Robotica (2004) volume 22, pp. 97–108. © 2004 Cambridge University Press
DOI: 10.1017/S0263574703005216 Printed in the United Kingdom

Optimum synthesis of planar parallel manipulators based on kinematic isotropy and force balancing

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(Received in Final Form: April 4, 2003)

SUMMARY

This paper deals with an optimum synthesis of planar parallel manipulators using two constrained optimisation procedures based on the minimization of: (i) the overall deviation of the condition number of manipulator Jacobian matrix from the ideal/isotropic condition number, and (ii) bearing forces throughout the manipulator workspace for force balancing. A revolute jointed planar parallel manipulator is used as an example to demonstrate the methodology. The parameters describing the manipulator geometry are obtained from the first optimisation procedure, and subsequently, the mass distribution parameters of the manipulator are determined from the second optimisation procedure based on force balancing. Optimisation results indicate that the proposed optimisation approach is systematic, versatile and easy to implement for the optimum synthesis of the parallel manipulator and other kinematic chains. This work contributes to previously published work from the point of view of being a systematic approach to the optimum synthesis of parallel manipulators, which is currently lacking in the literature.

KEYWORDS: Parallel manipulator; Optimum mechanism synthesis; Kinematic isotropy; Force balancing, kinematics analysis.

1. INTRODUCTION

Parallel manipulators or in-parallel actuated mechanisms, which consist of one or more closed kinematics chains, have the advantages of high stiffness, good dynamic characteristics, and precise positioning capability.[1] However, they have some disadvantages, mainly due to their parallel topology, including limited workspace, difficulties in their analysis, synthesis, control and trajectory planning. Of these, optimum synthesis of planar parallel manipulators exemplified with a planar parallel manipulator articulated with five revolute joints is the object of this paper. The proposed synthesis methodology consists of two constrained optimisation procedures. The first deals with the determination of manipulator link lengths from a kinematics point of view such that the resulting mechanism has a maximum of high mechanical advantage (a desirable range of transmission angle), high kinematic accuracy, dexterity, and singularity avoidance capabilities. The second is based on optimum force balancing yielding proper mass distribution parameters of the manipulator links with a minimum bearing and ground forces. The mass distributions of the links have a direct effect on the magnitude of inertial forces when the manipulator is in motion, especially for high-speed applications. The resulting manipulator is optimum with respect to kinematic isotropy and is in equilibrium in all configurations of the manipulator with zero actuator forces. Less powerful and smaller actuators can be employed to move such a manipulator with a high mechanical advantage, kinematic accuracy, dexterity, and singularity avoidance capabilities. It has been reported[2–6] that the planar parallel manipulator considered in this study is the only one of the multi crank mechanisms having practical importance, especially for following any arbitrary planar curve precisely which can not be handled with single degree of freedom mechanisms such as four-bar and slider crank mechanisms.

Although there is a wealth of literature on analytical techniques for synthesis of mechanisms, the optimum synthesis of planar parallel manipulators based on the two subsequent optimisation procedures has not been addressed before. The work most relevant to this includes that of Lee and Freudenstein[5] who reported on the optimisation of the variation of the transmission angle from 90 degrees for unlimited rotations of geared cranks of a revolute-jointed five bar mechanism. Synthesis charts for unlimited crank rotations of the mechanism with various gear ratios were presented. In another relevant work, Tao and Hall[2] presented an analysis of a geared symmetrical (parallel) five-bar linkage whose output point drives another linkage – a Scotch-Yoke mechanism. Synthesis charts based on the ratio of the slider displacement of the Scotch-Yoke mechanism to the crank length of the five-bar mechanism versus the main crank angle were given for the purpose of selecting the link lengths of the five-bar mechanism. Later, Rose[3] employed a graphical method to synthesise revolute-jointed five-bar mechanisms: (i) generating a straight line, (ii) passing through six precision points, and (iii) one of their coupler links performing a 90-degrees dwell. Pollitt[4] studied kinematics analysis of a class of five-bar mechanisms using a graphical method.

Recently, significant efforts have been directed towards analysing their workspace, kinematics, singularities and solution space of five-bar manipulators.[6,7–10] For example, Feng et al.[9] reported on the performance evaluation of a 2 DOF planar parallel manipulator. Atlases of global conditioning and global velocity indices based on the condition number of the kinematic influence matrix of the manipulator were presented. Yet, no numerical results demonstrating the use of these indices to determine link lengths were given. It must be noted that the influence matrix they employed was a function of the angular positions of the two active (driving) joints and two passive (driven) joints. That matrix is not the manipulator Jacobian matrix, which is a function of the angular positions of the active joints only. Zhou and Cheung[11] have described a modified genetic algorithm to optimise a five-bar mechanism. Their objective function was to minimize the power requirement of real-time adjustable motors over a number of representative trajectory curves.

The link lengths based on the synthesis charts presented in references [2,5,10,11] may not yield mechanisms that have some desired benefits such as high mechanical advantage, high accuracy, dexterity and singularity avoidance capabilities. Although analytical techniques for the synthesis of mechanisms result in a well-defined solution space, they require a great deal of computation to find the link lengths of the mechanism maximizing the desired benefits. Therefore, it is of important practical interest to make use of optimisation methodologies providing the best possible mechanisms without a great deal of analysis. Gosselin and Angeles[12] have proposed a global condition/performance index, which is based on integrating the condition number of the manipulator Jacobian matrix over the entire workspace of the manipulator for kinematic optimisation of robot manipulators. Later, this method was applied to optimise a purely translational Stewart platform type parallel manipulator by Tsai and Joshi.[13] In this paper, a similar approach is taken to determine link lengths of the parallel manipulator such that the overall deviation of the condition number of the Jacobian matrix from the ideal/isotropic number is minimum throughout its workspace.

When the weights of the links of articulated mechanisms are not balanced, their performance is heavily impaired, and large actuator forces are demanded to move the links. Balancing of the link weights, known as force balancing,* can be accomplished by a proper selection of mass distribution of the links such that the mechanism under consideration has a constant potential energy for all its configurations. The practical consequence of this achievement is that zero actuator forces are required when the manipulator is at rest, and less powerful and smaller actuators can be employed to move the manipulator. Of course, distributing or redistributing (in the form of adding counterweights) the mass of the links increase the inertia forces as well as bearing and ground forces. It is, therefore, reasonable to optimise the mass distribution of the links in order to ensure that the manipulator is force-balanced and is optimum with respect to the bearing and ground forces. The outcomes of the optimum mass distribution procedure, hence, complete the geometric and inertial parameters needed for kinematic and dynamic analysis.

Although significant efforts have been directed towards force balancing of planar and spatial parallel manipulators,[14–18] optimisation procedures for minimization of the force and moment transmitted to the ground,[19–22] and balancing methodologies based on shaking force and moment transmitted to the frame of single degree of freedom mechanisms,[23–26] little has been published on the optimal force balancing of planar parallel manipulators based on the minimization of all bearing, support (ground) forces simultaneously. A wide account of these studies is provided in Lowen et al.[24] Yan and Soong,[23] and Conte et al.[20] have reported on a balancing method that combines kinematic synthesis, dynamic synthesis, and input speed trajectory synthesis to simultaneously satisfy the kinematic requirements and constraints for four-bar linkages. In a recent study,[27] the mass distribution of a four-bar mechanism with small clearances at its three passive joints has been optimized without adding any counterweights to any links. The change in the amplitude and direction of the joint forces is taken as the optimization function. Jean and Gosselin[14] have reported on the static balancing of planar parallel manipulators based on the mass distribution of links, without paying attention to optimum force balancing. Later, Laliberte et al.[15] studied static balancing of 3-DOF planar parallel manipulators and presented balancing conditions based on the mass distribution of the links and elastic elements for 3-DOF parallel manipulators. Static balancing of spatial parallel platform type mechanisms based on elastic elements only has been investigated by Ebert-Uhoff et al.,[18] and least restrictive balancing conditions are determined. Wang and Gosselin[16] have described static balancing conditions for spatial 3-DOF parallel mechanisms by mass distribution of the links and springs. The balancing of spatial mechanisms has also been studied by Bagci,[26] and Walsh et al.[17] The mathematical framework for employing elastic elements to statically balance mechanisms has been derived by Streit and Gilmore[28] and Walsh et al.[17] It has been stressed that when elastic elements such as spring together with some counterweights are exploited for balancing, the total potential energy of the system consisting of elastic and gravitational potential energies is constant for all the configurations of the mechanism, and most importantly, a much smaller counterweight is required for balancing.

