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Stable laws for random dynamical systems

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Abstract. In this paper, we consider random dynamical systems formed by concatenating maps acting on the unit interval [0, 1] in an independent and identically distributed (i.i.d.) fashion. Considered as a stationary Markov process, the random dynamical system possesses a unique stationary measure ν. We consider a class of non-square-integrable observables ϕ, mostly of form $\phi(x) = d(x, x_0)^{-1/\alpha}$, where x_0 is a non-recurrent point (in particular a non-periodic point) satisfying some other genericity conditions and, more generally, regularly varying observables with index $\alpha \in (0, 2)$. The two types of maps we concatenate are a class of piecewise C^2 expanding maps and a class of intermittent maps possessing an indifferent fixed point at the origin. Under conditions on the dynamics and α, we establish Poisson limit laws, convergence of scaled Birkhoff sums to a stable limit law, and functional stable limit laws in both the annealed and quenched case. The scaling constants for the limit laws for almost every quenched realization are the same as those of the annealed case and determined by ν. This is in contrast to the scalings in quenched central limit theorems where the centering constants depend in a critical way upon the realization and are not the same for almost every realization.

Key words: stable limit laws, random dynamical systems, Poisson limit laws
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1. *Introduction*

In this paper, we consider non-square-integrable observables $\phi : [0, 1] \to \mathbb{R}$ on two simple classes of random dynamical system. One consists of randomly choosing in an independent and identically distributed (i.i.d.) manner from a finite set of maps which are strictly polynomially mixing with an indifferent fixed point at the origin, the other consisting of randomly choosing from a finite set of maps which are uniformly expanding and exponentially mixing. The main type of observable we consider is of the form $\phi(x) = |x - x_0|^{-1/\alpha}$, $\alpha \in (0, 2)$ which, in the i.i.d. case, lies in the domain of attraction of a stable law of index α. For certain results, the point x_0 has to satisfy some non-genericity conditions and, in particular, not be a periodic point for almost every realization of the random system (see Definition 2.3). Some of our results, particularly those involving convergence to exponential and Poisson laws, hold for general observables that are regularly varying with index α.

The settings for investigations on stable limit laws for observables on dynamical systems tend to be of two broad types: (1) 'good observables' (typically Hölder) on slowly mixing non-uniformly hyperbolic systems; and (2) 'bad' observables (unbounded with fat tails) on fast mixing dynamical systems. As illustrative examples of both settings, we give two results.

Example of type (1): The LSV intermittent map $T_\gamma : [0, 1] \to [0, 1]$, $\gamma \in (0, 1)$, is defined by

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } 0 \le x \le \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

The map T_γ has a unique absolutely continuous invariant measure μ_γ.

Gouëzel [**Gou04**, Theorem 1.3] showed that if $\gamma > \frac{1}{2}$ and $\phi : [0, 1] \to \mathbb{R}$ is Hölder continuous with $\phi(0) \neq 0$, $E_{\mu_\gamma}(\phi) = 0$, then for $\alpha = 1/\gamma$,

$$\frac{1}{bn^{1/\alpha}} \sum_{j=0}^{n-1} \phi \circ T^j \to^d X_{\alpha,\beta}$$

(β has a complicated expression).

Example of type (2): Gouëzel [**Gou**, Theorem 2.1] showed that if $T : [0, 1] \to [0, 1]$ is the doubling map $T(x) = 2x \pmod 1$ with invariant measure m, Lebesgue, and $\phi(x) = x^{-1/\alpha}$, $\alpha \in (0, 2)$, then there exists a sequence c_n such that

$$\frac{2^{1/\alpha} - 1}{n^{1/\alpha}} \sum_{j=0}^{n-1} \phi \circ T^j - c_n \to^d X_{\alpha,1}.$$

For further results on the first type, we refer to the influential papers [**Gou04**, **Gou07**] and [**MZ15**]. In the setting of 'good observables' (typically Hölder) on slowly mixing non-uniformly hyperbolic systems, the technique of inducing on a subset of phase space and constructing a Young tower has been used with some success. 'Good' observables lift to well-behaved observables lying in a suitable Banach space on the Young tower. This is not the case with unbounded observables with fat tails, though in [**Gou04**], the induction technique allows an observable to be unbounded at the fixed point in a family of intermittent maps.

For further results on the second type, we refer to the papers by Marta Tyran-Kaminska [**TK10a**, **TK10b**]. In the setting of Gibbs–Markov maps, she shows, among other results,

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that functions which are measurable with respect to the Gibbs–Markov partition and in the domain of attraction of a stable law with index α converge (under the appropriate scaling) in the J_1 topology to a Lévy process of index α [**TK10b**, Theorem 3.3, Corollaries 4.1 and 4.2].

For recent results on limit laws, though not stable laws, in the setting of skew-products with an ergodic base map and uniformly hyperbolic fiber maps, see also [**DFGTV20a**]. For a still very useful survey of techniques and ideas in random dynamical systems, we refer to [**Kif98**].

Our main results are given in §2. An introduction to stable laws and a discussion of modes of convergence are given in §§3 and 4. The Poisson point approach and its application to our random setting are detailed in §5. Results on convergence of return times to an exponential law and our point processes to a Poisson process are given in §6 (though the proofs of these results are delayed until §§8.1, 8.2, 9.1, and 9.2). The proofs of the main results are given in §10. We conclude in §11 with results on stable laws for the corresponding annealed systems.

2. *Main results*

For the sake of concreteness, we restrict ourselves to observables of the form

$$\phi_{x_0}(x) = |x - x_0|^{-1/\alpha}, \quad x \in [0, 1], \tag{2.1}$$

where x_0 is a non-recurrent point (see Definition 2.3) and $\alpha \in (0, 2)$, but it is possible to consider more general regularly varying observables ϕ which are piecewise monotonic with finitely many branches, see for instance [**TK10b**, §4.2] in the deterministic case. Note that ϕ_{x_0} is regularity varying with index α.

We will be considering the following set-up with (Ω, σ) the full two-sided shift on finitely many symbols. In most of our settings, we take $Y = [0, 1]$.

Let $\sigma : \Omega \to \Omega$ be an invertible ergodic measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a measurable space $(Y, \mathcal{B})$, let $\sigma : \Omega \to \Omega$ be the usual full shift and define

$$F : \Omega \times Y \to \Omega \times Y$$

by

$$F(\omega, x) = (\sigma\omega, T_\omega(x)).$$

We assume F preserves a probability measure ν on $\Omega \times Y$. We assume that ν admits a disintegration given by $\nu(d\omega, dx) = \mathbb{P}(d\omega)\nu^\omega(dx)$. For all $n \geq 1$, we have

$$F^n(\omega, x) = (\sigma^n\omega, T_\omega^n x),$$

where

$$T_\omega^n = T_{\sigma^{n-1}\omega} \circ \cdots \circ T_\omega,$$

which satisfies the equivariance relations $(T_\omega^n)_*\nu^\omega = \nu^{\sigma^n\omega}$ for $\mathbb{P}$-almost every (a.e.) $\omega \in \Omega$.

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For each $\omega \in \Omega$, we denote by P_ω the transfer operator of T_ω with respect to the Lebesgue measure m: for all $\phi \in L^\infty(m)$ and $\psi \in L^1(m)$,

$$\int_{[0,1]} (\phi \circ T_\omega) \cdot \psi \, dm = \int_{[0,1]} \phi \cdot P_\omega \psi \, dm.$$

We can then form, for $\omega \in \Omega$ and $n \geq 1$, the cocycle

$$P_\omega^n = P_{\sigma^{n-1}\omega} \circ \cdots \circ P_\omega.$$

Definition 2.1. (Scaling constants) We consider a sequence $(b_n)_{n\geq 1}$ of positive real numbers such that

$$\lim_{n\to\infty} n\nu(\phi_{x_0} > b_n) = 1. \tag{2.2}$$

Definition 2.2. (Centering constants) We define the centering sequence $(c_n)_{n\geq 1}$ by

$$c_n = \begin{cases} 0 & \text{if } \alpha \in (0, 1); \\ n\mathbb{E}_\nu(\phi_{x_0} \mathbf{1}_{\{\phi_{x_0} \leq b_n\}}) & \text{if } \alpha = 1; \\ n\mathbb{E}_\nu(\phi_{x_0}) & \text{if } \alpha \in (1, 2). \end{cases}$$

We now introduce two classes of random dynamical system (RDS) for which we are able to establish stable limit laws.

2.1. *Random uniformly expanding maps.* We consider random i.i.d. compositions with additional assumptions of uniform expansion. Let $\mathcal{S}$ be a finite collection of m piecewise C^2 uniformly expanding maps of the unit interval [0, 1]. More precisely, we assume that for each $T \in \mathcal{S}$, there exist a finite partition $\mathcal{A}_T$ of [0, 1] into intervals, such that for each $I \in \mathcal{A}_T$, T can be continuously extended as a strictly monotonic C^2 function on $\bar{I}$ and

$$\lambda := \inf_{I\in\mathcal{A}_T} \inf_{x\in\bar{I}} |T'(x)| > 1.$$

The maps T_ω (determined by the zeroth coordinate of ω) are chosen from $\mathcal{S}$ in an i.i.d. fashion according to a Bernoulli probability measure $\mathbb{P}$ on $\Omega := \{1, \ldots, m\}^{\mathbb{Z}}$. We will denote by $\mathcal{A}_\omega$ the partition of monotonicity of T_ω, and by $\mathcal{A}_\omega^n = \vee_{k=0}^{n-1} (T_\omega^k)^{-1}(\mathcal{A}_{\sigma^k\omega})$ the partition associated to T_ω^n. We introduce

$$\mathcal{D} = \bigcup_{n\geq 0} \bigcup_{\omega\in\Omega} \partial\mathcal{A}_\omega^n,$$

the set of discontinuities of all the maps T_ω^n. Note that $\mathcal{D}$ is at most a countable set.

