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A VALUATION THEORETIC CHARACTERIZATION OF RECURSIVELY SATURATED REAL CLOSED FIELDS

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Abstract. We give a valuation theoretic characterization for a real closed field to be recursively saturated. This builds on work in [9], where the authors gave such a characterization for κ-saturation, for a cardinal $\kappa \geq \aleph_0$. Our result extends the characterization of Harnik and Ressayre [7] for a divisible ordered abelian group to be recursively saturated.

§1. Introduction. Recursive saturation was introduced by Barwise and Schlipf in [2].

DEFINITION 1.1. Let L be a computable language. An L-structure $\mathcal{M}$ is *recursively saturated* if for every computable set of L-formulas $\tau(x, \bar{y})$ and every tuple $\bar{a}$ in $\mathcal{M}$, if $\tau(x, \bar{a})$ is finitely satisfiable in $\mathcal{M}$, then $\tau(x, \bar{a})$ is realized in $\mathcal{M}$.

For an arbitrary infinite cardinal κ, κ-saturation has been investigated in terms of valuation theory for divisible ordered abelian groups in [10] and for real closed fields in [9] (see Section 3). In this paper we prove an analogous valuation theoretic characterization for recursively saturated real closed fields (see Section 5). Our result extend that of [7] for recursively saturated divisible ordered abelian groups (see Section 4). Countable recursively saturated real closed fields have already been described in terms of their integer parts and models of Peano Arithmetic in [4].

§2. Preliminaries.

2.1. Scott sets and recursive saturation. A subset $\mathcal{T} \subset 2^{<\omega}$ is a *tree* if every substring of an element of $\mathcal{T}$ is also an element of $\mathcal{T}$. If $\sigma, \tau \in 2^{<\omega}$, we let $\sigma \prec \tau$ denote that σ is a substring of τ. A sequence $f \in 2^{\omega}$ is a *path* through a tree $\mathcal{T}$ if for all $\sigma \in 2^{<\omega}$ with $\sigma \prec f$, we have $\sigma \in \mathcal{T}$. For any $\sigma \in 2^{<\omega}$, the length of σ, denoted by $length(\sigma)$, is the unique $n \in \omega$ satisfying $\sigma \in 2^n$.

DEFINITION 2.1. A nonempty set $S \subset \mathbb{R}$ is a *Scott set* if

(1) S is computably closed, i.e., if $r_1, \ldots r_n \in S$ and $r \in \mathbb{R}$ is computable from $r_1 \oplus \ldots \oplus r_n$ (the *Turing join* of $r_1, \ldots, r_n$), then $r \in S$.

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(2) If an infinite tree $\mathcal{T} \subset 2^{<\omega}$ is computable in some $r \in S$, then $\mathcal{T}$ has a path that is computable in some $r' \in S$.

The following characterization was given by Scott in [17]. A countable Scott set is the collection of subsets of ω coded in some nonstandard model of Peano Arithmetic. This characterization was extended to Scott sets of cardinality ω_1 by Knight and Nadel in [12]. Hence, under CH, all Scott sets arise as collections of subsets of ω coded in nonstandard models of Peano Arithmetic.

FACT 2.2. *Any Scott set S is an Archimedean real closed field.*

DEFINITION 2.3 ([13, Defintion 1.2]). Let L be a computable language and $S \subset P(\omega)$. An L-structure M is *S-saturated* if

(1) every type realized in M is computable from some $s \in S$, and
(2) if $\tau(x, \bar{y})$ is computable in some $s \in S$ and $\bar{m}$ is a tuple in M such that $\tau(x, \bar{m})$ is finitely satisfiable in M, then $\tau(x, \bar{m})$ is realized in M.

Scott sets are intimately connected with recursively saturated models in the following sense.

LEMMA 2.4 ([13, Lemma 1.3]). *Let L be a computable language. An L-structure M is recursively saturated if and only if M is S-saturated for some Scott set S.*

2.2. Some valuation theoretic notions. We first summarize some background on divisible ordered abelian groups (see [10] and [11]). Let $(G, +, 0, <)$ be a divisible ordered abelian group, i.e., an ordered $\mathbb{Q}$-vector space. Given $A \subset G$, we let $\langle A \rangle_{\mathbb{Q}}$ denote the smallest divisible ordered subgroup of G containing A. For any $x \in G$, let $|x| = \max\{x, -x\}$. For nonzero $x, y \in G$, we define $x \sim y$ if there exists $n \in \mathbb{N}$ such that $n|x| \geq |y|$ and $n|y| \geq |x|$. We write $x \ll y$ if $n|x| < |y|$ for all $n \in \mathbb{N}$. Clearly, $\sim$ is an equivalence relation, and $[x]$ denotes the equivalence class of any nonzero $x \in G$. Let $\Gamma := \{[x] : x \in G \setminus \{0\}\}$. We define a total order $<_\Gamma$ on Γ in terms of $\ll$ as follows $[y] <_\Gamma [x]$ if $x \ll y$ (notice the reversed order). Given a linear order $(A, <)$ and $A_1, A_2 \subset A$, we use the notation $A_1 < A_2$ to indicate that $a_1 < a_2$ for all $a_1 \in A_1$ and $a_2 \in A_2$.

DEFINITION 2.5. (a) We call Γ the *value set of G*.

(b) The map $v \colon G \to \Gamma \cup \{\infty\}$ defined by $v(0) := \infty$ and $v(x) = [x]$ if $x \neq 0$ is *a valuation on G* (and (G, v) is a *valued $\mathbb{Q}$-vector space*), i.e., for every $x, y \in G$, $v(x) = \infty$ if and only if $x = 0$, $v(nx) = v(x)$ for all $n \in \mathbb{Z}^\times$, and $v(x + y) \geq \min\{v(x), v(y)\}$. We call v the *natural valuation on G*.

(c) The *rational rank* of G, denoted $\operatorname{rk}(G)$, is the linear dimension of G as a $\mathbb{Q}$-vector space.

