nu,\nu(k+1)}^{k+1}(-\mu(t-\tau_*)^\nu) \times \lambda^{k+1}\tau_*^{\nu(k+1)} E_{\nu,\nu(k+1)+1}^{k+2}(-\lambda\tau_*^\nu) + \lambda^{k+1}\tau_*^{\nu(k+1)-1} E_{\nu,\nu(k+1)}^{k+1}(-\lambda\tau_*^\nu) \times \mu^{k+1}(t-\tau_*)^{\nu(k+1)} E_{\nu,\nu(k+1)+1}^{k+2}(-\mu(t-\tau_*)^\nu) \Big\},$$

$$\varphi_1(x,t) = \frac{1}{2(c+v)} \sum_{k=0}^{+\infty} \Big\{ \lambda^{k+1}(\tau_*-\eta_0)^{\nu(k+1)-1} E_{\nu,\nu(k+1)}^{k+1}(-\lambda(\tau_*-\eta_0)^\nu) \times \mu^k (t-\tau_*+\eta_0)^{\nu k} E_{\nu,\nu k+1}^{k+1}(-\mu(t-\tau_*+\eta_0)^\nu) \Big\},$$

where $\tau_* = \tau_*(x,t)$ *is given in (2.8), and* $\eta_0 = \alpha/(c+v)$.

Proof. It is a straightforward consequence of (2.15), by recalling that assumption $\alpha_k = \beta_k = \alpha > 0$, $k \in \mathbb{N}$, yields $\theta_k = 0$ and $\eta_k = \tilde{\eta}_k = \alpha/(c+v)$. □

It is interesting to note that the functions φ_i, $i = -1, 0, 1$, have a specific probabilistic meaning. Indeed, φ_{-1} (φ_1) represents a measure of the sample paths of the process that perform a number of downward (upward) jumps that is one more than the upward (downward) jumps in the interval $(0, t]$, whereas φ_0 refers to the case when the number of upward and downward jumps coincide.

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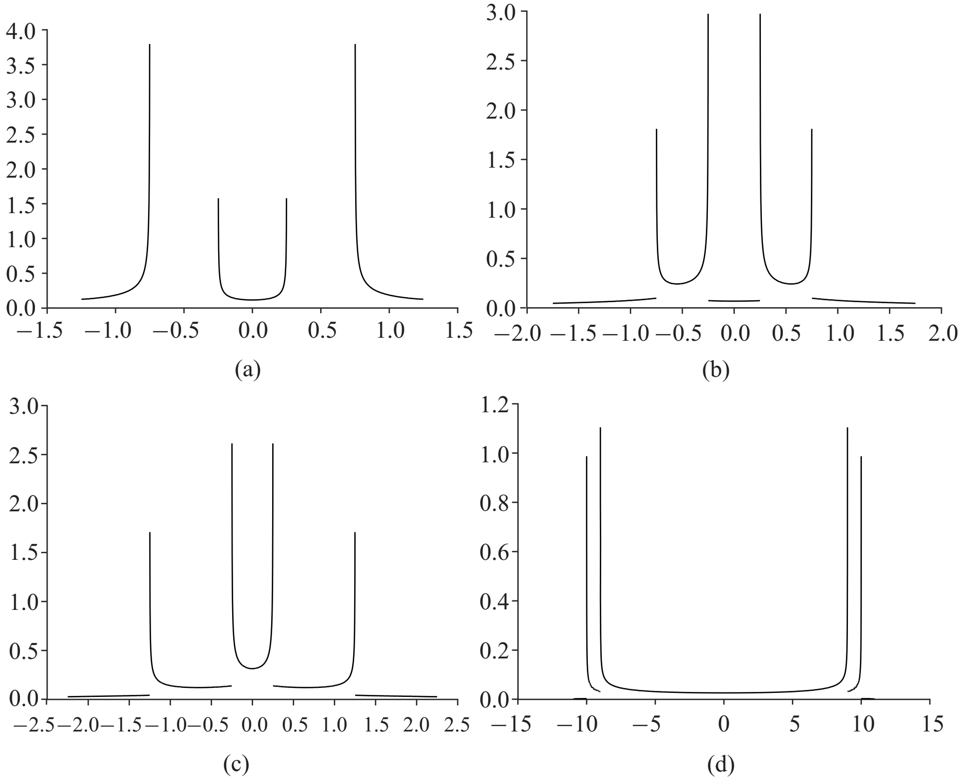

FIGURE 2: Plots of the density $p(x, t)$ obtained in Proposition 3.1 for $c = v = 1$, $\lambda = \mu = 1$, $\alpha = 1$, and $\nu = 0.5$ when (a) $t = 0.25$, (b) $t = 0.75$, (c) $t = 1.25$, and (d) $t = 10$.

