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RUIN PROBLEMS: SIMULATION OR CALCULATION?

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ABSTRACT

In this paper we use a case study of a non-life insurance portfolio to demonstrate how recent research in ruin theory can be applied to solvency problems. By approximating the aggregate claims distribution for the portfolio by a translated gamma distribution, we estimate ruin probabilities through a recursive procedure when the insurer earns investment income on its surplus. We also show the results of applying simulation techniques to this problem, and discuss some advantages and disadvantages of simulation as a means of assessing ruin probabilities. Finally, we discuss the calculation of the probability of ruin at the end of a specified time period.

KEYWORDS

Ruin Theory; Translated Gamma Distribution; Investment Income; Simulation; Recursive Calculation

1. INTRODUCTION

This paper is concerned with ruin theory and, in particular, the calculation/estimation of the probability of ruin. It is based on a case study of the solvency of a non-life insurance portfolio using data from a Danish insurance company. This paper has two purposes:

(1) to demonstrate how some recent research in ruin theory can be applied in a useful way; and
(2) to discuss some advantages/disadvantages of simulation as a methodology for assessing the probability of ruin.

Ruin theory has been an area of study for actuaries (and mathematicians) for many decades. A glance at the contents pages of actuarial research journals, for example the *ASTIN Bulletin*, shows that interest in this area remains as strong as ever. However, while the literature on ruin theory continues to grow and while the mathematics becomes ever more elegant, ruin theory does attract some negative comments. These range from the relatively lighthearted:

> ".... ruin theory, a topic about which it has been said that never have so many people written so much about such a small probability." Sundt (1993, p104)

to the more serious:

> "While ... (ruin) theory is well developed and well known, there are a number of respects in which it lacks realism to a point which militates against its practical use without substantial modification." Taylor & Buchanan (1988, p64)

> "Another serious shortcoming is that ... (Lundberg's upper bound for the probability of ruin in

> infinite time) provides non-zero survival probabilities only if the solvency margin tends to infinity." Daykin *et al.* (1993, p373)

To paraphrase Taylor & Buchanan's remark, (classical) ruin theory could be criticised for sacrificing realism for the sake of mathematical development. Some critics of ruin theory would advocate simulation as a more appropriate methodology for assessing solvency/probability of ruin. For example, Chapter 12 of Daykin *et al.* (1993), in which the assessment of the solvency of a general insurance operation is discussed, contains almost no mathematical development whatsoever, but uses simulation extensively.

Simulation is undoubtedly a powerful tool. Results can be obtained for extremely complex models, for example involving dynamic algorithms for management decisions, although the interpretation of these results is not always so easy! However, it is our view that, other things being equal, a result obtained by 'analytic' methods is to be preferred to one obtained by simulation. This accords with the views of Beard *et al.* (1983, p250):

> "Anyway, if direct analytic treatment is applicable it generally is more expedient, and the role of simulation is above all to deal with those cases where other techniques are impracticable."

Our views do not imply that we refuse to accept that simulation may be useful even in situations where analytic methods could be applied. However, we regard such situations as exceptions rather than the rule.

In Section 2 we give details of the problem that we are going to discuss. This is based on a study by Ramlau-Hansen (1988a & b) of non-life insurance data from a Danish insurance company, and concerns the solvency of a non-life insurance company/portfolio. In Section 3 we discuss the solution of our problem using only simulation. In Section 4 we present a method for solving our problem using analytic/numerical methods (and also, to a limited extent, simulation). In Section 5 we compare the two methods and draw some conclusions. Finally, in Section 6, we consider briefly a related problem.

2. THE PROBLEM

Ramlau-Hansen (1988a & b) published the results of a study of data from the years 1977-1981 from a Danish insurance company. The data related to policies covering single-family houses and dwellings against the risks of glass damage (i.e. damage to windows or sanitary fittings), fire damage and windstorm damage. In Part 1 of his study, Ramlau-Hansen fitted distributions to the claim numbers and claim amounts for each of these lines of insurance, and in Part 2 he used these results to discuss solvency requirements for portfolios of differing sizes and relative compositions.

For our purposes, we will consider a portfolio (without reinsurance) for which the expected claims outgo in one year has total 200 (in some suitable monetary units), of which 25% arises from glass claims and 75% from fire claims. For this portfolio, the standard deviation of the aggregate claims in one year is 20 and the

coefficient of skewness is 1.05 (see Ramlau-Hansen, 1988b, Table 2). Moreover, the aggregate claim amount in one year for this portfolio has a compound Poisson distribution, with the individual claim amount distribution being a weighted average of lognormal (for glass) and loggamma (for fire) distributions. (This is a slight simplification, since the claim number distribution for glass claims has a Poisson distribution with a parameter which can vary from year to year. However, the effect of this extra variability is small.)

