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Generalization of the two-dimensional Child–Langmuir law for non-zero injection velocities in a planar diode

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Abstract. A simple analytic expression of the two-dimensional Child–Langmuir law is derived for non-zero injection velocities and Lau's result is obtained in our model by setting the injection velocity equal to zero. The calculation results show that the modify term in our model is larger than Lau's with a non-zero electronic initial energy, and it is twice as large as Lau's when the electronic initial energy is much greater than the potential energy of the gap.

1. Introduction

The Child–Langmuir law is widely used in plasma physics and microwave electronics [1–4]. It gives the maximum current density allowed for steady-state electron beam transports across a gap. This maximum value is a result of the space charge effect.

Several studies have been carried out to extend the Child–Langmuir law to other situations. For example, Jaffé derived the one-dimensional Child–Langmuir law for non-zero injection velocities in [5]. The maximum current J_{SCL} that can be transmitted across the diode [6] and the current J_{BF}, marking the bifurcation of the state from completely to partially transmitting [7], can be derived from Jaffé's model.

The analytical solution of the two-dimensional limiting current in a diode is very difficult to obtain [8, 9]. However, many attempts have been made because the two-dimensional problem is of fundamental importance in microwave electronics. Luginsland et al. generalized the Child–Langmuir law to two dimensions via particle simulations [9]. Lau obtained a simple analytic theory for the two-dimensional Child–Langmuir law in 2001 [8], and the maximum current was given by

$$\frac{J(2)}{J(1)} \cong 1 + \frac{D}{4R}. \tag{1}$$

Here $J(2)$ is the maximum current for two-dimensional model, $J(1)$ is the maximum current for one-dimensional model, D is the gap separation and R is the beam radius. Lau's theory agrees well with Luginsland et al.'s simulation results [9].

However, Lau's model cannot be used to calculate the current J_{BF} with non-zero injection velocities.

In this paper, we have derived the analytic expression of the two-dimensional Child–Langmuir law for non-zero injection velocities for the first time. It is shown that Lau's result [8] is only a special case of our paper.

The paper is structured as follows. First, in Sec. 2 the one-dimensional Child–Langmuir law is reviewed, and we present the analytical forms of the two-dimensional Child–Langmuir law for non-zero injection velocities. In Sec. 3 we give some calculation results. A short summary is given in Sec. 4.

2. Derivation of the two-dimensional Child–Langmuir law for non-zero injection velocities

2.1. One-dimensional model for the Child–Langmuir law

It is generally assumed that the electrodes are infinite parallel planes and where all electrons are supposed to enter the space between the two planes with the same velocity v_0. Let the emitter plane, at $z = 0$, be kept at the potential $V = 0$, and the receiver plane, at $z = D$, be kept at the potential $V = V_D$. The equations for a one-dimensional diode can be written as follows [5]:

$$\frac{d^2V(z)}{dz^2} = -\frac{\rho}{\varepsilon_0}, \tag{2}$$

$$-\rho = \frac{J_0}{v_z}, \tag{3}$$

$$\frac{1}{2}mv_z^2 = eV + K, \tag{4}$$

where we define $V(z)$, J_0, ρ, v_z and K as the electronic potential, the current density, the space charge density, the electronic velocity and the electronic initial energy respectively; ε_0 and m are the dielectric constant in vacuum and electronic mass. Let $z_{\min}$ be the value of z for which the minimum $t_{\min}$ occurs. Assuming $\alpha = (J_0/\varepsilon_0)(m/2K)^{1/2}$, $t = [(eV/K)+1]^{1/2}$ and considering the condition of continuity at $z = z_{\min}$, we obtain the solution of the above equations as follows:

$$\frac{2}{3}(t-2c_1)(t+c_1)^{\frac{1}{2}} = \begin{cases} -\left(\dfrac{\alpha e}{K}\right)^{\frac{1}{2}} z + \dfrac{2}{3}(1-2c_1)(1+c_1)^{\frac{1}{2}} & 0 \leqslant z \leqslant z_{\min}, \quad (5) \\[2ex] \left(\dfrac{\alpha e}{K}\right)^{\frac{1}{2}} z - \dfrac{2}{3}(1-2c_1)(1+c_1)^{\frac{1}{2}} & z_{\min} \leqslant z \leqslant D. \quad (6) \end{cases}$$