In the uniformly expanding case, we also assume the conditions (LY), (Dec), and (Min). Condition (LY) is the usual Lasota–Yorke inequality while conditions (Dec) and (Min) were introduced by Conze and Raugi [**CR07**].

(LY) There exist $r \geq 1$, $M > 0$ and $D > 0$ and $\rho \in (0, 1)$ such that for all $\omega \in \Omega$ and all $f \in \mathrm{BV}$,

$$\|P_\omega f\|_{\mathrm{BV}} \leq M\|f\|_{\mathrm{BV}}$$

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and

$$\mathrm{Var}(P_\omega^r f) \leq \rho \mathrm{Var}(f) + D\|f\|_{L^1(m)}.$$

(Dec) There exist $C > 0$ and $\theta \in (0, 1)$ such that for all $n \geq 1$, all $\omega \in \Omega$, and all $f \in \mathrm{BV}$ with $\mathbb{E}_m(f) = 0$:

$$\|P_\omega^n f\|_{\mathrm{BV}} \leq C\theta^n \|f\|_{\mathrm{BV}}.$$

(Min) There exist $c > 0$ such that for all $n \geq 1$ and all $\omega \in \Omega$,

$$\inf_{x\in[0,1]} (P_\omega^n \mathbf{1})(x) \geq c > 0.$$

Definition 2.3. We say that x_0 is non-recurrent if x_0 satisfies the condition $T_\omega^n(x_0) \neq x_0$ for all $n \geq 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

THEOREM 2.4. *In the setting of expanding maps, assume conditions* (LY), (Min)*, and* (Dec)*. Suppose that $x_0 \notin \mathcal{D}$ is non-recurrent and consider the observable ϕ_{x_0}.*

If $\alpha \in (0, 1)$, then for $\mathbb{P}$-a.e. $\omega \in \Omega$, the functional stable limit holds:

$$X_n^\omega(t) := \frac{1}{b_n}\left[\sum_{j=0}^{\lfloor nt\rfloor-1} \phi_{x_0} \circ T_\omega^j - tc_n\right] \xrightarrow{d} X_{(\alpha)}(t) \quad \text{in } \mathbb{D}[0, \infty)$$

in the J_1 topology under the probability measure ν^ω, where $X_{(\alpha)}(t)$ is the α-stable process with Lévy measure $d\Pi_\alpha(dx) = \alpha|x|^{-(\alpha+1)}$ on $[0, \infty)$.

If $\alpha \in [1, 2)$, then the same result holds for m-a.e. x_0.

Example 2.5. (β-transformations) A simple example of a class of maps satisfying conditions (LY), (Dec), and (Min) [**CR07**] is to take m β-maps of the unit interval, $T_{\beta_i}(x) = \beta_i x \pmod 1$. We suppose $\beta_i > 1 + a$, $a > 0$, for all β_i, $i = 1, \ldots, m$.

2.2. *Random intermittent maps.* Now we consider a simple class of intermittent type maps.

Liverani, Saussol, and Vaienti [**LSV99**] introduced the map T_γ as a simple model for intermittent dynamics:

$$T_\gamma : [0, 1] \to [0, 1], \quad T_\gamma(x) := \begin{cases} (2^\gamma x^\gamma + 1)x & \text{if } 0 \leq x < \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

If $0 \leq \gamma < 1$, then T_γ has an absolutely continuous invariant measure μ_γ with density h_γ bounded away from zero and satisfying $h_\gamma(x) \sim Cx^{-\gamma}$ for x near zero.

We form a random dynamical system by selecting $\gamma_i \in (0, 1)$, $i = 1, \ldots, m$ in an i.i.d. fashion and setting $T_i := T_{\gamma_i}$. The associated Markov process on $[0, 1]$ has a stationary invariant measure ν which is absolutely continuous, with density h bounded away from zero.

We denote $\gamma_{\max} := \max_{1\leq i\leq m}\{\gamma_i\}$ and $\gamma_{\min} := \min_{1\leq i\leq m}\{\gamma_i\}$.

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THEOREM 2.6. *In the setting of an i.i.d. random composition of intermittent maps, suppose* $\alpha \in (0,1)$ *and* $\gamma_{\max} < \frac{1}{3}$*. Then, for m-a.e.* x_0, $(1/b_n)\sum_{j=0}^{n-1}\phi_{x_0}\circ T_\omega^j \xrightarrow{d} X_{(\alpha)}(1)$ *under the probability measure* ν^ω *for* $\mathbb{P}$*-a.e.* ω *(recall that* $c_n = 0$ *for* $\alpha \in (0,1)$*).*

Remark 2.7. (Convergence with respect to Lebesgue measure) We state our limiting theorems with respect to the fiberwise measures ν^ω, but by general results of Eagleson [**Eag76**] (see also [**Zwe07**]), the convergence holds with respect to any measure μ for which $\mu \ll \nu^\omega$, in particular, our convergence results hold with respect to Lebesgue measure m. Further details are given in the Appendix.

Our proofs are based on a Poisson process approach developed for dynamical systems by Tyran-Kaminska [**TK10a**, **TK10b**].

3. *Probabilistic tools*

In this section, we review some topics from probability theory.

3.1. *Regularly varying functions and domains of attraction.*

We refer to Feller [**Fel71**] or Bingham, Goldie, and Teugels [**BGT87**] for the relations between domains of attraction of stable laws and regularly varying functions. For ϕ regularly varying, we define the constants b_n and c_n as in the case of ϕ_{x_0}.

Remark 3.1. When $\alpha \in (0,1)$, then ϕ is not integrable and one can choose the centering sequence (c_n) to be identically 0. When $\alpha = 1$, it might happen that ϕ is not integrable, and it is then necessary to define c_n with suitably truncated moments as above. If ϕ is integrable, then center by $c_n = n\mathbb{E}_\nu(\phi)$.

We will use the following asymptotics for truncated moments, which can be deduced from Karamata's results concerning the tail behavior of regularly varying functions. Define p by $\lim_{x\to\infty}(\nu(\phi > x)/\nu(|\phi| > x)) = p$.

PROPOSITION 3.2. (Karamata) *Let* ϕ *be regularly varying with index* $\alpha \in (0,2)$*. Then, setting* $\beta := 2p - 1$ *and, for* $\varepsilon > 0$,

$$c_\alpha(\varepsilon) := \begin{cases} 0 & \text{if } \alpha \in (0,1); \\ -\beta \log \varepsilon & \text{if } \alpha = 1; \\ \varepsilon^{1-\alpha}\beta\alpha/(\alpha-1) & \text{if } \alpha \in (1,2), \end{cases} \tag{3.1}$$

and the following hold for all $\varepsilon > 0$*:*

(a) $\mathbb{E}_\nu(|\phi|^2 \mathbf{1}_{\{|\phi|\le \varepsilon b_n\}}) \sim \alpha/(2-\alpha)(\varepsilon b_n)^2 \nu(|\phi| > \varepsilon b_n)$;

(b) *if* $\alpha \in (0,1)$,

$$\mathbb{E}_\nu(|\phi|\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) \sim \frac{\alpha}{1-\alpha}\varepsilon b_n \nu(|\phi| > \varepsilon b_n);$$

(c) *if* $\alpha \in (1,2)$,

$$\lim_{n\to\infty}\frac{n}{b_n}\mathbb{E}_\nu(\phi\mathbf{1}_{\{|\phi|>\varepsilon b_n\}}) = c_\alpha(\varepsilon);$$

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(d) *if* $\alpha = 1$,

$$\lim_{n\to\infty} \frac{n}{b_n}\mathbb{E}_\nu(\phi \mathbf{1}_{\{\varepsilon b_n < |\phi| \le b_n\}}) = c_\alpha(\varepsilon);$$

(e) *if* $\alpha = 1$,

$$\frac{n}{b_n}\mathbb{E}_\nu(|\phi|\mathbf{1}_{\{|\phi|\le \varepsilon b_n\}}) \sim \widetilde{L}(n)$$

for a slowly varying function $\widetilde{L}$.

3.2. *Lévy α-stable processes.* A helpful and more detailed discussion can be found, e.g., in [**TK10a**, **TK10b**].

The $X(t)$ is a Lévy stable process if $X(0) = 0$, X has stationary independent increments, and $X(1)$ has an α-stable distribution.

The Lévy–Khintchine representation for the characteristic function of an α-stable random variable $X_{\alpha,\beta}$ with index $\alpha \in (0, 2)$ and parameter $\beta \in [-1, 1]$ has the form:

$$\mathbb{E}[e^{itX}] = \exp\bigg[ita_\alpha + \int (e^{itx} - 1 - itx1_{[-1,1]}(x))\Pi_\alpha(dx)\bigg],$$

where:

- $a_\alpha = \begin{cases} \beta(\alpha/(1-\alpha)) & \alpha \neq 1; \\ 0 & \alpha = 1; \end{cases}$
- Π_α is a Lévy measure given by
$$d\Pi_\alpha = \alpha(p1_{(0,\infty)}(x) + (1-p)1_{(-\infty,0)}(x))|x|^{-\alpha-1}\, dx;$$
- $p = (\beta + 1)/2$.

Note that p and β may equally serve as parameters for $X_{\alpha,\beta}$. We will drop the β from $X_{\alpha,\beta}$, as is common in the literature, for simplicity of notation and when it plays no essential role.