Ramlau-Hansen (1988b) studied the problem of determining the initial surplus required so that the probability of ruin *at the end of* t years (where $t = 1$, 2 and 5) is kept to some predetermined (and small) level. He did this for portfolios as described above, with the inclusion of factors such as business growth, claims inflation and investment income. We will consider a related, but different, problem. We will discuss the estimation of the probability of ruin *at the end of any one* of the next 10 years for a portfolio as described above, with the inclusion of stochastic investment income. In the language of ruin theory, our problem is the estimation of a finite and discrete time probability of ruin, where the time horizon is 10 years and the discrete time interval is 1 year. This is clearly a more challenging problem than the estimation of the probability of ruin at the end of a fixed time interval, and it could be argued that it is more relevant in practice.

Our problem is stated more precisely as follows. Let $X_1, X_2, ..., X_{10}$ be a sequence of independent and identically distributed random variables representing the aggregate claims from a portfolio in successive years. We assume that each X_t has the same distribution as $\kappa + Y(\alpha,\beta)$, where $Y(\alpha,\beta)$ has a gamma distribution with mean α/β and variance α/β^2. The parameters α, β and κ are chosen so that X_t has mean 200, standard deviation 20 and coefficient of skewness 1.05.

Let U denote the insurer's capital at the start of the 10-year period under consideration, and let P be the (assumed constant) annual premium income. We assume $P \geq 200$ and we will write $P = 200(1 + \theta)$, so that θ is the premium loading factor.

Let $\mathbf{i}_1, \mathbf{i}_2, ..., \mathbf{i}_{10}$ be a sequence of random variables representing the rates of interest earned by the insurer in successive years. These interest rates have distributions specified by the annual returns in successive years on United Kingdom equities (ignoring tax), as given by the investment model formulated by Wilkie (1986).

The insurer's capital at the end of year t, $t = 1, 2, ..., 10$, is a random variable $U(t)$, where:

$$U(t) = U(t-1) \times (1 + \mathbf{i}_t) + P - X_t \tag{2.1}$$

and $U(0) = U$. Our problem is to estimate the probability of ruin $\psi(U,10)$, defined by:

$$\psi(U, 10) = P(U(t) < 0 \text{ for some } t,\ t = 1, 2, ..., 10).$$

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Comments on the assumptions

(1) Our model assumes aggregate claims in different years are independent of each other. This agrees with Ramlau-Hansen's model.

(2) Our model assumes a degree of stationarity — aggregate claims in different years are identically distributed and the premium income is unchanged from year to year. These assumptions have been made to simplify the presentation. However, our analysis in the following sections could quite easily be extended to some situations where the claim number distribution and/or the premium income changed either deterministically or stochastically from year to year.

(3) In common with most classical risk theory models, our model assumes claims are paid without delay. However, this is not too unreasonable for the portfolio we are considering, since Ramlau-Hansen (1988a, pp8-9 and 1986, Table 4) shows that the vast majority of glass and fire claims are reported in the calendar year of occurrence.

(4) It can be seen from our recurrence relation for $U(t)$ that, in each year, we assume interest is earned on the capital at the start of the year, but not on the premium income for the year (or the claims outgo). There would be no extra difficulties in the following sections if we were to assume that premiums were paid at the start of the year (and so earned a whole year's interest) or that claims and/or premiums were paid in the middle of the year, for instance.

(5) We have used Wilkie's model for the rates of return $\mathbf{i}_t$ for two reasons. Firstly, it is commonly used in the literature. Secondly, it is difficult to handle analytically and most applications in which it is used are based solely on simulation. Although the $\mathbf{i}_t$s are modelled as the rates of return on equities, we do not suggest equities would necessarily be a suitable type of investment for the portfolio we are considering! One advantage (for our purposes) of using this particular model is that the means and standard deviations of the $\mathbf{i}_t$s are relatively large. (This point will be relevant in Section 5.) We have used Wilkie's Reduced Standard Basis for his model and $\mathbf{i}_1$ is the annual rate of return ten years after starting the model from neutral starting values. See Wilkie (1986) for details. This means that, for example, on the basis of 1,000 simulations:

$$\mathrm{E}[\mathbf{i}_1] = 13.8\% \qquad \mathrm{StDev}[\mathbf{i}_1] = 30.2\%$$
$$\mathrm{E}[\mathbf{i}_{10}] = 14.1\% \qquad \mathrm{StDev}[\mathbf{i}_{10}] = 29.5\%.$$

Note that the $\mathbf{i}_t$s are neither independent nor identically distributed.

(6) An important point to note is that our model uses a translated gamma distribution, fitted by moments, to approximate a compound Poisson distribution for the aggregate claims in one year. Recent research (Dickson & Waters, 1993 and 1994) has shown that this type of approximation can give

values for the probability of ruin (in both finite and infinite time) which are very close to the exact values.

