

# THE GAMMA CONSTRUCTION AND ASYMPTOTIC INVARIANTS OF LINE BUNDLES OVER ARBITRARY FIELDS

TAKUMI MURAYAMA 

**Abstract.** We extend results on asymptotic invariants of line bundles on complex projective varieties to projective varieties over arbitrary fields. To do so over imperfect fields, we prove a scheme-theoretic version of the gamma construction of Hochster and Huneke to reduce to the setting where the ground field is  $F$ -finite. Our main result uses the gamma construction to extend the ampleness criterion of de Fernex, Küronya, and Lazarsfeld using asymptotic cohomological functions to projective varieties over arbitrary fields, which was previously known only for complex projective varieties. We also extend Nakayama’s description of the restricted base locus to klt or strongly  $F$ -regular varieties over arbitrary fields.

## CONTENTS

1	Introduction	166
2	Definitions and preliminaries	170
2.1	Morphisms essentially of finite type	170
2.2	Base loci	170
2.3	$F$ -finite schemes	172
2.4	$F$ -singularities	172
2.5	Test ideals	173
3	The gamma construction of Hochster–Huneke	175
3.1	The construction and proof of Theorem A	175
3.2	Application to the minimal model program over imperfect fields	181
4	The ampleness criterion of de Fernex–Küronya–Lazarsfeld	183

---

Received September 25, 2018. Revised August 8, 2019. Accepted August 8, 2019.

2010 Mathematics subject classification. Primary 14C20, Secondary 13A35, 14F17, 14B05.

This material is based upon work supported by the National Science Foundation under grant nos. DMS-1501461 and DMS-1701622.

© 2019 *Foundation Nagoya Mathematical Journal*

4.1	Background on asymptotic cohomological functions	183
4.2	A lemma on base loci	185
4.3	Proof of Theorem B	189
5	Nakayama's theorem on restricted base loci	196
5.1	Background on restricted base loci and non-nef loci	196
5.2	Proof of Theorem C	198
	Appendix A. Some results on $F$ -injective rings	200
	Appendix B. Strong $F$ -regularity for non- $F$ -finite rings	203
	References	204

## §1. Introduction

Let  $X$  be a projective variety over a field  $k$ . When  $k$  is the field of complex numbers, results from the minimal model program can be used to understand the birational geometry of  $X$ . However, when  $k$  is an arbitrary field, many of these results and the tools used to prove them are unavailable. The most problematic situation is when  $k$  is an imperfect field of characteristic  $p > 0$ , in which case there are three major difficulties. To begin with, since  $k$  is of characteristic  $p > 0$ ,

- (I) Resolutions of singularities are not known to exist (see [32]), and
- (II) Vanishing theorems are false (Raynaud [56]).

A common workaround for (I) is to use de Jong's theory of alterations [38]. Circumventing (II), on the other hand, is more difficult. One useful approach is to exploit the Frobenius morphism  $F: X \rightarrow X$  and its Grothendieck trace  $F_*\omega_X^\bullet \rightarrow \omega_X^\bullet$ ; see [54]. However, for imperfect fields, this approach runs into another problem:

- (III) Most applications of Frobenius techniques require  $k$  to be  $F$ -finite, that is, satisfy  $[k : k^p] < \infty$ .

This last issue arises since Grothendieck duality cannot be applied to the Frobenius if it is not finite. Recent advances in the minimal model program over imperfect fields due to Tanaka [64; 65] suggest that it would be worthwhile to develop a systematic way to deal with (III).

Our first goal is to provide such a systematic way to reduce to the case when  $k$  is  $F$ -finite. While passing to a perfect closure of  $k$  fixes the  $F$ -finiteness issue, this operation can change the singularities of  $X$  drastically.

To preserve singularities, we prove the following scheme-theoretic version of the gamma construction of Hochster and Huneke [34].

**THEOREM A.** *Let  $X$  be a scheme essentially of finite type over a field  $k$  of characteristic  $p > 0$ , and let  $\mathcal{Q}$  be a set of properties in the following list: local complete intersection, Gorenstein, Cohen–Macaulay,  $(S_n)$ , regular,  $(R_n)$ , normal, weakly normal, reduced, strongly  $F$ -regular,  $F$ -pure,  $F$ -rational,  $F$ -injective. Then, there exists a purely inseparable field extension  $k \subseteq k^\Gamma$  such that  $k^\Gamma$  is  $F$ -finite and such that the projection morphism*

$$\pi^\Gamma : X \times_k k^\Gamma \longrightarrow X$$

*is a homeomorphism that identifies  $\mathcal{P}$  loci for every  $\mathcal{P} \in \mathcal{Q}$ .*

See Section 2.4 for definitions of  $F$ -singularities in the non- $F$ -finite setting. We in fact prove a slightly stronger version of Theorem A that allows  $k$  to be replaced by a complete local ring and allows finitely many schemes instead of just one; see Theorem 3.4. We use this added flexibility to prove that klt and log canonical pairs can be preserved under the gamma construction for surfaces and threefolds (Corollary 3.7), providing alternative proofs for the reduction steps in [64, Theorem 3.8] and [65, Theorem 4.12].

We note that parts of Theorems A and 3.4 are new even if  $X$  is affine. Namely, the statements for weak normality are completely new, and the statements for  $F$ -purity and  $F$ -injectivity in Theorem 3.4 were previously only known when the scheme  $X$  is the spectrum of a complete local ring [17, Lemma 2.9; 45, Proposition 5.6].

In the remainder of this paper, we give applications of the gamma construction (Theorem A) to the theory of asymptotic invariants of line bundles over arbitrary fields, in the spirit of recent work of Cutkosky [8], Fulger–Kollár–Lehmann [21], Birkar [4], and Burgos Gil–Gubler–Jell–Künnemann–Martin [6]. See [14] for a survey of the theory for smooth complex varieties. While the main difficulty lies in positive characteristic, we also prove statements over fields of characteristic zero that are not necessarily algebraically closed.

Our first application provides a characterization of ampleness based on the asymptotic growth of higher cohomology groups. It is well known that if  $X$  is a projective variety of dimension  $n > 0$ , then  $h^i(X, \mathcal{O}_X(mL)) = O(m^n)$  for every Cartier divisor  $L$ ; see [43, Example 1.2.20]. It is natural to ask when cohomology groups have submaximal growth. The following result

says that ample Cartier divisors  $L$  are characterized by having submaximal growth of higher cohomology groups for small perturbations of  $L$ .

**THEOREM B.** *Let  $X$  be a projective variety of dimension  $n > 0$  over a field  $k$ . Let  $L$  be an  $\mathbf{R}$ -Cartier divisor on  $X$ , and consider the following property:*

- ( $\star$ ) *There exists a very ample Cartier divisor  $A$  on  $X$  and a real number  $\varepsilon > 0$  such that*

$$\widehat{h}^i(X, L - tA) := \limsup_{m \rightarrow \infty} \frac{h^i(X, \mathcal{O}_X(\lceil m(L - tA) \rceil))}{m^n/n!} = 0$$

*for all  $i > 0$  and for all  $t \in [0, \varepsilon]$ .*

*Then,  $L$  is ample if and only if  $L$  satisfies ( $\star$ ) for some pair  $(A, \varepsilon)$ .*

Here, the functions  $\widehat{h}^i(X, -)$  are the *asymptotic higher cohomological functions* introduced by Küronya [41]. Theorem B was first proved by de Fernex, Küronya, and Lazarsfeld over the complex numbers [12, Theorem 4.1]. In positive characteristic, an interesting aspect of our proof is that it requires the gamma construction (Theorem A) to reduce to the case when  $k$  is  $F$ -finite. The main outline of the proof follows that in [12], although overcoming the three problems described above requires care.

We note that our motivation for Theorem B comes from studying Seshadri constants, where Theorem B can be used to show that Seshadri constants and moving Seshadri constants of  $\mathbf{Q}$ -Cartier divisors can be described via jet separation on projective varieties over arbitrary fields, without smoothness assumptions [50, Proposition 7.2.10]. As a result, one can extend [49, Theorem A] to varieties with either singularities of dense  $F$ -injective type in characteristic zero, or varieties with  $F$ -injective singularities in characteristic  $p > 0$ , without any assumptions on the ground field [50, Theorem 7.3.1]. In [50, Theorem D], we use a similar argument to prove a new, local version of the Angehrn–Siu theorem [1, Theorem 0.1] in characteristic zero without the use of Kodaira-type vanishing theorems.

Our second application is a special case of a conjecture of Boucksom, Broustet, and Pacienza [5]. If  $D$  is a pseudoeffective  $\mathbf{R}$ -Cartier divisor on a projective variety  $X$ , then both the *restricted base locus*  $\mathbf{B}_-(D)$  of  $D$ , which is a lower approximation of the stable base locus  $\mathbf{B}(D)$  of  $D$ , and the *non-nef locus*  $\text{NNeft}(D)$  of  $D$ , which is defined in terms of divisorial valuations, are empty if and only if  $D$  is nef. See Section 5.1 for definitions

of both invariants. Boucksom, Broustet, and Pacienza conjectured that these two invariants of  $D$  are equal for all pseudoeffective  $\mathbf{R}$ -Cartier divisors on normal projective varieties [5, Conjecture 2.7]. We extend the known cases of their conjecture to projective varieties over arbitrary fields.

**THEOREM C.** *Let  $X$  be a normal projective variety over a field  $k$ , and let  $D$  be a pseudoeffective  $\mathbf{R}$ -Cartier divisor on  $X$ . If  $\text{char } k = 0$  and the non-klt locus of  $X$  is at most zero-dimensional, or if  $\text{char } k = p > 0$  and the non-strongly  $F$ -regular locus of  $X$  is at most zero-dimensional, then*

$$\mathbf{B}_-(D) = \text{Nef}(D).$$

This extends theorems of Nakayama [52, Lemma V.1.9(1)] (in the smooth case) and Cacciola–Di Biagio [7, Corollary 4.9] over the complex numbers, and of Mustață [51, Theorem 7.2] (in the regular case) and Sato [57, Corollary 4.8] over  $F$ -finite fields of characteristic  $p > 0$ .

**Outline**

This paper is structured as follows: in Section 2, we review some basic material, including the necessary background on  $F$ -finiteness,  $F$ -singularities, and test ideals. In Section 3, we prove Theorem 3.4, which is a stronger version of Theorem A. We use this stronger version in some applications to the minimal model program over imperfect fields in Section 3.2. The last two sections are devoted to our applications of the gamma construction. In Section 4, we prove Theorem B after reviewing some background on asymptotic cohomological functions. An important ingredient is a lemma on base loci (Proposition 4.6) analogous to [12, Proposition 3.1]. In Section 5, we prove Theorem C after giving some background on restricted base loci and non-nef loci. Finally, in Appendix A, we prove some results on  $F$ -injective rings for which we could not find a suitable reference, and in Appendix B, we describe different notions of strong  $F$ -regularity for non- $F$ -finite rings.

**Notation**

All rings will be commutative with identity. If  $R$  is a ring, then  $R^\circ$  denotes the complement of the union of the minimal primes of  $R$ . A *variety* is a reduced and irreducible scheme that is separated and of finite type over a field. A *complete scheme* is a scheme that is proper over a field. Intersection products are defined using Euler characteristics; see [40, Appendix B].

Let  $\mathbf{k} \in \{\mathbf{Q}, \mathbf{R}\}$ . A  $\mathbf{k}$ -Cartier divisor (resp.  $\mathbf{k}$ -Weil divisor) is an element of  $\text{Div}_{\mathbf{k}}(X) := \text{Div}(X) \otimes_{\mathbf{Z}} \mathbf{k}$  (resp.  $\text{WDiv}_{\mathbf{k}}(X) := \text{WDiv}(X) \otimes_{\mathbf{Z}} \mathbf{k}$ ). We denote  $\mathbf{k}$ -linear equivalence (resp.  $\mathbf{k}$ -numerical equivalence) by  $\sim_{\mathbf{k}}$  (resp.  $\equiv_{\mathbf{k}}$ ). We then set  $N_{\mathbf{k}}^1(X) := \text{Div}_{\mathbf{k}}(X) / \equiv_{\mathbf{k}}$ , which is a finite-dimensional  $\mathbf{k}$ -vector space if  $X$  is a complete scheme [8, Proposition 2.3]. We fix compatible norms  $\|\cdot\|$  on  $N_{\mathbf{k}}^1(X)$  for  $\mathbf{k} \in \{\mathbf{Q}, \mathbf{R}\}$ .

If  $X$  is a scheme of prime characteristic  $p > 0$ , then we denote by  $F: X \rightarrow X$  the (absolute) Frobenius morphism, which is given by the identity map on points, and the  $p$ -power map

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow F_*\mathcal{O}_X(U) \\ f &\longmapsto f^p \end{aligned}$$

on structure sheaves, where  $U \subseteq X$  is an open subset. If  $R$  is a ring of prime characteristic  $p > 0$ , we denote the corresponding ring homomorphism by  $F: R \rightarrow F_*R$ . For every integer  $e \geq 0$ , the  $e$ th iterate of the Frobenius morphisms for schemes or rings is denoted by  $F^e$ .

## §2. Definitions and preliminaries

### 2.1 Morphisms essentially of finite type

Recall that a ring homomorphism  $A \rightarrow B$  is *essentially of finite type* if  $B$  is isomorphic (as an  $A$ -algebra) to a localization of an  $A$ -algebra of finite type. The corresponding scheme-theoretic notion is the following:

DEFINITION 2.1. [53, Definition 2.1(a)] Let  $f: X \rightarrow Y$  be a morphism of schemes. We say that  $f$  is *locally essentially of finite type* if there is an affine open covering  $Y = \bigcup_i \text{Spec } A_i$  such that for every  $i$ , there is an affine open covering

$$f^{-1}(\text{Spec } A_i) = \bigcup_j \text{Spec } B_{ij}$$

for which the corresponding ring homomorphisms  $A_i \rightarrow B_{ij}$  are essentially of finite type. We say that  $f$  is *essentially of finite type* if it is locally essentially of finite type and quasicompact.

The class of morphisms (locally) essentially of finite type is closed under composition and base change [53, (2.2)].

### 2.2 Base loci

In this subsection, we define the base ideal of a Cartier divisor and related objects.

DEFINITION 2.2. (see [43, Definition 1.1.8]) Let  $X$  be a complete scheme over a field  $k$ , and let  $D$  be a Cartier divisor. The *complete linear series* associated to  $D$  is the projective space  $|D| := \mathbf{P}(H^0(X, \mathcal{O}_X(D)))^\vee$  of one-dimensional subspaces of  $H^0(X, \mathcal{O}_X(D))$ . The *base ideal* of  $D$  is

$$(1) \quad \mathfrak{b}(|D|) := \text{im}(H^0(X, \mathcal{O}_X(D)) \otimes_k \mathcal{O}_X(-D) \xrightarrow{\text{eval}} \mathcal{O}_X).$$

The *base scheme*  $\text{Bs}(|D|)$  of  $D$  is the closed subscheme of  $X$  defined by  $\mathfrak{b}(|D|)$ , and the *base locus* of  $D$  is the underlying closed subset  $\text{Bs}(|D|)_{\text{red}}$ .

We need the following description for how base ideals transform under birational morphisms.

LEMMA 2.3. *Let  $f: X' \rightarrow X$  be a birational morphism between complete varieties, where  $X$  is normal. Then, for every Cartier divisor  $D$  on  $X$ , we have  $f^{-1}\mathfrak{b}(|D|) \cdot \mathcal{O}_{X'} = \mathfrak{b}(|f^*D|)$ .*

*Proof.* Since  $X$  is normal, we have  $f_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$  [29, Proof of Corollary III.11.4]. By the projection formula, we then have

$$H^0(X, \mathcal{O}_X(D)) \simeq H^0(X', \mathcal{O}_{X'}(f^*D)),$$

and the lemma then follows by pulling back the evaluation map (1). □

Next, we define a stable version of the base locus.