We also remark that Proposition 3.1 is an immediate extension of Proposition 4.1 of [9], which concerns the case of exponentially distributed interarrival times. In Figures 2–5 we present plots of the density $p(x, t)$ obtained in Proposition 3.1 for two choices of ν and two choices of the switching intensities. We first observe that the vertical asymptotes of density $p(x, t)$ are due to the singular behaviour at 0^+ of the Mittag-Leffler distribution. Similar to the case of exponentially distributed interarrival times (see Figures 2 and 3 of [9]), at the beginning of the motion the probability mass is concentrated in a neighbourhood of the origin and of $\pm\alpha$ (due to the occurrence of a small number of jumps). As time grows larger, the singularities are shifted towards the endpoints of the spatial interval and the effect of further velocity changes and jumps makes the density smoother and smoother, so that the probability mass is spread over the whole diffusion domain. Moreover, when $c = v$ and $\lambda = \mu$ it is easy to show that the density $p(x, t)$ given in Proposition 3.1 is an even function in x (see Figures 2 and 4). Instead, if $\lambda < \mu$ then the forward motion prevails over the backward motion, and thus the probability mass is more scattered over positive values of x (see Figures 3 and 5).

We recall that the traditional telegraph process governed by exponentially distributed interarrivals possesses a steady-state property that causes short-memory behaviour, whereas the present model based on Mittag-Leffler distributions is characterized by stronger memory properties. Indeed, if ν increases then the particle motion undergoes more rapid direction changes (and jumps), and thus the density $p(x, t)$ becomes more smooth. This can be noted by comparing

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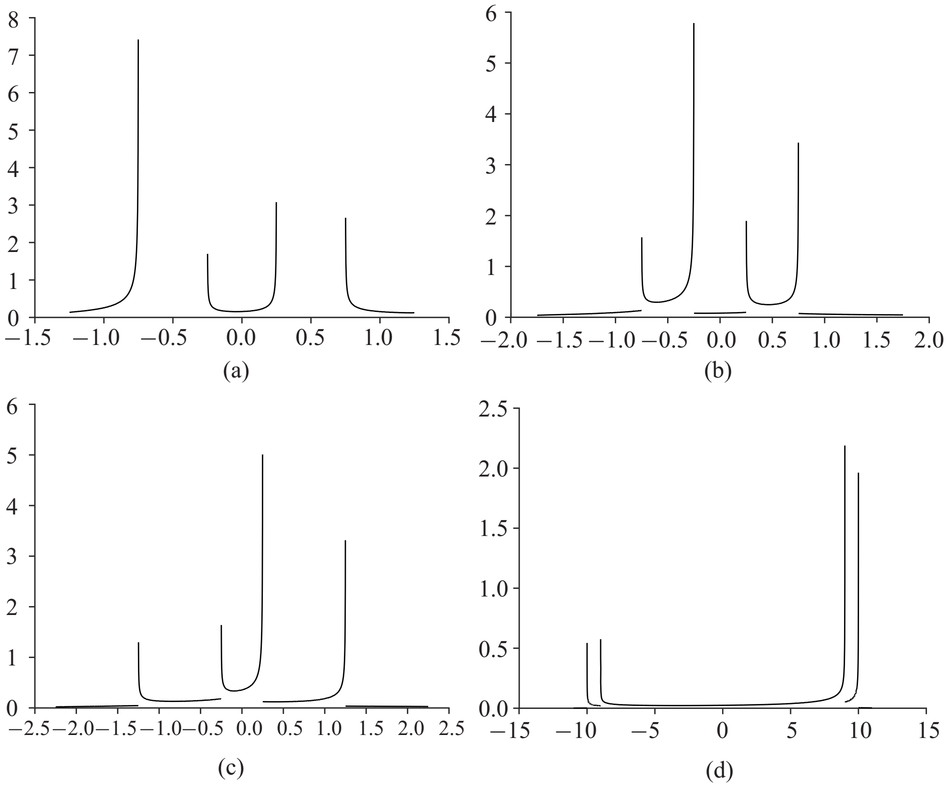

FIGURE 3: As in Figure 2 with $\mu = 2$.