3. A Simulation Approach to the Problem

Simulation is an approach that has often been applied in the past to ruin problems. This is not surprising, as the simulation approach to estimating the probability of ruin is a simple one. To estimate the probability of ruin for the process described in the previous section, we will follow the procedure described by Seal (1969). All we have to do is to simulate a large number of realisations, say n, of the process, and count the number which result in ruin according to our definition. If this latter number is l, then our estimate of the probability of ruin is l/n.

Formally, let n denote the number of realisations of the surplus process that we simulate, and let L denote the number which result in ruin. Then $L \sim B(n, \psi)$, where $\psi = \psi(U, 10)$ is the true (unknown) probability of ruin for this risk process. Our estimate of ψ is $\tilde{\psi} = L/n$. Assuming that n is large, the distribution of L is approximately normal, and hence the distribution of $\tilde{\psi}$ is approximately $N(\psi, \psi(1 - \psi)/n)$.

It is particularly easy to simulate realisations of our risk process. In formula (2.1) we must input values of $\mathbf{i}_t$ whose simulation requires values of standard normal variables, and values of X_t whose simulation requires values from a gamma distribution. Values from each of these distributions are produced by standard computer libraries such as IMSL or NAG. Successive values of the surplus process can then be calculated from formula (2.1), and hence we can determine whether or not ruin occurs for each simulated realisation of the process.

Table 1 shows estimates of $\psi(U, 10)$ for $U = 0, 10, 20, \ldots, 100$ with premium loading factors $\theta = 0, 0.05, 0.1, \ldots, 0.25$ when $n = 1{,}000$. Also shown are estimates of the standard error of $\tilde{\psi}$ when $\tilde{\psi}$ is non-zero. The estimated standard error is calculated as $(\tilde{\psi}(1 - \tilde{\psi})/1{,}000)^{1/2}$.

Two obvious questions we can ask about simulation as a means of estimating are:

(1) How many realisations of the surplus process should we simulate?
(2) How reliable are our estimates?

We chose to simulate 1,000 realisations of the surplus process, as we considered this number to be sufficiently large to give estimates of the correct magnitude. However, it is clear from the standard errors that there is considerable uncertainty about our estimates. For example, when $U = 0$ and $\theta = 20\%$, our estimate of ψ is 0.054 with a standard error of 0.007, so that an approximate 95% confidence interval for $\tilde{\psi}$ is (0.040, 0.068). This uncertainty over the value of ψ is the price that must be paid for selecting 1,000 as the number of simulations. The standard error of our estimate clearly reduces as the value of n increases.

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If we wish our estimate to be more precise, we could specify a criterion which would dictate the value of n. For example, if we state that out estimate should be within 5% of the true ruin probability with probability 0.95, then we find that the minimum value of n is $1{,}537(1 - \psi)/\psi$. If we replace ψ by $\tilde{\psi}$ and consider the previous combination of $U = 0$ and $\theta = 20\%$, then the minimum value of n is 26,926. Thus, an increase in confidence in our estimate is achieved at the expense of considerably greater computer run time. As our aim is to compare methods of calculating ψ, we will not produce estimates using larger values of n. We have, however, shown in Table 1 the minimum number of simulations, denoted N, required if the estimate of ψ is to be within 5% of the true value with probability 0.95. As a final comment on simulation, we note that simulation has produced a number of estimates which are zero. Whilst these estimates may be close to the true values, we should be cautious in interpreting them. For example, if the true value of ψ is 0.002, then the probability of 1,000 realisations of the process resulting in non-ruin is 0.135. Thus, when the ruin probability is small, and when the number of simulations is relatively small, there can be a significant probability of no realisations resulting in ruin.

4. An Analytic (/Simulation) Approach to the Problem

In the previous section our problem was 'solved' using only simulation. The solution was relatively straightforward, but somewhat imprecise and/or time consuming. In this section we present an alternative method of solution. This method will still involve an element of simulation — avoiding simulation entirely while working with Wilkie's investment model is not easy! However, it will also make use of analytic/numerical methods which have recently been investigated in the actuarial literature.