3.3. *Poisson point processes.* Let $(T_n)_{n\ge1}$ be a sequence of measurable transformations on a probability space $(Y, \mathcal{B}, \mu)$. For $n \ge 1$, we denote

$$T_1^n := T_n \circ \cdots \circ T_1. \tag{3.2}$$

Given $\phi : Y \to \mathbb{R}$ measurable, recall that we define the scaled Birkhoff sum by

$$S_n := \frac{1}{b_n}\bigg[\sum_{j=0}^{n-1} \phi \circ T_1^j - c_n\bigg] \tag{3.3}$$

for some real constants $b_n > 0$, c_n and the scaled random process $X_n(t)$, $n \ge 1$, by

$$X_n(t) := \frac{1}{b_n}\bigg[\sum_{j=0}^{\lfloor nt\rfloor-1} \phi \circ T_1^j - tc_n\bigg], \quad t \ge 0, \tag{3.4}$$

For $X_\alpha(t)$, a Lévy α-stable process, and $B \in \mathcal{B}((0,\infty) \times (\mathbb{R} \setminus \{0\}))$, define

$$N_{(\alpha)}(B) := \#\{s > 0 : (s, \Delta X_\alpha(s)) \in B\},$$

where $\Delta X_\alpha(t) := X_\alpha(t) - X_\alpha(t^-)$.

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The random variable $N_{(\alpha)}(B)$, which counts the jumps (and their time) of the Lévy process that lie in B, is finite almost surely (a.s.) if and only if $(m \times \Pi_\alpha)(B) < \infty$. In that case, $N_{(\alpha)}(B)$ has a Poisson distribution with mean $(m \times \Pi_\alpha)(B)$.

Similarly define

$$N_n(B) := \#\bigg\{ j \geq 1 : \bigg(\frac{j}{n}, \frac{\phi \circ T_1^{j-1}}{b_n} \bigg) \in B \bigg\}, \quad n \geq 1,$$

$N_n(B)$ counts the jumps of the process in equation (3.4) that lie in B. When a realization $\omega \in \Omega$ is fixed, we define

$$N_n^\omega(B) := \#\bigg\{ j \geq 1 : \bigg(\frac{j}{n}, \frac{\phi \circ T_\omega^{j-1}}{b_n} \bigg) \in B \bigg\}, \quad n \geq 1.$$

Definition 3.3. We say N_n converges in distribution to $N_{(\alpha)}$ and write

$$N_n \xrightarrow{d} N_{(\alpha)}$$

if and only if $N_n(B) \xrightarrow{d} N_{(\alpha)}(B)$ for all $B \in B((0, \infty) \times (\mathbb{R} \setminus \{0\}))$ with $(m \times \Pi_\alpha)(B) < \infty$ and $(m \times \Pi_\alpha)(\partial B) = 0$.

4. *Modes of convergence*

Consider the process X_α determined by the observable ϕ (that is, an i.i.d. version of ϕ which regularly varies with the same index α and parameter p). We are interested in the following limits.

(A) *Poisson point process convergence.*

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}$$

with respect to ν^ω for $\mathbb{P}$ a.e. ω, where $N_{(\alpha)}$ is the Poisson point process of an α-stable process with parameter determined by ν, the annealed measure.

(B) *Stable law convergence.*

$$S_n^\omega := \frac{1}{b_n}\bigg[\sum_{j=0}^{n-1} \phi \circ T_\omega^j - c_n \bigg] \xrightarrow{d} X_\alpha(1)$$

for $\mathbb{P}$-a.e. ω, with respect to ν^ω, for ϕ regularly varying with index α and $X_\alpha(t)$ the corresponding α-stable process, for suitable scaling and centering constants b_n and c_n.

(C) *Functional stable law convergence.*

$$X_n^\omega(t) := \frac{1}{b_n}\bigg[\sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \circ T_\omega^j - tc_n \bigg] \xrightarrow{d} X_\alpha(t)$$

in $\mathbb{D}[0, \infty)$ in the J_1 topology $\mathbb{P}$-a.e. ω, with respect to ν^ω for ϕ regularly varying with index α and $X_\alpha(t)$ the corresponding α-stable process.

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For the cases we are considering, the scaling constants b_n are given by equation (2.2) in Definition 2.1, and the centering constants c_n are given in Definition 2.2 (see also Remark 3.1).

Remark 4.1. In the limit laws for quenched systems that we obtain of type (B) and (C), the centering sequence c_n *does not depend on the realization* ω. This is in contrast to the case of the central limit theorem (CLT), where a random centering is necessary; see [**AA16**, Theorem 9] and [**NPT21**, Theorem 5.3].

5. *A Poisson point process approach to random and sequential dynamical systems*

Our results are based on the Poisson point process approach developed by Tyran-Kamińska [**TK10a**, **TK10b**] adapted to our random setting (see Theorems 5.1 and 5.3). Namely, convergence to a stable law or a Lévy process follows from the convergence of the corresponding (Poisson) jump processes, and control of the small jumps.

A key role is played by Kallenberg's theorem [**Kal76**, Theorem 4.7] to check convergence of the Poisson point processes, $N_n \xrightarrow{d} N_{(\alpha)}$. Kallenberg's theorem does not assume stationarity and hence we may use it in our setting.

In this section, we provide general conditions ensuring weak convergence to Lévy stable processes for non-stationary dynamical systems, following closely the approach of Tyran-Kamińska [**TK10b**]. We start from the very general setting of non-autonomous sequential dynamics and then specialize to the case of quenched random dynamical systems, which will be useful to treat i.i.d. random compositions in the later sections.

5.1. *Sequential transformations.*

Recall the notation introduced in §3.3. Here, $(T_n)_{n\geq 1}$ is a sequence of measurable transformations on a probability space $(Y, \mathcal{B}, \mu)$. For $n \geq 1$, recall we define

$$T_1^n = T_n \circ \cdots \circ T_1.$$

The proof of the following statement is essentially the same as the proof of [**TK10b**, Theorem 1.1].

Note that the measure μ does not have to be invariant. Moreover (see [**TK10b**, Remark 2.1]), the convergence $X_n \xrightarrow{d} X_{(\alpha)}$ holds even without the condition $\mu(\phi \circ T_1^j \neq 0) = 1$, which is used only for the converse implication of the 'if and only if'.

THEOREM 5.1. (Functional stable limit law, [**TK10b**, Theorem 1.1]) *Let* $\alpha \in (0, 2)$ *and suppose that* $\mu(\phi \circ T_1^j \neq 0) = 1$ *for all* $j \geq 0$. *Then* $X_n \xrightarrow{d} X_{(\alpha)}$ *in* $\mathbb{D}[0, \infty)$ *under the probability measure* μ *for some constants* $b_n > 0$ *and* c_n *if and only if:*

- $N_n \xrightarrow{d} N_{(\alpha)}$ *and*
- *for all* $\delta > 0$, $\ell \geq 1$, *with* $c_\alpha(\varepsilon)$ *given by equation (3.1),*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mu\bigg(\sup_{0\leq t\leq \ell} \bigg| \frac{1}{b_n} \bigg[\sum_{j=0}^{\lfloor nt\rfloor -1} \phi \circ T_1^j \mathbf{1}_{\{|\phi\circ T_1^j| \leq \varepsilon b_n\}} - t(c_n - b_n c_\alpha(\varepsilon)) \bigg] \bigg| \geq \delta \bigg) = 0. \tag{5.1}$$

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Remark 5.2. In some cases, the convergence $N_n \xrightarrow{d} N_{(\alpha)}$ does not hold, but one has convergence of the marginals, $N_n((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$. In this case, although unable to obtain a functional stable law convergence of type (C), we can in some settings prove the convergence to a stable law for the Birkhoff sums (convergence of type (B)).

In particular, we are unable to prove $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ for the case of random intermittent maps. However, in the setting of random uniformly expanding maps, we use the spectral gap to show that $N_n^\omega \xrightarrow{d} N_{(\alpha)}$, and then obtain the functional stable limit law.

The next statement is [**TK10b**, Lemma 2.2, part (2)], which follows from [**TK10a**, Theorem 3.2]. Again, the measure does not have to be invariant.

THEOREM 5.3. (Stable limit law, [**TK10b**, Lemma 2.2]) *For $\alpha \in (0,2)$, consider an observable ϕ on the probability measure μ, and $c_\alpha(\varepsilon)$ given by equation (3.1).*

If

$$N_n((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$$

and, for all $\delta > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mu\bigg(\bigg|\frac{1}{b_n}\bigg[\sum_{j=0}^{n-1}\phi\circ T_1^j\mathbf{1}_{\{|\phi\circ T_1^j|\le\varepsilon b_n\}} - (c_n - b_n c_\alpha(\varepsilon))\bigg]\bigg| \ge \delta\bigg) = 0, \tag{5.2}$$

then

$$\frac{1}{b_n}\bigg[\sum_{j=0}^{n-1}\phi\circ T_1^j - c_n\bigg] \xrightarrow{d} X_{(\alpha)}(1)$$

under the probability measure μ.

5.2. *Random dynamical systems.* Let $\phi : Y \to \mathbb{R}$ be a measurable function such that $\nu^\omega(\phi \ne 0) = 1$.

PROPOSITION 5.4. [**TK10b**, proof of Theorem 1.2] *Let $\alpha \in (0,1)$. With b_n as in Definition 2.1 and $c_n = 0$, suppose that for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{b_n}\sum_{j=0}^{n\ell-1}\mathbb{E}_{\nu^{\sigma^j\omega}}(|\phi|\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) = 0 \quad \text{for all } \ell \ge 1 \tag{5.3}$$

and

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}.$$

Then $X_n^\omega \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. We will check that the hypothesis of Theorem 5.1 is met for $\mathbb{P}$-a.e. ω with $T_n = T_{\sigma^{n-1}\omega}$, $\mu = \nu^\omega$. Recall that $c_n = c_\alpha(\varepsilon) = 0$ when $\alpha \in (0,1)$. Using [**KW69**, Theorem 1] (see Theorem 5.6) and the equivariance of the family of measures $\{\nu^\omega\}_{\omega\in\Omega}$, we have

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$$\nu^\omega\left(\sup_{0\le t\le \ell}\left|\frac{1}{b_n}\sum_{j=0}^{\lfloor nt\rfloor-1}\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}\right|\ge\delta\right)\le\frac{1}{\delta b_n}\sum_{j=0}^{n\ell-1}\mathbb{E}_{\nu^{\sigma^j\omega}}(|\phi|\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}),$$

which shows that the condition in equation (5.3) implies the condition in equation (5.1) for all $\delta>0$ and $\ell\ge 1$. □

Remark 5.5. One could replace the condition in equation (5.3) by one similar to that in equation (5.5), and use the argument in the proof of Proposition 5.7.