(d) For every $\gamma \in \Gamma$, fix $x \in \gamma$ and choose a maximal Archimedean subgroup A_γ of G containing x. We call A_γ the *Archimedean component associated to γ*. For each γ, A_γ is isomorphic to an ordered subgroup of $(\mathbb{R}, +, 0, <)$. Furthermore, we can calculate the isomorphism type of A_γ in terms of any $x \in \gamma$. Given $x, y \in \gamma$, we let $\frac{y}{x} = \sup\{r \in \mathbb{Q} \mid rx < y\}$, and let $A_{\gamma,x} = \{\frac{y}{x} \mid y \in \gamma\} \cup \{0\}$. Then, $A_\gamma \cong A_{\gamma,x}$.

(e) Given Γ a linearly ordered set and $\{B_\gamma \mid \gamma \in \Gamma\}$ a family of (additive) Archimedean groups, the *Hahn group* $G = \oplus_{\gamma \in \Gamma} B_\gamma$ is the set of functions

$f : \Gamma \to \cup_{\gamma\in\Gamma} B_\gamma$ such that $f(\gamma) \in B_\gamma$ and the support of f, $\mathrm{supp}(f) = \{\gamma \in \Gamma \mid f(\gamma) \neq 0\}$, is finite. The operation of G is componentwise addition and the order is the lexicographical order.

(f) A set $\{g_1, \dots, g_n\} \subset G$ is called *valuation independent* if for all $q_1, \dots, q_n \in \mathbb{Q}$, $v(\sum q_i g_i) = \min\{v(g_i) \mid q_i \neq 0\}$. A $\mathbb{Q}$-basis $\{g_1, \dots, g_n\}$ for G is called a *valuation basis* if it is valuation independent.

We need the following theorem from [3].

THEOREM 2.6 (Brown). *Every valued vector space of countable dimension admits a valuation basis.*

DEFINITION 2.7. Let λ be an infinite ordinal. A sequence $(a_\rho)_{\rho<\lambda}$ contained in G is *pseudo-Cauchy* if for every $\rho < \sigma < \tau < \lambda$ we have $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$. We say that $x \in G$ is a *pseudo-limit* of the pseudo-Cauchy sequence $(a_\rho)_{\rho<\lambda}$ if $v(x - a_\rho) = v(a_{\rho+1} - a_\rho)$ for all $\rho < \lambda$. Note that if $(a_\rho)_{\rho<\lambda}$ is a pseudo-Cauchy sequence then for all $\rho < \sigma < \lambda$, $v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho)$.

Now we recall analogous notions for real closed fields. Any real closed field $(R, +, \cdot, 0, 1, <)$ has a natural valuation v, that is the natural valuation associated with the divisible ordered abelian group $(R, +, 0, <)$. We define an addition on the value set $v(R^\times)$ by setting $v(x) + v(y)$ equal to $v(xy)$ for all $x, y \in v(R^\times)$. The value set, equipped with this addition, is a divisible ordered abelian group.

DEFINITION 2.8. (a) We call $G = v(R^\times)$ the *value group* of R.

(b) The *valuation ring of* R is $\mathcal{O}_R := \{r \in R \mid v(r) \geq 0\}$, and the *valuation ideal of* R is $\mu_R := \{r \in R \mid v(r) > 0\}$.

(c) The *residue field* k of R, denoted $\overline{R}$, is the quotient $\mathcal{O}_R/\mu_R$. The field k is a real closed Archimedean field, i.e., isomorphic to a subfield of $\mathbb{R}$. Given any $a \in \mathcal{O}_R$, we denote the *residue of* a by $\overline{a} \in k$. Notice that $(R, +, 0, <)$ has a unique Archimedean component up to isomorphism, which is isomorphic to $(k, +, 0, <)$.

(d) Given an ordered abelian group G and an Archimedean field k, the *field of generalized power series over* G, denoted $\mathbb{K} = k((G))$, consists of formal expressions of the form $\Sigma_{g\in G} c_g t^g$, where $c_g \in k$ and $\{g \in G \mid c_g \neq 0\}$ is well-ordered. Again, the addition is pointwise, the order is lexicographic, and the multiplication is given by the convolution formula. Note that $v(\mathbb{K}^\times) = G$ and $\overline{\mathbb{K}} = k$.

(e) If R is a real closed field, given $X \subset R$, we let $\langle X\rangle^{rc}$ denote the real closure of $\mathbb{Q}(X)$ in R.

§3. Background on κ-saturated structures. We now recall the characterization of $\aleph_\alpha$-saturation for divisible ordered abelian groups given in [10]. We need the notion of η_α-sets (see [15]). An η_α-set is a linear order $(A, <)$ such that, whenever $A_1, A_2 \subset A$ have cardinality less than $\aleph_\alpha$ and $A_1 < A_2$, then there is an $a \in A$ such that $A_1 < a < A_2$. Observe that an η_0-set is simply a dense linear order without endpoints.

THEOREM 3.1 ([10]). *Let G be a divisible ordered abelian group, and let $\aleph_\alpha \geq \aleph_0$. Then G is $\aleph_\alpha$-saturated in the language of ordered groups if and only if*

(i) *the value set of G is an η_α-set,*
(ii) *all the Archimedean components of G are isomorphic to $\mathbb{R}$, and*
(iii) *every pseudo-Cauchy sequence in a divisible subgroup of G with a value set of cardinality less than $\aleph_\alpha$ has a pseudo-limit in G.*

Notice that in the case of $\aleph_0$-saturation the necessary and sufficient conditions reduce only to (i) and (ii).

The following characterization of $\aleph_\alpha$-saturated real closed fields was obtained in [9].

Theorem 3.2 ([9, Theorem 6.2]). *Let R be a real closed field, v its natural valuation, G its value group, and k its residue field. Let $\aleph_\alpha \geq \aleph_0$. Then R is $\aleph_\alpha$-saturated in the language of ordered rings if and only if*

(i) *G is $\aleph_\alpha$-saturated,*
(ii) *$k \cong \mathbb{R}$,*
(iii) *every pseudo-Cauchy sequence in a subfield of R of absolute transcendence degree less than $\aleph_\alpha$ has a pseudo-limit in R.*

In the proof of Theorem 3.2 the *dimension inequality* (see [5]) is crucially used in the case of $\aleph_0$-saturation. This says that the rational rank of the value group of a finite transcendental extension of a real closed field is bounded by the transcendence degree of the extension.

§4. Recursively saturated divisible ordered abelian groups.