Let $\underline{i}_j(=(i_{1,j}, i_{2,j}, ..., i_{10,j}))$. be the jth simulation out of a total of n simulations of the sequence of interest rates $\mathbf{i}_1, \mathbf{i}_2, ..., \mathbf{i}_{10}$. Let $\psi(U, 10 | \underline{i}_j)$ denote the probability of ruin as defined in Section 2, *given* the (deterministic) sequence of interest rates $i_{1,j}, i_{2,j}, ..., i_{10,j}$, in other words, *given* that $\mathbf{i}_t = i_{t,j}$ for $t = 1, 2, ..., 10$. Provided we can calculate $\psi(U,10 | \underline{i}_j)$, we can estimate $\psi(U,10)$ using the sample mean of the n values of $\psi(U, 10 | \underline{i}_j)$, and we can estimate the standard error of this estimate from the sample standard error. If n is large — we will take $n = 1{,}000$ in our examples — our estimate will have approximately a normal distribution. To summarise, an estimate of $\psi(U,10)$ is $\hat{\psi}$, where:

$$\hat{\psi} = \frac{1}{n} \sum_{j=1}^{n} \psi(U, 10 | \underline{i}_j)$$

and, approximately,

$$\hat{\psi} \sim N(\psi(U, 10), \sigma^2/n)$$

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where σ^2 is the sample variance of $\{\psi(U, 10|\underline{i}_j)\}_{j=1}^{n}$. The only remaining problem is the calculation of $\psi(U, 10|\underline{i}_j)$.

Table 1. Estimates of $\psi(U, 10)$ from simulation

U		θ=0%	θ=5%	θ=10%	θ=15%	θ=20%	θ=25%
0	$\tilde{\psi}$	0.741	0.420	0.212	0.098	0.054	0.029
	s.e. ($\tilde{\psi}$)	0.014	0.016	0.013	0.009	0.007	0.005
	N	538	2,123	5,714	14,147	26,926	51,464
10	$\tilde{\psi}$	0.595	0.280	0.117	0.056	0.027	0.012
	s.e. ($\tilde{\psi}$)	0.016	0.014	0.010	0.007	0.005	0.003
	N	1,047	3,953	11,600	25,910	55,389	126,547
20	$\tilde{\psi}$	0.474	0.199	0.069	0.025	0.014	0.006
	s.e. ($\tilde{\psi}$)	0.016	0.013	0.008	0.005	0.004	0.002
	N	1,706	6,187	20,739	59,943	108,249	254,630
30	$\tilde{\psi}$	0.360	0.124	0.036	0.016	0.007	0.002
	s.e. ($\tilde{\psi}$)	0.015	0.010	0.006	0.004	0.003	0.001
	N	2,733	10,859	41,158	94,526	218,035	766,963
40	$\tilde{\psi}$	0.289	0.087	0.022	0.006	0.002	0
	s.e. ($\tilde{\psi}$)	0.014	0.009	0.005	0.002	0.001	
	N	3,782	16,130	68,327	254,630	766,963	
50	$\tilde{\psi}$	0.218	0.060	0.011	0.003	0	0
	s.e. ($\tilde{\psi}$)	0.013	0.008	0.003	0.002		
	N	5,514	24,080	138,191	510,797		
60	$\tilde{\psi}$	0.166	0.032	0.006	0.002	0	0
	s.e. ($\tilde{\psi}$)	0.012	0.006	0.002	0.001		
	N	7,723	46,495	254,630	766,963		
70	$\tilde{\psi}$	0.119	0.022	0.006	0	0	0
	s.e. ($\tilde{\psi}$)	0.010	0.005	0.002			
	N	11,379	68,327	254,630			
80	$\tilde{\psi}$	0.099	0.017	0.001	0	0	0
	s.e. ($\tilde{\psi}$)	0.009	0.004	0.001			
	N	13,989	88,875	1,535,463			
90	$\tilde{\psi}$	0.071	0.007	0	0	0	0
	s.e. ($\tilde{\psi}$)	0.008	0.003				
	N	20,111	218,035				
100	$\tilde{\psi}$	0.055	0.005	0	0	0	0
	s.e. ($\tilde{\psi}$)	0.007	0.002				
	N	26,409	305,864				

In principle, $\psi(U, 10|\underline{i}_j)$ can be calculated recursively as follows. Define $\psi(U, m|\underline{i}_j)$ to be the probability of ruin at any of the time points $11 - m$, $12 - m, \ldots, 10$ starting from surplus U at time $10 - m$ and given the set of interest rates $\underline{i}_j$, i.e. $\psi(U, m|\underline{i}_j) = \Pr(U(t) < 0$ for some t, $t = 11 - m, 12 - m, \ldots, 10|U(10 - m) = U, \underline{i}_j)$.

Then:

$$\psi(U, 1|\underline{i}_j) = 1 - F(U(1 + i_{10,j}) + P)$$

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and, for $m = 1, 2, ..., 9$:

$$\psi(U, m+1 \mid \underline{i}_j) = 1 - F(U(1+i_{10-m,j}) + P) + \int_K^{U(1+i_{10-m,j})+P} f(x)\psi(U(1+i_{10-m,j}) + P - x, m \mid \underline{i}_j)dx$$

where $F(x)$ and $f(x)$ are the distribution function and density function, respectively, of the random variable X_t .