We will follow the mathematical formulation derived by Berkof and Lowen[25] to determine the necessary optimisation constraints in order to keep the total potential energy of the system constant. Other constraints are related to the size and dimensions of the mass distribution of the links. Static force analysis of the manipulator is accomplished by using matrix method, which requires less mathematical manipulation. The objective function employed is the mean-square root of the sum-squared values of the bearing and ground forces calculated for a practical range of operation of the mechanism.

The optimisation procedures are implemented in MATLAB using the *constr()* function (In the new version of MATLAB, this function is superseded by *fmincon()*), which performs nonlinear optimisation based on a sequential quadratic programming method.[29] Sets of optimisation trials were accomplished to prove that the proposed optimisation methodology was an efficient, versatile and systematic procedure, and could be employed to determine the best possible link lengths and mass distribution parameters of a

* Force balancing, gravity balancing and static balancing are used interchangeably in the mechanism literature.

five-bar parallel manipulator, without going through a great deal of analysis. However, it must be kept in mind that the optimum solutions depend on the initial values assigned to the synthesis parameters and they do not necessarily represent a global minimum.

2. FIVE-BAR MANIPULATOR WITH REVOLUTE JOINTS

The five-bar planar manipulator considered in this study is shown in Figure 1, where its two joints (**A** and **E**) connected to the ground are active and the others are passive joints. The input motions of the active joints can be independent from each other or be provided via a set of gears maintaining a specified phase angle between the two active joints in order to generate an infinite number of different motions from the output point **C**.[5] Analytical expressions for the coordinates of the output point **C**, where the end effector is mounted, are obtained for the provided joint inputs θ_1 and θ_4, and the specified link lengths $\mathbf{L_0}$, $\mathbf{L_1}$, $\mathbf{L_2}$, $\mathbf{L_3}$, $\mathbf{L_4}$ and the angle α_0.[7] It must be noted that for a parallel RRRRR manipulator $\mathbf{L_1 = L_4}$ and $\mathbf{L_2 = L_3}$. The angle α_1 and the radial distance $\mathbf{R_i}$ describe the center of mass $\mathbf{G_i}$ of the $\mathbf{i^{th}}$ link.

3. KINEMATIC ANALYSIS AND JACOBIAN MATRIX

Referring to Figure 2(a), the coordinates of **B** and **D**, which are the **x** and **y** components of the $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_4$ vectors, can be considered as the coordinates of the centers of two circles of radii $\mathbf{L}_2$ and $\mathbf{L}_3$. The centres of two circles are expressed as functions of the inputs provided by the actuators fixed to the ground. It is well known that the intersection of the two circles gives a maximum of two solutions, which are the possible locations of point **C**. Referring to Figure 2(b), the analytical expressions for these two solutions are obtained using the following algorithm;

$$\begin{aligned} \vec{\mathbf{r}}_1 &= \mathbf{L}_1 \cos\theta_1\, \vec{\mathbf{i}} + \mathbf{L}_1 \sin\theta_1\, \vec{\mathbf{j}}, \\ \vec{\mathbf{r}}_4 &= (\mathbf{L}_0 + \mathbf{L}_4 \cos\theta_4)\vec{\mathbf{i}} + \mathbf{L}_4 \sin\theta_4\, \vec{\mathbf{j}}, \end{aligned} \tag{1}$$

$$\vec{\mathbf{q}} = \vec{\mathbf{r}}_4 - \vec{\mathbf{r}}_1 = \bar{\mathbf{x}}\vec{\mathbf{i}} + \bar{\mathbf{y}}\vec{\mathbf{j}}, \tag{2}$$

$$\bar{\mathbf{x}} = \mathbf{r}_{4x} - \mathbf{r}_{1x},\ \bar{\mathbf{y}} = \mathbf{r}_{4y} - \mathbf{r}_{1y},\ \mathbf{q} = \sqrt{\bar{\mathbf{x}}^2 + \bar{\mathbf{y}}^2}, \tag{3}$$

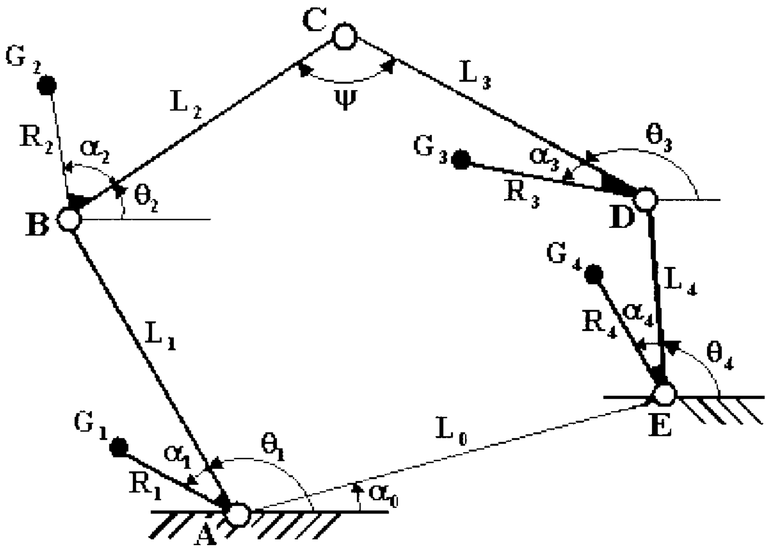

Fig. 1. Planar manipulator with all synthesis parameters. Note that the angle **α** is from the longitudinal axis of the $\mathbf{i^{th}}$ link.

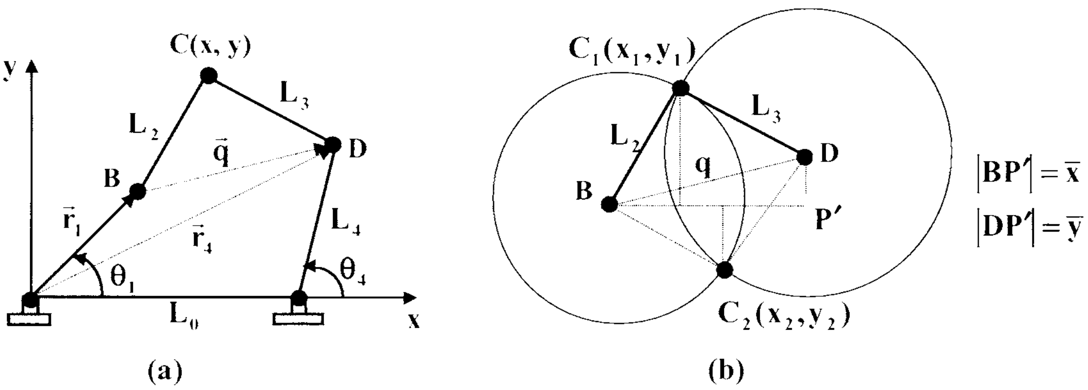

Fig. 2. (a) Geometric model of the manipulator for $a_0 = 0^0$, (b) representation of two possible forward kinematics solutions.