Harnik and Ressayre state the following result in [7] and sketch a proof just for the necessity of condition (ii). We include here a complete proof.

Theorem 4.1. *Let G be a divisible ordered abelian group. Then G is recursively saturated in the language of ordered groups if and only if*

(i) *the value set Γ of G is a dense linear order without endpoints, i.e., the value set of G is an η_0-set, and*
(ii) *all Archimedean components of G equal a common Scott set S.*

Proof. Suppose G is recursively saturated. We show (i) and (ii).
(i) Let $g, g' \in G$ such that $g, g' > 0$ and $v(g) < v(g')$. The partial type

$$\beta(x, g, g') = \{ng' < x \mid n \in \mathbb{N}\} \cup \{x < ng \mid n \in \mathbb{N}\}$$

is computable and finitely satisfiable (since $v(g) < v(g')$ and G is divisible). By recursive saturation, there is some $h \in G$ such that h realizes $\beta(x, g, g')$ in G. Hence, $v(g) < v(h) < v(g')$. A similar argument shows that Γ has no greatest or least element.
(ii) Since G is recursively saturated, G is S-saturated for some Scott set S by Lemma 2.4. Let $g \in G$ be nonzero. We show that $A_{[g],g} = S$. For any real r, the partial type $\delta_r(x, y)$ consisting of the formulas

$$qy < x < q'y \text{ for all } q, q' \in \mathbb{Q} \text{ satisfying } q < r < q'$$

has the same Turing degree as r. Since G is S-saturated and divisible and S is computably closed, $r \in S$ if and only if the type $\delta_r(x, g)$ is realized in G. By definition, $\delta_r(x, g)$ is realized in G if and only if $r \in A_{[g],g}$. Hence, $A_{[g],g} = S$.

Now, let G be a divisible ordered abelian group satisfying (i) and (ii). Let $S \subset \mathbb{R}$ be the Scott set such that $S = A_{[g],g}$ for all $g \in G$ by (ii). We show that G is

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recursively saturated. Let $\tau_p(x, \bar{y})$ be a computable partial type with $|\bar{y}| = n$, and let $\bar{g} = (g_1, \dots, g_n)$ be an n-tuple from G so that $\tau_p(x, \bar{g})$ is finitely satisfiable in G. Let $\delta(\bar{y})$ be the complete type of $\bar{g}$ in G. The proof follows the structure of the proof of Theorem 3.1 for the case of $\aleph_0$-saturation. However, now we also ensure that $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$ can be extended, computably in some $r \in S$, to a complete type $\tau(x, \bar{y})$ such that $\tau(x, \bar{g})$ is also finitely satisfiable in G.

Set $G' = \langle \bar{g} \rangle_{\mathbb{Q}}$. By Theorem 2.6, we may assume that $\bar{g}$ is a valuation basis for G'; it is computable to substitute each occurrence of g_i in $\tau_p(x, \bar{g})$ with its definition over such a basis. Similarly, we may assume that $g_1, g_2, \dots, g_n > 0$ and $v(g_1) \le v(g_2) \le \dots \le v(g_n)$.

We regroup $g_1, \dots, g_n$ in blocks

$$g_{11}, \dots, g_{1l_1}, g_{21}, \dots, g_{2l_2}, \dots, g_{k1}, \dots, g_{kl_k}$$

so that $v(g_{ij}) = v(g_{i1})$ and $v(g_{ij}) \neq v(g_{i'j'})$ for $i \neq i'$. Let $r_{ij} := \frac{g_{ij}}{g_{i1}}$ for $1 \le j \le l_i$ and $1 \le i \le k$.

Claim 4.2. *The complete n-type $\delta(\bar{y})$ of a valuation basis $\bar{g}$ for a finite dimensional divisible ordered abelian group is computable from an element in the Scott set S. Specifically, $\delta(\bar{y})$ is computable from the Turing join r' of r_{ij} for all $1 \le i \le k$ and $1 \le j \le l_i$.*

Proof. By (ii), $r_{ij} \in A_{[g_{i1}], g_{i1}} = S$. So, $r' \in S$. We isolate a subset $\delta_p(\bar{g})$ of $\delta(\bar{g})$ that is computable in r' and show that we can computably deduce the order of any two terms in $\bar{g}$ from $\delta_p(\bar{g})$. Since the theory of divisible ordered abelian groups admits effective quantifier elimination, we conclude that $\delta(\bar{g})$ is computable in r'. (For ease of reading, we describe the type δ_p and prove our claim in terms of the parameters $\bar{g}$ rather than some free variables $\bar{y}$.)

Let $\delta_p(\bar{g})$ be the partial type consisting of the formulas:

(a) $0 < g_{i1} \ \wedge \ g_{i1} > ng_{(i+1)1}$ for all $1 \le i < k$ and all $n \in \mathbb{N}$, and
(b) $qg_{i1} < g_{ij}$ (if $q < r_{ij}$) or $qg_{i1} > g_{ij}$ (if $q > r_{ij}$) for all $q \in \mathbb{Q}$, $1 \le i \le k$, and $1 \le j \le l_i$.

Note that $\delta_p(\bar{g})$ is computable from r'. Let $t(\bar{g}) = \sum_{1 \le i \le k} \sum_{1 \le j \le l_i} s_{ij} g_{ij}$ be a term in $\bar{g}$ where all $s_{ij} \in \mathbb{Z}$. It suffices to show that we can determine whether $t(\bar{g}) > 0$. Let i be the least index such that $s_{ij} \neq 0$. Since $\bar{g}$ is a valuation basis, $t(\bar{g}) > 0$ if and only if $t_i(\bar{g}) := \sum_{1 \le j \le l_i} s_{ij} g_{ij} > 0$. But $t_i(\bar{g}) > 0$ if and only if $\sum_{1 \le j \le l_i} s_{ij} r_{ij}$ is positive (see also [6], Propositions 12 and 13). Hence, we can r'-computably determine whether term $t(\bar{g}) > 0$. ⊣

Since $\tau_p(x, \bar{y})$ is computable, Claim 4.2 guarantees that $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$ is computable in r'. Then, there is an r'-computable infinite tree $\mathcal{T}$ such that any path through $\mathcal{T}$ encodes a complete consistent type $\tau(x, \bar{y})$ extending $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$. Since S is a Scott set and $\mathcal{T}$ is computable in $r' \in S$, there is some $r \in S$ such that r computes a complete extension $\tau(x, \bar{y})$ of $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$. By construction of $\mathcal{T}$, $\tau(x, \bar{g})$ is finitely satisfiable in G. We show that $\tau(x, \bar{g})$ is realized in G.