The rationale behind these formulae is as follows. For ruin to occur at time 10 starting from surplus U at time 9, the aggregate claims in the 10th year must exceed $U(1+i_{10,j}) + P$. This explains the formula for $\psi(U, 1|\underline{i}_j)$. The formula for $\psi(U, m + 1|\underline{i}_j)$ can be explained by considering the aggregate claim amount in the $(10 - m)$th year. If this exceeds $U(1+ i_{10-m,j}) + P$, then ruin occurs at time $(10 - m)$. This gives the term $1 - F(U(1 + i_{10-m,j}) + P)$. Otherwise, ruin must occur starting from surplus $U(1+ i_{10-m,j}) + P - x$ at time $10 - m$, where x is the aggregate claim amount in the $(10 - m)$th year — giving the integral term.

The model we are using is almost identical to that used by Beard *et al.* (1983, p.230, formula (6.7.7)) — the only differences being that their model used deterministic interest and a (deterministically) variable premium rate (which we could easily incorporate). The recursive formulae for $\psi(U, m + 1|\underline{i}_j)$ above correspond (with minor modifications) to Beard *et al.*'s formula (6.7.9). However, Beard *et al.* are sceptical about the possibility of obtaining good numerical results from such formulae:

> "... even if feasible ... this method may be laborious. If some approximation is to be used for F, e.g. the N(ormal) P(ower) or Γ formula, then the accumulation of inaccuracy as well as the normal rounding-off errors under the rather long sequence of computations may be difficult to control." Beard *et al.* (1983, p.231)

It is our contention that:

(a) this method need not be laborious; and
(b) there is evidence in the actuarial literature that it could produce very good numerical answers.

The remainder of this section is devoted to presenting the evidence to support these two points.

Let us first consider how laborious the calculations need to be using the above formulae. The amount of numerical work involved in using these formulae can be reduced, possibly considerably, using an intuitively appealing and simple procedure originally proposed by De Vylder & Goovaerts (1988). In outline, this procedure is as follows: for very large values of x, $F(x)$ will be very close to 1 and $\psi(x, m|\underline{i}_j)$ will be very close to 0. If, for suitably large values of x, we set $F(x) = 1$, $f(x) = 0$ and $\psi(x, m|\underline{i}_j) = 0$, then the upper limit of the range of integration in the above formula for $\psi(U, m + 1|\underline{i}_j)$ may be reduced and the

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lower limit may be increased. This procedure can be formalised as follows:

Let ε, $0 < \varepsilon < 1$, be some suitably small number and let ν_0 be a number such that:

$$F(x) \geq 1 - \varepsilon \text{ for } x \geq \nu_0.$$

Now define:

$$\begin{aligned} F_\varepsilon(x) &= F(x) \quad \text{for } x < \nu_0 \\ &= 1 \quad \text{for } x \geq \nu_0 \end{aligned}$$

$$\begin{aligned} f_\varepsilon(x) &= f(x) \quad \text{for } x < \nu_0 \\ &= 0 \quad \text{for } x \geq \nu_0 \end{aligned}$$

$$\psi_\varepsilon(U, 1 \mid \underline{i}_j) = 1 - F_\varepsilon(U(1 + i_{10,j}) + P)$$

$$\nu_1 = (\nu_0 - P)/(1 + i_{10,j}).$$

Note that, in all practical cases, ν_1 will be greater than 0, since $\Pr(X_t < P)$ will not be close to 1, and also that $\psi_\varepsilon(U, 1 \mid \underline{i}_j)$ will be equal to 0 whenever U is greater than ν_1.

Now define recursively for $m = 1, 2, \ldots, 9$:

$$\psi_\varepsilon(U, m+1 \mid \underline{i}_j) = 1 - F_\varepsilon(U(1 + i_{10-m,j}) + P) + \int_\kappa^{U(1+i_{10-m,j})+P} f_\varepsilon(x)\psi_\varepsilon(U(1 + i_{10-m,j}) + P - x, m \mid \underline{i}_j)dx$$

provided the right hand side of this expression is greater than ε. Define ν_{m+1} to be the value of U for which the right hand side of this expression equals ε. The definition of $\psi_\varepsilon(U, m + 1 \mid \underline{i}_j)$ is completed by setting it equal to 0 for $U \geq \nu_{m+1}$.

Calculating $\psi_\varepsilon(U, m + 1 \mid \underline{i}_j)$ rather than $\psi(U, m + 1 \mid \underline{i}_j)$, and using the former as an approximation to the latter, has the important computational advantage that the integrand in the expression for the former is zero outside the range $\max(\kappa, U(1 + i_{10\text{-}m,j}) + P - \nu_m)$ to $\min(U(1 + i_{10\text{-}m,j}) + P, \nu_0)$. Not only does this save (a possibly considerable amount of) computational time, but it does this in a controlled way, as the following result shows.