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and the coordinates of $\mathbf{C_1}$ and $\mathbf{C_2}$ are;

$$\mathbf{x_1 = x_B + Q\bar{x} - \bar{y}\sqrt{\frac{L_2^2}{q^2} - Q^2}},$$

$$\mathbf{y_1 = y_B + Q\bar{y} - \bar{x}\sqrt{\frac{L_2^2}{q^2} - Q^2}}, \tag{4}$$

$$\mathbf{x_2 = x_B + Q\bar{x} - \bar{y}\sqrt{\frac{L_2^2}{q^2} - Q^2}},$$

$$\mathbf{y_2 = y_B + Q\bar{y} - \bar{x}\sqrt{\frac{L_2^2}{q^2} - Q^2}}, \tag{5}$$

and

$$\mathbf{Q = \frac{L_2^2 + q^2 - L_3^2}{2q^2}}, \tag{6}$$

So, it is now possible to determine the position of the output point for given joint inputs. Depending on the link lengths, $\mathbf{(x_1, y_1)}$ and $\mathbf{(x_2, y_2)}$ can have real and imaginary values. If they are imaginary, some kinematics constraints are not satisfied; the mechanism cannot adopt those configurations. This is one of the constraints that must be satisfied by the link lengths.

By following the algorithm given above, the forward kinematics equations for the point $\mathbf{C_1}$ of the manipulator are obtained as;

$$\mathbf{x = L_1 \cos\theta_1 + \frac{1}{2}[L_0 + L_1(\cos\theta_4 - \cos\theta_1)]}$$
$$\mathbf{- L_1(\sin\theta_4 - \sin\theta_1)\sqrt{\frac{L_2^2}{q^2} - \frac{1}{4}}} \tag{7}$$
$$\mathbf{y = L_1 \cos\theta_1 + \frac{L_1}{2}(\sin\theta_4 - \sin\theta_1)]}$$
$$\mathbf{+ [L_0 + L_1(\cos\theta_4 - \cos\theta_1)]\sqrt{\frac{L_2^2}{q^2} - \frac{1}{4}}}$$

where

$$\mathbf{q^2 = L_0^2 + 2L_0L_1(\cos\theta_4 - \cos\theta_1) + 2L_1^2[1 - \cos(\theta_1 - \theta_4)]}$$

The inverse kinematics solution of the manipulator based on Sylvester's dialytic elimination method has been reported before.[8]

Equation (7) can be expressed as;

$$\mathbf{F(\Theta, X) = 0} \tag{8}$$

where $\mathbf{F}$ is two dimensional for the problem at hand and is a function of inputs and the outputs. Taking the first time derivative of Equation (8) leads to the relationship between the input velocity vector, $\mathbf{\dot{\Theta} = [\dot{\theta}_1, \dot{\theta}_4]^T}$, and the output velocity vector, $\mathbf{\dot{X} = [\dot{x}, \dot{y}]^T}$, as follows;

$$\mathbf{\frac{\partial F}{\partial \Theta}\dot{\Theta} + \frac{\partial F}{\partial X}\dot{X} = 0}$$

$$\Rightarrow \left.\begin{aligned}\mathbf{a_{11}\dot{\theta}_1 + a_{12}\dot{\theta}_4 + b_{11}\dot{x} + b_{12}\dot{y} = 0}\\ \mathbf{a_{12}\dot{\theta}_1 + a_{22}\dot{\theta}_4 + b_{21}\dot{x} + b_{22}\dot{y} = 0}\end{aligned}\right\} \Rightarrow \mathbf{A\dot{\Theta} + B\dot{X} = 0} \tag{9}$$

where $\mathbf{A}$ and $\mathbf{B}$ are configuration dependent $\mathbf{2 \times 2}$ forward and inverse Jacobian matrices, respectively. From Equation (9), the generalised velocity equation based on the manipulator Jacobian matrix $\mathbf{J}$ is formed as;

$$\mathbf{\dot{X} = \underbrace{-B^{-1}A}_{J}\dot{\Theta} \Rightarrow \dot{X} = J\dot{\Theta}} \tag{10}$$

where

$$\mathbf{a_{11} = 0.5KL_1q^4 \sin\theta_1 - (2EL_0 + q^4K^2)L_1 \cos\theta_1}$$
$$\mathbf{- 2EL_1^2 \sin(\theta_1 - \theta_2)}$$
$$\mathbf{a_{12} = 0.5KL_1q^4 \sin\theta_4 + (2EL_0 + q^4K^2)L_1 \cos\theta_4}$$
$$\mathbf{+ 2EL_1^2 \sin(\theta_1 - \theta_4)}$$
$$\mathbf{a_{21} = -0.5KL_1q^4 \cos\theta_1 - q^4K^2L_1 \sin\theta_1 + 2FL_0L_1 \cos\theta_1}$$
$$\mathbf{+ 2FL_1^2 \sin(\theta_1 - \theta_4)}$$
$$\mathbf{a_{22} = K^2L_1q^4 \sin\theta_4 - (2FL_0 + 0.5q^4K)L_1 \cos\theta_4}$$
$$\mathbf{- 2FL_1^2 \sin(\theta_1 - \theta_4)}$$

and

$$\mathbf{b_{11} = -Kq^2,\ b_{12} = 0,\ b_{21} = 0,\ b_{22} = -Kq^2}$$

$$\mathbf{K = \sqrt{\frac{L_2^2}{q^2} - \frac{1}{4}},\ E = 0.5L_1L_2^2 \sin(\theta_4 - \theta_1)},$$

$$\mathbf{F = 0.5L_2^2[L_0 + L_1(\cos\theta_4 - \cos\theta_1)]}.$$

In our previous study,[8] the expressions for the determinants of the $\mathbf{A}$ and $\mathbf{B}$ Jacobian matrices are used to generate singularity contours for a given set of geometric parameters of the manipulator, and the specified range of motion for θ_1 and θ_4.

3. STATIC FORCE ANALYSIS

The aim of static force analysis is to obtain analytical expressions for the forces acting on the bearings A, B, C, D, E while the manipulator is in equilibrium in all of the manipulator configurations. It is assumed that there is no friction in the system. The forces acting at the joints are found from the equilibrium equations written for each link of the manipulator. The equations describing the forces acting on links 1, 2, 3, and 4 are put in a matrix-vector form;

$$\mathbf{[M][F] = [F_g]} \tag{11}$$

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where **M**, **F**, and $\mathbf{F_g}$ denote the matrix of known mechanism dimensional parameters and joint angles, the vector of unknown forces, and the gravitational force vector, respectively. As seen from the determinant of **M** given below, it is singular when $\theta_4 = \frac{\theta_2+\theta_3}{2} + \frac{\pi}{2}(\mathbf{2k+1})$ for $\mathbf{k=0, 1}$ . . . or $\theta_1=\theta_2$ or $\theta_2=\theta_3$.

$$|\mathbf{M}| = -\mathbf{2L_2^2L_4^2 \cos}\left(\theta_4 - \left(\frac{\theta_2+\theta_3}{2}\right)\right)$$
$$\times \mathbf{sin}(\theta_1-\theta_2)\mathbf{sin}\left(\frac{\theta_2-\theta_3}{2}\right) \qquad (12)$$

Assuming that the matrix **M** is non-singular throughout the operation range of the manipulator. Equation (11) is solved for the unknown forces, $\mathbf{F_{Ax}, F_{Ay}, F_{Bx}, F_{By}, F_{Cx}, F_{Cy}, F_{Dx}, F_{Dy}}$, for which mathematical expressions are given in Appendix A.