Recall that $G' = \langle \bar{g} \rangle_{\mathbb{Q}}$, and let Γ' be the value set for G'. We let

$$B := \{b \in G' \mid \tau(x, \bar{g}) \vdash b \le x\} \text{ and } C := \{c \in G' \mid \tau(x, \bar{g}) \vdash x \le c\}.$$

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By quantifier elimination for divisible ordered abelian groups, to realize the type $\tau(x, \bar{g})$, it suffices to realize the r-computable partial type

$$\{b \leq x \mid b \in B\} \cup \{x \leq c \mid c \in C\}. \tag{1}$$

If $\tau(x, \bar{g}) \vdash x = a$ for any $a \in B \cup C$, then the type in (1) is realized by $a \in B \cup C \subset G$. So, suppose otherwise. Let $G'' \succ G$ be such that there is some $x_0 \in G''$ that realizes $\tau(x, \bar{g})$ in G''. Consider the set $\Delta = \{v(d - x_0) \mid d \in G'\}$.

Since G' has finite rank and $\langle \bar{g}, x_0 \rangle_{\mathbb{Q}} \supseteq G'$, Δ is finite and so it has a maximum element. We fix $d_0 \in G'$ such that $v(d_0 - x_0)$ is the maximum of Δ. Suppose that $d_0 \in B$ (the case that $d_0 \in C$ is symmetric). We show that the following r-computable partial type (2) is realized in G:

$$\{b - d_0 < x' \mid b \in B\} \cup \{x' < c - d_0 \mid c \in C\}. \tag{2}$$

Clearly, x' satisfies this cut if and only if $x' + d_0$ satisfies (1). We need to examine two cases below (we omit the proofs of Claims 4.3 and 4.4; see the analogous proofs in [10]).

Case 1 (*Residue Transcendental*) - Δ has a maximum, which is in Γ'.

Claim 4.3 (see [10, Theorem C]). *There exist $b_0 \in B$ and $c_0 \in C$ such that for all $b \in B$ and $c \in C$ with $b_0 \leq b$ and $c \leq c_0$,*

$$v(b - d_0) = v(x_0 - d_0) = v(c - d_0) \textit{ and, hence,}$$
$$v(b - x_0) = v(x_0 - d_0) = v(c - x_0).$$

It follows that for all $b \in B$ and $c \in C$ with $b \geq b_0$ and $c \leq c_0$,

$$\frac{b - d_0}{b_0 - d_0} < \frac{x_0 - d_0}{b_0 - d_0} < \frac{c - d_0}{b_0 - d_0}.$$

Then, let $\hat{r} \in \mathbb{R}$ fill the following r-computable cut in the reals:

$$\left\{\frac{b - d_0}{b_0 - d_0} < x' \mid b_0 \leq b \in B\right\} \cup \left\{x' < \frac{c - d_0}{b_0 - d_0} \mid c_0 \geq c \in C\right\}. \tag{3}$$

Also, $\hat{r} \in S$ since $r \in S$. By (ii), $S = A_{[b_0 - d_0], b_0 - d_0}$. So, there is some $\hat{g} \in G$ such that $\frac{\hat{g}}{b_0 - d_0} = \hat{r}$, and $\hat{g}$ fills the cut in (2) since $\hat{r}$ fills the cut in (3).

Case 2 (*Value Transcendental*) - Δ has a maximum, not in Γ'.

Set $\Delta_1 := \{v(c - d_0) \mid c \in C\}$ and $\Delta_2 := \{v(b - d_0) \mid b \in B \;\&\; b > d_0\}$.

Claim 4.4 (see 10, Theorem C]). $\Delta_1 < v(d_0 - x_0) < \Delta_2$.

Since G' has finite rank, Δ_1 and Δ_2 are finite and form a cut in the value set Γ. By (i), Γ is a dense linear order without endpoints, so there is some $y \in G$ with $y > 0$ that fills this cut in Γ. Then, for all $c \in C$ and $b \in B$ with $b > d_0$, we have $v(b - d_0) > v(y) > v(c - d_0)$ so $b - d_0 < y < c - d_0$. Hence, y fills the cut given in (2). This completes the proof that G is recursively saturated. $\dashv$

Remark 4.5. Given any Scott set S and any dense linear order without endpoints Γ, the Hahn group $G = \oplus_{\Gamma} S$ is thus an example of a recursively saturated divisible ordered abelian group. Hence, every Scott set S appears as the Archimedean component of a recursively saturated divisible ordered abelian group.

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For countable groups, these are the only examples:

Corollary 4.6. *A countable divisible ordered abelian group G is recursively saturated if and only if G is isomorphic to $\oplus_{\mathbb{Q}} S$ for some countable Scott set S.*

Proof. By Theorem 2.6, every countable divisible ordered abelian group admits a valuation basis. Therefore, G is isomorphic to $\oplus_{\gamma\in\Gamma} A_\gamma$ (see [11, Corollary 0.5]). By Theorem 4.1, the value set Γ of G is a countable dense linear order without endpoints (and therefore, $\Gamma \cong \mathbb{Q}$) and each $A_\gamma \cong S$ for some countable Scott set S. ⊣

§5. Recursively saturated real closed fields. We now give an analogous characterization of recursively saturated real closed fields. We need the following definition.

Definition 5.1. Let R be a real closed field and $r \in \mathbb{R}$. Let $\bar{d}$ be a finite tuple of parameters in R. We say that a length ω sequence of elements $(a_i)_{i<\omega}$ in $\langle \bar{d}\,\rangle^{rc}$ is *computable in r over $\bar{d}$* if there is an r-computable sequence of formulas $(\theta_i(x,\bar{y}))_{i<\omega}$ such that $\theta_i(x,\bar{d})$ defines a_i in R for all $i < \omega$.