DEFINITION 2.4. (see [43, Definition 2.1.20]) Let  $X$  be a complete scheme over a field, and let  $D$  be a Cartier divisor on  $X$ . The *stable base locus* of  $D$  is the closed subset

$$(2) \quad \mathbf{B}(D) := \bigcap_m \text{Bs}(|mD|)_{\text{red}}$$

of  $X$ , where the intersection runs over every integer  $m > 0$ . The Noetherian property implies  $\mathbf{B}(D) = \mathbf{B}(nD)$  for every integer  $n > 0$  [43, Example 2.1.23], hence the formula (2) can be used for  $\mathbf{Q}$ -Cartier divisors  $D$  by taking the intersection over every integer  $m > 0$  such that  $mD$  is integral.

The stable base locus is not a numerical invariant of  $D$  [44, Example 10.3.3]. In Section 5.1, we define the *restricted base locus*  $\mathbf{B}_-(D)$ , which is a numerically invariant approximation of  $\mathbf{B}(D)$ .

### 2.3 $F$ -finite schemes

As mentioned in Section 1, in positive characteristic, one often needs to restrict or reduce to the case when the Frobenius morphism is finite. We isolate this class of schemes.

**DEFINITION 2.5.** Let  $X$  be a scheme of prime characteristic  $p > 0$ . We say that  $X$  is  $F$ -finite if the (absolute) Frobenius morphism  $F: X \rightarrow X$  is finite. We say that a ring  $R$  of prime characteristic  $p > 0$  is  $F$ -finite if  $\text{Spec } R$  is  $F$ -finite, or equivalently if  $F: R \rightarrow F_*R$  is module-finite.

Note that a field  $k$  is  $F$ -finite if and only if  $[k : k^p] < \infty$ .  $F$ -finite schemes are ubiquitous in geometric contexts because of the following:

**EXAMPLE 2.6.** (see [42, p. 999]) If  $X$  is a scheme that is locally essentially of finite type over an  $F$ -finite scheme of prime characteristic  $p > 0$ , then  $X$  is  $F$ -finite. In particular, schemes essentially of finite type over perfect or  $F$ -finite fields are  $F$ -finite.

If a scheme  $X$  of prime characteristic  $p > 0$  is  $F$ -finite, then Grothendieck duality can be applied to the Frobenius morphism since it is finite [28, III.6]. The  $F$ -finiteness condition implies other desirable conditions as well.

**THEOREM 2.7.** [42, Theorem 2.5; 22, Remark 13.6] *Let  $R$  be a Noetherian  $F$ -finite ring of prime characteristic  $p > 0$ . Then,  $R$  is excellent and is isomorphic to a quotient of a regular ring of finite Krull dimension. In particular,  $R$  admits a dualizing complex  $\omega_R^\bullet$ .*

### 2.4 $F$ -singularities

We review some classes of singularities defined using the Frobenius morphism. See [62] for a survey, and see Appendix B for more material on strong  $F$ -regularity for non- $F$ -finite rings. Recall that a ring homomorphism  $R \rightarrow S$  is *pure* if the homomorphism  $R \otimes_R M \rightarrow S \otimes_R M$  is injective for every  $R$ -module  $M$ .

**DEFINITION 2.8.** [30, Definition 3.3; 36, p. 121] Let  $R$  be a Noetherian ring of prime characteristic  $p > 0$ . For every  $c \in R$  and every integer  $e > 0$ , we denote by  $\lambda_c^e$  the composition

$$R \xrightarrow{F^e} F_*^e R \xrightarrow{F_*^e(-\cdot c)} F_*^e R.$$

If  $c \in R$ , then following [10, Definition 6.1.1] we say that  $R$  is  *$F$ -pure along  $c$*  if  $\lambda_c^e$  is pure for some  $e > 0$ , and that



- (a)  $R$  is *strongly  $F$ -regular* if every localization  $R_{\mathfrak{p}}$  of  $R$  is  $F$ -pure along every  $c \in R_{\mathfrak{p}}^{\circ}$ ; and
- (b)  $R$  is  *$F$ -pure* if  $R$  is  $F$ -pure along  $1 \in R$ .

Note that (a) is not the usual definition (Definition B.1(a)) for strong  $F$ -regularity, which coincides with ours for  $F$ -finite rings. See Appendix B for a description of the relationship between different notions of strong  $F$ -regularity for non- $F$ -finite rings.

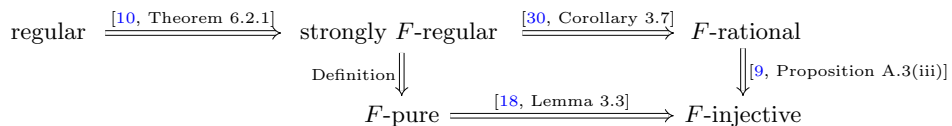
To define  $F$ -rationality, we recall that if  $R$  is a Noetherian ring, then a sequence of elements  $x_1, x_2, \dots, x_n \in R$  is a *sequence of parameters* if for every prime ideal  $\mathfrak{p}$  containing  $(x_1, x_2, \dots, x_n)$ , the images of  $x_1, x_2, \dots, x_n$  in  $R_{\mathfrak{p}}$  are part of a system of parameters in  $R_{\mathfrak{p}}$  [33, Definition 2.1].

DEFINITION 2.9. [19, Definition 1.10] A Noetherian ring of prime characteristic  $p > 0$  is  *$F$ -rational* if every ideal generated by a sequence of parameters in  $R$  is tightly closed in  $R$ .

See [33, Definition 3.1] for the definition of tight closure. Finally, we define  $F$ -injective singularities.

DEFINITION 2.10. [18, Definition on p. 473] A Noetherian ring  $R$  of prime characteristic  $p > 0$  is  *$F$ -injective* if, for every maximal ideal  $\mathfrak{m} \subseteq R$ , the  $R_{\mathfrak{m}}$ -module homomorphism  $H_{\mathfrak{m}}^i(F): H_{\mathfrak{m}}^i(R_{\mathfrak{m}}) \rightarrow H_{\mathfrak{m}}^i(F_*R_{\mathfrak{m}})$  induced by Frobenius is injective for all  $i$ .

We prove some basic results about  $F$ -injective rings in Appendix A. The relationship between these classes of singularities can be summarized as follows:



### 2.5 Test ideals

We review the theory of test ideals, which are the positive characteristic analogues of multiplier ideals. We recall that following [43, Definition 2.4.14], a collection  $\mathfrak{a}_{\bullet} := \{\mathfrak{a}_m\}_{m=1}^{\infty}$  of coherent ideal sheaves  $\mathfrak{a}_m \subseteq \mathcal{O}_X$  on a locally Noetherian scheme  $X$  is a *graded family of ideals* if  $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$  for all  $m, n \geq 1$ . We now fix the following notational conventions for pairs.

DEFINITION 2.11. (cf. [59, Definition 2.3]) A pair  $(X, \mathfrak{a}_\bullet^\lambda)$  consists of

- (i) an excellent reduced Noetherian scheme  $X$ ; and
- (ii) a symbol  $\mathfrak{a}_\bullet^\lambda$  where  $\mathfrak{a}_\bullet$  is a graded family of ideals on  $X$  such that for every open affine subset  $U = \text{Spec } R \subseteq X$ , we have  $\mathfrak{a}_m(U) \cap R^\circ \neq \emptyset$  for some  $m > 0$ , and where  $\lambda$  is a positive real number.

We drop  $\lambda$  from our notation if  $\lambda = 1$ . If  $\mathfrak{a}_\bullet = \{\mathfrak{a}^m\}_{m=1}^\infty$  for some fixed ideal sheaf  $\mathfrak{a}$ , then we denote the pair by  $(X, \mathfrak{a}^t)$ . If  $X = \text{Spec } R$  for a ring  $R$ , then we denote the pair by  $(R, \mathfrak{a}^\lambda)$ .

We now define test ideals for  $F$ -finite schemes of prime characteristic  $p > 0$ . See [61] and [62, Section 5] for overviews of the theory. We take Schwede’s characterization of test ideals via  $F$ -compatibility [59] as our definition.

DEFINITION 2.12. [59, Definition 3.1 and Theorem 6.3] Let  $(R, \mathfrak{a}^t)$  be a pair such that  $R$  is an  $F$ -finite ring of prime characteristic  $p > 0$ . An ideal  $J \subseteq R$  is *uniformly  $(\mathfrak{a}^t, F)$ -compatible* if for every integer  $e > 0$  and every  $\varphi \in \text{Hom}_R(F_*^e R, R)$ , we have

$$\varphi(F_*^e(J \cdot \mathfrak{a}^{\lceil t(p^e - 1) \rceil})) \subseteq J.$$

Now let  $(X, \mathfrak{a}^t)$  be a pair such that  $X$  is an  $F$ -finite scheme of prime characteristic  $p > 0$ . The *test ideal*  $\tau(X, \mathfrak{a}^t)$  is defined locally on each affine open subset  $U = \text{Spec } R \subseteq X$  as the smallest ideal that is uniformly  $(\mathfrak{a}^t, F)$ -compatible and intersects  $R^\circ$ .

The test ideal is well defined by [60, Proposition 3.23(ii)], and exists by [60, Theorem 3.18]. Formal properties analogous to those for multiplier ideals hold for test ideals; see [62, Proposition 5.6]. We can therefore define the following asymptotic version of test ideals:

DEFINITION 2.13. [57, Proposition-Definition 2.16] Let  $(X, \mathfrak{a}_\bullet^\lambda)$  be a pair such that  $X$  is  $F$ -finite of prime characteristic  $p > 0$ . The *asymptotic test ideal*  $\tau(X, \mathfrak{a}_\bullet^\lambda)$  is defined to be  $\tau(X, \mathfrak{a}_m^{\lambda/m})$  for sufficiently large and divisible  $m$ .

We often drop  $X$  from our notation if it is clear from context. Similarly, we drop  $\mathfrak{a}^t$  or  $\mathfrak{a}_\bullet^\lambda$  from our notation when working with the scheme itself.

The subadditivity theorem holds for both test ideals and asymptotic test ideals on regular complete local rings [27, Theorem 6.10(2)], and therefore

also holds on all regular  $F$ -finite schemes  $X$ , since for  $F$ -finite schemes, the formation of test ideals is compatible with localization and completion [26, Propositions 3.1 and 3.2].

The following example will be the most important in our applications.

EXAMPLE 2.14. (see [57, Definition 2.36]) Let  $X$  be a complete reduced scheme over an  $F$ -finite field of characteristic  $p > 0$ . If  $D$  is a Cartier divisor such that  $H^0(X, \mathcal{O}_X(mD)) \neq 0$  for some positive integer  $m$ , then for every real number  $t > 0$ , we set

$$\tau(X, t \cdot |D|) := \tau(X, \mathfrak{b}(|D|)^t).$$

If  $D$  is a  $\mathbf{Q}$ -Cartier divisor such that  $H^0(X, \mathcal{O}_X(mD)) \neq 0$  for some sufficiently divisible  $m > 0$ , then for every real number  $\lambda > 0$ , we set

$$\tau(X, \lambda \cdot \|D\|) := \tau(X, \mathfrak{a}_\bullet(D)^\lambda),$$

where  $\mathfrak{a}_m(D) = \mathfrak{b}(|mD|)$  if  $mD$  is integral, and 0 otherwise. See [43, Example 2.4.16(ii)].

### §3. The gamma construction of Hochster–Huneke

Our goal in this section is to prove Theorem A, which is a scheme-theoretic version of the gamma construction of Hochster and Huneke [34]. Hochster and Huneke first introduced the gamma construction in order to prove that test elements (in the sense of tight closure) exist for rings that are essentially of finite type over an excellent local ring of prime characteristic  $p > 0$ . However, to the best of our knowledge, their construction has not been applied explicitly in a geometric context.

As mentioned in Section 1, Theorem A provides a systematic way to reduce to the case when the ground field  $k$  is  $F$ -finite. We in fact show a more general result (Theorem 3.4), which allows for the ground field  $k$  to be replaced by a complete local ring, and allows finitely many schemes instead of just one. After proving Theorems A and 3.4 in Section 3.1, we prove that the  $F$ -pure locus is open in schemes essentially of finite type over excellent local rings (Corollary 3.5). We then give some basic applications of Theorem 3.4 to the minimal model program over imperfect fields in Section 3.2.

#### 3.1 The construction and proof of Theorem A

We start with the following account of Hochster and Huneke’s construction.

CONSTRUCTION 3.1. [34, (6.7) and (6.11)] Let  $(A, \mathfrak{m}, k)$  be a Noetherian complete local ring of prime characteristic  $p > 0$ . By the Cohen structure theorem, we may identify  $k$  with a coefficient field  $k \subseteq A$ . Moreover, by Zorn’s lemma (see [48, p. 202]), we may choose a  $p$ -basis  $\Lambda$  for  $k$ , which is a subset  $\Lambda \subseteq k$  such that  $k = k^p(\Lambda)$ , and such that for every finite subset  $\Sigma \subseteq \Lambda$  with  $s$  elements, we have  $[k^p(\Sigma) : k^p] = p^s$ .

Now let  $\Gamma \subseteq \Lambda$  be a cofinite subset, that is, a subset  $\Gamma$  of  $\Lambda$  such that  $\Lambda \setminus \Gamma$  is a finite set. For each integer  $e \geq 0$ , consider the subfield

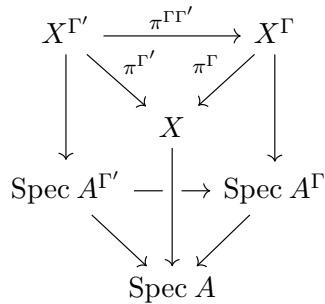
$$k_e^\Gamma = k[\lambda^{1/p^e}]_{\lambda \in \Gamma} \subseteq k_{\text{perf}}$$

of a perfect closure  $k_{\text{perf}}$  of  $k$ . These form an ascending chain, and we then set

$$A^\Gamma := \varinjlim_e (k_e^\Gamma \llbracket A \rrbracket),$$

where  $k_e^\Gamma \llbracket A \rrbracket$  is the completion of  $k_e^\Gamma \otimes_k A$  at the extended ideal  $\mathfrak{m} \cdot (k_e^\Gamma \otimes_k A)$ . Note that if  $A = k$  is a field, then  $A^\Gamma = k^\Gamma$  is a field by construction.

Finally, let  $X$  be a scheme essentially of finite type over  $A$ , and consider two cofinite subsets  $\Gamma \subseteq \Lambda$  and  $\Gamma' \subseteq \Lambda$  such that  $\Gamma \subseteq \Gamma'$ . We then have the following commutative diagram whose vertical faces are cartesian:



We list some elementary properties of the gamma construction.

LEMMA 3.2. Fix notation as in Construction 3.1, and let  $\Gamma \subseteq \Lambda$  be a cofinite subset.

- (i) The ring  $A^\Gamma$  and the scheme  $X^\Gamma$  are Noetherian and  $F$ -finite.
- (ii) The morphism  $\pi^\Gamma$  is a faithfully flat universal homeomorphism with local complete intersection fibers.
- (iii) Given a cofinite subset  $\Gamma \subseteq \Gamma'$ , the morphism  $\pi^{\Gamma'}$  is a faithfully flat universal homeomorphism.

*Proof.* The ring  $A^\Gamma$  is Noetherian and  $F$ -finite [34, (6.11)], hence  $X^\Gamma$  is also by Example 2.6 and the fact that morphisms essentially of finite type are preserved under base change [53, (2.2)]. The ring extensions  $A \subseteq A^\Gamma$  and  $A^\Gamma \subseteq A^{\Gamma'}$  are purely inseparable and faithfully flat [34, (6.11)], hence induce faithfully flat universal homeomorphisms on spectra [24, Proposition 2.4.5(i)]. Thus, the morphisms  $\pi^\Gamma$  and  $\pi^{\Gamma'}$  are faithfully flat universal homeomorphisms by base change. Finally, the ring extension  $A \subseteq A^\Gamma$  is flat with local complete intersection fibers [30, Lemma 3.19], hence  $\pi^\Gamma$  is also by base change [2, Corollary 4].  $\square$

Our goal now is to prove that if a local property of schemes satisfies certain conditions, then the property is preserved when passing from  $X$  to  $X^\Gamma$  for “small enough”  $\Gamma$ . For a scheme  $X$  and a property  $\mathcal{P}$  of local rings on  $X$ , the  $\mathcal{P}$  locus of  $X$  is  $\mathcal{P}(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ is } \mathcal{P}\}$ .