4. STAGE 1: LINK LENGTH OPTIMISATION FROM KINEMATIC ISOTROPY

The parameters defining the geometry of the manipulator depicted in Figure 1 are the link lengths $\mathbf{L_0, L_1, L_2, L_3, L_4}$. It must be recalled that the configuration dependent Jacobian matrix of a robot manipulator is involved in the force and motion transfer between the actuators and the end effector. Depending on the Jacobian matrix either relating the end effector velocity to the actuator velocity vector and vice versa, the motion and force errors propagated from the actuators to the end effector is bounded by the condition number of the Jacobian matrix.[30,31] The condition number of the Jacobian matrix is not only a measure of how accurate the force and motion transfer between the actuators and the end effector is accomplished, but also the measure of the ease with which the manipulator can arbitrarily change its pose (position and orientation) and apply forces in arbitrary directions. Because of this fact, the condition number of the manipulator Jacobian matrix has been recognized as an optimisation criterion in structural synthesis of robot manipulators even though it is not the only synthesis criterion.[32,33] The Jacobian matrix **J** relating the velocity vector $\dot{\mathbf{X}}$ of the end effector to the velocity vector $\dot{\Theta}$ of the actuated joints is;

$$\dot{\mathbf{X}} = \mathbf{J}\dot{\Theta} \qquad (13)$$

The error in the end effector velocity is bounded by

$$\mathbf{C(J)}\frac{\|\delta\dot{\Theta}\|}{\|\dot{\Theta}\|} \geq \frac{\|\delta\dot{\mathbf{X}}\|}{\|\dot{\mathbf{X}}\|}$$

where $\delta\dot{\mathbf{X}}$ and $\delta\dot{\Theta}$ are the error vectors for the end effector and actuator velocities, respectively, **C(J)** is the condition number of the Jacobian matrix given by $\|\mathbf{J}\|\|\mathbf{J}^{-1}\|$, and $\|.\|$ denotes norm. It must be emphasised that the condition number varies in the range of $\mathbf{1.0 \leq C(J) < \infty}$, and the manipulator configurations resulting in $\mathbf{C(J) = 1.0}$ are known as isotropic configurations where the manipulator has the best kinematic accuracy, dexterity and singularity avoidance. This follows that the propagation of uncertainties from joint positions to Cartesian positions is minimum. It is the object of the optimisation procedure carried out in this section to minimize the overall deviation of the Jacobian condition number from the isotropic configuration. As the condition number of Jacobian matrix changes with the manipulator configuration, it is appropriate to decide on an objective function covering all parts of the workspace while searching for the synthesis parameters satisfying the objective function and some synthesis constraints. With this in mind, a sum-squared deviation of the Jacobian condition number from that of the isotropic configuration throughout the workspace of the manipulator is adopted as the objective function (**OF**):

$$\mathbf{OF} = \frac{\mathbf{1}}{\mathbf{n}}\sqrt{\sum_{\mathbf{i,j=1}}^{\mathbf{n}} [\mathbf{1 - C(J)}]^{\mathbf{2}}} \qquad (14)$$

where **i** and **j** are the number of increments for θ_1 and θ_4, respectively. The goal of the optimisation is to determine the numerical values of the link lengths of the manipulator minimizing the objective function and satisfying the constraints given in the next subsection.

4.1. Constraints

In order to limit the solution, the objective function is subjected to the following constraints:

(i) The link lengths are non-zero and positive quantities,
(ii) Although a designer is free to choose any ratio among mechanism link lengths, the practical ratios of $\frac{\mathbf{L_0}}{\mathbf{L_1}} \leq \mathbf{4.0}$ and $\frac{\mathbf{L_0}}{\mathbf{L_2}} \leq \mathbf{2.0}$ are chosen to guarantee proportionate link lengths making mechanism production and testing easy and error-free.
(iii) The mechanism must obey the assemblability condition,[5,34] which requires that $\mathbf{L_0 < L_m + L_{m-1} + \cdots L_1}$, provided that $\mathbf{L_0}$ and $\mathbf{L_m}$ are the longest and shortest links of the mechanism, respectively. It must be noted that when $\mathbf{L_0 = L_m + L_{m-1} + \cdots L_1}$, all the links are collinear and the mechanism is a change-point mechanism. This is known as zero mobility condition.[35] The mechanism will not move unless a special precaution is taken to pass through the change-position configuration.
(iv) The transmission angle ψ, which is the angle between $\mathbf{L_2}$ and $\mathbf{L_3}$ of Figure 1, must be $\mathbf{50°} \leq \psi \leq \mathbf{130°}$, ideally be $\mathbf{90°}$, for effective force transmission from two coupler links to the output point **C**. Especially when its deviation from $\mathbf{90°}$ is greater than $\mp\mathbf{50^0}$, it causes unacceptable higher acceleration and jerk, and objectionable noise at high speeds. It must also be continuous between the desirable ranges.[1,5] Referring to Figure 2(a), it is mathematically expressed as;

$$\cos\psi = 1 - \frac{q^2}{2L_2^2} \tag{15}$$

(v) For the sake of generality, it is required that the links connected to the active joints make unlimited rotations. This constraint is based on the "the theorem of full rotatability of linkages".[34,35] which states that if the sum of the longest link length $\mathbf{L_0}$ and link $\mathbf{L_k}$ is smaller than the sum of the lengths of the rest of the mechanism links, the link $\mathbf{L_k}$ will make unlimited rotations. For the manipulator at hand, the links connected to the active joints /actuators are link **1** and link **4**. Therefore, the full rotatability constraint for the mechanism in Figure 2(a) requires $\mathbf{L_0+L_1<L_2+L_3+L_4}$ and $\mathbf{L_4+L_4<L_1+L_2+L_3}$. However, for a parallel five-bar manipulator, these two conditions reduce to $\mathbf{L_0<2L_2}$.

When $\mathbf{L_0+L_1=L_2+L_3+L_4}$ and $\mathbf{L_0+L_4=L^1+L_2+L_3}$, the centerlines of all links are collinear, the mechanism is a change-point mechanism.

On the other hand, when $\mathbf{L_0+L_1>L_2+L_3+L_4}$ and $\mathbf{L_0+L_4>L_1+L_2+L_3}$, the mechanism is a double-rocker mechanism; the active joints cannot make a full rotation.[35] Each of the active joints can move between two limit positions shown in Figure 3. Such a five bar mechanism can be called *quadruple-rocker* mechanism – drawing the same analogy from a double-rocker-four-bar mechanism known as triple-rocker mechanism.[37]

Hence, the problem is formulated as a constrained nonlinear optimisation problem. A computer program based on a sequential quadratic programming method is prepared in MATLAB to accomplish the constrained minimization of the **OF** as a function of the synthesis parameters, starting with an initial value for each parameter.

4.2. Optimisation results

For $\mathbf{30^0 \le \theta_1 \le 390^0}$ and $\theta_4=\sigma\theta_1$, many optimisation trials with different initial values for the synthesis parameters were conducted. It must be noted that Frobenius norm defined as $\sqrt{\mathbf{sum(diag(J^TJ))}}$ is used for the norm of the Jacobian matrix. For every trial, the initial value of only one of the parameters was varied with a step size of **0.1** while the others were kept constant. The range of initial values for the parameters was $\le\mathbf{L_0, L_1, L_2\le 2.0}$ and $\sigma=\mathbf{0.8}$. Different initial values resulted in different solutions satisfying the objective function and the synthesis constraints. Sets of optimised link lengths are given in Table I. The variation of the transmission angle and the condition number of the Jacobian matrix for the solution number 1 given in Table I are depicted in Figure 4. The three configurations of the resulting five bar parallel manipulator are shown in Figure 5.

The results given in Table I are in agreement with "Five-Bar Grashof Criteria" proposed by Ting,[35] which states that when $\mathbf{L_{max}+L_{min\,1}+L_{min\,2}>L_{other\,1}+L_{other\,2}}$, the mechanism has at most two cranks. This is also required by Constraint V given above; while $\mathbf{L_1}$ and $\mathbf{L_4}$ are making full rotations, the transmission angle is satisfying Constraint IV, as shown in the top plot of Figure 4. Other linkages satisfying the "Five-Bar Grashof Criteria" can be obtained by introducing appropriate constraints into the optimisation procedure.