Theorem 5.2. *If R is a real closed field with natural valuation v, value group G, and residue field k, then R is recursively saturated in the language of ordered rings if and only if there is a Scott set S such that*

(i) *G is recursively saturated with Archimedean components all equal to S;*
(ii) $(k, +, \cdot, 0, 1, <) \cong (S, +, \cdot, 0, 1, <)$;
(iii) *every pseudo-Cauchy sequence of length ω that is computable in an element of S over some finite tuple of parameters in R has a pseudo-limit in R; and*
(iv) *every type realized by some finite tuple $\bar{a}$ in R is computable in an element of S.*

Proof. Suppose that R is recursively saturated. By Lemma 2.4, R is S-saturated for some Scott set S, and hence, (iv) holds for this S. We show that conditions (i), (ii), and (iii) also hold with respect to S.
(ii) Since R is S-saturated as an ordered field, the reduct $(R, +, 0, <)$ is also S-saturated (and so recursively saturated by Lemma 2.4) as a divisible ordered abelian group. By Theorem 4.1, the Archimedean components $A_{[r],r}$ of $(R, +, 0, <)$ equal S for all nonzero $r \in R$. In particular, $A_{[1],1} = S$. Hence, $(k, +, 0, <) \cong (S, +, 0, <)$. Since R is a real closed field, k is isomorphic (as an ordered field) to a real closed field $K \subset \mathbb{R}$. By Hölder's Theorem, the resulting ordered group isomorphism ϕ from $(S, +, 0, <)$ to $(K, +, 0, <)$ has the form $\phi(x) = sx$ for some $s \in \mathbb{R}$. We show that ϕ is the identity function and hence a field isomorphism. Since S and K are fields, $1 \in S \cap K$ so $s \in K$ and $\frac{1}{s} \in S$. Then, $s, \frac{1}{s} \in S \cap K$. Then, $S = K$ (given $t \in S$, $\frac{t}{s} \in S$ so $\phi(\frac{t}{s}) = t \in K$, and the other inclusion is similar). Since ϕ is an ordered group isomorphism, ϕ is the identity.
(i) We now show that G is S-saturated. Then, the necessity of (i) follows from Theorem 4.1.

We prove that (2) of Definition 2.3 holds. Let $\bar{g} = (g_1, \dots, g_n)$ be an n-tuple from G. Suppose that $\beta(x, \bar{y})$ is a partial type in the language of ordered groups computable from an element in S. Further suppose that $\beta(x, \bar{g})$ is finitely satisfiable in G. Let $G' = \langle \bar{g} \rangle_{\mathbb{Q}}$. We may also assume that $\beta(x, \bar{y})$ is a complete r-computable

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type for some $r \in S$ and that $\bar{g}$ is a basis for G', as in the proof of sufficiency of Theorem 4.1.

If $\beta(x, \bar{g}) \vdash x = g$ for any $g \in G'$, then we are done. Otherwise, let $B := \{b \in G' \mid \beta(x, \bar{g}) \vdash b < x\}$ and $C := \{c \in G' \mid \beta(x, \bar{g}) \vdash x < c\}$. It suffices to fill the following r-computable cut in G:

$$\{b < x \mid b \in B\} \cup \{x < c \mid c \in C\}. \tag{4}$$

Take a tuple $\bar{d} = (d_1, \dots, d_n)$ in R so that $d_i > 0$ and $v(d_i) = g_i$ for $1 \leq i \leq n$. Then, the multiplicative subgroup

$$\left\{ \prod_{i=1}^{n} d_i^{q_i} \in R \mid q_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n \right\}$$

is a section for G' in R. We show that there is an r-computable partial type in the language of ordered rings $\tilde{\beta}(x, \bar{y})$ such that if $\tilde{\beta}(x, \bar{d})$ is realized in R, then the cut described by $\beta(x, \bar{g})$ is filled in G. Since $\bar{g}$ is a basis for G', every element of G' can be expressed as some $\sum_{i=1}^{n} q_i g_i$ for some $q_1, \dots, q_n \in \mathbb{Q}$.

Since $r \in S$ computes the complete type $\beta(x, \bar{y})$, we can r-computably decide for a given $(q_1, \dots, q_n) \in \mathbb{Q}^n$ if $\sum_{i=1}^{n} q_i g_i \in B$ or $\sum_{i=1}^{n} q_i g_i \in C$. We let the partial type $\tilde{\beta}(x, \bar{y})$ consist of all formulas of the form

$$\prod_{i=1}^{n} y_i^{q_i} < x \text{ if } \sum_{i=1}^{n} q_i g_i \in C \text{ or } x < \prod_{i=1}^{n} y_i^{q_i} \text{ if } \sum_{i=1}^{n} q_i g_i \in B$$

for all $(q_1, \dots, q_n) \in \mathbb{Q}^n$.

The partial type $\tilde{\beta}(x, \bar{y})$ is computable in $r \in S$, and $\tilde{\beta}(x, \bar{d})$ is finitely satisfiable in R because R is a dense linear order without endpoints. Since R is S-saturated, there exists some $d \in R$ such that d realizes $\tilde{\beta}(x, \bar{d})$ in R. By definition of $\tilde{\beta}(x, \bar{d})$, we have that $B < v(d) < C$. So, $v(d)$ realizes $\beta(x, \bar{g})$ in G, as desired.

The argument that (1) of Definition 2.3 holds is similar. Indeed, given $\bar{g}$ an n-tuple in G that, without loss of generality, is a basis for $\langle \bar{g} \rangle_{\mathbb{Q}}$ and $\bar{d}$ a tuple in R such that $v(d_i) = g_i$ for $1 \leq i \leq n$, we can compute the type of $\bar{g}$ in G from the type of $\bar{d}$ in R using the translation described above. Since R is S-saturated, the type of $\bar{d}$ is computable in some $r \in S$, so the type of $\bar{g}$ is computable in S. Thus, G is S-saturated.