Figures 2 and 4 (respectively, 3 and 5) with Figure 2 (respectively, Figure 3) of [9], which is concerned with the $\nu = 1$ case. This is also confirmed by the fact that, due to (2.14), the probability

$$\mathbb{P}[N_t = 0] = \mathbb{P}[X_t \in \{-v t, ct\} \mid X_0 = 0] = \tfrac{1}{2}[E_{\nu,1}(-\lambda t^{\nu}) + E_{\nu,1}(-\mu t^{\nu})] \tag{3.1}$$

is decreasing in $\nu \in (0, 1)$ for suitably large values of t (see, for example, Figure 6).

Let us now analyse the mean displacement of the particle, described by X_t, when the upward and downward jumps are constant, and possibly different.

Proposition 3.2. *Let $\alpha_j = \alpha$ and $\beta_j = \beta$ for all $j \in \mathbb{N}$, and let both U and D be Mittag-Leffler-distributed with parameters (λ, ν). Let $\mathbb{P}(V_0 = c) = \mathbb{P}(V_0 = -v) = \frac{1}{2}$. Then, for $t > 0$, we have*

$$\mathbb{E}(X_t \mid X_0 = 0) = \frac{1}{2}\bigg[(c - v)t + (\alpha - \beta)\frac{\lambda t^{\nu}}{\Gamma(\nu + 1)}\bigg]. \tag{3.2}$$

Proof. The proof is similar to that of Proposition 4.5 of [9]. Indeed, denoting by $\mathbb{E}_y(\cdot)$ the mean conditional on $V_0 = y \in \{-v, c\}$, from (2.1), we need to compute

$$\mathbb{E}(X_t \mid X_0 = 0) = \frac{1}{2}\bigg[\int_0^t \mathbb{E}_c(V_s)\,\mathrm{d}s + \int_0^t \mathbb{E}_{-v}(V_s)\,\mathrm{d}s + \mathbb{E}_c\bigg(\sum_{k=1}^{N_t} w_k\bigg) + \mathbb{E}_{-v}\bigg(\sum_{k=1}^{N_t} w_k\bigg)\bigg],$$

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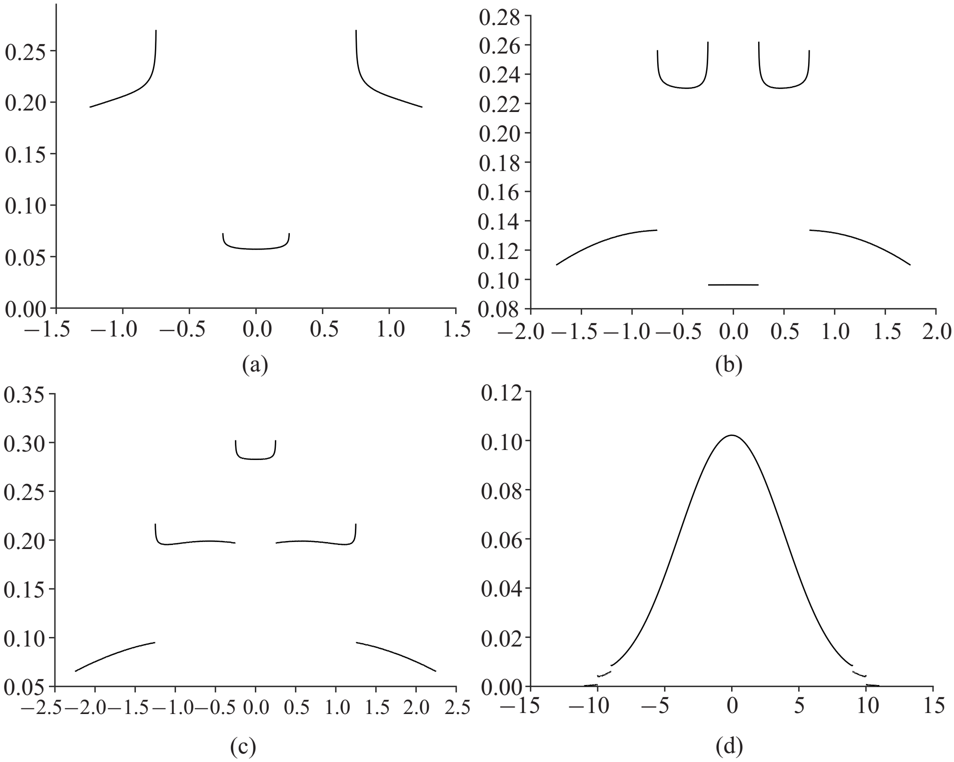

FIGURE 4: Plots of the density $p(x, t)$ obtained in Proposition 3.1 for $c = v = 1$, $\lambda = \mu = 1$, $\alpha = 1$, and $\nu = 0.95$ when (a) $t = 0.25$, (b) $t = 0.75$, (c) $t = 1.25$, and (d) $t = 10$.