PROPOSITION 3.3. *Fix notation as in Construction 3.1, and let  $\mathcal{P}$  be a property of local rings of prime characteristic  $p > 0$ .*

- (i) *Suppose that for every flat local homomorphism  $B \rightarrow C$  of Noetherian local rings with local complete intersection fibers, if  $B$  is  $\mathcal{P}$ , then  $C$  is  $\mathcal{P}$ . Then,  $\pi^\Gamma(\mathcal{P}(X^\Gamma)) = \mathcal{P}(X)$  for every cofinite subset  $\Gamma \subseteq \Lambda$ .*
- (ii) *Consider the following conditions:*
  - ( $\Gamma 1$ ) *If  $B$  is a Noetherian  $F$ -finite ring of prime characteristic  $p > 0$ , then  $\mathcal{P}(\text{Spec } B)$  is open.*
  - ( $\Gamma 2$ ) *For every flat local homomorphism  $B \rightarrow C$  of Noetherian local rings of prime characteristic  $p > 0$  with zero-dimensional fibers, if  $C$  is  $\mathcal{P}$ , then  $B$  is  $\mathcal{P}$ .*
  - ( $\Gamma 3$ ) *For every local ring  $B$  essentially of finite type over  $A$ , if  $B$  is  $\mathcal{P}$ , then there exists a cofinite subset  $\Gamma_1 \subseteq \Lambda$  such that  $B^\Gamma$  is  $\mathcal{P}$  for every cofinite subset  $\Gamma \subseteq \Gamma_1$ .*
  - ( $\Gamma 3'$ ) *For every flat local homomorphism  $B \rightarrow C$  of Noetherian local rings of prime characteristic  $p > 0$  such that the closed fiber is a field, if  $B$  is  $\mathcal{P}$ , then  $C$  is  $\mathcal{P}$ .*

*If  $\mathcal{P}$  satisfies ( $\Gamma 1$ ), ( $\Gamma 2$ ), and one of either ( $\Gamma 3$ ) or ( $\Gamma 3'$ ), then there exists a cofinite subset  $\Gamma_0 \subseteq \Lambda$  such that  $\pi^\Gamma(\mathcal{P}(X^\Gamma)) = \mathcal{P}(X)$  for every cofinite subset  $\Gamma \subseteq \Gamma_0$ .*

*Proof.* For (i), it suffices to note that  $\pi^\Gamma$  is faithfully flat with local complete intersection fibers by Lemma 3.2(ii).

For (ii), we first note that (Γ3′) implies (Γ3), since there exists a cofinite subset  $\Gamma_1 \subseteq \Lambda$  such that the closed fiber is a field for every cofinite subset  $\Gamma \subseteq \Gamma_1$  by [34, Lemma 6.13(b)]. From now on, we therefore assume that  $\mathcal{P}$  satisfies (Γ1), (Γ2), and (Γ3).

For every cofinite subset  $\Gamma \subseteq \Lambda$ , the set  $\mathcal{P}(X^\Gamma)$  is open by (Γ1) since  $X^\Gamma$  is Noetherian and  $F$ -finite by Lemma 3.2(i). Moreover, the morphisms  $\pi^\Gamma$  and  $\pi^{\Gamma'}$  are faithfully flat universal homeomorphisms for every cofinite subset  $\Gamma' \subseteq \Lambda$  such that  $\Gamma \subseteq \Gamma'$  by Lemmas 3.2(ii) and 3.2(iii), hence by (Γ2), we have the inclusions

$$(3) \quad \mathcal{P}(X) \supseteq \pi^\Gamma(\mathcal{P}(X^\Gamma)) \supseteq \pi^{\Gamma'}(\mathcal{P}(X^{\Gamma'}))$$

in  $X$ , where  $\pi^\Gamma(\mathcal{P}(X^\Gamma))$  and  $\pi^{\Gamma'}(\mathcal{P}(X^{\Gamma'}))$  are open. Since  $X$  is Noetherian, it satisfies the ascending chain condition on the open sets  $\pi^\Gamma(\mathcal{P}(X^\Gamma))$ , hence we can choose a cofinite subset  $\Gamma_0 \subseteq \Lambda$  such that  $\pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$  is maximal with respect to inclusion.

We claim that  $\mathcal{P}(X) = \pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$  for every cofinite subset  $\Gamma \subseteq \Gamma_0$ . By (3), it suffices to show the inclusion  $\subseteq$ . Suppose there exists  $x \in \mathcal{P}(X) \setminus \pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$ . By (Γ3), there exists a cofinite subset  $\Gamma_1 \subseteq \Lambda$  such that  $(\pi^\Gamma)^{-1}(x) \in \mathcal{P}(X^\Gamma)$  for every cofinite subset  $\Gamma \subseteq \Gamma_1$ . Choosing  $\Gamma = \Gamma_0 \cap \Gamma_1$ , we have  $x \in \pi^\Gamma(\mathcal{P}(X^\Gamma)) \setminus \pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$ , contradicting the maximality of  $\pi^{\Gamma_0}(\mathcal{P}(X^{\Gamma_0}))$ . □

We now prove that the properties in Theorem A are preserved when passing to  $X^\Gamma$ . Special cases of the following result appear in [34, Lemma 6.13], [66, Theorem 2.2], [17, Lemma 2.9], [30, Lemmas 3.23 and 3.30], and [45, Proposition 5.6].

**THEOREM 3.4.** *Fix notation as in Construction 3.1.*

- (i) *For every cofinite subset  $\Gamma \subseteq \Lambda$ , the map  $\pi^\Gamma$  identifies local complete intersection, Gorenstein, Cohen–Macaulay, and  $(S_n)$  loci.*
- (ii) *There exists a cofinite subset  $\Gamma_0 \subseteq \Lambda$  such that  $\pi^\Gamma$  identifies regular (resp.  $(R_n)$ , normal, weakly normal, reduced, strongly  $F$ -regular,  $F$ -pure,  $F$ -rational,  $F$ -injective) loci for every cofinite subset  $\Gamma \subseteq \Gamma_0$ .*

Note that Theorem 3.4 implies Theorem A since if  $A$  is a field, then  $A^\Gamma$  is also by Construction 3.1, and moreover if one wants to preserve more than one property at once, then it suffices to intersect the various  $\Gamma_0$  for the different properties.

*Proof.* For (i), it suffices to note that these properties satisfy the condition in Proposition 3.3(i) by [2, Corollary 2] and [48, Theorem 23.4, Corollary to Theorem 23.3, and Theorem 23.9(iii)], respectively.

We now prove (ii). We first note that (ii) holds for regularity since (Γ1) holds by the excellence of  $X^\Gamma$ , and (Γ2) and (Γ3') hold by [48, Theorem 23.7]. Since  $\pi^\Gamma$  preserves the dimension of local rings, we therefore see that (ii) holds for  $(R_n)$ . (ii) for normality and reducedness then follows from (i) since they are equivalent to  $(R_1) + (S_2)$  and  $(R_0) + (S_1)$ , respectively.

To prove (ii) holds in the remaining cases, we check the conditions in Proposition 3.3(ii). For weak normality, (Γ1) holds by [3, Theorem 7.1.3], and (Γ2) holds by [47, Corollary II.2]. To show that (Γ3) holds, recall by [47, Theorem I.6] that a reduced ring  $B$  is weakly normal if and only if

$$(4) \quad B \longrightarrow B^\nu \begin{array}{c} \xrightarrow{b \rightarrow b \otimes 1} \\ \xrightarrow{b \rightarrow 1 \otimes b} \end{array} (B^\nu \otimes_B B^\nu)_{\text{red}}$$

is an equalizer diagram, where  $B^\nu$  is the normalization of  $B$ . Now suppose  $B$  is weakly normal, and let  $\Gamma_1 \subseteq \Lambda$  be a cofinite subset such that  $B^\Gamma$  is reduced,  $(B^\nu)^\Gamma$  is normal, and  $((B^\nu \otimes_B B^\nu)_{\text{red}})^\Gamma$  is reduced for every cofinite subset  $\Gamma \subseteq \Gamma_1$ ; such a  $\Gamma_1$  exists by the previous paragraph. We claim that  $B^\Gamma$  is weakly normal for every  $\Gamma \subseteq \Gamma_1$  cofinite in  $\Lambda$ . Since (4) is an equalizer diagram and  $A \subseteq A^\Gamma$  is flat, the diagram

$$B^\Gamma \longrightarrow (B^\nu)^\Gamma \begin{array}{c} \xrightarrow{b \rightarrow b \otimes 1} \\ \xrightarrow{b \rightarrow 1 \otimes b} \end{array} ((B^\nu \otimes_B B^\nu)_{\text{red}})^\Gamma$$

is an equalizer diagram. Moreover, since  $B^\Gamma \subseteq (B^\nu)^\Gamma$  is an integral extension of rings with the same total ring of fractions, and  $(B^\nu)^\Gamma$  is normal, we see that  $(B^\nu)^\Gamma = (B^\Gamma)^\nu$ . Finally,  $((B^\nu \otimes_B B^\nu)_{\text{red}})^\Gamma$  is reduced, hence we have the natural isomorphism

$$((B^\nu \otimes_B B^\nu)_{\text{red}})^\Gamma \simeq ((B^\Gamma)^\nu \otimes_{B^\Gamma} (B^\Gamma)^\nu)_{\text{red}}.$$

Thus, since the analogue of (4) with  $B$  replaced by  $B^\Gamma$  is an equalizer diagram, we see that  $B^\Gamma$  is weakly normal for every  $\Gamma \subseteq \Gamma_1$  cofinite in  $\Lambda$ , hence (Γ3) holds for weak normality.

We now prove (ii) for strong  $F$ -regularity,  $F$ -purity, and  $F$ -rationality. First, (Γ1) holds for strong  $F$ -regularity by [30, Lemma 3.29], and the same argument shows that (Γ1) holds for  $F$ -purity since the  $F$ -pure and

$F$ -split loci coincide for  $F$ -finite rings [36, Corollary 5.3]. Next, (Γ1) for  $F$ -rationality holds by [66, Theorem 1.11] since the reduced locus is open and reduced  $F$ -finite rings are admissible in the sense of [66, Definition 1.5] by Theorem 2.7. It then suffices to note that (Γ2) holds by [30, Lemma 3.17], [36, Proposition 5.13], and [66, (6) on p. 440], respectively, and (Γ3) holds by [30, Corollary 3.31], [45, Proposition 5.4], and [66, Lemma 2.3], respectively.

Finally, we prove (ii) for  $F$ -injectivity. First, (Γ1) and (Γ2) hold by Lemmas A.2 and A.3, respectively. The proof of [17, Lemma 2.9(b)] implies (Γ3), since the residue field of  $B$  is a finite extension of  $k$ , hence socles of Artinian  $B$ -modules are finite-dimensional  $k$ -vector spaces. □

We have the following consequence of Theorem 3.4, which was first attributed to Hoshi in [31, Theorem 3.2]. Note that the analogous statements for strong  $F$ -regularity and  $F$ -rationality appear in [30, Proposition 3.33] and [66, Theorem 3.5], respectively.

**COROLLARY 3.5.** *Let  $X$  be a scheme essentially of finite type over a local  $G$ -ring  $(A, \mathfrak{m})$  of prime characteristic  $p > 0$ . Then, the  $F$ -pure locus is open in  $X$ .*

Recall that a Noetherian ring  $R$  is a  $G$ -ring if, for every prime ideal  $\mathfrak{p} \subseteq R$ , the completion homomorphism  $R_{\mathfrak{p}} \rightarrow \widehat{R}_{\mathfrak{p}}$  is regular; see [48, pp. 255–256] for the definitions of  $G$ -rings and of regular homomorphisms. Excellent rings are  $G$ -rings by definition; see [48, Definition on p. 260].

*Proof.* Let  $A \rightarrow \widehat{A}$  be the completion of  $A$  at  $\mathfrak{m}$ , and let  $\Lambda$  be a  $p$ -basis for  $\widehat{A}/\mathfrak{m}\widehat{A}$  as in Construction 3.1. For every cofinite subset  $\Gamma \subseteq \Lambda$ , consider the commutative diagram

$$\begin{CD} X \times_A \widehat{A}^\Gamma @>\pi^\Gamma>> X \times_A \widehat{A} @>\pi>> X \\ @VVV @VVV @VVV \\ \text{Spec } \widehat{A}^\Gamma @>>> \text{Spec } \widehat{A} @>>> \text{Spec } A \end{CD}$$

where the squares are cartesian. By Theorem 3.4, there exists a cofinite subset  $\Gamma \subseteq \Lambda$  such that  $\pi^\Gamma$  is a homeomorphism identifying  $F$ -pure loci. Since  $X \times_A \widehat{A}^\Gamma$  is  $F$ -finite, the  $F$ -pure locus in  $X \times_A \widehat{A}$  is therefore open by the fact that (Γ1) holds for  $F$ -purity (see the proof of Theorem 3.4(ii)).

Now let  $x \in X \times_A \widehat{A}$ . Since  $A \rightarrow \widehat{A}$  is a regular homomorphism, the morphism  $\pi$  is also regular by base change [24, Proposition 6.8.3(iii)]. Thus,  $\mathcal{O}_{X \times_A \widehat{A}, x}$  is  $F$ -pure if and only if  $\mathcal{O}_{X, \pi(x)}$  is  $F$ -pure by [36, Proposition



5.13] and [30, Propositions 2.4(4) and 2.4(6)]. Denoting the  $F$ -pure locus in  $X$  by  $W$ , we see that  $\pi^{-1}(W)$  is the  $F$ -pure locus in  $X \times_A \widehat{A}$ . Since  $\pi^{-1}(W)$  is open and  $\pi$  is quasicompact and faithfully flat by base change, the  $F$ -pure locus  $W \subseteq X$  is open by [24, Corollaire 2.3.12].  $\square$

REMARK 3.6. Although Lemma A.2 shows that the  $F$ -injective locus is open under  $F$ -finiteness hypotheses, and the gamma construction (Theorem 3.4) implies that the  $F$ -injective locus is open for schemes essentially of finite type over *complete* local rings, the fact that the  $F$ -injective locus is open under the hypotheses of Corollary 3.5 is a recent result due to Rankeya Datta and the author [9, Theorem B].