(iii) Let $(a_i)_{i<\omega}$ be a pseudo-Cauchy sequence that is computable in some $r \in S$ over a finite tuple of parameters $\bar{d}$ in R. By Definition 5.1, there is an r-computable sequence of formulas $(\theta_i(x, \bar{y}))_{i<\omega}$ such that $\theta_i(x, \bar{d})$ defines a_i in R. The partial type $\nu(x, \bar{y})$ consisting of the formulas

$$(\exists z_i, z_{i+1})(\theta_i(z_i, \bar{y}) \wedge \theta_{i+1}(z_{i+1}, \bar{y})) \wedge n|x - z_{i+1}| < |z_i - z_{i+1}|)$$

for all $n \in \mathbb{N}$ is computable in $r \in S$. We show $\nu(x, \bar{d})$ is finitely satisfiable in R. Given a finite set of formulas $D \subset \nu(x, \bar{d})$, let $j < \omega$ be the largest number such that $\theta_j(x, \bar{d})$ appears as a subformula of an element in D. Then, $a_{j+1} \in R$ satisfies all formulas in D since $(a_i)_{i<\omega}$ is pseudo-Cauchy. Since R is S-saturated, there is some $a \in R$ such that a realizes $\nu(x, \bar{d})$ in R. This implies that $v(a - a_{i+1}) > v(a_{i+1} - a_i)$ for all $i < \omega$. Since $v(a - a_i) \geq \min\{v(a - a_{i+1}), v(a_{i+1} - a_i)\}$, it follows that

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$v(a - a_i) = v(a_{i+1} - a_i)$ for all $i < \omega$. Hence, a is a pseudo-limit of $(a_i)_{i<\omega}$, as required. So, (i), (ii), (iii), and (iv) are necessary if R is recursively saturated.

We assume that there is a Scott set S for which conditions (i), (ii), (iii), and (iv) hold for R. We show that R is S-saturated, and hence, recursively saturated by Lemma 2.4.

Let $\bar{a} = (a_1, \dots, a_n)$ be a finite tuple from R, and let $\tau_p(x, \bar{y})$ be an r'-computable set of formulas for some $r' \in S$ such that $\tau_p(x, \bar{a})$ is finitely satisfiable in R. Without loss of generality, we may assume that $\bar{a}$ is a transcendence basis for $\langle \bar{a} \rangle^{rc}$; it is computable to substitute each occurrence of a_i in $\tau_p(x, \bar{a})$ with its definition over such a basis. We first extend $\tau_p(x, \bar{y})$ to a complete type $\tau(x, \bar{y})$ computable in an element of S so that $\tau(x, \bar{a})$ is also finitely satisfiable in R. Let $\delta(\bar{y})$ be the complete type of $\bar{a}$ in R. By (iv), the type $\delta(\bar{y})$ is computable in some $s \in S$, so $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$ is computable in $r' \oplus s \in S$. Since $\tau_p(x, \bar{a}) \cup \delta(\bar{a})$ is finitely satisfiable in R, there is an $(r' \oplus s)$-computable infinite tree such that any path through it encodes a complete consistent type extending $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$. Since S is a Scott set and this tree is computable in $r' \oplus s \in S$, there is some $r \in S$ such that r computes a complete consistent extension $\tau(x, \bar{y})$ of $\tau_p(x, \bar{y}) \cup \delta(\bar{y})$. We may suppose that $\tau(x, \bar{y}) \vdash x > 0$.

Let $R' = \langle \bar{a} \rangle^{rc}$, and let G' be the value group of R'. Let $B := \{b \in R' \mid \tau(x, \bar{a}) \vdash b \le x\}$ and $C := \{c \in R' \mid \tau(x, \bar{a}) \vdash x \le c\}$. The theory of real closed fields, like the theory of divisible ordered abelian groups, admits quantifier elimination in the language of ordered rings. Hence, to realize the type $\tau(x, \bar{a})$, it suffices to realize the r-computable partial type

$$\{b \le x \mid b \in B\} \cup \{x \le c \mid c \in C\}. \tag{5}$$

If $\tau(x, \bar{a}) \vdash x = a$ for any $a \in B \cup C$, then the type in (5) is realized by $a \in B \cup C \subset R$. Hence, suppose not. Let $R'' \succ R$ be such that there is some $x_0 \in R''$ realizing $\tau(x, \bar{a})$ in R''. Consider the set $\Delta = \{v(d - x_0) \mid d \in R'\}$. We examine three cases, as we did for Theorem 4.1.

CASE 1 (*Immediate Transcendental*) - Δ has no maximum.

Let $\Delta_B := \{v(d - x_0) \mid d \in B\}$ and $\Delta_C := \{v(d - x_0) \mid d \in C\}$. At least one of Δ_B and Δ_C is cofinal in Δ. Suppose that Δ_B is cofinal in Δ (as the other case is symmetric). We construct a pseudo-Cauchy sequence in B that is computable in some element of S over parameters $\bar{a}$ and has a pseudo-limit $a \in R$ satisfying $B < a < C$. Since $\bar{a}$ is a transcendence basis of R', there is a computable enumeration of formulas $\{\psi_i(x, \bar{y})\}_{i<\omega}$ such that every element in R' is defined by exactly one formula in this sequence over the parameters $\bar{a}$ (by effective quantifier elimination for real closed fields). Let $\tilde{a}_i$ denote the element in R' defined by $\psi_i(x, \bar{a})$ in R. Note that it is r-computable to determine whether $\tilde{a}_i < \tilde{a}_j$ or $\tilde{a}_j < \tilde{a}_i$ in R' and whether $\tilde{a}_i \in B$ or $\tilde{a}_i \in C$.

We define a tree $\mathcal{T} \subset 2^{<\omega}$ computable in r so that every path through $\mathcal{T}$ corresponds to a pseudo-Cauchy sequence in B whose limit realizes the partial type in (5). Condition (iii) and the fact that S is a Scott set imply that the partial type in (5) is realized in R. For any $\sigma \in 2^{<\omega}$, we put $\sigma \in \mathcal{T}$ if the following three properties hold.

(I) For all $i < length(\sigma)$, if $\tilde{a}_i \in C$, then $\sigma(i) = 0$;

(II) For all $i < length(\sigma)$, set $a'_i := 0$ if for all $j \le i$, $\sigma(j) = 0$, and otherwise, set $a'_i := \tilde{a}_{j'}$ where $j' = \max\{j \le i \mid \sigma(j) = 1\}$. (Note that $a'_i \in B$ since $x_0 > 0$.) Then,

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$$(\forall j \leq i < length(\sigma))(\tilde{a}_j \in B \implies \tilde{a}_j \leq a'_i), \text{ i.e.,}$$
$$\{\tilde{a}_j \mid \tilde{a}_j \in B \ \&\ j \leq i\} \leq a'_i \leq \{\tilde{a}_j \mid \tilde{a}_j \in C \ \&\ j \leq i\}.$$

(III) $(\forall i < j < k < m = length(\sigma))$
$(\sigma(i) = \sigma(j) = \sigma(k) = 1 \implies m|\tilde{a}_k - \tilde{a}_j| < |\tilde{a}_j - \tilde{a}_i|)$.