where w_k is defined in (2.2). By setting

$$p_n(t) = \mathbb{P}(N_t = n) = (\lambda t^\nu)^n E^{n+1}_{\nu,\nu n+1}(-\lambda t^\nu), \qquad n \in \mathbb{N}_0,\ t > 0,$$

we have (see Equation (2.33) and Remark 3.4 of [3])

$$\begin{aligned}\mathbb{E}_c\left(\sum_{k=1}^{N_t} w_k\right) &= \frac{\alpha-\beta}{2}\sum_{n=0}^{+\infty} n\, p_n(t) + \frac{\alpha+\beta}{2}\sum_{n=0}^{+\infty} p_{2n+1}(t)\\ &= \frac{\alpha-\beta}{2}\frac{\lambda t^\nu}{\Gamma(\nu+1)} + \frac{\alpha+\beta}{2}\lambda t^\nu E_{\nu,\nu+1}(-2\lambda t^\nu). \end{aligned} \tag{3.3}$$

Moreover, due to Equation (2.30) of [3], we have

$$\mathbb{E}[(-1)^{N_\lambda(s)}] = E_{\nu,1}(-2\lambda s^\nu), \tag{3.4}$$

and then

$$\int_0^t \mathbb{E}_c(V_s)\,\mathrm{d}s = \frac{c-v}{2}t + \frac{c+v}{2}t E_{\nu,2}(-2\lambda t^\nu). \tag{3.5}$$

Similarly to (3.3) and (3.5), we have:

$$\mathbb{E}_{-v}\left(\sum_{k=1}^{N_t} w_k\right) = \frac{\alpha-\beta}{2}\frac{\lambda t^\nu}{\Gamma(\nu+1)} - \frac{\alpha+\beta}{2}\lambda t^\nu E_{\nu,\nu+1}(-2\lambda t^\nu) \tag{3.6}$$

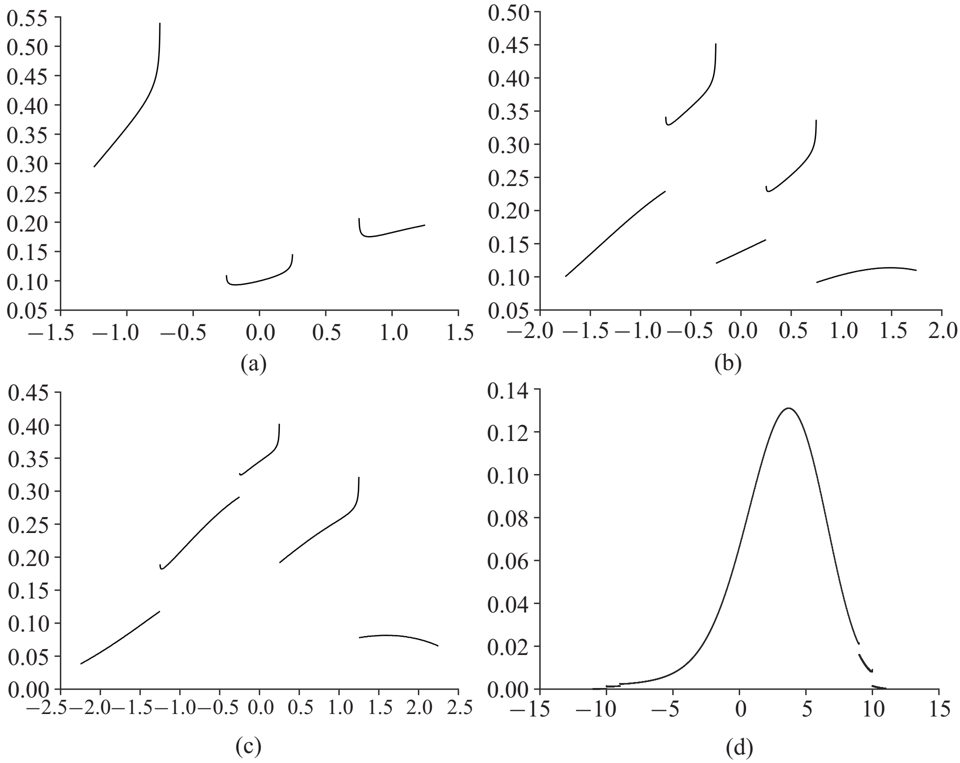

FIGURE 5: As in Figure 4 with $\mu = 2$.