Result: For $m = 1, 2, \ldots, 10$:

$$0 \leq \psi(U, m \mid \underline{i}_j) - \psi_\varepsilon(U, m \mid \underline{i}_j) \leq 3m\varepsilon.$$

Proof: The proof of this result follows precisely the proof given by De Vylder & Goovaerts (1988, Section 5), although the setting is somewhat different in their case.

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Now let us turn to the accuracy of numerical results obtained using the above formulae for $\psi_\varepsilon(U, m|\underline{i}_j)$. De Vylder & Goovaerts (1988) originally proposed a method similar to the above for calculating the probability of ruin in finite and *continuous* time. Their method, as sharpened somewhat by Dickson & Waters (1991, Section 2), had the extra complication that, being a discrete time approximation to a continuous time process, the interval between 'checks' on the surplus process, which is 1 year in our model, had to be very short to obtain good approximations to the continuous time process. This, in turn, meant that, for any sensible time horizon, the number of steps in the recursive calculation of the probability of ruin could be very large, possibly several thousands, rather than the 10 steps in our calculation of $\psi_\varepsilon(U,10|\underline{i}_j)$. Despite the inevitable consequences for the accuracy of the calculations of a large number of recursive calculations, the results obtained, using this method for calculating the probability of finite and continuous time ruin are very good. See Dickson & Waters (1991, Table 1). It would be expected that, using the method with a smaller number of recursions, i.e. 10, as in the calculation of $\psi_\varepsilon(U, m|\underline{i}_j)$, would give even more accurate results.

Table 2 shows results corresponding to those in Table 1, but calculated using the methods described in this section. In all cases the number of simulations used is 1,000, and the 'truncation control' parameter ε has been set equal to $(3 \times 10^6)^{-1}$, so that, using the result above, the maximum numerical error resulting from the truncation procedure is 10^{-5}. The integral in the expression for $\psi_\varepsilon(U, 10|\underline{i}_j)$ has been calculated using the repeated trapezium rule with an integer step size. The value of N, the minimum number of simulations required to achieve the specified degree of accuracy, has been calculated in Table 2 as it was in Table 1, i.e. assuming the number of simulations is sufficiently large for the estimate $\hat{\psi}$ to have approximately a normal distribution. For $U = 0$, the values of N in Table 2 are sufficiently small to make this assumption doubtful. However, these values do indicate that very few simulations are needed to achieve the required degree of accuracy.

5. Some Comments on Sections 3 and 4

The most startling conclusion from a comparison of Tables 1 and 2 is the difference in the numbers of simulations required to achieve a given level of accuracy for each of the two methods presented above. For example, to estimate $\psi(50, 10)$ for $\theta = 0.1$ to within 5% of the correct value with probability 0.95 requires about 138,000 simulations using pure simulation, as in Section 3, but less than 1,000 using simulation/calculation, as in Section 4. For this combination of U and θ, the ratio of the numbers of simulations required by the two methods to achieve any given level of accuracy is about 155. The absolute difference in the numbers of simulations required for the two methods becomes more extreme if we require a higher level of accuracy. For example, changing 5% and 0.95 to 1% and 0.99 in this case increases the numbers of simulations required to about

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6 million and 38,000, respectively! (Such comparisons are slightly biased in favour of the latter method, since we are ignoring the, probably very small, error resulting from the numerical integration needed to calculate $\psi_\varepsilon(U, 10|\underline{i}_j)$.)