In order to demonstrate the effectiveness of the optimisation procedure, we chose a set of manipulator link lengths

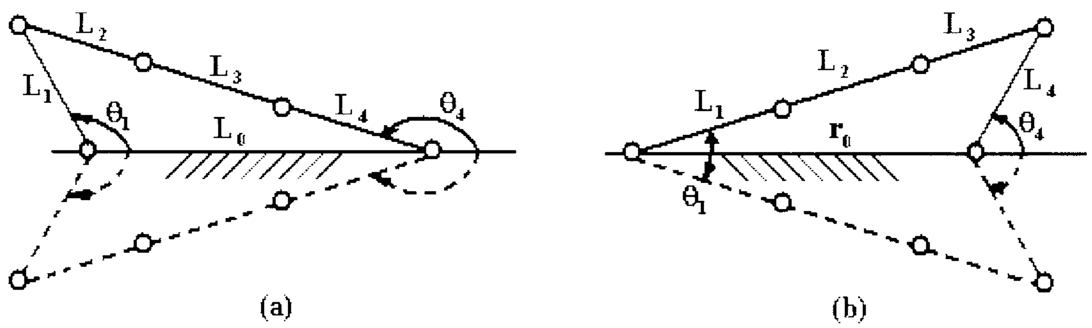

Fig. 3. Limit positions of the mechanism for links **1** and **4** connected to the ground link.

Table I. The solutions obtained from the constrained optimisation of the mechanism. The units for link lengths are arbitrary length units.

Solution No.	Initial Values (L_0, L_1, L_2)	Optimised Values (L_0, L_1, L_2)	Transmission Angles (deg.) (ψ_{min}, ψ_{max})	Overall Objective Function
1	1.7, 1.4, 1.7	8.0745, 3.2950, 5.4365	51.7765, 120.1068	1.5594
2	1.6, 1.2, 1.5	13.4005, 4.305, 9.0	60.3734, 114.6718	2.2431
3	1.0, 1.4, 1.0	10.9380, 2.9036, 6.9654	70.0873, 120.8616	1.844
4	1.7, 1.4, 1.6	29.9748, 9.6113, 23.9729	50.0023, 89.9458	1.3218

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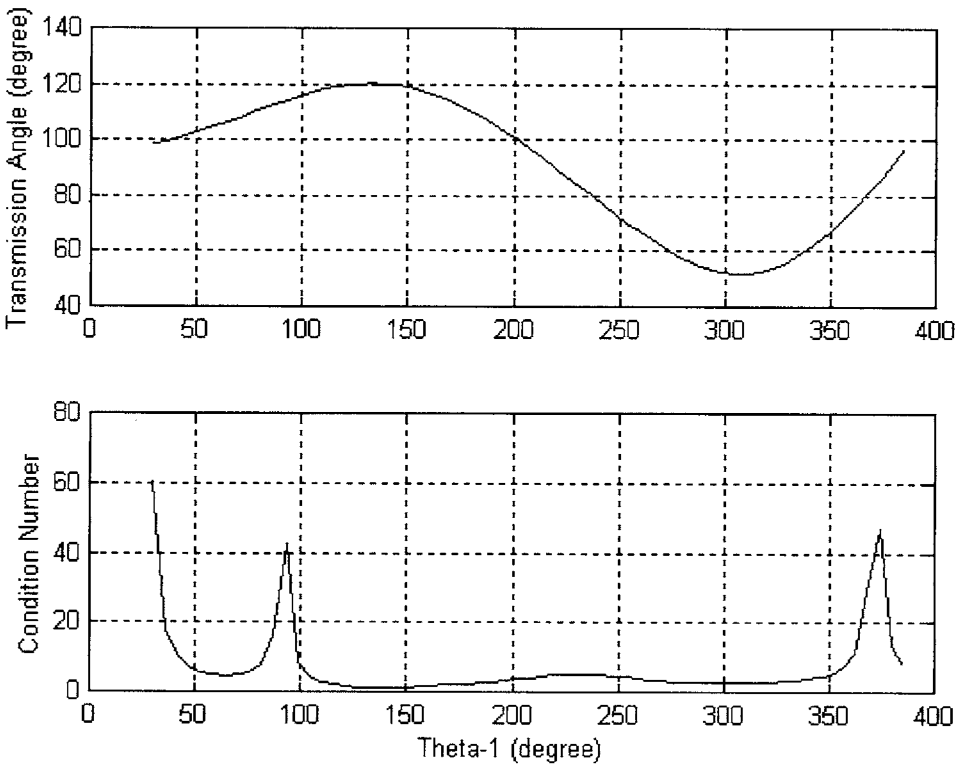

Fig. 4. Variation of the transmission angle (top plot), and of the Jacobian matrix condition number (bottom plot) with $\boldsymbol{\theta_1}$.

(**L_0=9.0, L_1=4.0, L_2=6.0**) satisfying all of the constraints expressed, except Constraint IV related to the transmission angle. The transmission angle and **OF** are calculated from Equation (14) and Equation (15), respectively, and are shown in Figure 6. Note that although the transmission angle almost satisfies Constraint IV by chance (**ψ_{min}=48.8539^0, ψ_{max}=124.8693^0**), the overall deviation of the Jacobian matrix condition number from the isotropic configuration is quite high (**OF=220.4423**) when compared to those of the solutions given in Table I. This follows that although the mechanism satisfy all the synthesis constraints, its accuracy, dexterity and singularity avoidance capabilities are lower than the ones synthesised using the objective function and the synthesis constraints.

5. STAGE 2: MASS DISTRIBUTION OPTIMISATION FROM FORCE BALANCING

Mass distributions of the links of an articulated mechanism are as important as its geometry for a good global performance. The aim of this stage of the optimum

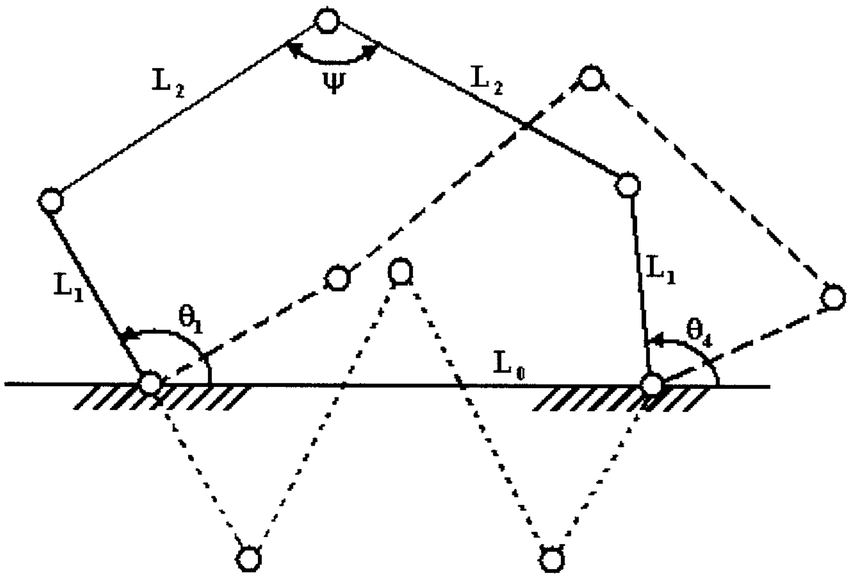

Fig. 5. The manipulator with optimised link lengths given in the first row of Table 1; ". ." is for $\boldsymbol{\theta_1=120^0}$, $\boldsymbol{\theta_4=96^0}$, $\boldsymbol{\psi=119.3881^0}$, "- - -" is for $\boldsymbol{\theta_1=30^0}$, $\boldsymbol{\theta_4=24^0}$, $\boldsymbol{\psi=98.4969^0}$, and ". . .."is for $\boldsymbol{\theta_1=300^0}$, $\boldsymbol{\theta_4=240^0}$, $\boldsymbol{c=52.1535^0}$.