Considering $dt/dz|_{z=z_{\min}} = 0$, we obtain $z_{\min}$ and $t_{\min}$ as

$$z_{\min} = \frac{2}{3}\left(\frac{K}{\alpha e}\right)^{\frac{1}{2}}(1-2c_1)(1+c_1)^{\frac{1}{2}}, \tag{7}$$

$$t_{\min} = -c_1. \tag{8}$$

The constant c_1 is determined from the boundary condition $V(D) = V_D$. From equation (8) and $t = [(eV/K)+1]^{1/2}$, it is found that we can obtain the current

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$J_{\rm BF}$ and $z_{\min}$ with $c_1 = 0$:

$$J_{\rm BF} = \frac{4\varepsilon_0}{9D^2 e}\sqrt{\frac{2}{m}}K^{\frac{3}{2}}\left[\left(\frac{V_D}{V_0}+1\right)^{\frac{3}{4}}+1\right]^2, \tag{9}$$

$$z_{\min} = \frac{D}{1+(1+V_D/V_0)^{\frac{3}{4}}}, \tag{10}$$

where $V_0 = mv_0^2/2e$.

2.2. Modifications to the two-dimensional Child–Langmuir law

Let $\rho(r,\varphi,z)$ be the charge density within the gap. Considering the case where electron emission is restricted to a circular patch of radius R on the cathode, the space charge yields an electric field at $(0,0,z_{\min})$ with the magnitude

$$|\Delta E| = \begin{cases} \dfrac{\rho(r,\varphi,z)r\Delta r\Delta\varphi\Delta z}{4\pi\varepsilon_0[r^2+(z-z_{\min})^2]}, & 0 \leqslant z \leqslant z_{\min}, \quad (11) \\[2ex] \dfrac{\rho(r,\varphi,z)r\Delta r\Delta\varphi\Delta z}{4\pi\varepsilon_0[r^2+(z_{\min}-z)^2]}, & z_{\min} \leqslant z \leqslant D. \quad (12) \end{cases}$$

We consider the z component of the electric field on the axis line. By multiplying equations (11) and (12) by the directional cosine and summing over the space charge within the gap, the space charge field at $(0,0,z_{\min})$ is given by

$$\begin{aligned} |E| = {} & \int_0^{z_{\min}} dz \int_0^{2\pi} d\varphi \int_0^R dr \frac{\rho(r,\varphi,z)(z_{\min}-z)r}{4\pi\varepsilon_0[r^2+(z_{\min}-z)^2]^{\frac{3}{2}}} \\ & - \int_{z_{\min}}^D dz \int_0^{2\pi} d\varphi \int_0^R dr \frac{\rho(r,\varphi,z)(z-z_{\min})r}{4\pi\varepsilon_0[r^2+(z-z_{\min})^2]^{\frac{3}{2}}}. \end{aligned} \tag{13}$$

Assuming $R \geqslant D$ and the charge density ρ is independent of r and φ, we can write (13) as follows by integrating over variables r and φ:

$$\begin{aligned} |E| = {} & \frac{1}{2\varepsilon_0}\left\{\int_0^{z_{\min}} dz\,\rho(z)\left[1-\frac{1}{R}\frac{(z_{\min}-z)}{\sqrt{1+((z_{\min}-z)/R)^2}}\right]\right. \\ & \left. - \int_{z_{\min}}^D dz\,\rho(z)\left[1-\frac{1}{R}\frac{(z-z_{\min})}{\sqrt{1+((z-z_{\min})/R)^2}}\right]\right\}, \\ \approx {} & \frac{1}{2\varepsilon_0}\left\{\int_0^{z_{\min}} dz\,\rho(z)\left[1-\frac{(z_{\min}-z)}{R}\right] - \int_{z_{\min}}^D dz\,\rho(z)\left[1-\frac{(z-z_{\min})}{R}\right]\right\}. \end{aligned} \tag{14}$$