### 3.2 Application to the minimal model program over imperfect fields

With notation as in Construction 3.1, let  $\{X_i\}$  be a finite set of schemes essentially of finite type over  $A$ . For each  $i$ , Theorem 3.4 produces a cofinite subset  $\Gamma_0^i \subseteq \Lambda$  such that properties of  $X_i$  are inherited by  $X_i^\Gamma$  for every  $\Gamma \subseteq \Gamma_0^i$  cofinite in  $\Lambda$ . Setting  $\Gamma_0 = \bigcap_i \Gamma_0^i$  gives a cofinite subset of  $\Lambda$  which works for every scheme in the set  $\{X_i\}$  at once. We illustrate this strategy with the following:

COROLLARY 3.7. *Let  $(X, \Delta)$  be a pair consisting of a normal variety  $X$  over a field  $k$  of characteristic  $p > 0$  and an  $\mathbf{R}$ -Weil divisor  $\Delta$  on  $X$ . Fix notation as in Construction 3.1, where we set  $A = k$ .*

- (i) *If  $X$  is a regular variety and  $\Delta$  has simple normal crossing support, then there exists a cofinite subset  $\Gamma_0 \subseteq \Lambda$  such that  $X^\Gamma$  is a regular variety and  $(\pi^\Gamma)^*\Delta$  has simple normal crossing support for every cofinite subset  $\Gamma \subseteq \Gamma_0$ .*
- (ii) *If  $\dim X \leq 3$  and  $(X, \Delta)$  is klt (resp. log canonical), then there exists a cofinite subset  $\Gamma_0 \subseteq \Lambda$  such that  $(X^\Gamma, (\pi^\Gamma)^*\Delta)$  is klt (resp. log canonical) for every cofinite subset  $\Gamma \subseteq \Gamma_0$ .*

*Proof.* For (i), first write  $\Delta = \sum a_i D_i$ , where  $a_i \in \mathbf{R}$  and  $D_i$  are prime divisors. By Theorem 3.4 applied to the regular locus of  $X$  and of every set of intersections of the  $D_i$ , we see that there exists a cofinite subset  $\Gamma_0 \subseteq \Lambda$  such that  $X^\Gamma$  is a regular variety and  $(\pi^\Gamma)^*\Delta = \sum a_i (\pi^\Gamma)^* D_i$  has simple normal crossing support for every  $\Gamma \subseteq \Gamma_0$  cofinite in  $\Lambda$ . (ii) then follows by applying (i) to a log resolution of  $(X, \Delta)$  while simultaneously choosing  $\Gamma_0$  such that  $X^\Gamma$  is normal for every cofinite subset  $\Gamma \subseteq \Gamma_0$ .  $\square$

Corollary 3.7(ii) easily provides another method for proving the reduction step in [64, Theorem 3.8]. It can also be used to prove the more subtle reduction step in the following result of Tanaka.

**THEOREM 3.8.** [65, Theorem 4.12] *Let  $k$  be a field of characteristic  $p > 0$ . Let  $(X, \Delta)$  be a log canonical surface over  $k$ , where  $\Delta$  is a  $\mathbf{Q}$ -Weil divisor. Let  $f: X \rightarrow S$  be a projective morphism to a separated scheme  $S$  of finite type over  $k$ . If  $K_X + \Delta$  is  $f$ -nef, then  $K_X + \Delta$  is  $f$ -semi-ample.*

The first step of the proof in [65] is to reduce to the case where  $k$  is  $F$ -finite and contains an infinite perfect field in order to apply [63, Theorem 1]. We illustrate how one can use the gamma construction (Theorem 3.4) to make this reduction.

*Proof of reduction.* Note that the formation of  $K_X$  is compatible with ground field extensions [28, Corollary V.3.4(a)], and that  $f$ -nefness is preserved under base change since  $f$ -ampleness is. By flat base change and the fact that field extensions are faithfully flat,  $f$ -semi-ampleness can be checked after a ground field extension. Since  $k(x^{1/p^\infty})$  contains the infinite perfect field  $\mathbf{F}_p(x^{1/p^\infty})$  and applying the gamma construction to  $k(x^{1/p^\infty})$  results in an  $F$ -finite field (Construction 3.1 and Lemma 3.2(i)), it therefore suffices to show that for some choice of  $\Gamma$ , the base change of  $(X, \Delta)$  under the sequence of ground field extensions

$$k \subseteq k(x^{1/p^\infty}) \subseteq (k(x^{1/p^\infty}))^\Gamma$$

is a log canonical surface. Moreover, Corollary 3.7(ii) implies it suffices to prove that the base change of  $(X, \Delta)$  to  $k(x^{1/p^\infty})$  is a log canonical surface.

Fix a log resolution  $\mu: Y \rightarrow X$  for  $(X, \Delta)$ , and write  $K_Y - \mu^*(K_X + \Delta) = \sum_i a_i E_i$ . Note that  $k(x^{1/p^\infty}) = \bigcup_e k(x^{1/p^e})$ , and that each field  $k(x^{1/p^e})$  is isomorphic to  $k(x)$ . Since integrality, normality, and regularity are preserved under limits of schemes with affine and flat transition morphisms [24, Corollaire 5.13.4 and Proposition 5.13.7], it suffices to show that  $X \times_k k(x)$  is a normal variety,  $Y \times_k k(x)$  is a regular variety, and each  $E_i \times_k k(x)$  is a regular variety such that every intersection of the  $E_i \times_k k(x)$ 's is regular. This follows for  $X \times_k k(x)$ , since if  $\bigcup_j U_j$  is an affine open covering of  $X$ , then  $X \times_k k(x)$  is covered by affine open subsets that are localizations of the normal varieties  $U_j \times_k \mathbf{A}_k^1$ , which pairwise intersect. A similar argument works for  $Y$ , the  $E_i$ 's, and the intersections of the  $E_i$ 's. □

**§4. The ampleness criterion of de Fernex–Küronya–Lazarsfeld**

We now come to our first application of the gamma construction, Theorem B. Let  $X$  be a projective variety of dimension  $n > 0$ . For every Cartier divisor  $L$  on  $X$ , we have

$$h^i(X, \mathcal{O}_X(mL)) = O(m^n)$$

for every  $i$ ; see [43, Example 1.2.20]. In [12, Theorem 4.1], de Fernex, Küronya, and Lazarsfeld asked when the higher cohomology groups have submaximal growth, that is, when  $h^i(X, \mathcal{O}_X(mL)) = o(m^n)$ . They proved that over the complex numbers, ample Cartier divisors  $L$  are characterized by having submaximal growth of higher cohomology groups for small perturbations of  $L$ . The content of Theorem B is that their characterization holds for projective varieties over arbitrary fields. Note that one can have  $\widehat{h}^i(X, L) = 0$  for all  $i > 0$  without  $L$  being ample, or even pseudoeffective, hence the perturbation by  $A$  is necessary; see [41, Section 3.1] or [14, Example 4.4].

After reviewing some background material on asymptotic cohomological functions in Section 4.1 following [41, Section 2; 6, Section 3], we prove an analogue of a lemma on base loci [12, Proposition 3.1] in Section 4.2. This latter subsection is where asymptotic test ideals are used. Finally, we prove Theorem B in Section 4.3 using the gamma construction and alterations.

Before getting into the details of the proof, we briefly describe the main difficulties in adapting the proof of [12, Theorem 4.1] to positive characteristic. First, the proof of [12, Proposition 3.1] requires resolutions of singularities, and because of this, we can only prove a version of this lemma (Proposition 4.6) under the additional hypothesis that a specific pair has (a weak version of) a log resolution. This weaker result suffices for Theorem B since we can reduce to this situation by taking the Stein factorization of an alteration. Second, [12] uses the assumption that the ground field is uncountable to choose countably many very general divisors that facilitate an inductive argument. We reduce to the setting where the ground field is uncountable by adjoining uncountably many indeterminates to our ground field and then applying the gamma construction (Theorem A) to reduce to the  $F$ -finite case; see Lemma 4.9.

**4.1 Background on asymptotic cohomological functions**

We first review Küronya’s asymptotic cohomological functions with suitable modifications to work over arbitrary fields, following [41, Section 2; 6, Section 3]. Asymptotic cohomological functions are defined as follows:

DEFINITION 4.1. [6, Definition 3.4.6] Let  $X$  be a projective scheme of dimension  $n$  over a field. For every integer  $i \geq 0$ , the *ith asymptotic cohomological function* on  $X$  is the function defined by setting

$$\widehat{h}^i(X, D) := \limsup_{m \rightarrow \infty} \frac{h^i(X, \mathcal{O}_X(\lceil mD \rceil))}{m^n/n!}$$

for an  $\mathbf{R}$ -Cartier divisor  $D$  on  $X$ , where the round-up is defined by writing  $D = \sum_i a_i D_i$  as an  $\mathbf{R}$ -linear combination of Cartier divisors and setting  $\lceil mD \rceil := \sum_i \lceil ma_i \rceil D_i$ ; see [6, Definition 3.4.1]. The numbers  $\widehat{h}^i(X, D)$  only depend on the  $\mathbf{R}$ -linear equivalence class of  $D$  and are independent of the decomposition  $D = \sum_i a_i D_i$  by [6, Remark 3.4.5], hence  $\widehat{h}^i(X, -)$  gives rise to well-defined functions  $\text{Div}_{\mathbf{R}}(X) \rightarrow \mathbf{R}$  and  $\text{Div}_{\mathbf{R}}(X)/\sim_{\mathbf{R}} \rightarrow \mathbf{R}$ .

A key property of asymptotic cohomological functions is the following:

PROPOSITION 4.2. [6, Proposition 3.4.8] *Let  $X$  be a projective scheme of dimension  $n$  over a field. For every  $i \geq 0$ , the function  $\widehat{h}^i(X, -)$  on  $\text{Div}_{\mathbf{R}}(X)$  is homogeneous of degree  $n$ , and is continuous on every finite-dimensional  $\mathbf{R}$ -subspace of  $\text{Div}_{\mathbf{R}}(X)$  with respect to every norm.*

Proposition 4.2 shows that Definition 4.1 is equivalent to Küronya’s original definition in [41], and allows us to prove that asymptotic cohomological functions behave well with respect to generically finite morphisms.

PROPOSITION 4.3. (cf. [41, Proposition 2.9(1)]) *Let  $f: Y \rightarrow X$  be a surjective morphism of projective varieties, and consider an  $\mathbf{R}$ -Cartier divisor  $D$  on  $X$ . Suppose  $f$  is generically finite of degree  $d$ . Then, for every  $i$ , we have*

$$\widehat{h}^i(Y, f^*D) = d \cdot \widehat{h}^i(X, D).$$

*Proof.* The proof of [41, Proposition 2.9(1)] works in our setting with the additional hypothesis that  $D$  is a Cartier divisor. It therefore suffices to reduce to this case. If the statement holds for integral  $D$ , then it also holds for  $D \in \text{Div}_{\mathbf{Q}}(X)$  by homogeneity of  $\widehat{h}^i$  (Proposition 4.2). Moreover, the subspace of  $\text{Div}_{\mathbf{R}}(X)$  spanned by the Cartier divisors appearing in  $D$  is finite-dimensional, hence by approximating each coefficient in  $D$  by rational numbers, Proposition 4.2 implies the statement for  $D \in \text{Div}_{\mathbf{R}}(X)$  by continuity. □

REMARK 4.4. We repeatedly use the same steps as in the proof of Proposition 4.3 to prove statements about  $\widehat{h}^i(X, D)$  for arbitrary  $\mathbf{R}$ -Cartier

divisors by reducing to the case when  $D$  is a Cartier divisor. If  $D$  is an  $\mathbf{R}$ -Cartier divisor, we can write  $D$  as the limit of  $\mathbf{Q}$ -Cartier divisors by approximating each coefficient in a decomposition of  $D$  by rational numbers, and continuity of asymptotic cohomological functions (Proposition 4.2) then allows us to reduce to the case when  $D$  is a  $\mathbf{Q}$ -Cartier divisor. By homogeneity of asymptotic cohomology functions (Proposition 4.2), one can then reduce to the case when  $D$  is a Cartier divisor.

We also need the following:

**PROPOSITION 4.5.** (Asymptotic Serre duality; cf. [41, Corollary 2.11])  
*Let  $X$  be a projective variety of dimension  $n$ , and let  $D$  be an  $\mathbf{R}$ -Cartier divisor on  $X$ . Then, for every  $0 \leq i \leq n$ , we have*

$$\widehat{h}^i(X, D) = \widehat{h}^{n-i}(X, -D).$$

*Proof.* By Remark 4.4, it suffices to consider the case when  $D$  is integral. Let  $f: Y \rightarrow X$  be a regular alteration of degree  $d$  [38, Theorem 4.1]. We then have

$$\widehat{h}^i(Y, f^*D) = \limsup_{m \rightarrow \infty} \frac{h^{n-i}(Y, \mathcal{O}_Y(K_Y - f^*(mD)))}{m^n/n!} = \widehat{h}^{n-i}(Y, -f^*D)$$

by Serre duality and [6, Lemma 3.2.1], respectively. By Proposition 4.3, the left-hand side is equal to  $d \cdot \widehat{h}^i(X, D)$  and the right-hand side is equal to  $d \cdot \widehat{h}^{n-i}(X, -D)$ , hence the statement follows after dividing by  $d$ .  $\square$

**4.2 A lemma on base loci**

A key ingredient in our proof of Theorem B is the following result on base loci, which is an analogue of [12, Proposition 3.1] over more general fields. In positive characteristic, we use asymptotic test ideals instead of asymptotic multiplier ideals, which requires working over an  $F$ -finite field.

**PROPOSITION 4.6.** *Let  $V$  be a normal projective variety of dimension at least two over an infinite field  $k$ , where if  $\text{char } k = p > 0$ , then we also assume that  $k$  is  $F$ -finite. Let  $D$  be a Cartier divisor on  $V$ . Assume there exists a closed subscheme  $Z \subseteq V$  of pure dimension 1 such that*

- (i)  $D \cdot Z_\alpha < 0$  for every irreducible component  $Z_\alpha$  of  $Z$ , and
- (ii) *There exists a projective birational morphism  $\mu: V' \rightarrow V$  such that  $V'$  is regular and  $(\mu^{-1}(Z))_{\text{red}}$  is a simple normal crossing divisor.*

Let  $\mathfrak{a} \subseteq \mathcal{O}_V$  be the ideal sheaf of  $Z$ . Then, there exist positive integers  $q$  and  $c$  such that for every integer  $m \geq c$ , we have

$$\mathfrak{b}(|mqD|) \subseteq \mathfrak{a}^{m-c}.$$

Here,  $\mathfrak{b}(|D|)$  denotes the *base ideal* of the Cartier divisor  $D$ ; see Definition 2.2. We note that the  $Z_\alpha$  can possibly be nonreduced.

In the proof below, we use the fact [40, Lemma B.12] that if  $W$  is a one-dimensional subscheme of a complete scheme  $X$  over a field, and if  $D$  is a Cartier divisor on  $X$ , then

$$(5) \quad (D \cdot W) = \sum_{\alpha} \text{length}_{\mathcal{O}_{X, \eta_\alpha}}(\mathcal{O}_{W_\alpha, \eta_\alpha}) \cdot (D \cdot W_\alpha),$$

where the  $W_\alpha$  are the one-dimensional components of  $W$  with generic points  $\eta_\alpha \in W_\alpha$ .

*Proof.* The statement is trivial if  $H^0(V, \mathcal{O}_V(mD)) = 0$  for every integer  $m > 0$ , since in this case  $\mathfrak{b}(|mqD|) = 0$  for all positive integers  $m, q$ . We therefore assume  $H^0(V, \mathcal{O}_V(mD)) \neq 0$  for some integer  $m > 0$ . We prove the statement in positive characteristic; see Remark 4.7 for the characteristic zero case.

We fix some notation. Set  $D' = \mu^*D$  and set  $E = (\mu^{-1}(Z))_{\text{red}}$ . We fix a very ample Cartier divisor  $H$  on  $V'$ , and set  $A = K_{V'} + (\dim V' + 1)H$ . For every subvariety  $W \subseteq V'$ , a *complete intersection curve* is a curve formed by taking the intersection of  $\dim W - 1$  hyperplane sections in  $|H|_W|$ , and a *general complete intersection curve* is one formed by taking these hyperplane sections to be general in  $|H|_W|$ . For each positive integer  $q$ , we consider the asymptotic test ideal

$$\tau(V', \|qD'\|) = \tau(\|qD'\|) \subseteq \mathcal{O}_{V'}.$$

By uniform global generation for test ideals [57, Proposition 4.1], the sheaf

$$(6) \quad \tau(\|qD'\|) \otimes \mathcal{O}_{V'}(qD' + A)$$

is globally generated for every integer  $q > 0$ .