It is clear that $\mathcal{T}$ is a tree and computable in r by definition.

We now show that $\mathcal{T}$ is infinite. Since Δ_B has no largest element, there exists a cofinal sequence in Δ_B (which is then cofinal in Δ). Moreover, since R' is countable and $B < x_0 < C$ in R'', we can choose the cofinal sequence $(v(\tilde{a}_{i_l} - x_0))_{l<\omega} \in \Delta_B$ so that $(i_l)_{l<\omega}$ and $(v(\tilde{a}_{i_l} - x_0))_{l<\omega}$ are increasing and satisfying the following property (reminiscent of (II)):

(II′) for each $n < \omega$, set $a'_n := 0$ if no $j \leq n$ equals some i_l and otherwise set $a'_n := \tilde{a}_{i_{l'}}$ where index $i_{l'} = \max\{i_l \mid i_l \leq n\}$. Then,

$$(\forall j \leq n)(\tilde{a}_j \in B \implies \tilde{a}_j \leq a'_n).$$

We prove that there is a path $\mathcal{P}'$ through $\mathcal{T}$. Define $\mathcal{P}' \in 2^\omega$ so that $\mathcal{P}'(j) = 1$ if and only if $j = i_l$ for some $l \in \omega$. Let $\sigma_n = \mathcal{P}' \upharpoonright n$. It is clear that σ_n satisfies (I) and (II) by definition. We show that σ_n satisfies (III). Suppose $i < j < k < n$ with

$$\sigma_n(i) = \sigma_n(j) = \sigma_n(k) = 1,$$

i.e., $i = i_l$, $j = i_{l'}$ and $k = i_{l''}$ with $l < l' < l''$. It suffices to show that $v(\tilde{a}_j - \tilde{a}_i) < v(\tilde{a}_k - \tilde{a}_j)$. We have that

$$v(\tilde{a}_{i_l} - x_0) < v(\tilde{a}_{i_{l'}} - x_0) < v(\tilde{a}_{i_{l''}} - x_0) \text{ and}$$
$$v(\tilde{a}_j - \tilde{a}_i) = \min\{v(\tilde{a}_{i_{l'}} - x_0), v(\tilde{a}_{i_l} - x_0)\} = v(\tilde{a}_{i_l} - x_0). \text{ Hence,}$$
$$v(\tilde{a}_k - \tilde{a}_j) = \min\{v(\tilde{a}_{i_{l''}} - x_0), v(\tilde{a}_{i_{l'}} - x_0)\} = v(\tilde{a}_{i_{l'}} - x_0) \text{ so}$$
$$v(\tilde{a}_j - \tilde{a}_i) < v(\tilde{a}_k - \tilde{a}_j), \text{ as desired.}$$

Hence, $\mathcal{T}$ is an infinite tree computable in r. Since S is a Scott set, there exists a path $\mathcal{P}$ through $\mathcal{T}$ computable in some $t \in S$. Note that B has no maximum element in R' since Δ_B has no largest element. Hence, there are infinitely many $j < \omega$ such that $\mathcal{P}(j) = 1$ by property (II) of $\mathcal{T}$. Uniformly in $l < \omega$, we can t-computably find the index k_l such that $\mathcal{P}(k_l) = 1$ and $|\{j \leq k_l \mid \mathcal{P}(j) = 1\}| = l$. By property (III) of the definition of $\mathcal{T}$, the sequence $(\tilde{a}_{k_l})_{l<\omega}$ is pseudo-Cauchy. Since $(\psi_{k_l}(x, \bar{y}))_{l<\omega}$ is t-computable, the sequence $(\tilde{a}_{k_l})_{l<\omega}$ has a pseudo-limit $a \in R$ by (iii).

Since Δ_B is cofinal in Δ and $(\tilde{a}_{k_l})_{l<\omega}$ is cofinal in B by property (II) of $\mathcal{T}$, we have that $(v(\tilde{a}_{k_l} - x_0))_{l<\omega}$ is cofinal in Δ. We have that

$$v(a - x_0) \geq \min\{v(a - \tilde{a}_{k_l}), v(\tilde{a}_{k_l} - x_0)\},$$
$$v(a - \tilde{a}_{k_l}) = v(\tilde{a}_{k_{l+1}} - \tilde{a}_{k_l}) \text{ (since } a \text{ is a pseudo-limit), and}$$
$$v(\tilde{a}_{k_{l+1}} - \tilde{a}_{k_l}) \geq \min\{v(\tilde{a}_{k_{l+1}} - x_0), v(\tilde{a}_{k_l} - x_0)\} = v(\tilde{a}_{k_l} - x_0)$$

(since $\tilde{a}_{k_l} < \tilde{a}_{k_{l+1}} < x_0$ by definition of $\mathcal{T}$). We can conclude that $v(a - x_0) \geq v(\tilde{a}_{k_l} - x_0)$ for all $l < \omega$ and, thus, that $v(a - x_0) > v(d - x_0)$ for all $d \in R'$.

We show that $B < a < C$ holds in R, and so a realizes the type in (5). Let $b \in B \subset R'$. We claim that $b < a$. If not, $a \leq b < x_0$, and hence, $v(b - x_0) \geq v(a - x_0)$, a contradiction. The argument that $a < c$ for any $c \in C$ is similar. Similarly, $a < C$.

We now suppose that Δ has a maximum element g. Let $d_0 \in R'$ be such that $v(d_0 - x_0) = g$. The remaining two cases, corresponding to $g \in G'$ and $g \notin G'$, are proved essentially as in Theorem 3.2 [9, Theorem 6.2]. We include the outline of these proofs for completeness but omit the proofs of Claims 5.3 and 5.4, which correspond to Claims 4.3 and 4.4, respectively in the group case.