Since the injection current density J_0 is constant, we can write (14) as

$$|E| = G\left(\frac{J}{2\varepsilon_0}\right)\left\{\int_0^{z_{\min}} dz \frac{1}{v(z)}\left[1-\frac{(z_{\min}-z)}{R}\right] - \int_{z_{\min}}^D dz \frac{1}{v(z)}\left[1-\frac{(z-z_{\min})}{R}\right]\right\}. \tag{15}$$

The multiplication factor G is used to account for the image charges in (15). The 2D limiting current density $J_{\rm BF}$ is reached when this total space charge field equals

$(V_0/z_{\min}) - ((V_D + V_0)/(D - z_{\min}))$. This gives

$$G\left(\frac{J_{\mathrm{BF}}(2)}{2\varepsilon_0}\right)\left\{\int_0^{z_{\min}} dz\frac{1}{v(z)}\left[1-\frac{z_{\min}-z}{R}\right]-\int_{z_{\min}}^{D} dz\frac{1}{v(z)}\left[1-\frac{(z-z_{\min})}{R}\right]\right\}$$
$$= \frac{V_0}{z_{\min}} - \frac{V_D + V_0}{D - z_{\min}}. \tag{16}$$

Letting R approach infinity, we obtain the one-dimensional result from (16):

$$G\left(\frac{J_{\mathrm{BF}}(1)}{2\varepsilon_0}\right)\left\{\int_0^{z_{\min}} dz\frac{1}{v(z)}-\int_{z_{\min}}^{D} dz\frac{1}{v(z)}\right\} = \frac{V_0}{z_{\min}} - \frac{V_D + V_0}{D - z_{\min}}. \tag{17}$$

Assuming that G is applicable for any value of R, we can obtain the ratio of $J_{\mathrm{BF}}(2)$ to $J_{\mathrm{BF}}(1)$ as follows:

$$\frac{J_{\mathrm{BF}}(2)}{J_{\mathrm{BF}}(1)} \cong 1 + \frac{1}{R}\left[\frac{\int_0^{z_{\min}} dz((z_{\min}-z)/v(z)) - \int_{z_{\min}}^{D} dz((z-z_{\min})/v(z))}{\int_0^{z_{\min}} dz(1/v(z)) - \int_{z_{\min}}^{D} dz(1/v(z))}\right]. \tag{18}$$

From (5)–(8) and $c_1 = 0$, we can obtain the electronic velocity as follows:

$$v_z = \begin{cases} \beta[z_{\min} - z]^{\frac{2}{3}} & 0 \leqslant z \leqslant z_{\min}, \quad (19) \\ \beta[z - z_{\min}]^{\frac{2}{3}} & z_{\min} \leqslant \mathrm{z} \leqslant \mathrm{D}, \quad (20) \end{cases}$$

where $\beta^2 = (2K/m)(1/z_{\min})^{4/3}$.
Substituting (19) and (20) into (18), we obtain

$$\frac{J_{\mathrm{BF}}(2)}{J_{\mathrm{BF}}(1)} \cong 1 + \frac{D}{4R}\left[\frac{(z_{\min}/D)^{\frac{4}{3}} - (1 - z_{\min}/D)^{\frac{4}{3}}}{(z_{\min}/D)^{\frac{1}{3}} - (1 - z_{\min}/D)^{\frac{1}{3}}}\right], \tag{21}$$

which is the analytical expression for two-dimensional Child–Langmuir law for non-zero injection velocities.