*Step 1.* There exists an integer  $\ell_0 > 0$  such that for every integer  $\ell > \ell_0$  and for every irreducible component  $F$  of  $E$  that dominates  $(Z_\alpha)_{\text{red}}$  for some  $\alpha$ , we have

$$\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-F).$$

Let  $C \subseteq F$  be a general complete intersection curve; note that  $C$  is integral by Bertini’s theorem [20, Theorem 3.4.10 and Corollary 3.4.14] and dominates  $(Z_\alpha)_{\text{red}}$  for some  $\alpha$ , hence  $(D' \cdot C) < 0$  by the projection formula and (5). If for some integer  $q > 0$ , the curve  $C$  is not contained in the zero locus of  $\tau(\|qD'\|)$ , then the fact that the sheaf (6) is globally generated implies

$$((qD' + A) \cdot C) \geq 0.$$

Letting  $\ell_{0F} = -(A \cdot C)/(D' \cdot C)$ , we see that the ideal  $\tau(\|\ell D'\|)$  vanishes everywhere along  $C$  for every integer  $\ell > \ell_{0F}$ . By varying  $C$ , the ideal  $\tau(\|\ell D'\|)$  must vanish everywhere along  $F$  for every integer  $\ell > \ell_{0F}$ , hence we can set  $\ell_0 = \max_F\{\ell_{0F}\}$ .

*Step 2.* Let  $E_i$  be an irreducible component of  $E$  not dominating  $Z_\alpha$  for every  $\alpha$ . Suppose  $E_j$  is another irreducible component of  $E$  such that  $E_i \cap E_j \neq \emptyset$  and for which there exists an integer  $\ell_j$  such that for every integer  $\ell > \ell_j$ , we have

$$\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E_j).$$

Then, there is an integer  $\ell_i \geq \ell_j$  such that for every integer  $\ell > \ell_i$ , we have

$$\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E_i).$$

Let  $C \subseteq E_i$  be a complete intersection curve. By the assumption that  $E$  is a simple normal crossing divisor, there exists at least one closed point  $P \in C \cap E_j$ . For every  $\ell > \ell_j$  and every  $m > 0$ , we have the sequence of inclusions

$$\begin{aligned} & (\tau(\|m\ell D'\|) \otimes \mathcal{O}_{V'}(m\ell D' + A)) \cdot \mathcal{O}_C \\ (7) \quad & \subseteq (\tau(\|\ell D'\|)^m \otimes \mathcal{O}_{V'}(m\ell D' + A)) \cdot \mathcal{O}_C \\ & \subseteq (\mathcal{O}_{V'}(-mE_j) \otimes \mathcal{O}_{V'}(m\ell D' + A)) \cdot \mathcal{O}_C \\ & \subseteq \mathcal{O}_C(A|_C - mP), \end{aligned}$$

where the first two inclusions follow from subadditivity [27, Theorem 6.10(2)] and by assumption, respectively. The last inclusion holds since  $C$  maps to a closed point in  $V$ , hence  $\mathcal{O}_C(D') = \mathcal{O}_C$ . By the global generation of the sheaf in (6) for  $q = m\ell$ , the inclusion (7) implies that for every integer  $\ell > \ell_j$ , if  $\tau(\|m\ell D'\|)$  does not vanish everywhere along  $C$ , then  $(A \cdot C) \geq m$ . Choosing  $\ell_i = \ell_j \cdot ((A \cdot C) + 1)$ , we see that  $\tau(\|\ell D'\|)$  vanishes everywhere along  $C$  for every integer  $\ell > \ell_i$ . By varying  $C$ , we have  $\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E_i)$  for every integer  $\ell > \ell_i$ .

*Step 3.* There exists an integer  $a > 0$  such that

$$\mathfrak{b}(|maD'|) \subseteq \mathcal{O}_{V'}(-mE)$$

for every integer  $m > 0$ .

Write

$$E = \bigcup_j \bigcup_{i \in I_j} E_{ij},$$

where the  $E_{ij}$  are the irreducible components of  $E$ , and the  $\bigcup_{i \in I_j} E_{ij}$  are the connected components of  $E$ . Since  $V$  is normal, each preimage  $\mu^{-1}(Z_\alpha)$  is connected by Zariski's main theorem [29, Corollary III.11.4], hence each connected component  $\bigcup_{i \in I_j} E_{ij}$  of  $E$  contains an irreducible component  $E_{i_0j}$  that dominates  $(Z_\alpha)_{\text{red}}$  for some  $\alpha$ . By Step 1, there exists an integer  $\ell_0$  such that for every  $j$ , we have  $\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E_{i_0j})$  for every integer  $\ell > \ell_0$ . For each  $j$ , by applying Step 2 ( $|I_j| - 1$ ) times to the  $j$ th connected component  $\bigcup_{i \in I_j} E_{ij}$  of  $E$ , we can find  $\ell_j$  such that  $\tau(\|\ell D'\|) \subseteq \mathcal{O}_{V'}(-E_{ij})$  for every  $i \in I_j$  and for every integer  $\ell > \ell_j$ . Setting  $a = \max_j \{\ell_j\} + 1$ , we have  $\tau(\|aD'\|) \subseteq \mathcal{O}_{V'}(-E)$ . Thus, for every integer  $m > 0$ , we have

$$\mathfrak{b}(|maD'|) \subseteq \tau(|maD'|) \subseteq \tau(\|\ell maD'\|) \subseteq \tau(\|aD'\|)^m \subseteq \mathcal{O}_{V'}(-mE),$$

where the first inclusion follows by the fact that  $V'$  is regular hence strongly  $F$ -regular [62, Propositions 5.6(1) and 5.6(5)], the second inclusion is by definition of the asymptotic test ideal, and the third inclusion is by subadditivity [27, Theorem 6.10(2)].

*Step 4.* Conclusion of proof of Proposition 4.6.

Let  $\pi: V'' \rightarrow V'$  be the normalized blowup of the ideal  $\mu^{-1}\mathfrak{a} \cdot \mathcal{O}_{V'}$ , and write  $(\mu \circ \pi)^{-1}\mathfrak{a} \cdot \mathcal{O}_{V''} = \mathcal{O}_{V''}(-E'')$  for a Cartier divisor  $E''$  on  $V''$ . Note that since  $\mathcal{O}_{V''}(-(\pi^*E)_{\text{red}})$  is the radical of  $\mathcal{O}_{V''}(-E'')$ , there exists an integer  $b > 0$  such that  $\mathcal{O}_{V''}(-b(\pi^*E)_{\text{red}}) \subseteq \mathcal{O}_{V''}(-E'')$ . We then have

$$\begin{aligned} \mathfrak{b}(|mab \pi^*D'|) &= \pi^{-1}\mathfrak{b}(|mabD'|) \cdot \mathcal{O}_{V''} \\ &\subseteq \mathcal{O}_{V''}(-mb \pi^*E) \\ &\subseteq \mathcal{O}_{V''}(-mb(\pi^*E)_{\text{red}}) \\ &\subseteq \mathcal{O}_{V''}(-mE'') \end{aligned}$$



by Step 3, where the first equality holds by Lemma 2.3. Setting  $q = ab$  and pushing forward by  $\mu \circ \pi$ , we have

$$\begin{aligned} \mathfrak{b}(|mqD|) &\subseteq (\mu \circ \pi)_* \mathfrak{b}(|mq \pi^* D'|) \\ &\subseteq (\mu \circ \pi)_* \mathcal{O}_{V''}(-mE'') = \overline{\mathfrak{a}^m}, \end{aligned}$$

where the first inclusion follows from Lemma 2.3, and where  $\overline{\mathfrak{a}^m}$  is the integral closure of  $\mathfrak{a}^m$  [44, Remark 9.6.4]. Finally, given any ideal  $\mathfrak{a} \subseteq \mathcal{O}_V$ , there exists an integer  $c$  such that  $\overline{\mathfrak{a}^{\ell+1}} = \mathfrak{a} \cdot \overline{\mathfrak{a}^\ell}$  for all  $\ell \geq c$  [44, Proof of Proposition 9.6.6], hence  $\overline{\mathfrak{a}^m} \subseteq \mathfrak{a}^{m-c}$  for all  $m \geq c$ .  $\square$

REMARK 4.7. When  $\text{char } k = 0$ , one can prove the stronger statement of [12, Proposition 3.1] using resolutions of singularities and the asymptotic multiplier ideals  $\mathcal{J}(\|D\|)$  defined in [44, Definition 11.1.2] by replacing [62, Proposition 5.6] and [27, Theorem 6.10(2)] with [13, Proposition 2.3] and [39, Theorem A.2], respectively. To replace [57, Proposition 4.1], one can pass to the algebraic closure (since the formation of multiplier ideals is compatible with ground field extensions [39, Proposition 1.9]) to deduce uniform global generation from the algebraically closed case [44, Corollary 11.2.13].

Before moving on to the proof of Theorem B, we note that after a preprint of this paper was posted, the authors of [46] informed us that they had proved an asymptotic nonvanishing statement [46, Lemma 4.2] using techniques similar to ours in Proposition 4.6 and Theorem B. By combining the methods in this paper and in [46, Lemmas 4.3 and 4.4], one can prove the full analogue of [12, Proposition 3.1], namely:

PROPOSITION 4.8. *Let  $V$  be a normal projective variety of dimension at least two over a field  $k$ . Let  $D$  be a Cartier divisor on  $V$ , and suppose there exists an integral curve  $Z \subseteq V$  such that  $(D \cdot Z) < 0$ . Denote by  $\mathfrak{a} \subseteq \mathcal{O}_V$  the ideal sheaf defining  $Z$ . Then, there exist positive integers  $q$  and  $c$  such that for every integer  $m \geq c$ , we have*

$$\mathfrak{b}(|mqD|) \subseteq \mathfrak{a}^{m-c}.$$

We will not use Proposition 4.8 in the sequel. See [50, Proposition 6.2.1] for a proof.

### 4.3 Proof of Theorem B

We now prove Theorem B. We first note that the direction  $\Rightarrow$  in Theorem B follows from existing results.

*Proof of  $\Rightarrow$  in Theorem B.* Let  $A$  be a very ample Cartier divisor. Then, for all  $t$  such that  $L - tA$  is ample, we have  $\widehat{h}^i(X, L - tA) = 0$  by Serre vanishing and by homogeneity and continuity; see Remark 4.4.  $\square$

For the direction  $\Leftarrow$ , it suffices to show Theorem B for Cartier divisors  $L$  by continuity and homogeneity; see Remark 4.4. We also make the following two reductions. Recall that an  $\mathbf{R}$ -Cartier divisor  $L$  on  $X$  satisfies  $(\star)$  for a pair  $(A, \varepsilon)$  consisting of a very ample Cartier divisor  $A$  on  $X$  and a real number  $\varepsilon > 0$  if  $\widehat{h}^i(X, L - tA) = 0$  for all  $i > 0$  and all  $t \in [0, \varepsilon)$ .

LEMMA 4.9. *To prove the direction  $\Leftarrow$  in Theorem B, we may assume that the ground field  $k$  is uncountable. In positive characteristic, we may also assume that  $k$  is  $F$ -finite.*

*Proof.* We first construct a sequence

$$k \subseteq k' \subseteq K$$

of two field extensions such that  $X \times_k K$  is integral, where  $k'$  is uncountable and  $K$  is  $F$ -finite in positive characteristic. If  $k$  is already uncountable, then let  $k' = k$ . Otherwise, consider a purely transcendental extension

$$k' := k(x_\alpha)_{\alpha \in A},$$

where  $\{x_\alpha\}_{\alpha \in A}$  is an uncountable set of indeterminates; note that  $k'$  is uncountable by construction. To show that  $X \times_k k'$  is integral, let  $\bigcup_j U_j$  be an affine open covering of  $X$ . Then,  $X \times_k k'$  is covered by affine open subsets that are localizations of the integral varieties  $U_j \times_k \text{Spec}(k[x_\alpha]_{\alpha \in A})$ , which pairwise intersect, hence  $X \times_k k'$  is integral. We set  $K = k'$  in characteristic zero, and in positive characteristic, the gamma construction (Theorem A) shows that there is a field extension  $k' \subseteq K$  such that  $K$  is  $F$ -finite and the scheme  $X \times_k K$  is integral. Note that  $K$  is uncountable since it contains the uncountable field  $k'$ .

Now suppose  $X$  is a projective variety over  $k$ , and let  $L$  be an Cartier divisor satisfying  $(\star)$  for some pair  $(A, \varepsilon)$ . Let

$$\pi: X \times_k K \longrightarrow X$$

be the first projection map, which we note is faithfully flat by base change. Then, the pullback  $\pi^*A$  of  $A$  is very ample, and to show that  $L$  is ample, it suffices to show that  $\pi^*L$  is ample by flat base change and Serre’s criterion

for ampleness. By the special case of Theorem B over the ground field  $K$ , it therefore suffices to show that  $\pi^*L$  satisfies  $(\star)$  for the pair  $(\pi^*A, \varepsilon)$ .

We want to show that for every  $i > 0$  and for all  $t \in [0, \varepsilon)$ , we have

$$(8) \quad \widehat{h}^i(X, L - tA) = \widehat{h}^i(X \times_k K, \pi^*(L - tA)) = 0.$$

For every  $D \in \text{Div}(X)$  and every  $i \geq 0$ , the number  $h^i(X, \mathcal{O}_X(D))$  is invariant under ground field extensions by flat base change, hence  $\widehat{h}^i(X, D)$  is also. By homogeneity and continuity (see Remark 4.4), the number  $\widehat{h}^i(X, D)$  is also invariant under ground field extensions for  $D \in \text{Div}_{\mathbf{R}}(X)$ , hence (8) holds.  $\square$

REMARK 4.10. We note that if  $k$  is  $F$ -finite or perfect, then one can construct a field extension  $k \subseteq K$  as in Lemma 4.9 in a more elementary manner. When  $k$  is  $F$ -finite of characteristic  $p > 0$ , then one can set  $K$  to be  $k(x_\alpha^{1/p^\infty})_{\alpha \in A}$  for an uncountable set of indeterminates  $\{x_\alpha\}_{\alpha \in A}$ , since integrality and normality are preserved under limits of schemes with affine and flat transition morphisms [24, Corollaire 5.13.4]. When  $k$  is perfect, then one can set  $K$  to be a perfect closure of  $k(x_\alpha)_{\alpha \in A}$ . In this case,  $X$  is geometrically reduced, and the morphism  $X \times_k K \rightarrow X \times_k k(x_\alpha)_{\alpha \in A}$  is a homeomorphism since  $k(x_\alpha)_{\alpha \in A} \subseteq K$  is purely inseparable [24, Proposition 2.4.5(i)]. Thus, the base extension  $X \times_k K$  is integral.

LEMMA 4.11. *To prove the direction  $\Leftarrow$  in Theorem B, it suffices to show that every Cartier divisor satisfying  $(\star)$  is nef.*

*Proof.* Suppose  $L$  is a Cartier divisor satisfying  $(\star)$  for a pair  $(A, \varepsilon)$ . Choose  $\delta \in (0, \varepsilon) \cap \mathbf{Q}$  and let  $m$  be a positive integer such that  $m\delta$  is an integer. Then, the Cartier divisor  $m(L - \delta A)$  is nef since

$$\begin{aligned} \widehat{h}^i(X, m(L - \delta A) - tA) &= \widehat{h}^i(X, mL - (t + m\delta)A) \\ &= m \cdot \widehat{h}^i\left(X, L - \left(\frac{t}{m} + \delta\right)A\right) = 0 \end{aligned}$$

for all  $t \in [0, m\varepsilon - \delta)$  by homogeneity (Proposition 4.2). Thus, the Cartier divisor  $L = (L - \delta A) + \delta A$  is ample by [43, Corollary 1.4.10].  $\square$

We also need the following result to allow for an inductive proof. Note that the proof in [12] works in our setting.

LEMMA 4.12. [12, Lemma 4.3] *Let  $X$  be a projective variety of dimension  $n > 0$  over an uncountable field, and let  $L$  be a Cartier divisor on  $X$ .*

Suppose  $L$  satisfies  $(\star)$  for a pair  $(A, \varepsilon)$ , and let  $E \in |A|$  be a very general divisor. Then, the restriction  $L|_E$  satisfies  $(\star)$  for the pair  $(A|_E, \varepsilon)$ .