THEOREM 5.6. (Kounias and Weng [**KW69**, special case of Theorem 1 therein]) *Assume the random variables X_k are in $L^1(\mu)$. Then*

$$\mu\left(\max_{1\le k\le n}\left|\sum_{\ell=1}^{k}X_\ell\right|\ge\delta\right)\le\frac{1}{\delta}\sum_{k=1}^{n}\mathbb{E}_\mu(|X_k|).$$

PROPOSITION 5.7. *Let $\alpha\in[1,2)$.*

With b_n and c_n as in Definitions 2.1 and 2.2, and $c_\alpha(\varepsilon)$ as in equation (3.1), suppose that for all $\varepsilon>0$ and all $\ell\ge 1$,

$$\lim_{n\to\infty}\sup_{0\le t\le\ell}\left|\frac{1}{b_n}\left[\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})-t(c_n-b_nc_\alpha(\varepsilon))\right]\right|=0\quad \textit{for } \mathbb{P}\textit{-a.e. } \omega\in\Omega, \tag{5.4}$$

and that for all $\delta>0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\operatorname*{esssup}_{\omega\in\Omega}\nu^\omega\left(\max_{1\le k\le n}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})]\right|\ge\delta\right)=0. \tag{5.5}$$

If $N_n^\omega\xrightarrow{d}N_{(\alpha)}$ for $\mathbb{P}$-a.e. $\omega\in\Omega$, then $X_n^\omega\xrightarrow{d}X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Proof. As in the proof of Proposition 5.4, we check the hypothesis of Theorem 5.1 with $T_n=T_{\sigma^{n-1}\omega}$, $\mu=\nu^\omega$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. We will see that equation (5.1) follows from equations (5.4) and (5.5).

Using the equivariance of $\{\nu^\omega\}_{\omega\in\Omega}$, we see that the condition in equation (5.1) is implied by the equation (5.4) and

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu^\omega\left(\sup_{1\le k\le n\ell}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\left[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\right]\right|\ge\delta\right)=0. \tag{5.6}$$

We next show that the condition in equation (5.5) implies equation (5.6).

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Since

$$\left\{ \sup_{1\le k\le n\ell} \left| \frac{1}{b_n} \sum_{j=0}^{k-1} [\phi\circ T_\omega^j \mathbf{1}_{\{|\phi\circ T_\omega^j|\le \varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi \mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})] \right| \ge \delta \right\}$$
$$\subset \bigcup_{i=0}^{\ell-1} \left\{ \sup_{in<k\le(i+1)n} \left| \frac{1}{b_n} \sum_{j=in}^{k-1} [\phi\circ T_\omega^j \mathbf{1}_{\{|\phi\circ T_\omega^j|\le \varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi \mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})] \right| \ge \frac{\delta}{\ell} \right\},$$

we obtain that, using again the equivariance, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\nu^\omega\left(\sup_{1\le k\le n\ell} \left| \frac{1}{b_n} \sum_{j=0}^{k-1} [\phi\circ T_\omega^j \mathbf{1}_{\{|\phi\circ T_\omega^j|\le \varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi \mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})] \right| \ge \delta \right)$$
$$\le \sum_{i=0}^{\ell-1} \nu^{\sigma^{in}\omega}\left(\sup_{1\le k\le n} \left| \frac{1}{b_n} \sum_{j=0}^{k-1} [\phi\circ T_{\sigma^{in}\omega}^j \mathbf{1}_{\{|\phi\circ T_{\sigma^{in}\omega}^j|\le \varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j(\sigma^{in}\omega)}}(\phi \mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})] \right| \ge \frac{\delta}{\ell} \right)$$
$$\le \ell\cdot \operatorname*{esssup}_{\omega'\in\Omega} \nu^{\omega'}\left(\max_{1\le k\le n} \left| \frac{1}{b_n} \sum_{j=0}^{k-1} [\phi\circ T_{\omega'}^j \mathbf{1}_{\{|\phi\circ T_{\omega'}^j|\le \varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j\omega'}}(\phi \mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})] \right| \ge \frac{\delta}{\ell} \right).$$

Thus, the condition in equation (5.5) implies equation (5.6), which concludes the proof. □

The analog for the convergence to a stable law is the following proposition.

PROPOSITION 5.8. *Suppose that for $\mathbb{P}$-a.e. $\omega\in\Omega$, we have*

$$N_n^\omega((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot).$$

If $\alpha\in(0,1)$ (so $c_n=0$), we require in addition that

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty} \frac{1}{b_n}\sum_{j=0}^{n-1} \mathbb{E}_{\nu^{\sigma^j\omega}}(|\phi|\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) = 0. \tag{5.7}$$

If $\alpha\in[1,2)$, we require instead of equation (5.7) that for all $\varepsilon>0$,

$$\lim_{n\to\infty} \left| \frac{1}{b_n}\left[\sum_{j=0}^{n-1} \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) - (c_n - b_n c_\alpha(\varepsilon)) \right] \right| = 0$$

and

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty} \nu^\omega\left(\left| \frac{1}{b_n}\sum_{j=0}^{n-1} \left[\phi\circ T_\omega^j \mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) \right] \right| \ge \delta \right) = 0.$$

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Then

$$\frac{1}{b_n}\left[\sum_{j=0}^{n-1}\phi\circ T_\omega^j - c_n\right]\xrightarrow{d} X_{(\alpha)}(1)$$

under the probability measure ν^ω *for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$.

Proof. We check the conditions of Theorem 5.3.

The proof for $\alpha\in(0,1)$ is similar to the proof of Proposition 5.4, the proof of the case $\alpha\in[1,2)$ is similar to the proof of Proposition 5.7. □

5.3. *The annealed transfer operator.* We assume that the random dynamical system $F:\Omega\times[0,1]\to\Omega\times[0,1]$,

$$F(\omega,x)=(\sigma\omega,T_\omega(x)),$$

which can also be viewed as a Markov process on $[0,1]$, has a stationary measure ν with density h. The map $F:\Omega\times[0,1]\to\Omega\times[0,1]$ will preserve $\mathbb{P}\times\nu$. Recall that $\mathbb{P}:=\{(p_1,\dots,p_m)\}^{\mathbb{Z}}$.

We use the notation $P_{\mu,i}$ for the transfer operator of $T_i:[0,1]\to[0,1]$ with respect to a measure μ on $[0,1]$, that is,

$$\int f\cdot g\circ T_i\,d\mu=\int(P_{\mu,i}f)g\,d\mu\quad\text{for all } f\in L^1(\mu),\, g\in L^\infty(\mu).$$

The annealed transfer operator is defined by

$$P_\mu(f):=\sum_{i=1}^m p_iP_{\mu,i}(f)$$

with adjoint

$$U(f):=\sum_{i=1}^m p_i f\circ T_i$$

which satisfies the duality relation

$$\int f(g\circ U)\,d\mu=\int(P_\mu f)g\,d\mu\quad\text{for all } f\in L^1(\mu),\, g\in L^\infty(\mu).$$

As above, we assume there are sample measures $d\nu^\omega=h_\omega\,dx$ on each fiber $[0,1]$ of the skew product such that

$$P_\omega h_\omega=h_{\sigma\omega},$$

where P_ω is the transfer operator of T_{ω_0} with respect to the Lebesgue measure.

Therefore,

$$\nu(A)=\int_\Omega\left[\int_A h_\omega\,dx\right]d\mathbb{P}(\omega)$$

for all Borel sets $A\subset[0,1]$.

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5.4. *Decay of correlations.* We now consider the decay of correlation properties of the annealed systems associated to maps satisfying conditions (LY), (Dec), and (Min) and intermittent maps.

By [**ANV15**, Proposition 3.1] in the setting of maps satisfying conditions (LY), (Dec), and (Min), we have exponential decay in BV against L^1: there are $C > 0$, $0 < \lambda < 1$ such that

$$\left| \int fg \circ U^n \, d\nu - \int f \, d\nu \int g \, d\nu \right| \leq C\lambda^n \|f\|_{\mathrm{BV}} \|g\|_{L^1(\nu)}.$$

In the setting of intermittent maps, by [**BB16**, Theorem 1.2], we have polynomial decay in Hölder against L^∞: there exists $C > 0$ such that

$$\left| \int fg \circ U^n \, d\nu - \int f \, d\nu \int g \, d\nu \right| \leq Cn^{1-1/\gamma_{\min}} \|f\|_{\text{Hölder}} \|g\|_{L^\infty(\nu)}.$$

We now consider a useful property satisfied by our class of random uniformly expanding maps.

Definition 5.9. (Condition U) We assume that almost each ν^ω is absolutely continuous with respect to the Lebesgue measure m, and

$$\text{for some } C > 0, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega \implies C^{-1} \leq h_\omega := \frac{d\nu^\omega}{dm} \leq C, \; m\text{-a.e.} \tag{5.8}$$

$$\text{the map } \omega \in \Omega \mapsto h_\omega \in L^\infty(m) \text{ is Hölder continuous.} \tag{5.9}$$

Consequently, the stationary measure ν is also absolutely continuous with respect to m, with density $h \in L^\infty(m)$ given by $h(x) = \int_\Omega h_\omega(x)\mathbb{P}(d\omega)$ and satisfying equation (5.8).