CASE 2 (*Residue Transcendental*) - Δ has a maximum $g \in G'$.

Let $a \in R'$ be such that $a > 0$ and $v(d_0 - x_0) = g = v(a)$.

CLAIM 5.3 (see [10, Theorem 6.2]). *There exist $b_0 \in B$ and $c_0 \in C$ such that for all $b \in B$ with $b \geq b_0$ and for all $c \in C$ with $c \leq c_0$, we have*

$$v(b - d_0) = g = v(a) = v(c - d_0) \text{ and, hence,}$$
$$v(b - x_0) = g = v(a) = v(x_0 - d_0) = v(c - x_0).$$

Consider the r-computable partial type:

$$\left\{\frac{b - d_0}{a} < x \mid b \in B \,\&\, b \geq b_0\right\} \cup \left\{x < \frac{c - d_0}{a} \mid c \in C \,\&\, c \leq c_0\right\}. \tag{6}$$

It suffices to find some $x' \in R$ realizing the type in (6) since then $x = a \cdot x' + d_0$ realizes the type in (5). From Claim 5.3, one can prove that the sets

$$\left\{q \in \mathbb{Q} \mid q < \frac{b - d_0}{a} \,\&\, b_0 \leq b \in B\right\} \text{ and } \left\{q \in \mathbb{Q} \mid \frac{c - d_0}{a} < q \,\&\, c_0 \geq c \in C\right\}$$

form an r-computable cut in $\mathbb{R}$ filled by some $t' \in \mathbb{R}$. Since t' is computable in r, we have $t' \in S \cong k$ by (ii) and so there is some $x' \in R$ with $\bar{x'} = t'$. Then, $x' \in R$ realizes the partial type in (6).

CASE 3 (*Value Transcendental*) - Δ has a maximum $g \notin G'$.

We suppose that $d_0 \in B$; the case that $d_0 \in C$ is similar. Consider the sets $\Delta_1 = \{v(c - d_0) \mid c \in C\}$ and $\Delta_2 = \{v(b - d_0) \mid b \in B \,\&\, b > d_0\}$.

CLAIM 5.4 (see [10, Theorem 6.2]). $\Delta_1 < g < \Delta_2$.

The real closed field R' has finite absolute transcendence degree, so G' has finite rational rank (see [16], Section 10). Take $d_1, \ldots, d_n \in R'$ so that $\{v(d_i) \mid 1 \leq i \leq n\}$ is a basis for G' and the multiplicative subgroup

$$\left\{\prod_{i=1}^{n} d_i^{q_i} \in R' \mid q_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n\right\}$$

is a section for G' in R'. We show that there is an r-computable partial type $\beta(y, v(d_1), \ldots, v(d_n))$ (in the language of ordered groups) that describes the cut given by $\Delta_1 \cup \Delta_2$ in G'.

Since $\{v(d_i) \mid 1 \leq i \leq n\}$ is a basis for G', every element in G' equals $\sum_{i=1}^{n} q_i v(d_i)$ for some $(q_1, \ldots, q_n) \in \mathbb{Q}^n$. Since $r \in S$ computes Δ_1 and Δ_2, it is r-computable to determine whether a given $(q_1, \ldots, q_n) \in \mathbb{Q}^n$ satisfies $\sum_{i=1}^{n} q_i v(d_i) \in \Delta_1$ or

$\sum_{i=1}^n q_i v(d_i) \in \Delta_2$. We let $\beta(x, y_1, \ldots, y_n)$ be the r-computable partial type consisting of the formulas

$$\sum_{i=1}^n q_i y_i < x \text{ if } \sum_{i=1}^n q_i y_i \in \Delta_1 \text{ or } x < \sum_{i=1}^n q_i y_i \text{ if } \sum_{i=1}^n q_i y_i \in \Delta_2.$$

for all $(q_1, \ldots, q_n) \in \mathbb{Q}^n$. Since G is divisible, $\beta(x, v(d_1), \ldots, v(d_n))$ is finitely satisfiable in G.

By (i), Lemma 2.4, and Theorem 4.1, G is S-saturated. Since $r \in S$, there exists some $h \in G$ realizing $\beta(x, v(d_1), \ldots, v(d_n))$ in G. Thus, $\Delta_1 < h < \Delta_2$ in G. Let $a \in R$ satisfy $v(a) = h$ and $a > 0$. Then, for all $c \in C$ and $b \in B$ with $b > d_0$,

$$v(c - d_0) < v(a) < v(b - d_0) \text{ and so, } b - d_0 < a < c - d_0.$$

Thus, $B < a + d_0 < C$, so $a + d_0$ realizes the type given in (5).

Therefore, in each of the three cases, $\tau(x, \bar{a})$ is realized in R. This completes the proof that R is recursively saturated. ⊣

Remark 5.5. (1) For countable real closed fields, the four conditions in Theorem 5.2 are equivalent to the existence of an integer part that is a nonstandard model of Peano Arithmetic by Theorems 5.1 and 5.2 in [4].

(2) Let S be a Scott set. Proposition 3.1 in [4] together with the results mentioned in Section 2.1 give a recursively saturated real closed field with residue field S (under the Continuum Hypothesis, CH). Marker (personal communication) has recently observed that for any Scott set S there is a recursively saturated real closed field R with residue field S, regardless of the status of CH.

(3) Condition (iv) does not follow from the other three conditions in Theorem 5.2, as witnessed by the following example of D. Marker [14].

Let S be a countable Scott set. By Corollary 4.6, $G = \oplus_{\mathbb{Q}} S$ is recursively saturated. Then, $R = S((G))$ satisfies the first three conditions listed in Theorem 5.2, but R realizes 2-types of arbitrary complexity. Let $f \in 2^\omega$, and let $a = \sum_{n<\omega} f(n)t^{ng}$ for some $g \in G$ with $g > 0$. Then, f is computable in the complete type $\delta(x, y)$ of (a, g). So, R does not satisfy (iv).

(4) In Theorem 4.1, we use a valuation basis for G to avoid the need for a condition like (iv). However, a real closed field of finite absolute transcendence degree need not admit a valuation transcendency basis; see [8] for a precise definition of valuation transcendency basis and counterexamples.

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