and

$$\int_0^t \mathbb{E}_{-v}(V_s)\,\mathrm{d}s = \frac{c-v}{2}t - \frac{c+v}{2}t\,E_{\nu,2}(-2\lambda t^{\nu}). \tag{3.7}$$

Therefore, the proof follows from (3.3)–(3.7). □

Let us now discuss some features of the mean given in (3.2). First of all, we note that the linear term $(c-v)t$ has to be attributed to the alternating motion, the term $(\alpha-\beta)\lambda t^{\nu}/\Gamma(\nu+1)$ is related to the jump component of the motion characterized by Mittag-Leffler distributed intertimes, and the factor $\frac{1}{2}$ is due to the random initial velocity V_0. Clearly, result (3.2) is in agreement with Equation (39) of [9], for the jump-telegraph process with exponentially distributed intertimes, that we recover for $\nu = 1$.

In Table 1 we provide expressions for the means of some fractional stochastic processes that can be compared with the conditional mean of X_t determined in (3.2). We note that cases (vi), (vii), and (viii) refer to birth–death-type processes with n_0 progenitors. Moreover, the fractional Poisson process in case (i) becomes identical to X_t for $c = v = 0$ (no telegraph component), $\alpha = 1$ and $\beta = -1$ (upward jumps of size 1 occur at every event of the underlying fractional Poisson process). A similar remark holds for the process with multiple jumps in case (iv). Clearly, the mean of the jump-telegraph process X_t, under the assumptions of Proposition 3.2, vanishes in the symmetric case when $c = v$ and $\alpha = \beta$, as well as for the symmetric fractional (telegraph) process studied in [26] (case (ix) of Table 1). In Figure 7 we present the means considered in cases (i)–(viii) of Table 1, for two choices of ν, with $\lambda = 1$, $\mu = 0.5$ and $n_0 = 10$,

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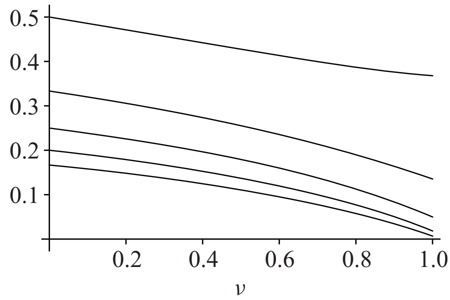

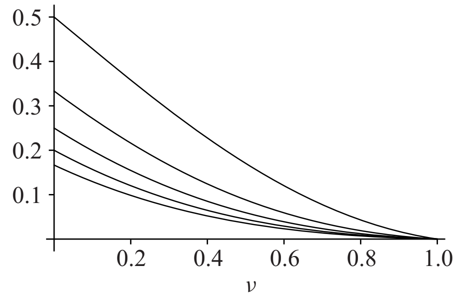

FIGURE 6: Probabilities calculated from (3.1) for $\lambda = \mu = 1, 2, 3, 4, 5$ (*from top to bottom*) with (a) $t = 1$ and (b) $t = 10$ for $0 < \nu < 1$.

TABLE 1: Mean values of some processes of interest.

Fractional process	Mean value	Ref.
(i) Poisson process	$\dfrac{\lambda t^{\nu}}{\Gamma(\nu+1)}$	[3]
(ii) Alternative Poisson process	$\dfrac{\lambda t}{\nu}\dfrac{E_{\nu,\nu}(\lambda t)}{E_{\nu,1}(\lambda t)}$	[2]
(iii) Second-order Poisson process	$\dfrac{\lambda t^{\nu}}{2\Gamma(\nu+1)} - \dfrac{\lambda t^{\nu}}{2} E_{\nu,\nu+1}(-2\lambda t^{\nu})$	[3]
(iv) Poisson process with jumps $1, \dots, k$	$\dfrac{\sum_{j=1}^{k} j\lambda_j t^{\nu}}{\Gamma(\nu+1)}$	[11]
(v) Linear birth process	$E_{\nu,1}(\lambda t^{\nu})$	[27]
(vi) Linear birth-death process	$n_0 E_{\nu,1}((\lambda-\mu)t^{\nu})$	[28]
(vii) Linear death process	$n_0 E_{\nu,1}(-\mu t^{\nu})$	[30]
(viii) Sublinear death process	$\sum_{j=1}^{n_0} \binom{n_0+1}{j+1} (-1)^{j+1} E_{\nu,1}(-\mu j t^{\nu})$	[30]
(ix) Telegraph process	0	[26]

and for $k = 3$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in case (iv). The asymptotic behaviour of the means given in Table 1 is finally provided in Table 2, obtained thanks to Equation (4.4.16) of [16] and Equation (A.3) of [37]. In case (viii) of Table 2, $\gamma \simeq 0.577\,216$ is the Euler constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, i.e. the logarithmic derivative of the gamma function.