LEMMA 5.10. *Properties* (LY), (Min), *and* (Dec) *imply Condition U. Namely, there exists a unique Hölder map* $\omega \in \Omega \mapsto h_\omega \in \mathrm{BV}$ *such that* $P_\omega h_\omega = h_{\sigma\omega}$ *and equations (5.8) and (5.9) are satisfied by* [**ANV15**].

Proof. By condition (Dec), and as all the operators P_ω are Markov with respect to m, we have

$$\|P^{n+k}_{\sigma^{-(n+k)}\omega}\mathbf{1} - P^n_{\sigma^{-n}\omega}\mathbf{1}\|_{\mathrm{BV}} \leq C\kappa^n \|\mathbf{1} - P^k_{\sigma^{-(n+k)}\omega}\mathbf{1}\|_{\mathrm{BV}} \leq C\kappa^n, \tag{5.10}$$

which proves that $(P^n_{\sigma^{-n}\omega}\mathbf{1})_{n\geq 0}$ is a Cauchy sequence in BV converging to a unique limit $h_\omega \in BV$ satisfying $P_\omega h_\omega = h_{\sigma\omega}$ for all ω. The lower bound in equation (5.8) follows from the condition (Min), while the upper bound is a consequence of the uniform Lasota–Yorke inequality of condition (LY), as actually the family $\{h_\omega\}_{\omega\in\Omega}$ is bounded in BV. To prove the Hölder continuity of $\omega \mapsto h_\omega$ with respect to the distance d_θ, we remark that if ω and ω' agree in coordinates $|k| \leq n$, then

$$\|h_\omega - h_{\omega'}\|_{\mathrm{BV}} = \|P^k_{\sigma^{-k}\omega}(h_{\sigma^{-k}\omega} - h_{\sigma^{-k}\omega'})\|_{\mathrm{BV}} \leq C\theta^n \leq Cd_\theta(\omega, \omega'). \qquad \square$$

Remark 5.11. Note that the density h of the stationary measure ν also belongs to BV and is uniformly bounded from above and below, as the average of h_ω over Ω.

5.4.1. *The sample measures h_ω.* The regularity properties of the sample measures h_ω, both as functions of ω and as functions of x on $[0, 1]$, play a key role in our estimates. We will first recall how the sample measures are constructed. Suppose $\omega := (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots, \omega_n, \ldots,)$ and define $h_n(\omega) = P_{\omega_{-1}} \ldots P_{\omega_{-n}} 1$ as a sequence of functions on the fiber I above ω. In the setting both of random uniformly expanding maps and of intermittent maps $\{h_n(\omega)\}$ is a Cauchy sequence and has a limit h_ω.

In the setting of random expanding maps, h_ω is uniformly BV in ω as

$$\|h_n(\omega) - h_{n+1}(\omega)\|_{\mathrm{BV}} \le \|P_{\omega_{-1}} P_{\omega_{-2}} \ldots P_{\omega_{-n}}(1 - P_{\omega_{-n-1}} 1)\|_{\mathrm{BV}} \le C\lambda^n.$$

In the setting of intermittent maps with $\gamma_{\max} = \max_{1\le i\le m}\{\gamma_i\}$, the densities h_ω lie in the cone

$$L := \big\{ f \in \mathcal{C}^0((0, 1]) \cap L^1(m), \quad f \ge 0,\ f \text{ non-increasing}, \\ X^{\gamma_{\max}+1} f \text{ increasing},\ f(x) \le a x^{-\gamma_{\max}} m(f) \big\}$$

where $X(x) = x$ is the identity function and $m(f)$ is the integral of f with respect to m. In [**AHN+15**], it is proven that for a fixed value of $\gamma_{\max} \in (0, 1)$, provided that the constant a is big enough, the cone L is invariant under the action of all transfer operators P_{γ_i} with $0 < \gamma_i \le \gamma_{\max}$ and so (see e.g. [**NPT21**, Proposition 3.3], which summarizes results of [**NTV18**])

$$\|h_n(\omega) - h_{n+k}(\omega)\|_{L^1(m)} \le \|P_{\omega_{-1}} P_{\omega_{-2}} \ldots P_{\omega_{-n}}(1 - P_{\omega_{-n-1}} \ldots P_{\omega_{-n-k}} 1)\|_{L^1(m)} \\ \le C_{\gamma_{\max}} n^{1-1/\gamma_{\max}} (\log n)^{1/\gamma_{\max}},$$

whence $h_\omega \in L^1(m)$. In later arguments, we will use the approximation

$$\|h_n(\omega) - h_\omega\|_{L^1(m)} \le C_{\gamma_{\max}} n^{1-1/\gamma_{\max}} (\log n)^{1/\gamma_{\max}}. \tag{5.11}$$

We mention also the recent paper [**KL21**], where the logarithm term in equation (5.11) is shown to be unnecessary and moment estimates are given.

We now show that h_ω is a Hölder function of ω on (Ω, d_θ) in the setting of random expanding maps.

For $\theta \in (0, 1)$, we introduce on Ω the symbolic metric

$$d_\theta(\omega, \omega') = \theta^{s(\omega,\omega')},$$

where $s(\omega, \omega') = \inf\{k \ge 0 : \omega_\ell \ne \omega'_\ell \text{ for some } |\ell| \le k\}$.

Suppose ω, ω' agree in coordinates $|k| \le n$ (that is, backwards and forwards in time) so that $d_\theta(\omega, \omega') \le \theta^n$ in the symbolic metric on Ω. Then

$$\|h_\omega - h_{\omega'}\|_{\mathrm{BV}} \le \|P_{\omega_{-1}} P_{\omega_1} \ldots P_{\omega_{-n+1}}(h_{(\sigma^{-n+1}\omega)} - h_{(\sigma^{-n+1}\omega')})\|_{\mathrm{BV}} \\ \le C\lambda^{n-1} = C' d_\theta(\omega, \omega')^{\log_\theta \lambda}.$$

Recall that $\|f\|_\infty \le C\|f\|_{\mathrm{BV}}$, see e.g. [**BG97**, Lemma 2.3.1].

That is, Condition U (see Definition 5.9) holds for random expanding maps.

The map $\omega \mapsto h_\omega$ is not Hölder in the setting of intermittent maps; in several arguments, we will use the regularity properties of the approximation $h_n(\omega)$ for h_ω.

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However, on intervals that stay away from zero, all functions in the cone L are comparable to their mean. Therefore, on sets that are uniformly away from zero, all the above densities/measures ($d\nu = h\,dx$, h_ω, $h_n(\omega)$) are still comparable.

Namely,

$$\text{for any } \delta \in (0,1), \text{ there is } C_\delta > 0 \text{ such that}$$
$$h \in L \implies 1/C_\delta < h(x)/m(h) < C_\delta \text{ for } x \in [\delta, 1]. \tag{5.12}$$

Indeed, $h/m(h)$ is bounded below by [**LSV99**, Lemma 2.4], and the upper bound follows from the definition of the cone.

6. *Ancilliary results*

Let $x_0 \in [0,1]$, and, for $\alpha \in (0,2)$, recall we define the function $\phi_{x_0}(x) = |x - x_0|^{-1/\alpha}$. It is easy to see that ϕ_{x_0} is regularity varying with index α and that $p = 1$.

6.1. *Exponential law and point process results.*

We denote by $\mathcal{J}$ the family of all finite unions of intervals of the form $(x, y]$, where $-\infty \le x < y \le \infty$ and $0 \notin [x, y]$.

For a measurable subset $U \subset [0,1]$, we define the hitting time of $(\omega, x) \in \Omega \times [0,1]$ to U by

$$R_U(\omega)(x) := \inf\{k \ge 1 : T_\omega^k(x) \in U\}. \tag{6.1}$$

Recall that $\phi_{x_0}(x) := d(x, x_0)^{-1/\alpha}$ depends on the choice of $x_0 \in [0,1]$. Recall also that

$$\mathcal{D} = \bigcup_{n \ge 0} \bigcup_{\omega \in \Omega} \partial \mathcal{A}_\omega^n,$$

the set of discontinuities of all the maps T_ω^n.

THEOREM 6.1. *In the setting of §2.1, assume conditions* (LY), (Min), *and* (Dec). *If $x_0 \notin \mathcal{D}$ is non-recurrent, then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $0 \le s < t$,*

$$\lim_{n\to\infty} \nu^{\sigma^{\lfloor ns \rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor n(t-s) \rfloor) = e^{-(t-s)\Pi_\alpha(J)},$$

where $A_n := \phi_{x_0}^{-1}(b_n J)$, $J \in \mathcal{J}$.

THEOREM 6.2. *In the setting of intermittent maps, assume that $\gamma_{\max} < \frac{1}{3}$. Then for m-a.e. x_0 for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $0 \le s < t$,*

$$\lim_{n\to\infty} \nu^{\sigma^{\lfloor ns \rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor n(t-s) \rfloor) = e^{-(t-s)\Pi_\alpha(J)},$$

where $A_n := \phi_{x_0}^{-1}(b_n J)$, $J \in \mathcal{J}$.

THEOREM 6.3. *In the setting of §2.1, assume conditions* (LY), (Min), *and* (Dec). *If $x_0 \notin \mathcal{D}$ is non-recurrent, then for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$N_n^\omega \xrightarrow{d} N_{(\alpha)},$$

under the probability ν^ω.

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THEOREM 6.4. *In the setting of intermittent maps for m-a.e. x_0 for $\mathbb{P}$-a.e. ω,*

$$N_n^\omega((0,1]\times\cdot)\xrightarrow{d}N_{(\alpha)}((0,1]\times\cdot).$$

After some preliminary lemmas and results, Theorem 6.1 is proved in §8.1, Theorem 6.2 in §8.2, Theorem 6.3 in §9.1, and Theorem 6.4 in §9.2.

7. *Scheme of proofs*

7.1. *Two useful lemmas.* We now proceed to the proofs of the main results. We will use the following technical propositions which are a form of spatial ergodic theorem which allows us to prove exponential and Poisson limit laws.