Table 2. Estimates of $\psi(U, 10)$ from calculation

U		$\theta = 0\%$	$\theta = 5\%$	$\theta = 10\%$	$\theta = 15\%$	$\theta = 20\%$	$\theta = 25\%$
0	$\hat{\psi}$	0.73989	0.40535	0.19981	0.09762	0.04768	0.02319
	s.e. $(\hat{\psi})$	0.00125	0.00094	0.00042	0.00016	0.00006	0.00002
	N	5	9	7	5	3	2
10	$\hat{\psi}$	0.59635	0.26565	0.11284	0.05021	0.02310	0.01078
	s.e. $(\hat{\psi})$	0.00210	0.00131	0.00061	0.00029	0.00014	0.00007
	N	20	38	45	51	56	62
20	$\hat{\psi}$	0.46522	0.17090	0.06332	0.02592	0.01133	0.00511
	s.e. $(\hat{\psi})$	0.00291	0.00150	0.00064	0.00029	0.00013	0.00006
	N	61	118	157	189	215	237
30	$\hat{\psi}$	0.35528	0.10947	0.03575	0.01356	0.00566	0.00248
	s.e. $(\hat{\psi})$	0.00333	0.00140	0.00053	0.00022	0.00010	0.00005
	N	136	253	339	409	465	509
40	$\hat{\psi}$	0.26758	0.07029	0.02041	0.00721	0.00288	0.00123
	s.e. $(\hat{\psi})$	0.00340	0.00119	0.00040	0.00015	0.00007	0.00003
	N	249	441	584	699	791	862
50	$\hat{\psi}$	0.19974	0.04540	0.01181	0.00390	0.00150	0.00062
	s.e. $(\hat{\psi})$	0.00324	0.00096	0.00029	0.00010	0.00004	0.00002
	N	404	686	892	1,055	1,185	1,284
60	$\hat{\psi}$	0.14830	0.02955	0.00693	0.00214	0.00079	0.00032
	s.e. $(\hat{\psi})$	0.00294	0.00075	0.00020	0.00007	0.00003	0.00001
	N	604	995	1,268	1,475	1,641	1,770
70	$\hat{\psi}$	0.10979	0.01938	0.00412	0.00119	0.00042	0.00017
	s.e. $(\hat{\psi})$	0.00259	0.00058	0.00014	0.00004	0.00002	0.00001
	N	854	1,378	1,722	1,964	2,160	2,318
80	$\hat{\psi}$	0.08120	0.01283	0.00248	0.00067	0.00023	0.00009
	s.e. $(\hat{\psi})$	0.00223	0.00045	0.00010	0.00003	0.00001	0.00000
	N	1,162	1,850	2,268	2,529	2,738	2,859
90	$\hat{\psi}$	0.06006	0.00856	0.00151	0.00038	0.00013	0.00005
	s.e. $(\hat{\psi})$	0.00190	0.00034	0.00007	0.00002	0.00001	0.00000
	N	1,537	2,428	2,927	3,184	3,330	3,292
100	$\hat{\psi}$	0.04447	0.00576	0.00093	0.00022	0.00007	0.00003
	s.e. $(\hat{\psi})$	0.00160	0.00026	0.00005	0.00001	0.00000	0.00000
	N	1,990	3,135	3,723	3,891	3,856	3,524

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A point to note here is that we have deliberately chosen a stochastic investment model with relatively high variability. (See comment (5) in Section 2.) Intuitively, the higher the variability resulting from the interest rate model, the more attractive the pure simulation approach of Section 3 should be. The reason for this is that the variability in $\psi_\varepsilon(U, 10|\underline{i}_j)$ is due solely to the interest rate model, whereas the estimates in Section 3 have this same variability and, in addition, variability resulting from the aggregate claim amounts.

A feature of Table 2 is the very small numbers of simulations required to achieve the given level of accuracy for $U = 0$, although, as explained at the end of Section 4, the precise values for N shown in Table 2 should be treated with caution in cases where N is very small. The explanation for this feature is that from $U = 0$, ruin, if it occurs at all, is likely to occur very quickly, i.e. at the end of the first year. However, because we have assumed interest is not earned or paid on premiums and claims within the year they are paid, the interest rate model has no effect on the surplus at the end of the first year when $U = 0$.

Finally, since the main thrust of this paper has been to extol the virtues of calculation as opposed to simulation, a comment in favour of simulation is in order! We have already commented, in Section 1, that simulation can deal with very complex models, and this point is easily illustrated by considering a simple extension of the model we have used in this paper. Suppose the premium charged for the year t to $t + 1$ depends on the surplus at time $t - 1$, i.e. on $U(t - 1)$. Intuitively, this may be quite reasonable, since premiums may well be revised to take account of the solvency level of the insurer, and there will inevitably be a time delay before any premium changes can be introduced. The simulation approach of Section 3 would have no difficulty in coping with this extra complication. In contrast, the approach of Section 4 could not so easily be adapted. The reason for this is that the recursive algorithm for calculating $\psi_\varepsilon(U, 10|\underline{i}_j)$ works *backwards* from time 9, giving $\psi_\varepsilon(U, 1|\underline{i}_j)$, through time $10 - m$, giving $\psi_\varepsilon(U, m|\underline{i}_j)$, to time 0, giving $\psi_\varepsilon(U, 10|\underline{i}_j)$. Hence, at time $10 - m$ it would be difficult to take into account in the calculation of $\psi_\varepsilon(U, m|\underline{i}_j)$ the level of surplus at any earlier time. However, since the set of interest rates $\underline{i}_j$ is fixed for the calculation of $\psi_\varepsilon(U, 10|\underline{i}_j)$ it would be possible to use the approach of Section 4 (and also of Section 3) for a model where the premium in any year depended on the interest rates in previous years.

6. Ruin at the End of a Specified Period

In this section we consider, very briefly, the estimation of the probability of ruin for our portfolio *at the end of a specified number of years*. We denote this probability $\phi(U, m)$, so that $\phi(U, m)$ is the probability that the surplus *at the end of* m years, $U(m)$ in the notation of Section 2, is negative. Two reasons why this probability is of some interest are:

(1) It corresponds to the approach taken by Ramlau-Hansen (1988b, Sections 3 and 4) when he considers ruin over periods longer than one year.