https://doi.org/10.1017/S0263574703005216 Published online by Cambridge University Press

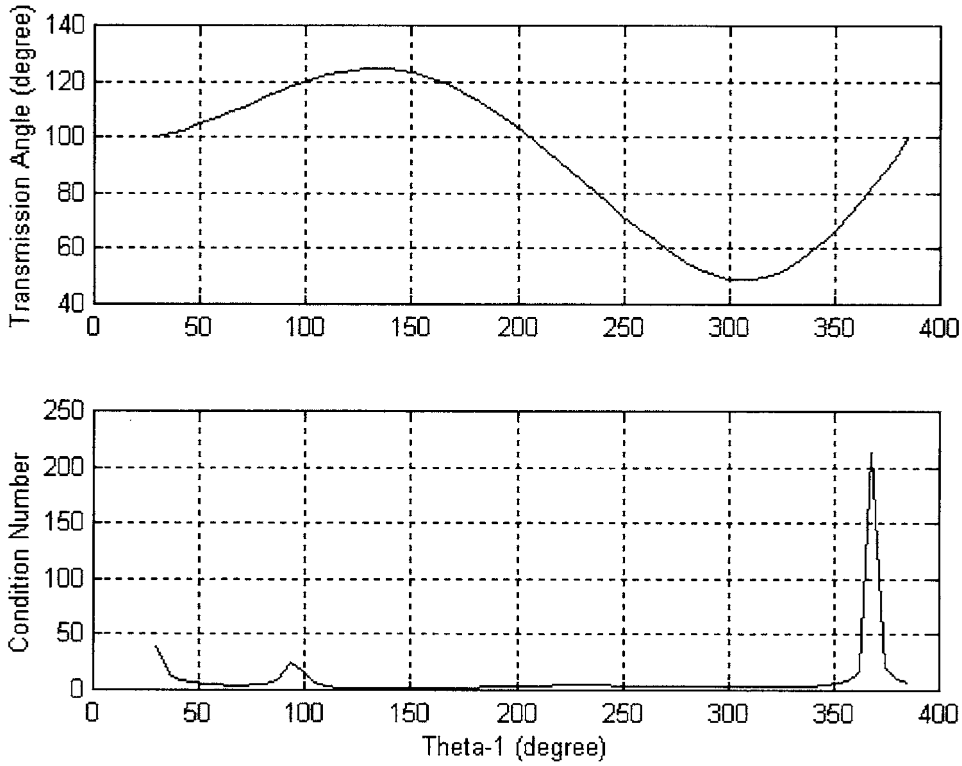

Fig. 6. Variation of the transmission angle (top plot), and of the Jacobian matrix condition number (bottom plot) with $\boldsymbol{\theta_1}$. Link lengths are chosen arbitrarily, without considering the objective function.

synthesis methodology is to determine mass distribution parameters of the manipulator for force balancing, assuming that the links lengths determined from Stage 1 are the specified manipulator parameters. Although the aim of a typical balancing procedure is to minimize the forces transmitted to the ground, that of the optimisation procedure presented in this section is to minimize all the reaction forces.

5.1. Balancing conditions

A force-balanced mechanism requires that its potential energy consisting of gravitational and elastic potential energies is constant for all configurations of the mechanism.[17] The total potential energy is formulated as

$$\mathbf{V} = \mathbf{M_T g R_g} + \mathbf{V_e} \tag{16}$$

where $\mathbf{M_T}$ is the total mass of the moving links, $\mathbf{R_g}$ is the vertical component of the position vector $\vec{\mathbf{R}}$, given by Equation (20), describing the mass center of the mechanism, and $\mathbf{V_e}$ is the elastic potential energy of the system, which is not applicable for the balancing approach adopted in this paper.

Using the notation given in Figure 1, the position vector $\vec{\mathbf{R}}$ describing the mass center of the mechanism is[38]

$$\vec{\mathbf{R}} = \frac{1}{\mathbf{M_T}} \sum_{i=0}^{4} \mathbf{m_i} \vec{\mathbf{r}}_i \tag{17}$$

where $\vec{\mathbf{r}}_i$ is the position vector describing the mass center $\mathbf{G_i}$ of the $\mathbf{i^{th}}$ moving link with a mass of $\mathbf{m_i}$ with respect to the reference point $\mathbf{A}$. The individual position vectors are expressed in complex numbers as

$$\begin{aligned}
\vec{\mathbf{r}}_1 &= \mathbf{R_1 e^{j(\theta_1+\alpha_1)}}, \\
\vec{\mathbf{r}}_2 &= \mathbf{L_1 e^{j\theta_1} + R_2 e^{j(\theta_2+\alpha_2)}}, \\
\vec{\mathbf{r}}_3 &= \mathbf{L_0 e^{j\alpha_0} + L_4 e^{J\theta_4} + R_3 e^{j(\theta_3+\alpha_3)}}, \\
\vec{\mathbf{r}}_4 &= \mathbf{L_0 e^{j\theta_0} + R_4 e^{j(\theta_4+\alpha_4)}},
\end{aligned} \tag{18}$$

The exponential terms are related to each other by the loop closure equation

$$\mathbf{L_1 e^{j\theta_1} + L_2 e^{j\theta_2} = L_0 e^{j\alpha_0} + L_3 e^{j\theta_3} + L_4 e^{j\theta_4}} \tag{19}$$

The exponential term $\mathbf{e^{j\theta_3}}$ (or $\mathbf{e^{j\theta_2}}$) is extracted from Equation (19), and then substituted into Equation (18). Consequently, the position vectors in Equation (18) can be put into Equation (17) from which the position vector describing the overall mass center of the mechanism is obtained as

$$\vec{\mathbf{R}} = \frac{\mathbf{1}}{\mathbf{M_T}} \left[\begin{array}{l}
\mathbf{e^{j\theta_1}} \left(\mathbf{m_1 R_1 e^{j\alpha_1}} + \dfrac{\mathbf{m_3 R_3 L_1}}{\mathbf{L_3}} \mathbf{e^{j\alpha_3}} + \mathbf{m_2 L_1} \right) \\
+ \mathbf{e^{j\theta_2}} \left(\mathbf{m_2 R_2 e^{j\alpha_2}} + \dfrac{\mathbf{m_3 R_3 L_2}}{\mathbf{L_3}} \mathbf{e^{j\alpha_3}} \right) \\
+ \mathbf{e^{j\theta_4}} \left(\mathbf{m_4 R_4 e^{j\alpha_4}} - \dfrac{\mathbf{m_3 R_3 L_4}}{\mathbf{L_3}} \mathbf{e^{j\alpha_3}} + \mathbf{m_3 L_4} \right) \\
+ \mathbf{L_0 e^{j\alpha_0}} \left(- \dfrac{\mathbf{m_3 R_3}}{\mathbf{L_3}} \mathbf{e^{j\alpha_3}} + \mathbf{m_3} + \mathbf{m_4} \right)
\end{array} \right] \tag{20}$$