LEMMA 7.1. *Assume Condition U and let $\chi_n : Y \to \mathbb{R}$ be a sequence of functions in $L^1(m)$ such that $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1}\widetilde{L}(n))$ for some slowly varying function $\widetilde{L}$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all $\ell \geq 1$,*

$$\lim_{n\to\infty}\sup_{0\le k\le \ell}\left|\sum_{j=0}^{kn-1}(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n)-\mathbb{E}_\nu(\chi_n))\right|=0.$$

Therefore, given $(s,t]\subset[0,\infty)$ and $\varepsilon>0$, for $\mathbb{P}$-a.e. ω, there exists $N(\omega)$ such that

$$\left|\sum_{r=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n)-\mathbb{E}_\nu(\chi_n))\right|\le\varepsilon$$

for all $n \geq N(\omega)$.

Proof. We obtain the second claim by taking the difference between two values of ℓ in the first claim.

Fix $\ell \geq 1$. For $\delta > 0$, let

$$U_k^n(\delta)=\left\{\omega\in\Omega:\left|\sum_{j=0}^{kn-1}(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n)-\mathbb{E}_\nu(\chi_n))\right|\ge\delta\right\}$$

and

$$B^n(\delta)=\left\{\omega\in\Omega:\sup_{0\le k\le\ell}\left|\sum_{j=0}^{kn-1}(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n)-\mathbb{E}_\nu(\chi_n))\right|\ge\delta\right\}.$$

Note that

$$B^n(\delta)=\bigcup_{k=0}^{\ell}U_k^n(\delta).$$

We define $f_n(\omega)=\mathbb{E}_{\nu^\omega}(\chi_n)$ and $\overline{f}_n=\mathbb{E}_{\mathbb{P}}(f_n)$. We claim that $f_n:\Omega\to\mathbb{R}$ is Hölder with norm $\|f_n\|_\theta=\mathcal{O}(n^{-1}\widetilde{L}(n))$. Indeed, for $\omega\in\Omega$, we have

$$|f_n(\omega)|=\left|\int_Y\chi_n(x)\,d\nu^\omega(x)\right|\le\|h_\omega\|_{L_m^\infty}\|\chi_n\|_{L_m^1}\le\frac{C}{n}\widetilde{L}(n),$$

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and for $\omega, \omega' \in \Omega$, we have

$$\begin{aligned}|f_n(\omega) - f_n(\omega')| &= \left| \int_Y \chi_n(x)\, d\nu^{\omega}(x) - \int_Y \chi_n(x)\, d\nu^{\omega'}(x) \right| \\ &\leq \int_Y |\chi_n(x)| \cdot |h_\omega(x) - h_{\omega'}(x)|\, dm(x) \\ &\leq \|h_\omega - h_{\omega'}\|_{L^\infty_m} \|\chi_n\|_{L^1_m} \\ &\leq \frac{C}{n} \widetilde{L}(n) d_\theta(\omega, \omega'),\end{aligned}$$

since $\omega \in \Omega \mapsto h_\omega \in L^\infty(m)$ is Hölder continuous. In particular, we also have that $\overline{f}_n = \mathcal{O}(n^{-1}\widetilde{L}(n))$.

We have, using Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}(U_k^n(\delta)) &= \mathbb{P}\left(\left\{\omega \in \Omega : \left| \sum_{j=0}^{kn-1} (f_n \circ \sigma^j - \overline{f}_n) \right| \geq \delta \right\}\right) \\ &\leq \frac{1}{\delta^2} \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{j=0}^{kn-1} (f_n \circ \sigma^j - \overline{f}_n)\right)^2\right) \\ &\leq \frac{1}{\delta^2}\left[\sum_{j=0}^{kn-1} (\mathbb{E}_{\mathbb{P}} |f_n \circ \sigma^j - \overline{f}_n|^2 \right. \\ &\quad \left. + 2 \sum_{0 \leq i < j \leq kn-1} \mathbb{E}_{\mathbb{P}}((f_n \circ \sigma^i - \overline{f}_n)(f_n \circ \sigma^j - \overline{f}_n))\right].\end{aligned}$$

By the σ-invariance of $\mathbb{P}$, we have

$$\mathbb{E}_{\mathbb{P}} |f_n \circ \sigma^j - \overline{f}_n|^2 = \mathbb{E}_{\mathbb{P}} |f_n - \overline{f}_n|^2,$$

and, since $(\Omega, \mathbb{P}, \sigma)$ admits exponential decay of correlations for Hölder observables, there exist $\lambda \in (0, 1)$ and $C > 0$ such that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}((f_n \circ \sigma^i - \overline{f}_n)(f_n \circ \sigma^j - \overline{f}_n)) &= \mathbb{E}_{\mathbb{P}}((f_n - \overline{f}_n)(f_n \circ \sigma^{j-i} - \overline{f}_n)) \\ &\leq C\lambda^{j-i} \|f_n - \overline{f}_n\|_\theta^2.\end{aligned}$$

We then obtain that

$$\begin{aligned}\mathbb{P}(U_k^n(\delta)) &\leq \frac{C}{\delta^2}\left[kn\|f_n - \overline{f}_n\|_{L^2_m}^2 + 2 \sum_{0 \leq i < j \leq kn-1} \lambda^{j-i} \|f_n - \overline{f}_n\|_\theta^2\right] \\ &\leq C\frac{nk}{\delta^2} \|f_n\|_\theta^2 \\ &\leq C\frac{k}{n\delta^2} (\widetilde{L}(n))^2,\end{aligned}$$

which implies that

$$\mathbb{P}(B^n(\delta)) \leq C\frac{\ell^2}{n\delta^2} (\widetilde{L}(n))^2.$$

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Let $\eta > 0$. By the Borel–Cantelli lemma, it follows that for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $N(\omega, \delta) \geq 1$ such that $\omega \notin B^{\lfloor p^{1+\eta} \rfloor}(\delta)$ for all $p \geq N(\omega, \delta)$.

Let now $P := \lfloor p^{1+\eta} \rfloor < n \leq P' = \lfloor (p+1)^{1+\eta} \rfloor$ for p large enough. Let $0 \leq k \leq \ell$. Then, since $\| f_n \|_\infty = \mathcal{O}(n^{-1} \widetilde{L}(n))$,

$$\bigg| \sum_{j=0}^{kP-1} (f_n(\sigma^j \omega) - \overline{f}_n) - \sum_{j=0}^{kn-1} (f_n(\sigma^j \omega) - \overline{f}_n) \bigg| \leq \sum_{j=kP}^{kn-1} |f_n(\sigma^j \omega) - \overline{f}_n|$$

$$\leq C \frac{P' - P}{P} \widetilde{L}(n) \leq C \frac{\widetilde{L}(p^{1+\eta})}{p},$$

because on the one hand,

$$\frac{P' - P}{P} = \frac{\lfloor (p+1)^{1+\eta} \rfloor - \lfloor p^{1+\eta} \rfloor}{\lfloor p^{1+\eta} \rfloor} = \mathcal{O}\bigg(\frac{1}{p} \bigg),$$

and on the other hand, by Potter's bounds, for $\tau > 0$,

$$\widetilde{L}(n) \leq C \widetilde{L}(P) \bigg(\frac{n}{P} \bigg)^\tau \leq C \widetilde{L}(P) \bigg(\frac{P'}{P} \bigg)^\tau \leq C \widetilde{L}(P).$$

Since

$$\bigg| \sum_{j=0}^{kP-1} (f_n(\sigma^j \omega) - \overline{f}_n) \bigg| < \delta$$

for all $0 \leq k \leq \ell$, it follows that for $\mathbb{P}$-a.e. ω, there exists $N(\omega, \delta)$ such that $\omega \notin B^n(2\delta)$ for all $n \geq N(\omega, \delta)$, which concludes the proof. □

We now consider a corresponding result to Lemma 7.1 in the setting of intermittent maps.

LEMMA 7.2. *Assume that* $\gamma_{\max} < 1/2$, *and that* $\chi_n \in L^1(m)$ *is such that* $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1})$, $\|\chi_n\|_\infty = \mathcal{O}(1)$ *and there is* $\delta > 0$ *such that* $\operatorname{supp}(\chi_n) \subset [\delta, 1]$ *for all n.*

Then, for $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and for all* $\ell \geq 1$,

$$\lim_{n \to \infty} \sup_{0 \leq k \leq \ell} \bigg| \sum_{j=0}^{kn-1} (\mathbb{E}_{\nu_{\sigma^j \omega}}(\chi_n) - \mathbb{E}_\nu(\chi_n)) \bigg| = 0.$$

Proof. In the setting of intermittent maps, we must modify the argument of Lemma 7.1 slightly as h_ω is not a Hölder function of ω. Instead, we consider $h^i_\omega = P^i_{\sigma^{-i}\omega} \mathbf{1}$ and use that, by equation (5.11),

$$\|h^i_\omega - h_\omega\|_{L^1(m)} \leq C i^{1 - 1/\gamma_{\max}} \quad \text{(leaving out the log term).} \tag{7.1}$$

Note that h^i_ω is the ith approximate to h_ω in the pullback construction of h_ω. Let ν^i_ω be the measure such that $d\nu^i_\omega / dm = h^i_\omega$.

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Consider

$$f_n^i(\omega) = \mathbb{E}_{\nu_\omega^i}(\chi_n)\,, \quad f_n(\omega) = \mathbb{E}_{\nu^\omega}(\chi_n),$$

$$\overline{f}_n^i = \mathbb{E}_{\mathbb{P}}(f_n^i)\,, \quad \overline{f}_n = \mathbb{E}_{\mathbb{P}}(f_n).$$

By equation (5.12), on the set $[\delta, 1]$, the densities involved $(h_\omega^k, h_\omega, h = d\nu/dm)$ are uniformly bounded above and away from zero. Thus, $\|f_n^i\|_\infty = \mathcal{O}(n^{-1})$.