It should be appropriate to perform a qualitative comparison between the traditional telegraph process (based on exponentially distributed intertimes) and the process considered in this paper. For instance, the memory properties of X_t are expected to be stronger than the traditional model (corresponding to the $\nu = 1$ case). In general, it is interesting to measure the depth of the memory by means of time-covariances. Unfortunately, in this case the autocovariance of X_t is not easy to compute since the joint distribution of (X_s, X_t) does not seem achievable. For instance, for the random telegraph process with Mittag-Leffler distributed intertimes, the covariance was given in a formal series form by Ferraro *et al.* [13]. Generally, upper bounds

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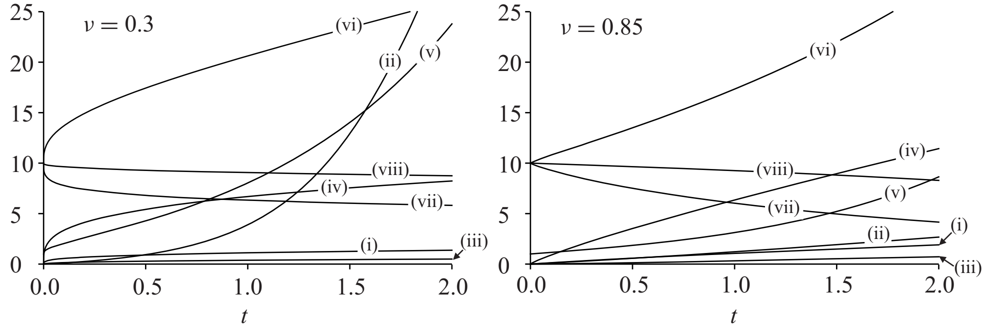

FIGURE 7: Plots of the mean values of the processes considered in Table 1.

TABLE 2: Asymptotic behaviour of the mean values of some processes of interest.

(i)	$\dfrac{\lambda t^\nu}{\Gamma(\nu+1)}$	(v)	$\dfrac{\exp\{\lambda^{1/\nu}t\}}{\nu}$
(ii)	$\dfrac{(\lambda t)^{1/\nu}}{\nu}$	(vi)	$n_0\dfrac{\exp\{(\lambda-\mu)^{1/\nu}t\}}{\nu}$ for $\lambda > \mu$
(iii)	$\dfrac{\lambda t^\nu}{2\Gamma(\nu+1)} - \dfrac{1}{4}$	(vii)	$n_0\dfrac{1}{\Gamma(1-\nu)\mu t^\nu}$
(iv)	$\dfrac{\sum_{j=1}^k j\lambda_j t^\nu}{\Gamma(\nu+1)}$	(viii)	$\dfrac{1}{\Gamma(1-\nu)\mu t^\nu}(n_0+1)[\gamma - 1 + \psi(n+2)]$

to $\mathrm{cov}(X_s, X_t)$ can be obtained by means of suitable inequalities, such as the Cauchy–Schwarz inequality, but unfortunately this approach does not lead to tight bounds in our case. Moreover, the underlying fractional alternating Poisson process N_t is nonstationary and this prevents us from using typical techniques for stationary processes.

We conclude this section with the following result, which gives the conditional distribution of X_t in the absence of jumps, recovered by applying Theorem 2.1 of [7].

Corollary 3.1. *Let U and D have Mittag-Leffler distribution with parameters (λ, ν) and (μ, ν), respectively, and let $\alpha = \beta = 0$. For all $t > 0$, we have*

$$\mathbb{P}[X_t = ct, V_t = c \mid X_0 = 0, V_0 = c] = E_{\nu,1}(-\lambda t^\nu);$$

moreover, for $-vt < x < ct$, it holds that

$$f(x,t \mid c) = \frac{1}{c+v}\sum_{k=1}^{+\infty}\{\mu^k (t-\tau_*)^{\nu k-1}E^k_{\nu,\nu k}(-\mu(t-\tau_*)^\nu)\lambda^k \tau_*^{\nu k}E^{k+1}_{\nu,\nu k+1}(-\lambda\tau_*^\nu)\},$$

$$b(x,t \mid c) = \frac{1}{c+v}\sum_{k=0}^{+\infty}\{\lambda^{k+1}\tau_*^{\nu(k+1)-1}E^{k+1}_{\nu,\nu(k+1)}(-\lambda\tau_*^\nu) \times \mu^k (t-\tau_*)^{\nu k}E^{k+1}_{\nu,\nu k+1}(-\mu(t-\tau_*)^\nu)\},$$