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When the component of $\vec{\mathbf{R}}$ associated with the gravitational acceleration is substituted into Equation (16), the total potential energy of the system becomes

$$\mathbf{V}=\mathbf{g}\left[\begin{array}{l}\sin\theta_1\left(\mathbf{m_1R_1}e^{j\alpha_1}+\dfrac{\mathbf{m_3R_3L_1}}{\mathbf{L_3}}e^{j\alpha_3}+\mathbf{m_2L_1}\right)\\ +\sin\theta_2\left(\mathbf{m_2R_2}e^{j\alpha_2}+\dfrac{\mathbf{m_3R_3L_2}}{\mathbf{L_3}}e^{j\alpha_3}\right)\\ +\sin\theta_4\left(\mathbf{m_4R_4}e^{j\alpha_4}-\dfrac{\mathbf{m_3R_3L_4}}{\mathbf{L_3}}e^{j\alpha_3}+\mathbf{m_3L_4}\right)\\ +\mathbf{L_0}\sin\alpha_0\left(-\dfrac{\mathbf{m_3R_3}}{\mathbf{L_3}}e^{j\alpha_3}+\mathbf{m_3}+\mathbf{m_4}\right)\end{array}\right] \tag{21}$$

If the coefficients of $\sin\theta_z$ (for $\mathbf{z=1, 2, 4}$) are zero, the potential energy of the system will be constant. After necessary mathematical operations, the following force balancing conditions are derived

$$\mathbf{g}\left(\mathbf{m_1R_1}\cos\alpha_1+\frac{\mathbf{m_3R_3L_1}}{\mathbf{L_3}}\cos\alpha_3+\mathbf{m_2L_1}\right)=0 \tag{22}$$

$$\mathbf{g}\left(\mathbf{m_1R_1}\sin\alpha_1+\frac{\mathbf{m_3R_3L_1}}{\mathbf{L_3}}\sin\alpha_3\right)=0 \tag{23}$$

$$\mathbf{g}\left(\mathbf{m_2R_2}\cos\alpha_2+\frac{\mathbf{m_3R_3L_2}}{\mathbf{L_3}}\cos\alpha_3\right)=0 \tag{24}$$

$$\mathbf{g}\left(\mathbf{m_2R_2}\sin\alpha_2+\frac{\mathbf{m_3R_3L_2}}{\mathbf{L_3}}\sin\alpha_3\right)=0 \tag{25}$$

$$\mathbf{g}\left(\mathbf{m_4R_4}\cos\alpha_4-\frac{\mathbf{m_3R_3L_4}}{\mathbf{L_3}}\cos\alpha_3+\mathbf{m_3L_4}\right)=0 \tag{26}$$

$$\mathbf{g}\left(\mathbf{m_4R_4}\sin\alpha_4-\frac{\mathbf{m_3R_3L_4}}{\mathbf{L_3}}\sin\alpha_3\right)=0 \tag{27}$$

From Equations (24) and (25),

$$\alpha_2=\alpha_3 \text{ and } \mathbf{R_3}=-\left(\frac{\mathbf{m_2}}{\mathbf{m_3}}\right)\left(\frac{\mathbf{L_3}}{\mathbf{L_2}}\right)\mathbf{R_2} \tag{28}$$

This is an important observation for the mass distribution of links 2 and 3, which have to obey Equation (28) for force balancing. Depending on the mass ratio and the link length ratio and $\mathbf{R_2}\neq\mathbf{0.0}$, $\mathbf{R_3}$ is evaluated in terms of negative $\mathbf{R_2}$, which implies that one of them is always negative, i.e., the locations of the mass centers for link 2 and link 3 are separated from each other by $\mathbf{180^0}$. This might restrict the usable workspace of the manipulator. However, this problem can be avoided by imposing tight constraints on the sizes of the radial distances $\mathbf{R_2}$ and $\mathbf{R_3}$.

The remaining four conditions (Equations (22), (23), (26), (27)) are taken as the constraints which must be satisfied by the balancing parameters while minimizing an optimisation function described in the next subsection.

5.2. *Objective function*

Similar to the optimisation functions used before by others[19–21,23,27] for optimum balancing of single degree freedom mechanisms, a mean-square root of the sum-squared discrete values of all reaction forces in the manipulator is adopted as the objective function (**OF**)

$$\mathbf{OF}=\frac{\mathbf{1}}{\mathbf{n}}\sqrt{\sum_{\mathbf{i=1}}^{\mathbf{n}}(\mathbf{F_A^2}+\mathbf{F_B^2}+\mathbf{F_C^2}+\mathbf{F_D^2}+\mathbf{F_E^2})} \tag{29}$$

where **n** is the number of the discrete values into which the operation range of manipulator divided. The goal of the optimisation is to determine the numerical values of $\mathbf{m_i}$, R_i, α_i (for $\mathbf{i=1, 2, 3, 4}$), minimizing the objective function and satisfying the constraints given in the next subsection.

5.3. *Constraints*

The objective function is subjected to the following constraints, in addition to the constraints imposed by Equations (22), (23), (26–28):

1. $-\mathbf{L_z}\leq\mathbf{R_z}\leq\mathbf{L_z}$ for $\mathbf{z=1\cdots4}$,
2. $\mathbf{1\leq m_z\leq5}$ for $\mathbf{z=1}$, $\mathbf{4}$ and $\mathbf{1\leq m_z\leq3}$ for $\mathbf{z=2, 3}$,
3. $\mathbf{0\leq\alpha_z\leq180^0}$ for $\mathbf{z=1\cdots4}$,

Hence, force balancing of the manipulator is also formulated as a constrained nonlinear optimisation problem. A computer program in MATLAB is employed to accomplish the constrained minimization of the **OF** as a function of the balancing parameters, starting with an initial value for each parameter.

5.4. *Numerical results*

The link lengths $\mathbf{L_0=3*8.0745=24.2235}$, $\mathbf{L_1=3*3.2950}$, and $\mathbf{L_2=3*5.4365}$ given in the first row of Table I are utilised here while determining the optimum values of mass distribution parameters. With reference to Equation (28), $\mathbf{m_3=m_2}$ gives $\mathbf{R_3=-R_2}$. This has reduced the number of balancing parameters to nine. For $\mathbf{30^0\leq\theta_1\leq390^0}$, $\mathbf{\theta_4=0.8\theta_1}$, with step sizes of **0.1 rad** (i.e., $\mathbf{n=63}$), and $\mathbf{\alpha_0=0^0}$, a number of optimisation trials with different initial values for the balancing parameters were conducted. Different initial values provided different solutions satisfying the objective function and the balancing constraints. Sets of optimised balancing parameters together with the initial values used are given in Table II. These values describe the mass distribution ($\mathbf{m_z}$, $\mathbf{R_z}$, α_z) parameters for optimum force balancing. The numerical values of mass distribution

https://doi.org/10.1017/S0263574703005216 Published online by Cambridge University Press

Table II. The solutions obtained from the constrained optimisation of force balancing. The units are arbitrary length and mass units. The units for the angles are in radians.

Solution No.	Initial Values $\begin{pmatrix} R_1, R_2, R_4, m_1, \\ m_2, m_1, \alpha_1, \alpha_2, \alpha_4 \end{pmatrix}$	Optimised Values $\begin{pmatrix} R_1, R_2, R_4, m_1 \\ m_2, m_4, \alpha_1, \alpha_2, \alpha_4 \end{pmatrix}$	Objective Function
1	**$L_1/6$, $L_2/6$, $L_4/6$, 3.0, 1.0, 3.0, $\pi/10$, $\pi/10$, $\pi/10$**	**9.8850, −2.8853, 8.1363, 1.1769, 1.0, 1.0, π, 2π, π**	**4.0914**
2	**$L_1/10$, $L_2/10$, $L_4/10$, 1.0, 1.0, 1.0, 0.0, 0.0, 0.0**	**−9.8850, 0.0, −9.8850, 1.0, 1.0, 1.0, 0.0, 0.0, 0.0**	**3.9084**

parameters change with their initial values as well as the chosen link lengths, and thus, they do not necessarily represent a global minimum.

The reaction forces acting at bearings/joints A, B, C, D, E, were found to have constant magnitudes of 23.0912, 11.5456, 1.7355, 8.0746, and 17.8846, respectively, for the solution given in the first row of Table II.