We can now show the direction  $\Leftarrow$  in Theorem B; by Lemma 4.11, we need to show that every Cartier divisor satisfying  $(\star)$  is nef. Recall that by Lemma 4.9, we may assume that the ground field  $k$  is uncountable, and in positive characteristic, we may assume that  $k$  is  $F$ -finite as well. Our proof follows that in [12, pp. 450–454] after reducing to a setting where Proposition 4.6 applies, although we have to be more careful in positive characteristic.

*Proof of  $\Leftarrow$  in Theorem B.* We proceed by induction on  $\dim X$ . Suppose  $\dim X = 1$ ; we show the contrapositive. If  $L$  is not nef, then  $\deg L < 0$  and  $-L$  is ample. Thus, by asymptotic Serre duality (Proposition 4.5), we have  $\widehat{h}^1(X, L) = \widehat{h}^0(X, -L) \neq 0$ , hence  $(\star)$  does not hold for every choice of  $(A, \varepsilon)$ .

We now assume  $\dim X \geq 2$ . Suppose by way of contradiction that there is a non-nef Cartier divisor  $L$  satisfying  $(\star)$ . We first claim that there exists a finite morphism  $\nu: \widetilde{X} \rightarrow X$  such that  $\nu^*L$  satisfies  $(\star)$ , and such that  $\widetilde{X}$  satisfies the hypotheses of Proposition 4.6 for  $D = \nu^*L$ . Choose an integral curve  $Z \subset X$  such that  $L \cdot Z < 0$ , and let  $\varphi: X' \rightarrow X$  be a regular alteration for the pair  $(X, Z)$  [38, Theorem 4.1], in which case  $(\varphi^{-1}(Z))_{\text{red}}$  is a simple normal crossing divisor. Consider the Stein factorization [29, Corollary III.11.5]

$$(9) \quad \begin{array}{ccc} X' & \xrightarrow{\mu} & \widetilde{X} \\ & \searrow \varphi & \downarrow \nu \\ & & X \end{array}$$

for the morphism  $\varphi$ , in which case  $\widetilde{X}$  is a normal projective variety. Now let  $\widetilde{Z}$  be the scheme-theoretic inverse image of  $Z$  under  $\nu$ , and write

$$\widetilde{Z} = \bigcup_{\alpha} \widetilde{Z}_{\alpha},$$

where  $\widetilde{Z}_{\alpha}$  are the irreducible components of  $\widetilde{Z}$ . Since  $\nu$  is finite, every  $\widetilde{Z}_{\alpha}$  is one-dimensional and dominates  $Z$ , hence the projection formula and (5) imply  $\nu^*L \cdot \widetilde{Z}_{\alpha} < 0$ . Finally,  $(\varphi^{-1}(Z))_{\text{red}} = (\mu^{-1}(\widetilde{Z}))_{\text{red}}$  is a simple normal crossing divisor by the factorization (9).

We now show that  $\nu^*L$  satisfies  $(\star)$ . Since  $\nu^*A$  is ample [43, Proposition 1.2.13], we can choose a positive integer  $a$  such that  $a\nu^*A$  is very ample.

Then, Proposition 4.3 implies

$$(10) \quad \widehat{h}^i(\widetilde{X}, \nu^*L - ta\nu^*A) = (\deg \nu) \cdot \widehat{h}^i(X, L - taA) = 0$$

for all  $i > 0$  and for all  $t \in [0, \varepsilon/a)$ . Replacing  $A$  by  $aA$ , we assume that  $\nu^*A$  is very ample.

For the rest of the proof, our goal is to show that

$$(11) \quad \widehat{h}^1(\widetilde{X}, \nu^*L - \delta\nu^*A) \neq 0$$

for  $0 < \delta \ll 1$ , contradicting (10). Let  $F \in |\nu^*A|$  be a very general divisor. By Bertini’s theorem [20, Theorem 3.4.10 and Corollary 3.4.14], we may assume that  $F$  is a subvariety of  $\widetilde{X}$ , in which case by inductive hypothesis and Lemma 4.12, we have that  $\nu^*L|_F$  is ample. Since ampleness is an open condition in families [25, Corollaire 9.6.4], there exists an integer  $b > 0$  such that  $b\nu^*L$  is very ample along the generic divisor  $F_\eta \in |\nu^*A|$ . By possibly replacing  $b$  with a multiple, we may also assume that  $mb\nu^*L|_{F_\eta}$  has vanishing higher cohomology for every integer  $m > 0$ . Since the ground field  $k$  is uncountable, we can then choose a sequence of very general Cartier divisors  $\{E_\beta\}_{\beta=1}^\infty \subseteq |\nu^*A|$  such that the following properties hold:

- (a)  $E_\beta$  is a subvariety of  $\widetilde{X}$  for all  $\beta$  (by Bertini’s theorem [20, Theorem 3.4.10 and Corollary 3.4.14]);
- (b) for all  $\beta$ ,  $b\nu^*L|_{E_\beta}$  is very ample and  $mb\nu^*L|_{E_\beta}$  has vanishing higher cohomology for every integer  $m > 0$  (by the constructibility of very ampleness in families [25, Proposition 9.6.3] and by semicontinuity); and
- (c) for every positive integer  $r$  and for all nonnegative integers  $j$  and  $m$ , the  $k$ -dimension of cohomology groups of the form

$$(12) \quad H^j(E_{\beta_1} \cap E_{\beta_2} \cap \dots \cap E_{\beta_r}, \mathcal{O}_{E_{\beta_1} \cap E_{\beta_2} \cap \dots \cap E_{\beta_r}}(mL))$$

is independent of the  $r$ -tuple  $(\beta_1, \beta_2, \dots, \beta_r)$  (by semicontinuity; see [41, Proposition 5.5]).

We denote by  $h^j(\mathcal{O}_{E_1 \cap E_2 \cap \dots \cap E_r}(mL))$  the dimensions of the cohomology groups (12). By homogeneity (Proposition 4.2), we can replace  $L$  by  $bL$  so that  $\nu^*L|_{E_\beta}$  is very ample with vanishing higher cohomology for all  $\beta$ .

To show (11), we now follow the proof in [12, pp. 453–454] with appropriate modifications. Given positive integers  $m$  and  $r$ , consider the

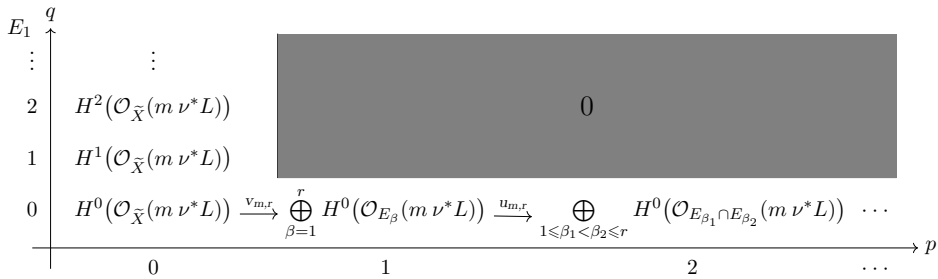
complex

$$\begin{aligned}
 K_{m,r}^\bullet &:= \left( \bigotimes_{\beta=1}^r (\mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{E_\beta}) \right) \otimes \mathcal{O}_{\tilde{X}}(m \nu^* L) \\
 &= \left\{ \begin{array}{l} \mathcal{O}_{\tilde{X}}(m \nu^* L) \longrightarrow \bigoplus_{\beta=1}^r \mathcal{O}_{E_\beta}(m \nu^* L) \\ \longrightarrow \bigoplus_{1 \leq \beta_1 < \beta_2 \leq r} \mathcal{O}_{E_{\beta_1} \cap E_{\beta_2}}(m \nu^* L) \longrightarrow \dots \end{array} \right\}.
 \end{aligned}$$

By [41, Corollary 4.2], this complex is acyclic away from  $\mathcal{O}_{\tilde{X}}(m \nu^* L)$ , hence is a resolution for  $\mathcal{O}_{\tilde{X}}(m \nu^* L - r \nu^* A)$ . In particular, we have

$$H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}(m \nu^* L - r \nu^* A)) = \mathbf{H}^j(\tilde{X}, K_{m,r}^\bullet).$$

The right-hand side is computed by an  $E_1$ -spectral sequence whose first page is



hence there is a natural inclusion

$$(13) \quad \frac{\ker(u_{m,r})}{\text{im}(v_{m,r})} \subseteq H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(m \nu^* L - r \nu^* A)).$$

We want to bound the left-hand side of (13) from below. First, there exists a constant  $C_1 > 0$  such that  $h^0(\mathcal{O}_{E_1 \cap E_2}(m \nu^* L)) \leq C_1 \cdot m^{n-2}$  for all  $m \gg 0$  [43, Example 1.2.20]. Thus, we have

$$\text{codim} \left( \ker(u_{m,r}) \subseteq \bigoplus_{\beta=1}^r H^0(E_\beta, \mathcal{O}_{E_\beta}(m \nu^* L)) \right) \leq C_2 \cdot r^2 m^{n-2}$$

for some  $C_2$  and for all  $m \gg 0$ . Now by Proposition 4.6, there are positive integers  $q$  and  $c$  such that  $\mathfrak{b}(|mq \nu^* L|) \subseteq \mathfrak{a}^{m-c}$  for all  $m > c$ , where  $\mathfrak{a}$  is

the ideal sheaf of  $\tilde{Z}$ . By replacing  $L$  by  $qL$ , we can assume that this inclusion holds for  $q = 1$ . The morphism  $v_{m,r}$  therefore fits into the following commutative diagram:

$$\begin{CD} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\nu^*L) \otimes \mathfrak{a}^{m-c}) @>v'_{m,r}>> \bigoplus_{\beta=1}^r H^0(E_\beta, \mathcal{O}_{E_\beta}(m\nu^*L) \otimes \mathfrak{a}^{m-c}) \\ @| @. @VVV \\ H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(m\nu^*L)) @>v_{m,r}>> \bigoplus_{\beta=1}^r H^0(E_\beta, \mathcal{O}_{E_\beta}(m\nu^*L)) \end{CD}$$

We claim that there exists a constant  $C_3 > 0$  such that for all  $m \gg 0$ ,

$$(14) \quad \text{codim}(H^0(\mathcal{O}_{E_\beta}(m\nu^*L) \otimes \mathfrak{a}^{m-c}) \subseteq H^0(\mathcal{O}_{E_\beta}(m\nu^*L))) \geq C_3 \cdot m^{n-1}.$$

Granted this, we have

$$\dim\left(\frac{\ker(u_{m,r})}{\text{im}(v_{m,r})}\right) \geq C_4 \cdot (rm^{n-1} - r^2m^{n-2})$$

for some constant  $C_4 > 0$  and for all  $m \gg 0$ . Fixing a rational number  $0 < \delta \ll 1$  and setting  $r = m\delta$  for an integer  $m > 0$  such that  $m\delta$  is an integer, we then see that there exists a constant  $C_5 > 0$  such that

$$h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(m(\nu^*L - \delta\nu^*A))) \geq C_5 \cdot \delta m^n$$

for all  $m \gg 0$ , contradicting (10).

It remains to show (14). Since the vanishing locus of  $\mathfrak{a}$  may have no  $k$ -rational points, we pass to the algebraic closure of  $k$  to bound the codimension on the left-hand side of (14) from below. Let  $\bar{E}_\beta := E_\beta \times_k \bar{k}$ , and denote by  $\pi: \bar{E}_\beta \rightarrow E_\beta$  the projection morphism. Note that

$$\begin{aligned} \text{codim}(H^0(\mathcal{O}_{E_\beta}(m\nu^*L) \otimes \mathfrak{a}^{m-c}) \subseteq H^0(\mathcal{O}_{E_\beta}(m\nu^*L))) \\ = \text{codim}(H^0(\mathcal{O}_{\bar{E}_\beta}(m\pi^*\nu^*L) \otimes \pi^{-1}\mathfrak{a}^{m-c} \cdot \mathcal{O}_{\bar{E}_\beta}) \subseteq H^0(\mathcal{O}_{\bar{E}_\beta}(m\pi^*\nu^*L))) \end{aligned}$$

by the flatness of  $k \subseteq \bar{k}$ . Since  $\mathcal{O}_{\bar{E}_\beta}(\pi^*\nu^*L)$  is very ample by base change, we can choose a closed point  $x \in Z(\pi^{-1}\mathfrak{a} \cdot \mathcal{O}_{\bar{E}_\beta}) \cap \bar{E}_\beta$ , in which case  $m\pi^*\nu^*L$  separates  $(m - c)$ -jets at  $x$  by [37, Proof of Lemma 3.7]. Finally, the dimension of the space of  $(m - c)$ -jets at  $x$  is at least that for a regular point of a variety of dimension  $n$ , hence

$$\text{codim}(H^0(\mathcal{O}_{\bar{E}_\beta}(m\pi^*\nu^*L) \otimes \pi^{-1}\mathfrak{a}^{m-c} \cdot \mathcal{O}_{\bar{E}_\beta}) \subseteq H^0(\mathcal{O}_{\bar{E}_\beta}(m\pi^*\nu^*L)))$$

$$\begin{aligned} &\geq \text{codim}(H^0(\mathcal{O}_{\overline{E}_\beta}(m \pi^* \nu^* L) \otimes \mathfrak{m}_x^{m-c}) \subseteq H^0(\mathcal{O}_{\overline{E}_\beta}(m \pi^* \nu^* L))) \\ &\geq \binom{m-c+n}{n-1} \geq C_3 \cdot m^{n-1} \end{aligned}$$

for some constant  $C_3 > 0$  and all  $m \gg 0$ , as required. □

**§5. Nakayama’s theorem on restricted base loci**

We now come to our second application of the gamma construction, Theorem C. This result extends known cases of the following conjecture due to Boucksom, Broustet, and Pacienza.

CONJECTURE 5.1. [5, Conjecture 2.7] *Let  $X$  be a normal projective variety, and let  $D$  be a pseudoeffective  $\mathbf{R}$ -divisor on  $X$ . Then, we have*

$$\mathbf{B}_-(D) = \text{Nef}(D).$$

In Section 5.1, we define the restricted base locus  $\mathbf{B}_-(D)$  and the non-nef locus  $\text{Nef}(D)$ . We then prove Theorem C in Section 5.2 using the gamma construction (Theorem 3.4) to reduce to the  $F$ -finite case, in which case it suffices to apply results in [57]. We recall that  $\|\cdot\|$  denotes a compatible choice of Euclidean norms on the vector spaces  $N_{\mathbf{Q}}^1(X)$  and  $N_{\mathbf{R}}^1(X)$ , which are finite-dimensional for complete schemes  $X$  by [8, Proposition 2.3].

**5.1 Background on restricted base loci and non-nef loci**

We start by defining the following numerically invariant approximation of the stable base locus defined in Definition 2.4.

DEFINITION 5.2. [15, Definition 1.12] *Let  $X$  be a projective scheme over a field, and let  $D$  be an  $\mathbf{R}$ -Cartier divisor on  $X$ . The *restricted base locus* of  $D$  is the subset*

$$\mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D + A)$$

of  $X$ , where the union runs over all ample  $\mathbf{R}$ -Cartier divisors  $A$  such that  $D + A$  is a  $\mathbf{Q}$ -Cartier divisor. Note that  $\mathbf{B}_-(D) = \emptyset$  if and only if  $D$  is nef [15, Example 1.18].

We need the following result, which says that the formation of restricted base loci is compatible with ground field extensions.