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and

$$b(x,t \mid -v) = \frac{1}{c+v} \sum_{k=1}^{+\infty} \{\lambda^k \tau_*^{\nu k-1} E_{\nu,\nu k}^k(-\lambda \tau_*^\nu) \mu^k (t-\tau_*)^{\nu k} E_{\nu,\nu k+1}^{k+1}(-\mu(t-\tau_*)^\nu)\},$$

$$f(x,t \mid -v) = \frac{1}{c+v} \sum_{k=0}^{+\infty} \{\mu^{k+1} (t-\tau_*)^{\nu(k+1)-1} E_{\nu,\nu(k+1)}^{k+1}(-\mu(t-\tau_*)^\nu) \times \lambda^k \tau_*^{\nu k} E_{\nu,\nu k+1}^{k+1}(-\lambda \tau_*^\nu)\},$$

where $\tau_* = \tau_*(x,t)$ *is given in (2.8).*

4. First-passage time problem

In this section we study the distribution of the (upward) first-passage time of X_t through a constant barrier, say $\gamma > 0$, conditional on the initial state

$$\tau_\gamma = \inf\{t \ge 0 \colon X_t \ge \gamma\}, \qquad X_0 = 0, \qquad V_0 = c. \tag{4.1}$$

The downward case can be treated similarly. Hereafter, we express the probability distribution of (4.1) in terms of the following subdensity functions:

$$g_\gamma(t;n) := \frac{\mathbb{P}(\tau_\gamma \in \mathrm{d}t, N_t = n)}{\mathrm{d}t}, \qquad n \in \mathbb{N}. \tag{4.2}$$

Proposition 4.1. *For* $t > 0$*, it holds that*

$$\mathbb{P}(\tau_\gamma \in \mathrm{d}t) = E_{\nu,1}(-\lambda t^\nu)\delta_{\gamma/c}(\mathrm{d}t) + \sum_{k=0}^{+\infty} g_\gamma(t;2k+1)\,\mathrm{d}t + \sum_{k=1}^{+\infty} g_\gamma(t;2k)\,\mathrm{d}t, \tag{4.3}$$

where

$$g_\gamma(t;\ 2k+1)\,\mathrm{d}t = \mathbb{P}\big(cU^{(i)} - vD^{(i-1)} + \alpha^{(i)} - \beta^{(i-1)} < \gamma,\ i = 1,\dots,k, \\ U^{(k+1)} + D^{(k)} \in \mathrm{d}t, cU^{(k+1)} - vD^{(k)} + \alpha^{(k+1)} - \beta^{(k)} \ge \gamma\big) \tag{4.4}$$

and

$$g_\gamma(t;2k)\,\mathrm{d}t = \mathbb{P}\bigg(cU^{(i)} - vD^{(i-1)} + \alpha^{(i)} - \beta^{(i-1)} < \gamma,\ i = 1,\dots,k, \\ \frac{\gamma - \alpha^{(k)} + \beta^{(k)} + (c+v)D^{(k)}}{c} \in \mathrm{d}t\bigg). \tag{4.5}$$

Proof. By the law of total probability, we can express the conditional distribution of τ_γ in the form

$$\mathbb{P}(\tau_\gamma \in \mathrm{d}t) = \mathbb{P}(U_1 > t)\delta_{\frac{\gamma}{c}}(\mathrm{d}t) + \sum_{j=1}^{+\infty} \mathbb{P}(\tau_\gamma \in \mathrm{d}t, N_t = j), \tag{4.6}$$

where $\delta_{\gamma/c}$ is the Dirac delta measure at γ/c corresponding to the motion without any direction switchings. Moreover, in (4.6), the series on the right-hand side represents the absolutely continuous part of the distribution with the condition of at least one direction reversal, and N_t is the fractional alternating Poisson process introduced in Section 2. We also recall that U_1 has a Mittag-Leffler distribution with parameters (λ, ν). We consider two cases, namely when N_t