Pick $0 < a < 1$ such that $\beta := ((1/\gamma_{\max}) - 1)a - 1 > 0$.

For a given n, take $i = i_n = n^a$. By equation (7.1), for all ω, n, and $i = n^a$,

$$|f_n^i(\omega) - f_n(\omega)| \le \|h_\omega^i - h_\omega\|_{L^1(m)}\|\chi_n\|_{L^\infty(m)} = \mathcal{O}(n^{-(\beta+1)}).$$

Then

$$|\overline{f}_n^i - \overline{f}_n| = \mathcal{O}(n^{-(\beta+1)})$$

and

$$\left|\sum_{r=0}^{kn-1}[f_n^i(\sigma^r\omega) - f_n(\sigma^r\omega)]\right| \le C\ell n^{-\beta}.$$

Given ε, choose n large enough that for all $0 \le k \le \ell$,

$$\left\{\omega \in \Omega : \left|\sum_{r=0}^{kn-1}(f_n(\sigma^r\omega) - \overline{f}_n)\right| > \varepsilon\right\} \subset \left\{\omega \in \Omega : \left|\sum_{r=0}^{kn-1}(f_n^i(\sigma^r\omega) - \overline{f}_n^i)\right| > \frac{\varepsilon}{2}\right\}.$$

By Chebyshev,

$$\mathbb{P}\left(\left|\sum_{r=0}^{kn-1}(f_n^i \circ \sigma^r - \overline{f}_n^i)\right| > \frac{\varepsilon}{2}\right) \le \frac{4}{\varepsilon^2}\sum_{r=0}^{kn-1}\mathbb{E}_{\mathbb{P}}\left([f_n^i \circ \sigma^r - \overline{f}_n^i]^2\right)$$
$$+ \frac{4}{\varepsilon^2}\left[2\sum_{r=0}^{kn-1}\sum_{u=r+1}^{kn-1}|\mathbb{E}_{\mathbb{P}}[(f_n^i \circ \sigma^r - \overline{f}_n^i)(f_n^i \circ \sigma^u - \overline{f}_n^i)]|\right].$$

We bound

$$\sum_{r=0}^{kn-1}\mathbb{E}_{\mathbb{P}}([f_n^i - \overline{f}_n^i]^2) \le C\sum_{r=0}^{kn-1}\|f_n^i\|_\infty^2 \le \frac{C\ell}{n}$$

and note that if $|r - u| > n^a$, then by independence,

$$\mathbb{E}_{\mathbb{P}}[(f_n^i \circ \sigma^r - \overline{f}_n^i)(f_n \circ \sigma^u - \overline{f}_n^i)] = \mathbb{E}_{\mathbb{P}}[f_n^i \circ \sigma^r - \overline{f}_n^i]\mathbb{E}_{\mathbb{P}}[f_n^i \circ \sigma^u - \overline{f}_n^i] = 0$$

and hence, we may bound

$$\sum_{r=0}^{kn-1}\sum_{u=r+1}^{kn-1}|\mathbb{E}_{\mathbb{P}}[(f_n^i \circ \sigma^r - \overline{f}_n^i)(f_n^i \circ \sigma^u - \overline{f}_n^i)]| \le \frac{C\ell}{n^{1-a}}.$$

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Thus, for n large enough,

$$\mathbb{P}\bigg(\bigg\{\omega \in \Omega : \bigg|\sum_{r=0}^{kn-1}[f_n(\sigma^r\omega) - \overline{f}_n]\bigg| > \varepsilon\bigg\}\bigg) \leq \frac{C\ell}{n^{1-a}\varepsilon^2}.$$

The rest of the argument proceeds as in the case of Lemma 7.1 using a speedup along a sequence $n = p^{1+\eta}$, where $\eta > a/(1-a)$, since $\|f_n\|_\infty = \mathcal{O}(n^{-1})$ still holds. □

7.2. *Criteria for stable laws and functional limit laws.* The next theorem shows that for regularly varying observables, Poisson convergence and Condition U imply convergence in the J_1 topology if $\alpha \in (0, 1)$ and gives an additional condition to be verified in the case $\alpha \in [1, 2)$.

Note that equation (7.2) is essentially the condition in equation (5.5) of Proposition 5.7.

THEOREM 7.3. *Assume ϕ is regularly varying, Condition U holds, and that*

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

If $\alpha \in [1, 2)$, assume furthermore that for all $\delta > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \nu\bigg(\max_{1\leq k\leq n}\bigg|\frac{1}{b_n}\sum_{j=0}^{k-1}[\phi \circ T_\omega^j \mathbf{1}_{\{|\phi\circ T_\omega^j|\leq\varepsilon b_n\}} - \mathbb{E}_{\nu_{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\leq\varepsilon b_n\}})]\bigg| \geq \delta\bigg) = 0. \tag{7.2}$$

Then $X_n^\omega \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Remark 7.4. From equation (5.8) and Theorem 5.1, it follows that the convergence of X_n^ω also holds under the probability measure ν.

Proof of Theorem 7.3. When $\alpha \in (0, 1)$, we check the hypothesis of Proposition 5.4. Using equation (5.8), we have

$$\bigg|\frac{1}{b_n}\sum_{j=0}^{n\ell-1}\mathbb{E}_{\nu_{\sigma^j\omega}}(|\phi|\mathbf{1}_{\{|\phi|\leq\varepsilon b_n\}})\bigg| \leq C\frac{n\ell}{b_n}\mathbb{E}_\nu(|\phi|\mathbf{1}_{\{|\phi|\leq\varepsilon b_n\}}).$$

Using Proposition 3.2, we see that the condition in equation (5.3) is satisfied since $\alpha < 1$, thus proving the theorem in this case.

When $\alpha \in [1, 2)$, we consider instead Proposition 5.7. First, we remark that the condition in equation (5.5) is implied by equations (7.2) and (5.8). It remains to check the condition in equation (5.4), which constitutes the rest of the proof.

If $\alpha \in (1, 2)$, we have

$$\bigg|\frac{1}{b_n}\bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\mathbb{E}_{\nu_{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\leq\varepsilon b_n\}}) - t(c_n - b_n c_\alpha(\varepsilon))\bigg]\bigg| \leq A_n^\omega(t) + B_{n,\varepsilon}^\omega(t) + C_{n,\varepsilon}^\omega(t) \tag{7.3}$$

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with

$$A_n^\omega(t) = \left| \frac{1}{b_n} \left[\sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{E}_{\nu_{\sigma^j \omega}}(\phi) - tc_n \right] \right|,$$

$$B_{n,\varepsilon}^\omega(t) = \left| \frac{1}{b_n} \left[\sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{E}_{\nu_{\sigma^j \omega}}(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) - nt\mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) \right] \right|,$$

and

$$C_{n,\varepsilon}^\omega(t) = \left| \frac{nt}{b_n} \mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) - tc_\alpha(\varepsilon) \right|.$$

Since ϕ is regularity varying with index $\alpha > 1$, it is integrable and the function $\omega \mapsto \mathbb{E}_{\nu^\omega}(\phi)$ is Hölder. Hence, it satisfies the law of the iterated logarithm, and we have for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} \mathbb{E}_{\nu_{\sigma^j \omega}}(\phi) - \mathbb{E}_\nu(\phi) \right| = \mathcal{O}\left(\frac{\sqrt{\log \log k}}{\sqrt{k}} \right).$$

Thus, we have

$$\sup_{0 \le t \le \ell} A_n^\omega(t) = \mathcal{O}\left(\frac{\sqrt{n\ell}\sqrt{\log \log(n\ell)}}{b_n} \right).$$

As a consequence, we can deduce that $\lim_{n\to\infty} \sup_{0\le t\le \ell} A_n^\omega(t) = 0$ since $b_n = n^{\frac{1}{\alpha}} \widetilde{L}(n)$ for a slowly varying function $\widetilde{L}$, with $\alpha < 2$.

By Proposition 3.2, we also have

$$\lim_{n\to\infty} nb_n^{-1} \mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) = c_\alpha(\varepsilon).$$

In particular, we have

$$\lim_{n\to\infty} \sup_{0\le t\le \ell} C_{n,\varepsilon}^\omega(t) = 0.$$

This also implies that $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1})$ if we define $\chi_n = b_n^{-1} \phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}$. From Lemma 7.1, it follows that $\lim_{n\to\infty} \sup_{0\le t\le \ell} B_{n,\varepsilon}^\omega(t) = 0$.

Putting all these estimates together concludes the proof when $\alpha \in (1, 2)$.

When $\alpha = 1$, we estimate the right-hand side of equation (7.3) by $A_{n,\varepsilon}^\omega(t) + B_{n,\varepsilon}^\omega(t)$ with

$$A_{n,\varepsilon}^\omega(t) = \left| \frac{1}{b_n} \left[\sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{E}_{\nu_{\sigma^j \omega}}(\phi \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) - nt\mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) \right] \right|$$

and

$$B_{n,\varepsilon}^\omega(t) = \left| \frac{nt}{b_n} \mathbb{E}_\nu(\phi \mathbf{1}_{\{\varepsilon b_n < |\phi| \le b_n\}}) - tc_\alpha(\varepsilon) \right|.$$

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We define $\chi_n = b_n^{-1}\phi \mathbf{1}_{\{|\phi|\le \varepsilon b_n\}}$. By Proposition 3.2, we have $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1}\widetilde{L}(n))$ for some slowly varying function $\widetilde{L}$, and so by Lemma 7.1,

$$\lim_{n\to\infty} \sup_{0\le t\le \ell} A^{\omega}_{n,\varepsilon}(t) = 0.$$

However, by Proposition 3.2, we have

$$\lim_{n\to\infty} nb_n^{-1}\mathbb{E}_\nu(\phi \mathbf{1}_{\{\varepsilon b_n < |\phi| \le \varepsilon b_n\}}) = c_\alpha(\varepsilon)$$

and so $\lim_{n\to\infty} \sup_{0\le t\le \ell} B^{\omega}_{n,\varepsilon}(t) = 0$ which completes the proof. □

8. *An exponential law*

We denote by $\mathcal{J}$ the family of all finite unions of intervals of the form $(x, y]$, where $-\infty \le x < y \le \infty$ and $0 \notin [x, y]$. For $J \in \mathcal{J}$, we will establish a quenched exponential law for the sequence of sets $A_n = (\phi_{x_0})^{-1}(b_n J)$. Similar results were obtained in [**CF20**, **FFV17**, **HRY20**, **RSV14**, **RT15**].