3. Calculation results

We set the planar gap is equal to 1 cm, gap voltage is equal to 1 kV. Figure 1 shows $z_{\min}$ as a function of gap voltage V_D with $R = 8$ cm. We can see that $z_{\min}$ is less than $D/2$ with $V_D > 0$ and $z_{\min}$ is equal to zero with $V_D \gg V_0$ (see [9,10]). Assuming

$$f\left(\frac{z_{\min}}{D}\right) = \left[\frac{(z_{\min}/D)^{\frac{4}{3}} - (1 - z_{\min}/D)^{\frac{4}{3}}}{(z_{\min}/D)^{\frac{1}{3}} - (1 - z_{\min}/D)^{\frac{1}{3}}}\right] \tag{22}$$

and according to (10), we can see that $z_{\min}/D$ is equal to zero if $V_D \gg V_0$, then $f(z_{\min}/D) = 1$, which is shown in Fig. 2. This means that (21) agrees with (1) with $V_D \gg V_0$, and that the maximum value of $f(z_{\min}/D)$ is 2 with $z_{\min}/D = 0.5$ which means the modify term in (21) is twice as large as that in (1).

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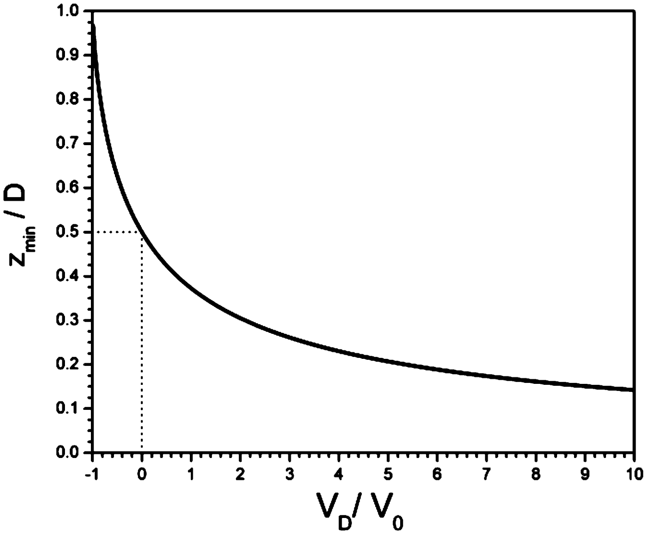

Figure 1. $z_{\min}$ as a function of V_D.

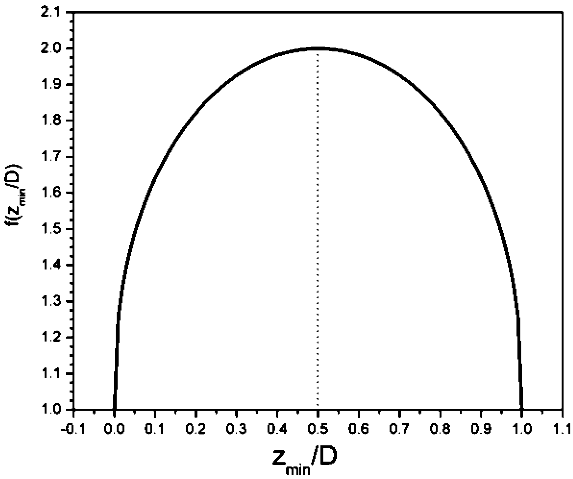

Figure 2. $f(z_{\min}/D)$ as a function of $z_{\min}/D$.

4. Summary

In this paper, the two-dimensional Child–Langmuir law for non-zero injection velocities in a planar diode is presented. The calculation results show that $z_{\min}$ is less than $D/2$ with $V_D > 0$ and the modify term is twice as large as Lau's model with $V_D = 0$. Our model degenerates to Lau's model with $V_D \gg V_0$.

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