LEMMA 5.3. *Let  $X$  be a projective scheme over a field  $k$ , and let  $D$  be an  $\mathbf{R}$ -Cartier divisor on  $X$ . Let  $k \subseteq k'$  be a field extension with corresponding*



projection morphism  $\pi: X \times_k k' \rightarrow X$ . Then,

$$\mathbf{B}_-(\pi^*D) = \pi^{-1}(\mathbf{B}_-(D)).$$

*Proof.* Let  $\{A_n\}_{n \geq 1}$  be a sequence of ample  $\mathbf{R}$ -Cartier divisors such that  $\lim_{n \rightarrow \infty} \|A_n\| = 0$  and such that  $D + A_n$  is a  $\mathbf{Q}$ -Cartier divisor for every  $n$ . By [15, Proposition 1.19], we have

$$\mathbf{B}_-(D) = \bigcup_{n \geq 1} \mathbf{B}(D + A_n).$$

By flat base change, we have  $\pi^{-1}(\mathbf{B}(D + A_n)) = \mathbf{B}(\pi^*(D + A_n))$ , hence

$$\pi^{-1}(\mathbf{B}_-(D)) = \bigcup_{n \geq 1} \mathbf{B}(\pi^*(D + A_n)) = \mathbf{B}_-(\pi^*D),$$

where the second equality follows from applying [15, Proposition 1.19] again to the sequence  $\{\pi^*A_n\}_{n \geq 1}$  of ample  $\mathbf{R}$ -Cartier divisors on  $X \times_k k'$ .  $\square$

Next, we want to define the non-nef locus.

DEFINITION 5.4. [52, Definition III.2.2; 7, Definition 2.11] Let  $X$  be a normal projective variety, and let  $D$  be a big  $\mathbf{R}$ -Cartier divisor. Consider a divisorial valuation  $v$  on  $X$ . The *numerical vanishing order* of  $D$  along  $v$  is

$$v_{\text{num}}(D) := \inf_{E \equiv_{\mathbf{R}} D} v(E),$$

where the infimum runs over all effective  $\mathbf{R}$ -Cartier divisors  $\mathbf{R}$ -numerically equivalent to  $D$ . When  $D$  is a pseudoeffective  $\mathbf{R}$ -Cartier divisor, we set

$$v_{\text{num}}(D) := \sup_A v_{\text{num}}(D + A),$$

where the supremum runs over all ample  $\mathbf{R}$ -Cartier divisors  $A$  on  $X$ , and where we note that  $D + A$  is a big  $\mathbf{R}$ -Cartier divisor by [43, Theorem 2.2.26]. The *non-nef locus* of a pseudoeffective  $\mathbf{R}$ -Cartier divisor  $D$  is

$$\text{NNef}(D) := \bigcup_v c_X(v),$$

where the union runs over all divisorial valuations such that  $v_{\text{num}}(D) > 0$ , and  $c_X(v)$  is the center of the divisorial valuation  $v$ . Note that  $\text{NNef}(D) = \emptyset$  if and only if  $D$  is nef [52, Remark III.2.8].

To prove Theorem C, we also use the following:

DEFINITION 5.5. [15, Definition 2.2 and Remark 2.3] Let  $X$  be a normal projective variety, and let  $D$  be a  $\mathbf{Q}$ -Cartier divisor. Consider a divisorial valuation  $v$  on  $X$ . The *asymptotic order of vanishing* of  $D$  along  $v$  is

$$v(\|D\|) := \inf_{E \sim_{\mathbf{Q}} D} v(E),$$

where the infimum runs over all effective  $\mathbf{Q}$ -Cartier divisors  $\mathbf{Q}$ -linearly equivalent to  $D$ .

**5.2 Proof of Theorem C**

We start by proving a version of Theorem C for arbitrary normal projective varieties. We fix some notation. Let  $X$  be a normal projective variety over a field  $k$ . If  $\text{char } k = 0$  and  $\Delta$  is an effective  $\mathbf{Q}$ -Weil divisor on  $X$  such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier, then the *non-klt locus* of the pair  $(X, \Delta)$  is  $\text{Nklt}(X, \Delta) := Z(\mathcal{J}(X, \Delta))$ , where  $\mathcal{J}(X, \Delta)$  is the multiplier ideal [44, Definition 9.3.56], and the *non-klt locus* of  $X$  is

$$\text{Nklt}(X) := \bigcap_{\Delta} \text{Nklt}(X, \Delta),$$

where the intersection runs over all effective  $\mathbf{Q}$ -Weil divisors  $\Delta$  such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier. If  $\text{char } k = p > 0$ , then the *non-strongly  $F$ -regular locus* of  $X$  is

$$\text{NSFR}(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ is not strongly } F\text{-regular}\}.$$

If  $X$  is  $F$ -finite, then  $\text{NSFR}(X) = Z(\tau(X))$ , since test ideals localize [60, Proposition 3.23(ii)], and since  $\tau(R) = R$  for an  $F$ -finite ring  $R$  if and only if  $R$  is strongly  $F$ -regular [62, Proposition 5.6(5)].

THEOREM 5.6. (cf. [7, Corollary 4.7; 57, Corollary 4.7]) *Let  $X$  be a normal projective variety over a field  $k$ , and let  $D$  be a pseudoeffective  $\mathbf{R}$ -Cartier divisor on  $X$ . If  $\text{char } k = 0$ , then*

$$(15) \quad \mathbf{B}_-(D) \setminus \text{Nklt}(X) = \text{Nef}(D) \setminus \text{Nklt}(X),$$

and if  $\text{char } k = p > 0$ , then

$$(16) \quad \mathbf{B}_-(D) \setminus \text{NSFR}(X) = \text{Nef}(D) \setminus \text{NSFR}(X).$$

*Proof.* We first prove that  $\text{NNeft}(D) \subseteq \mathbf{B}_-(D)$  for every pseudoeffective  $\mathbf{R}$ -Cartier divisor  $D$ , following [5, Lemma 2.6]. Let  $x \notin \mathbf{B}_-(D)$ , and let  $v$  be a divisorial valuation such that  $x \in c_X(v)$ . By definition of  $\mathbf{B}_-(D)$ , there exists an ample  $\mathbf{R}$ -Cartier divisor  $A$  such that  $D + A$  is  $\mathbf{Q}$ -Cartier divisor for which  $x \notin \mathbf{B}(D + A)$ . Thus, there exists an effective  $\mathbf{Q}$ -Cartier divisor  $E$  such that  $E \sim_{\mathbf{Q}} D + A$  and such that  $x \notin \text{Supp } E$ . We therefore have  $v_{\text{num}}(D + A) \leq v(E) = 0$ .

It remains to show the inclusions  $\subseteq$ . We first consider the case when  $\text{char } k = 0$ . Let  $\Delta$  be an effective  $\mathbf{Q}$ -Weil divisor such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier. The proof of [7, Theorem 4.5] holds in this setting after replacing the application of Nadel vanishing and Castelnuovo–Mumford regularity in the proof of [7, Lemma 4.1] with the uniform global generation statement mentioned in Remark 4.7, hence  $\mathbf{B}_-(D) \setminus \text{Nklt}(X, \Delta) \subseteq \text{NNeft}(D) \setminus \text{Nklt}(X, \Delta)$ . Taking the union over all  $\mathbf{Q}$ -Weil divisors  $\Delta$  such that  $K_X + \Delta$  is  $\mathbf{Q}$ -Cartier, we then see that the inclusion  $\subseteq$  holds in (15).

We now consider the characteristic  $p > 0$  case. By [7, Lemmas 2.12 and 2.13], there exists a sequence  $\{A_n\}_{n \geq 0}$  of ample  $\mathbf{R}$ -Cartier divisors on  $X$  such that  $D + A_n$  is a  $\mathbf{Q}$ -Cartier divisor for every  $n$ ,  $\lim_{n \rightarrow \infty} \|A_n\| \rightarrow 0$ , and

$$\mathbf{B}_-(D) = \bigcup_n \mathbf{B}_-(D + A_n) \quad \text{and} \quad \text{NNeft}(D) = \bigcup_n \text{NNeft}(D + A_n).$$

By proving the inclusion  $\subseteq$  in (16) for  $D + A_n$ , it therefore suffices to consider the case when  $D$  is a big  $\mathbf{Q}$ -Cartier divisor. Let  $x \in \mathbf{B}_-(D)$ , and consider a divisorial valuation  $v$  on  $X$  such that  $c_X(v) = \overline{\{x\}}$ , which is given by the order of vanishing along a prime Cartier divisor  $E$  on a normal birational model  $X'$  of  $X$ . By applying the gamma construction (Theorem 3.4) to  $X, X',$  and  $E$ , there exists a field extension  $k \subseteq k^\Gamma$  such that  $X \times_k k^\Gamma$  and  $X' \times_k k^\Gamma$  are normal varieties,  $E \times_k k^\Gamma$  is a prime divisor, and  $\pi^\Gamma(\text{NSFR}(X^\Gamma)) = \text{NSFR}(X)$ . Note that the order of vanishing along  $E \times_k k^\Gamma$  defines a divisorial valuation  $v^\Gamma$  on  $X^\Gamma$  extending  $v$ . Since  $\mathbf{B}_-((\pi^\Gamma)^*D) = (\pi^\Gamma)^{-1}(\mathbf{B}_-(D))$  by Lemma 5.3, we have  $(\pi^\Gamma)^{-1}(x) \in \mathbf{B}_-((\pi^\Gamma)^*D)$ , hence [57, Corollary 4.6] implies

$$(\pi^\Gamma)^{-1}(x) \in \bigcup_{m \geq 1} Z(\tau(X^\Gamma, m \cdot \|(\pi^\Gamma)^*D\|)).$$

By the proof of the implication (2)  $\Rightarrow$  (5) in [57, Proposition 3.17] (which does not use the assumption that  $K_X$  is  $\mathbf{Q}$ -Cartier), we see that  $v^\Gamma(\|(\pi^\Gamma)^*D\|) > 0$ , and by pulling back Cartier divisors in  $|D|$  to  $X^\Gamma$ ,

we have  $v(\|D\|) > 0$  as well. Finally, since  $D$  is big, [15, Lemma 3.3] implies  $v_{\text{num}}(D) = v(\|D\|) > 0$ , hence  $x \in \text{NNeF}(D)$ , and the inclusion  $\subseteq$  holds in (16).  $\square$

We now prove Theorem C.

*Proof of Theorem C.* As in the proof of Theorem 5.6, the inclusion  $\text{NNeF}(D) \subseteq \mathbf{B}_-(D)$  holds, hence it suffices to show the reverse inclusion. By Theorem 5.6, we have  $\mathbf{B}_-(D) \setminus \text{Nklt}(X) \subseteq \text{NNeF}(D)$  (resp.  $\mathbf{B}_-(D) \setminus \text{NSFR}(X) \subseteq \text{NNeF}(D)$ ). Now let  $\{A_n\}_{n \geq 1}$  be a sequence of ample  $\mathbf{R}$ -Cartier divisors such that  $\lim_{n \rightarrow \infty} \|A_n\| \rightarrow 0$  and such that  $D + A_n$  is a  $\mathbf{Q}$ -Cartier divisor for every  $n$ . By [15, Proposition 1.19], we have

$$\mathbf{B}_-(D) = \bigcup_{n \geq 1} \mathbf{B}(D + A_n).$$

Since each  $\mathbf{B}(D + A_n)$  does not contain any isolated points [16, Proposition 1.1], we see that  $\mathbf{B}_-(D)$  does not contain any isolated points. Finally, since  $\text{Nklt}(X)$  (resp.  $\text{NSFR}(X)$ ) is a discrete set of isolated closed points by assumption, we have  $\mathbf{B}_-(D) \subseteq \text{NNeF}(D)$  by Theorem 5.6.  $\square$

**Acknowledgments.** I am grateful to my advisor Mircea Mustața for his constant support and for several illuminating conversations, and to Mitsuyasu Hashimoto, Melvin Hochster, Emanuel Reinecke, Karen E. Smith, Daniel Smolkin, Matthew Stevenson, Shunsuke Takagi, and Farrah Yhee for helpful discussions. I am particularly grateful to Harold Blum for pointing out applications of [12], to Rankeya Datta for clarifications on strong  $F$ -regularity for non- $F$ -finite rings, to Alex Küronya for talking to me about his results in [41] and [12], and to Linquan Ma, Thomas Polstra, Karl Schwede, and Kevin Tucker for sharing a preliminary draft of [46] with me. I first proved Theorem B in order to prove some results in joint work with Mihai Fulger, and I would especially like to thank him for allowing me to write this standalone paper. Finally, I am indebted to the anonymous referee for useful suggestions that improved the quality of this paper.

### Appendix A. Some results on $F$ -injective rings

Let  $R$  be a Noetherian ring of prime characteristic  $p > 0$ . Recall from Definition 2.10 that  $R$  is  $F$ -injective if for every maximal ideal  $\mathfrak{m} \subseteq R$ , the  $R$ -module homomorphism  $H_{\mathfrak{m}}^i(F): H_{\mathfrak{m}}^i(R_{\mathfrak{m}}) \rightarrow H_{\mathfrak{m}}^i(F_*R_{\mathfrak{m}})$  induced by Frobenius is injective for all  $i$ .

In this appendix, we prove some facts about  $F$ -injective rings for which we could not find a reference. First, we characterize  $F$ -finite rings that are  $F$ -injective using Grothendieck duality. This characterization is already implicit in [18, Remark on p. 473] and the proof of [58, Proposition 4.3]. Note that if  $R$  is an  $F$ -finite ring, then the exceptional pullback  $F^!$  from Grothendieck duality exists [28, III.6], and  $R$  has a normalized dualizing complex  $\omega_R^\bullet$  by Theorem 2.7.

LEMMA A.1. (cf. [18, Remark on p. 473]) *Let  $R$  be an  $F$ -finite Noetherian ring of prime characteristic  $p > 0$ . Then,  $R$  is  $F$ -injective if and only if the  $R$ -module homomorphisms*

$$(A1) \quad \mathbf{h}^{-i}\mathrm{Tr}_F : \mathbf{h}^{-i}F_*F^!\omega_R^\bullet \longrightarrow \mathbf{h}^{-i}\omega_R^\bullet$$

*induced by the Grothendieck trace of Frobenius are surjective for all  $i$ .*

Lemma A.1 is most useful when  $R$  is essentially of finite type over an  $F$ -finite field, in which case  $F^!\omega_R^\bullet \simeq \omega_R^\bullet$  in the derived category  $\mathbf{D}_{\mathrm{qc}}^+(R)$  [53, Theorem 5.3], hence the homomorphisms in (A1) can be written as  $\mathbf{h}^{-i}F_*\omega_R^\bullet \rightarrow \mathbf{h}^{-i}\omega_R^\bullet$ .

*Proof.* By Grothendieck local duality [28, Corollary V.6.3],  $R$  is  $F$ -injective if and only if

$$F^* : \mathrm{Ext}_R^{-i}(F_*R, \omega_R^\bullet) \longrightarrow \mathrm{Ext}_R^{-i}(R, \omega_R^\bullet)$$

is surjective for all  $i$ . By Grothendieck duality for finite morphisms [28, Theorem III.6.7], this occurs if and only if

$$F_* \mathrm{Ext}_R^{-i}(R, F^!\omega_R^\bullet) \longrightarrow \mathrm{Ext}_R^{-i}(R, \omega_R^\bullet)$$

is surjective for all  $i$ . Since  $\mathrm{Ext}_R^{-i}(R, -) = \mathbf{h}^{-i}(-)$  and by the description of the Grothendieck duality isomorphism [28, Theorem III.6.7], this is equivalent to the surjectivity of (A1) for all  $i$ . □

Next, we prove that  $F$ -injectivity is an open condition on  $F$ -finite schemes. This extends [55, Proposition 3.12] to the nonlocal case.