In order to demonstrate the effectiveness of the optimisation procedure for mass distribution parameters, the numerical values of the balancing parameters* given for a planar parallel manipulator (basically the same manipulator considered in this study) by Jean and Gosselin[14] are used to calculate the bearing forces for an arbitrary $\mathbf{L=10}$ **units** and $\mathbf{m=2}$ **units**. It is determined that although the parameters satisfy the necessary balancing conditions expressed by Equations (22), (23), (26–28), the average values of the forces acting at the bearings A, B, C, D, E, (22.6511, 21.3253, 22.6570, 71.0258, and 81.1819, respectively) are significantly higher than those obtained from the optimisation procedure presented above.

As given in Equation (12), when $\theta_4=\dfrac{\theta_2+\theta_3}{2}+\dfrac{\pi}{2}(\mathbf{2k+1})$ for $\mathbf{k=0, 1\ldots}$ or $\theta_1=\theta_2$ or $\theta_2=\theta_3$, the forces in the mechanism tend to approach infinity. This is an undesirable situation that must be avoided by selecting proper link lengths during the synthesis stage. When the mechanism is in these singular configurations ($\theta_1=\theta_2$ or $\theta_2=\theta_3$), the transmission angle requirement is also violated. This problem was considered while determining the link lengths.

6. CONCLUSIONS

We have presented an optimum synthesis methodology for planar parallel manipulators. Once the parameters describing the geometry of the manipulator have been determined from an optimisation procedure based on kinematic isotropy, the remaining synthesis parameters, the mass distribution parameters, have been obtained from a subsequent optimisation procedure based on force balancing. The resulting manipulator is expected: (i) to have a good global performance based on high kinematic accuracy, dexterity and singularity avoidance capabilities with a high mechanical advantage, (ii) to require less powerful and smaller actuators to move its links, and (iii) to transmit less shaking force and moment to the ground. Optimisation results indicate that the proposed optimisation approach is systematic, versatile and easy to implement for the optimum synthesis of a parallel manipulator and other kinematic chains. This work contributes to previously published work from the point of view of being a simple and systematic approach to the optimum synthesis of parallel manipulators, which is currently lacking in the literature. Future work includes optimum synthesis of other planar parallel manipulators articulated with prismatic and revolute joints, using multi-objective functions based on kinematic isotropy, dynamic synthesis (e.g., minimization of angular accelerations) and *dynamic balancing*. These objectives will be formulated into a convex optimisation problem, which can be solved using either heuristic search algorithms such as genetic algorithms or interval analysis to obtain a global optimum solution.

* $\mathbf{R_1=0.577*L}$, $\mathbf{R_2=-R_3=1.155*L}$, $\mathbf{R_4=-1.041*L}$, $\alpha_1=\mathbf{90^0}$, $\alpha_2=\alpha_3=\mathbf{30^0}$, $\alpha_4=\mathbf{16.102^0}$.

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APPENDIX A: BEARING/JOINT FORCES

$$F_{Ax}=\frac{1}{L_2L_4\sin(\theta_1-\theta_2)}\left\{\begin{array}{l}gL_2m_1R_1\cos(\alpha_1+\theta_1)\cos\theta_2\\+L_4\cos\theta_1[gL_2m_2\cos\theta_2-gR_2m_2\cos(\alpha_2+\theta_2)]\end{array}\right\}\tag{A1}$$

$$F_{Ay}=\frac{1}{L_2L_4\sin(\theta_1-\theta_2)}\left\{\begin{array}{l}-gL_4m_2R_2\sin\theta_1\cos(\alpha_2+\theta_2)+gL_2R_1m_1\sin\theta_2\cos(\alpha_1+\theta_1)\\+L_2L_4\sin\theta_1\cos\theta_2[gm_1+gm_2]-gL_2L_4m_1\cos\theta_1\sin\theta_2\end{array}\right\}\tag{A2}$$

$$F_{Bx}=\frac{1}{L_2L_4\sin(\theta_1-\theta_2)}\left\{\begin{array}{l}-gL_2m_1R_1\cos(\alpha_1+\theta_1)\cos\theta_2\\+L_4\cos\theta_1[-gL_2m_2\cos\theta_2+gR_2m_2\cos(\alpha_2+\theta_2)]\end{array}\right\}\tag{A3}$$

$$F_{By}=\frac{1}{L_2L_4\sin(\theta_1-\theta_2)}\left\{\begin{array}{l}gL_4m_2R_2\sin\theta_1\cos(\alpha_2+\theta_2)-L_2L_4gm_2\sin\theta_1\cos\theta_2\\-gL_2R_1m_1\sin\theta_2\cos(\alpha_1+\theta_1)\end{array}\right\}\tag{A4}$$

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$$F_{Cx}=\frac{1}{L_2L_4\sin(\theta_1-\theta_2)}\left\{\begin{array}{l}L_4\cos\theta_1[-gL_2m_2\cos\theta_2+gR_2m_2\cos(\alpha_2+\theta_2)]\\-L_2gR_1m_1\cos(\alpha_1+\theta_1)\cos\theta_2\end{array}\right\} \tag{A5}$$

$$F_{Cy}=\frac{1}{L_2L_4\sin(\theta_1-\theta_2)}\left\{\begin{array}{l}gL_4m_2R_2\sin\theta_1\cos(\alpha_2+\theta_2)\\-L_2\sin\theta_2[gL_4m_2\cos\theta_1+gR_1m_1\cos(\alpha_1+\theta_1)]\end{array}\right\} \tag{A6}$$

$$F_{Dx}=\frac{g}{L_2L_4[\sin(\theta_4-\theta_2)-\sin(\theta_4-\theta_3)]}\left[\begin{array}{l}L_2m_4R_4\cos(\alpha_4+\theta_4)(\cos\theta_2-\cos\theta_3)\\+L_4\cos\theta_4\left[\begin{array}{l}L_2m_3\cos\theta_2+R_2m_2\cos(\alpha_2+\theta_2)\\-L_2m_3\cos\theta_3+R_3m_3\cos(\alpha_3+\theta_3)\end{array}\right]\end{array}\right] \tag{A7}$$

$$F_{Dy}=\frac{g}{L_2L_4[\sin(\theta_4-\theta_2)-\sin(\theta_4-\theta_3)]}\left[\begin{array}{l}L_2L_4m_3\cos\theta_2\sin\theta_4+L_4m_2R_2\sin\theta_4\cos(\alpha_2+\theta_2)\\-L_2L_4m_3\cos\theta_3\sin\theta_4+L_4m_3R_3\sin\theta_4\cos(\alpha_3+\theta_3)\\+L_2m_4R_4\sin\theta_2\cos(\alpha_4+\theta_4)\\-L_2m_4R_4\sin\theta_3\cos(\alpha_4+\theta_4)\end{array}\right] \tag{A8}$$

$$F_{Ex}=F_{Dx} \tag{A9}$$

$$F_{Ey}=\frac{g}{L_2L_4[\sin(\theta_4-\theta_{2\_}-\sin(\theta_4-\theta_3)]}\left[\begin{array}{l}L_2L_4(m_3+m_4)\cos\theta_2\sin\theta_4+L_4m_2R_2\sin\theta_4\cos(\alpha_2+\theta_2)\\-L_2L_4(m_3-m_4)\cos\theta_3\sin\theta_4+L_4m_3R_3\sin\theta_4\cos(\alpha_3+\theta_3)\\+L_2m_4L_4\cos\theta_4(\sin\theta_3-\sin\theta_2)\\+L_2m_4R_4\cos(\alpha_4+\theta_4)(\sin\theta_2-\sin\theta_3)\end{array}\right] \tag{A10}$$

https://doi.org/10.1017/S0263574703005216 Published online by Cambridge University Press