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(2) It is tempting to regard $\phi(U, m)$ as an approximation to $\psi(U, m)$. In relation to a similar, but not identical, model, Taylor & Buchanan (1988, ¶3.2.2.8) say that $\phi(U, m)$ is a "simple, and often adequate, approximation" to $\psi(U, m)$ for small values of m. With this comment in mind, it is of some interest to check the accuracy of this approximation for the particular model we are considering in this paper.

Let $\phi(U, m|\underline{i}_j)$ denote the probability that $U(m)$ is negative, *given* $\mathbf{i}_t = i_{t,j}$ for $t = 1, 2, ..., 10$. Note that $\phi(U, m)$ and $\phi(U, m|\underline{i}_j)$ must necessarily be less than or equal to $\psi(U,m)$ and $\psi(U, m|\underline{i}_j)$, respectively.

Given the sequence of interest rates $\underline{i}_j$, we can write down the following formula for $U(m)$:

$$U(m) = U \times \prod_{t=1}^{m}(1+i_{t,j}) + P \times \sum_{t=1}^{m}\prod_{k=t+1}^{m}(1+i_{k,j}) - \sum_{t=1}^{m} X_t \prod_{k=t+1}^{m}(1+i_{k,j})$$

where $\prod_{k=m+1}^{m}(1+i_{t,j})$ is taken to be equal to 1. Knowing the (first three) moments of X_t, it is elementary to calculate the (first three) moments of $\sum_{t=1}^{m} X_t \prod_{k=t+1}^{m}(1+i_{k,j})$, and hence, approximating the distribution of this random variable by a translated gamma distribution, to calculate approximately $\phi(U, m|\underline{i}_j)$. By simulating a large number of sequences of interest rates $\underline{i}_j$ we can calculate an estimate of $\phi(U,m)$, denoted $\hat{\phi}(U,m)$, together with an approximate standard error of the estimate exactly as we did in Section 4 for $\psi(U, m)$.

Table 3 shows, for $\theta = 5\%$, estimated values of $\phi(U, m)$, and the corresponding estimated values of $\psi(U, m)$, for $U = 0$, 50, 100 and $m = 2$, 5, 10, together with the standard errors of these estimates. Table 4 gives the corresponding information for $\theta = 15\%$. The values of $\hat{\phi}(U,m)$ have been calculated as described above, using the same set of 1,000 sequences of interest rates $\underline{i}_j$, as were used in Sections 3 and 4. The values $\hat{\psi}(U, m)$ have been calculated as described in Section 4. (The values of $\hat{\psi}(U, 10)$ have been taken directly from Table 2.)

Tables 3 and 4 are not very extensive, but they do show quite clearly, for the model we have been considering, that $\phi(U, m)$ cannot be considered a reliable approximation to $\psi(U, m)$.

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Table 3. Estimates of $\psi(U, m)$ and $\phi(U, m)$ when $\theta = 5\%$

	$\hat{\phi}(U,2)$	$\hat{\psi}(U,2)$	$\hat{\phi}(U,5)$	$\hat{\psi}(U,5)$	$\hat{\phi}(U,10)$	$\hat{\psi}(U,10)$
$U = 0$	0.22024	0.33252	0.13937	0.39013	0.07702	0.40535
s.e.	0.00004	0.00049	0.00015	0.00084	0.00027	0.00094
$U = 50$	0.01603	0.02132	0.01677	0.03926	0.01047	0.04540
s.e.	0.00038	0.00052	0.00035	0.00082	0.00021	0.00096
$U = 100$	0.00118	0.00147	0.00207	0.00422	0.00158	0.00576
s.e.	0.00006	0.00008	0.00010	0.00019	0.00007	0.00026

Table 4: Estimates of $\psi(U, m)$ and $\phi(U, m)$ when $\theta = 15\%$

	$\hat{\phi}(U,2)$	$\hat{\psi}(U,2)$	$\hat{\phi}(U,5)$	$\hat{\psi}(U,5)$	$\hat{\phi}(U,10)$	$\hat{\psi}(U,10)$
$U = 0$	0.03348	0.09374	0.00333	0.09767	0.00018	0.09762
s.e.	0.00004	0.00013	0.00003	0.00016	0.00001	0.00016
$U = 50$	0.00153	0.00347	0.00019	0.00401	0.00001	0.00390
s.e.	0.00004	0.00009	0.00001	0.00010	0.00000	0.00010
$U = 100$	0.00009	0.00017	0.00002	0.00023	0.00000	0.00022
s.e.	0.00001	0.00001	0.00000	0.00001	0.00000	0.00001

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