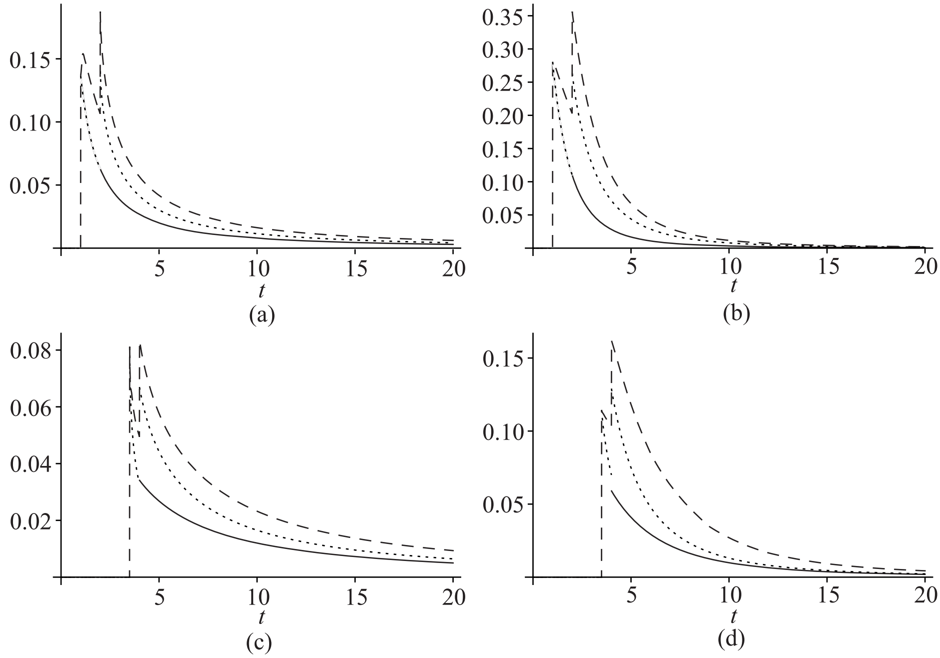

FIGURE 8: Plots of the bounds in (4.7) for $\gamma = 2$, $\alpha = \beta = 1$, $c = v = 1$, and $\lambda = \mu = 1$ with (a) $\nu = 0.5$ and (b) $\nu = 0.85$. Similarly, for $\gamma = 10$, $\alpha = 2$, $\beta = 1$, $c = 2$, $v = 1$, $\lambda = 2$, and $\mu = 1$ with (c) $\nu = 0.5$ and (d) $\nu = 0.85$. The cases $k = 1$ (*solid line*), $k = 2$ (*dotted line*), and $k = 3$ (*dashed line*) are considered.

is odd, and when N_t is even. If by time t there have been $2k + 1$, $k \geq 0$, changes of direction ($k + 1$ upward and k backward), then the particle crosses level γ for the first time due to the effect of the $(k + 1)$th upward jump. If by time t there have been $2k$, $k \geq 1$, changes of direction (k upward and k backward), then the first passage of the particle through level γ is due to the effect of the upward motion after the last renewal event. Recalling (2.5), this implies that density (4.6) becomes

$$\mathbb{P}(\tau_\gamma \in \mathrm{d}t) = \mathbb{P}(U_1 > t)\delta_{\gamma/c}(\mathrm{d}t) + \sum_{k=0}^{+\infty} g_\gamma(t;\ 2k+1)\,\mathrm{d}t + \sum_{k=1}^{+\infty} g_\gamma(t; 2k)\,\mathrm{d}t,$$

where $g_\gamma(t;\ 2k+1)\,\mathrm{d}t = \mathbb{P}(\tau_\gamma \in \mathrm{d}t, N_t = 2k+1)$ and $g_\gamma(t; 2k)\,\mathrm{d}t = \mathbb{P}(\tau_\gamma \in \mathrm{d}t, N_t = 2k)$ were expressed in (4.4) and (4.5). The final expression (4.3) thus follows. □

We remark that (4.3) is formally effective, but the determination of an explicit form of $g_\gamma(t; k)$ is difficult to obtain when k is large. Nevertheless, the above result is useful since

$$\hat{g}_\gamma(t; k) := \sum_{i=1}^{k} g_\gamma(t; i), \qquad k \in \mathbb{N}, \tag{4.7}$$

constitutes a sequence of increasing lower bounds for $\mathbb{P}(\tau_\gamma \in \mathrm{d}t)$ when k grows. In conclusion, in Figure 8 we present plots of $\hat{g}_\gamma(t; k)$ for $k = 1, 2, 3$ and for various choices of the parameters involved, these giving lower bounds for the distribution (4.6). From the performed computations

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we note that for small values of t the bound $\hat{g}_\gamma(t; k)$ increases as ν increases. This is in agreement with the expected behaviour of the functions (4.2).

Acknowledgements

This research was performed under partial support by the GNCS of INdAM. The authors thank the anonymous referees for useful comments that improved the paper.

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