LEMMA A.2. (cf. [55, Proposition 3.12; 58, Proposition 4.3]) *If  $R$  is an  $F$ -finite Noetherian ring of prime characteristic  $p > 0$ , then the locus*

$$\{\mathfrak{p} \in \mathrm{Spec} R \mid R_{\mathfrak{p}} \text{ is } F\text{-injective}\}$$

*is open. In particular,  $R$  is  $F$ -injective if and only if  $R_{\mathfrak{p}}$  is  $F$ -injective for every prime ideal  $\mathfrak{p} \subseteq R$ .*

*Proof.* For each integer  $i$ , let  $M_i$  be the cokernel of the  $R$ -module homomorphism in (A1). Since the Grothendieck trace is compatible with flat base change [28, Proposition III.6.6(2)] and Frobenius is compatible with localizations, the support of  $M_i$  is the locus where  $R$  is not  $F$ -injective by Lemma A.1. Now  $\omega_R^\bullet$  and  $F_*F^!\omega_R^\bullet$  have coherent cohomology that is nonzero in only finitely many degrees by definition of  $\omega_R^\bullet$  and [28, Proposition III.6.1], hence each  $M_i$  is finitely generated over  $R$ , has closed support, and is nonzero for only finitely many  $i$ . The locus

$$(A2) \quad \text{Spec } R \setminus \left( \bigcup_i \text{Supp } M_i \right) = \{ \mathfrak{p} \in \text{Spec } R \mid R_{\mathfrak{p}} \text{ is } F\text{-injective} \}$$

is therefore open, proving the first statement. The locus (A2) is in particular closed under generization, hence  $R$  is  $F$ -injective only if  $R_{\mathfrak{p}}$  is  $F$ -injective for every prime ideal  $\mathfrak{p} \subseteq R$ . The converse implication holds by definition, proving the second statement.  $\square$

Finally, to prove Theorem 3.4, we used the following descent property for  $F$ -injectivity.

LEMMA A.3. (cf. [30, Lemma 4.6]) *Let  $\varphi: R \rightarrow S$  be a pure homomorphism of rings of prime characteristic  $p > 0$ . If  $I \subseteq R$  is a finitely generated ideal such that the Frobenius action*

$$H_{IS}^i(F_S): H_{IS}^i(S) \longrightarrow H_{IS}^i(F_{S*}S)$$

*is injective for some  $i$ , then the Frobenius action*

$$H_I^i(F_R): H_I^i(R) \longrightarrow H_I^i(F_{R*}R)$$

*is also injective.*

*In particular, suppose that  $(R, \mathfrak{m})$  and  $(S, \mathfrak{n})$  are Noetherian local rings and that  $\varphi$  is a local homomorphism with zero-dimensional closed fiber. If  $S$  is  $F$ -injective, then  $R$  is  $F$ -injective.*

In the non-Noetherian case, we define local cohomology modules by taking injective resolutions in the category of sheaves of abelian groups on spectra, as is done in [23, Exposé I, Définition 2.1].

*Proof.* For each  $i$ , we have the following commutative square:

$$(A3) \quad \begin{array}{ccc} H_I^i(R) & \xrightarrow{H_I^i(F_R)} & H_I^i(F_{R*}R) \\ \downarrow & & \downarrow \\ H_{IS}^i(S) & \xrightarrow{H_{IS}^i(F_S)} & H_{IS}^i(F_{S*}S) \end{array}$$

Since  $\varphi$  is pure, the map  $H_I^i(R) \hookrightarrow H_{IS}^i(S)$  is injective by applying [35, Corollary 6.6] to the Čech complex, which computes local cohomology by [23, Exposé II, Proposition 5]. The bottom horizontal arrow in (A3) is injective by assumption, hence the top horizontal arrow  $H_I^i(F_R)$  in (A3) is also injective by the commutativity of the diagram. The second statement in the lemma is a special case of the first, since under the given assumptions, we have  $\sqrt{\mathfrak{m}_S} = \mathfrak{n}$ , hence  $H_{\mathfrak{m}_S}^i(S) \simeq H_{\mathfrak{n}}^i(S)$ .  $\square$

### Appendix B. Strong $F$ -regularity for non- $F$ -finite rings

In this appendix, we describe the relationship between different notions of strong  $F$ -regularity in the non- $F$ -finite setting. This material is based on [30, Section 3] and [10, Section 6].

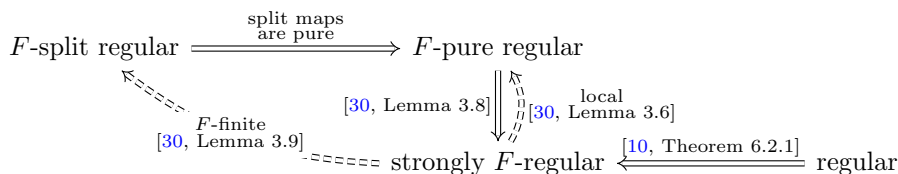
DEFINITION B.1. Let  $R$  be a Noetherian ring of prime characteristic  $p > 0$ . We follow the notation in Definition 2.8. For every  $c \in R$ , we also say that  $R$  is  $F$ -split along  $c$  if  $\lambda_c^e$  splits as an  $R$ -module homomorphism for some  $e > 0$ . We then say that

- (a)  $R$  is *split  $F$ -regular* if  $R$  is  $F$ -split along every  $c \in R^\circ$  [34, Definition 5.1];
- (b)  $R$  is  *$F$ -pure regular* if  $R$  is  $F$ -pure along every  $c \in R^\circ$  [34, Remark 5.3]; and
- (c)  $R$  is *strongly  $F$ -regular* if every inclusion of  $R$ -modules is tightly closed [30, Definition 3.3].

The definition in (c) is due to Hochster; see [33, Definition 8.2] for the definition of tight closure for modules. While (c) is not the definition used in the rest of this paper (Definition 2.8(a)), these two definitions are equivalent by [30, Lemma 3.6].

Note that (a) is the usual definition of strong  $F$ -regularity in the  $F$ -finite setting. The terminology in (a) and (b) is from [10, Definitions 6.6.1 and 6.1.1].  $F$ -pure regular rings are called *very strongly  $F$ -regular* in [30, Definition 3.4].

The relationship between these notions of strong  $F$ -regularity can be summarized as follows:



## REFERENCES

- [1] U. Angehrn and Y. T. Siu, *Effective freeness and point separation for adjoint bundles*, Invent. Math. **122** (1995), 291–308.
- [2] L. L. Avramov, *Flat morphisms of complete intersections*, Soviet Math. Dokl. **16** (1975), 1413–1417; Translated from the Russian by D. L. Johnson.
- [3] J. Bingen and H. Flenner, “*On the fibers of analytic mappings*”, in *Complex Analysis and Geometry*, Univ. Ser. Math., Plenum, New York, 1993, 45–101.
- [4] C. Birkar, *The augmented base locus of real divisors over arbitrary fields*, Math. Ann. **368** (2017), 905–921.
- [5] S. Boucksom, A. Broustet and G. Pacienza, *Uniruledness of stable base loci of adjoint linear systems via Mori theory*, Math. Z. **275** (2013), 499–507.
- [6] J. I. Burgos Gil, W. Gubler, P. Jell, K. Künnemann and F. Martin, *Differentiability of non-archimedean volumes and non-archimedean Monge–Ampère equations*, Algebr. Geom. (2019), to appear, With an appendix by R. Lazarsfeld, arXiv:1608.01919v6 [math.AG].
- [7] S. Cacciola and L. Di Biagio, *Asymptotic base loci on singular varieties*, Math. Z. **275** (2013), 151–166.
- [8] S. D. Cutkosky, *Teissier’s problem on inequalities of nef divisors*, J. Algebra Appl. **14** (2015), 1540002, 37 pp.
- [9] R. Datta and T. Murayama, *Permanence properties of  $F$ -injectivity*, preprint, 2019. arXiv:1906.11399v1 [math.AC].
- [10] R. Datta and K. E. Smith, *Frobenius and valuation rings*, Algebra Number Theory **10** (2016), 1057–1090; See also [11].
- [11] R. Datta and K. E. Smith, *Correction to the article “Frobenius and valuation rings”*, Algebra Number Theory **11** (2017), 1003–1007.
- [12] T. de Fernex, A. Küronya and R. Lazarsfeld, *Higher cohomology of divisors on a projective variety*, Math. Ann. **337** (2007), 443–455.
- [13] T. de Fernex and M. Mustață, *Limits of log canonical thresholds*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 491–515.
- [14] L. Ein, R. Lazarsfeld, M. Mustață, M. Nakamaye and M. Popa, *Asymptotic invariants of line bundles*, Pure Appl. Math. Q. **1** (2005), 379–403.
- [15] L. Ein, R. Lazarsfeld, M. Mustață, M. Nakamaye and M. Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier (Grenoble) **56** (2006), 1701–1734.
- [16] L. Ein, R. Lazarsfeld, M. Mustață, M. Nakamaye and M. Popa, *Restricted volumes and base loci of linear series*, Amer. J. Math. **131** (2009), 607–651.
- [17] F. Enescu and M. Hochster, *The Frobenius structure of local cohomology*, Algebra Number Theory **2** (2008), 721–754.



- [18] R. Fedder, *F-purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983), 461–480.
- [19] R. Fedder and K.-i. Watanabe, “A characterization of *F*-regularity in terms of *F*-purity”, in *Commutative Algebra (Berkeley, CA, 1987)*, Math. Sci. Res. Inst. Publ. **15**, Springer, New York, 1989, 227–245.
- [20] H. Flenner, L. O’Carroll and W. Vogel, *Joins and Intersections*, Springer Monogr. Math., Springer, Berlin, 1999.
- [21] M. Fulger, J. Kollár and B. Lehmann, *Volume and Hilbert function of  $\mathbb{R}$ -divisors*, Michigan Math. J. **65** (2016), 371–387.
- [22] O. Gabber, “Notes on some *t*-structures”, in *Geometric Aspects of Dwork Theory. Vol. II*, Walter de Gruyter, Berlin, 2004, 711–734.
- [23] A. Grothendieck, *Séminaire de géométrie algébrique du Bois Marie, 1962. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Doc. Math. (Paris), **4**, Soc. Math. France, Paris, 2005, With an exposé by Mme M. Raynaud, With a preface and edited by Y. Laszlo, Revised reprint of the 1968 French original.
- [24] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Publ. Math. Inst. Hautes Études Sci. **24** (1965), 1–231.
- [25] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 1–255.
- [26] N. Hara and S. Takagi, *On a generalization of test ideals*, Nagoya Math. J. **175** (2004), 59–74.
- [27] N. Hara and K. Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), 3143–3174.
- [28] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics, **20**, Springer, Berlin–New York, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne.
- [29] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, **52**, Springer, New York–Heidelberg, 1977.
- [30] M. Hashimoto, *F-pure homomorphisms, strong F-regularity, and F-injectivity*, Comm. Algebra **38** (2010), 4569–4596.
- [31] M. Hashimoto, “*F*-purity of homomorphisms, strong *F*-regularity, and *F*-injectivity”, in *The 31st Symposium on Commutative Algebra in Japan (Osaka, 2009)*, 2010, 9–16. <http://www.math.okayama-u.ac.jp/~hashimoto/paper/Comm-Alg/31-all.pdf>.
- [32] H. Hauser, *On the problem of resolution of singularities in positive characteristic (or: a proof we are still waiting for)*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), 1–30.
- [33] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon–Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.
- [34] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.
- [35] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay*, Adv. Math. **13** (1974), 115–175.
- [36] M. Hochster and J. L. Roberts, *The purity of the Frobenius and local cohomology*, Adv. Math. **21** (1976), 117–172.
- [37] A. Ito, *Okounkov bodies and Seshadri constants*, Adv. Math. **241** (2013), 246–262.
- [38] A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. Math. Inst. Hautes Études Sci. **83** (1996), 51–93.

- [39] M. Jonsson and M. Mustață, *Valuations and asymptotic invariants for sequences of ideals*, Ann. Inst. Fourier (Grenoble) **62** (2012), 2145–2209.
- [40] S. L. Kleiman, “*The Picard scheme*”, in *Fundamental Algebraic Geometry*, Math. Surveys Monographs, **123**, American Mathematical Society, Providence, RI, 2005, 235–321.
- [41] A. Küronya, *Asymptotic cohomological functions on projective varieties*, Amer. J. Math. **128** (2006), 1475–1519.
- [42] E. Kunz, *On Noetherian rings of characteristic  $p$* , Amer. J. Math. **98** (1976), 999–1013.
- [43] R. Lazarsfeld, *Positivity in Algebraic Geometry. I. Classical Setting: Line Bundles and Linear Series*, Ergeb. Math. Grenzgeb. (3), **48**, Springer, Berlin, 2004.
- [44] R. Lazarsfeld, *Positivity in Algebraic Geometry. II. Positivity for Vector Bundles, and Multiplier Ideals*, Ergeb. Math. Grenzgeb. (3), **49**, Springer, Berlin, 2004.
- [45] L. Ma, *Finiteness properties of local cohomology for  $F$ -pure local rings*, Int. Math. Res. Not. IMRN **20** (2014), 5489–5509.
- [46] L. Ma, T. Polstra, K. Schwede and K. Tucker,  *$F$ -signature under birational morphisms*, Forum Math. Sigma **7** (2019), e11, 20 pp.
- [47] M. Manaresi, *Some properties of weakly normal varieties*, Nagoya Math. J. **77** (1980), 61–74.
- [48] H. Matsumura, *Commutative Ring Theory*, 2nd ed., Cambridge Stud. Adv. Math., **8**, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.
- [49] T. Murayama, *Frobenius–Seshadri constants and characterizations of projective space*, Math. Res. Lett. **25** (2018), 905–936.
- [50] T. Murayama, *Seshadri constants and Fujita’s conjecture via positive characteristic methods*, Ph.D. thesis, University of Michigan, Ann Arbor, 2019, 206 pp.
- [51] M. Mustață, “*The non-nef locus in positive characteristic*”, in *A Celebration of Algebraic Geometry*, Clay Math. Proc., **18**, American Mathematical Society, Providence, RI, 2013, 535–551.
- [52] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Mem. **14**, Math. Soc. Japan, Tokyo, 2004.
- [53] S. Nayak, *Compactification for essentially finite-type maps*, Adv. Math. **222** (2009), 527–546.
- [54] Zs. Patakfalvi, K. Schwede and K. Tucker, “*Positive characteristic algebraic geometry*”, in *Surveys on Recent Developments in Algebraic Geometry*, Proc. Sympos. Pure Math., **95**, American Mathematical Society, Providence, RI, 2017, 33–80.
- [55] P. H. Quy and K. Shimomoto,  *$F$ -injectivity and Frobenius closure of ideals in Noetherian rings of characteristic  $p > 0$* , Adv. Math. **313** (2017), 127–166.
- [56] M. Raynaud, “*Contre-exemple au ‘vanishing theorem’ en caractéristique  $p > 0$* ”, in *C. P. Ramanujam—A Tribute*, Tata Inst. Fund. Res. Studies in Math., **8**, Springer, Berlin–New York, 1978, 273–278.
- [57] K. Sato, *Stability of test ideals of divisors with small multiplicity*, Math. Z. **288** (2018), 783–802.
- [58] K. Schwede,  *$F$ -injective singularities are Du Bois*, Amer. J. Math. **131** (2009), 445–473.
- [59] K. Schwede, *Centers of  $F$ -purity*, Math. Z. **265** (2010), 687–714.
- [60] K. Schwede, *Test ideals in non- $\mathbb{Q}$ -Gorenstein rings*, Trans. Amer. Math. Soc. **363** (2011), 5925–5941.
- [61] K. Schwede and K. Tucker, “*A survey of test ideals*”, in *Progress in Commutative Algebra 2*, Walter de Gruyter, Berlin, 2012, 39–99.

- [62] S. Takagi and K.-i. Watanabe, *F-singularities: applications of characteristic  $p$  methods to singularity theory*, Sugaku Expositions **31** (2018), 1–42; Translated from the Japanese by the authors.
- [63] H. Tanaka, *Semiample perturbations for log canonical varieties over an  $F$ -finite field containing an infinite perfect field*, Internat. J. Math. **28** (2017), 1750030, 13 pp.
- [64] H. Tanaka, *Minimal model program for excellent surfaces*, Ann. Inst. Fourier (Grenoble) **68** (2018), 345–376.
- [65] H. Tanaka, *Abundance theorem for surfaces over imperfect fields*, Math. Z. (2019), to appear arXiv:1502.01383v5 [math.AG], doi:10.1007/s00209-019-02345-2.
- [66] J. D. Vélez, *Openness of the  $F$ -rational locus and smooth base change*, J. Algebra **172** (1995), 425–453.

*Department of Mathematics*  
*Princeton University*  
*Princeton, NJ 08544-1000*  
*USA*

`takumim@math.princeton.edu`