Since ϕ is regularly varying, it is easy to verify that

$$\lim_{n\to\infty} n\nu(A_n) = \Pi_\alpha(J).$$

In particular, $m(A_n) = \mathcal{O}(n^{-1})$.

LEMMA 8.1. *Assume Condition U and that ϕ is regularly varying with index α.*

If $A_n \subset [0, 1]$ is a sequence of measurable subsets such that $m(A_n) = \mathcal{O}(n^{-1})$, then for all $0 \le s < t$,

$$\lim_{n\to\infty} \left(\left[\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n) \right] - n(t-s)\nu(A_n) \right) = 0.$$

The same result holds in the setting of intermittent maps if $A_n \subset [\delta, 1]$ for some $\delta > 0$ with $m(A_n) = \mathcal{O}(n^{-1})$. In particular, if $A_n = \phi_{x_0}^{-1}(b_n J)$ for $J \in \mathcal{J}$ and $x_0 \ne 0$, then for all $0 \le s < t$,

$$\lim_{n\to\infty} \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n) = (t-s)\Pi_\alpha(J).$$

Proof. For the first statement, it suffices to apply Lemma 7.1 or Lemma 7.2 with $\chi_n = \mathbf{1}_{A_n}$. The second statement immediately follows since $\lim_n n\nu(A_n) = \Pi_\alpha(J)$. □

COROLLARY 8.2. *Assume the hypothesis of Lemma 8.1.*

Let $J \in \mathcal{J}$ and set $A_n = \phi^{-1}(b_n J)$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $0 \le s < t$,

$$\lim_{n\to\infty} \prod_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (1 - \nu^{\sigma^j\omega}(A_n)) = e^{-(t-s)\Pi_\alpha(J)}.$$

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Proof. Since $\nu^\omega(A_n)$ is of order at most n^{-1} uniformly in $\omega \in \Omega$, it follows that

$$\log\left[\prod_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(1-\nu^{\sigma^j\omega}(A_n))\right] = -\left(\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\nu^{\sigma^j\omega}(A_n)\right)+\mathcal{O}(n^{-1}).$$

By Lemma 8.1,

$$\lim_{n\to\infty}\sum_{j=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1}\nu^{\sigma^j\omega}(A_n) = (t-s)\Pi_\alpha(J),$$

which yields the conclusion. □

Definition 8.3. For a measurable subset $U \subset Y = [0, 1]$, we define the hitting time of $(\omega, x) \in \Omega \times Y$ to U by

$$R_U(\omega)(x) := \inf\{k \geq 1 : T_\omega^k(x) \in U\},$$

and the induced measure by ν on U by

$$\nu_U(A) := \frac{\nu(A \cap U)}{\nu(U)}.$$

To establish our exponential law, we will first obtain a few estimates, based on the proof of [**HSV99**, Theorem 2.1], to relate $\nu^\omega(R_{A_n}(\omega) > \lfloor nt\rfloor)$ to $\sum_{j=0}^{\lfloor nt\rfloor-1}\nu^{\sigma^j\omega}(A_n)$ so that we are able to invoke Corollary 8.2.

The next lemma is basically [**RSV14**, Lemma 6].

LEMMA 8.4. *For every measurable set $U \subset [0, 1]$, we have the bound*

$$\begin{aligned}\left|\nu^\omega(R_U(\omega) > k) - \prod_{j=1}^{k}(1-\nu^{\sigma^j\omega}(U))\right| &\leq \sum_{j=1}^{k}\nu^{\sigma^j\omega}(U)\; c_{\sigma^j\omega}(k-j, U)\prod_{i=1}^{j-1}(1-\nu^{\sigma^i\omega}(U))\\ &\leq \sum_{j=1}^{k}\nu^{\sigma^j\omega}(U)\; c_{\sigma^j\omega}(U),\end{aligned}$$

where

$$c_\omega(k, U) := |\nu_U^\omega(R_U(\omega) > k) - \nu^\omega(R_U(\omega) > k)|$$

and

$$c_\omega(U) := \sup_{k\geq 0} c_\omega(k, U).$$

Proof. Note that $\{R_U(\omega) > k\} = [T_\omega^1]^{-1}(U^c \cap \{R_U(\sigma\omega) > k-1\})$ and so, using the equivariance of $\{\nu^\omega\}_{\omega\in\Omega}$,

$$\nu^\omega(R_U(\omega) > k) = \nu^{\sigma\omega}(U^c \cap \{R_U(\sigma\omega) > k-1\}).$$

Hence,

$$\nu^\omega(R_U(\omega) > k) = \nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) - \nu^{\sigma\omega}(U \cap \{R_U(\sigma\omega) > k-1\}).$$

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We note that

$$\begin{aligned}\nu^{\omega}(R_U(\omega) > k) &= \nu^{\sigma\omega}(R_U(\sigma\omega) > k-1)\\ &\quad - \nu^{\sigma\omega}(U)[\nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) + c_{\sigma\omega}(k-1, U)]\\ &= (1-\nu^{\sigma\omega}(U))\nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) - \nu^{\sigma\omega}(U)c_{\sigma\omega}(k-1, U).\end{aligned}$$

Iterating we obtain, using the fact that for $\mathbb{P}$-a.e. ω, $\nu^{\omega}(R_U(\omega) \geq 1) = 1$,

$$\nu^{\omega}(R_U(\omega) > k) = \prod_{j=1}^{k}(1-\nu^{\sigma^j\omega}(U)) - \sum_{j=1}^{k}\nu^{\sigma^j\omega}(U)c_{\sigma^j\omega}(k-j, U)\prod_{i=1}^{j-1}(1-\nu^{\sigma^i\omega}(U)),$$

which yields the conclusion. □

We will estimate now the coefficients $c_{\omega}(U)$.

LEMMA 8.5. *For any measurable subset $U \subset Y$ such that $\mathbf{1}_U \in \mathrm{BV}$, we have, for all N,*

$$c_{\omega}(U) \leq \nu_U^{\omega}(R_U(\omega) \leq N) + \nu^{\omega}(R_U(\omega) \leq N) + \frac{1}{\nu^{\omega}(U)}\|P_{\omega}^{N}([\mathbf{1}_U - \nu^{\omega}(U)]h_{\omega})\|_{L^1(m)} \tag{8.1}$$

and

$$\nu_U^{\omega}(R_U(\omega) \leq N) \leq \frac{1}{\nu^{\omega}(U)}\nu^{\omega}(R_U(\omega) \leq N), \quad \nu^{\omega}(R_U(\omega) \leq N) \leq \sum_{i=1}^{N}\nu^{\sigma^i\omega}(U). \tag{8.2}$$

Proof. The estimates in equation (8.2) follow from

$$\{R_U(\omega) \leq N\} = \bigcup_{i=1}^{N}(T_{\omega}^{i})^{-1}(U),$$

and therefore

$$\nu^{\omega}(R_U(\omega) \leq N) \leq \sum_{i=1}^{N}\nu^{\sigma^i\omega}(U).$$

For equation (8.1), note that

$$c_{\omega}(U) = |\nu_U^{\omega}(R_U(\omega) \leq j) - \nu^{\omega}(R_U(\omega) \leq j)|.$$

If $j \leq N$, then

$$c_{\omega}(U) \leq \nu_U^{\omega}(R(\omega) \leq N) + \nu^{\omega}(R(\omega) \leq N).$$

If $j > N$, we write

$$\begin{aligned}&\nu_U^{\omega}(R_U(\omega) \leq j) - \nu^{\omega}(R_U(\omega) \leq j)\\ &\quad = \nu_U^{\omega}(R_U(\omega) \leq j) - \nu_U^{\omega}(T_{\omega}^{-N}(R_U(\sigma^N\omega) \leq j-N))\\ &\qquad + \nu_U^{\omega}(T_{\omega}^{-N}(R_U(\sigma^N\omega) \leq j-N)) - \nu^{\omega}(T_{\omega}^{-N}(R_U(\sigma^N\omega) \leq j-N))\\ &\qquad + \nu^{\omega}(T_{\omega}^{-N}(R_U(\sigma^N\omega) \leq j-N)) - \nu^{\omega}(R_U(\omega) \leq j)\\ &\quad = (a) + (b) + (c).\end{aligned}$$

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To bound terms (a) and (c), note that

$$\{R_U(\omega) \le j\} = \{R_U(\omega) \le N\} \cup T_\omega^{-N}(\{R_U(\sigma^N\omega) \le j - N)\})$$

so

$$|\nu^\omega(R_U(\omega) \le j) - \nu^\omega(T_\omega^{-N}(R_U(\sigma^N\omega) \le j - N))| \le \nu^\omega(R_U(\omega) \le N) \tag{8.3}$$

and similarly for ν_U^ω.

To bound term (b), we use the decay of P_ω^k. Setting $V = \{R_U(\sigma^N\omega) \le j - N\}$, we have

$$\begin{aligned}
&|\nu_U^\omega(T_\omega^{-N}(V)) - \nu^\omega(T_\omega^{-N}(V))| \\
&\quad = \frac{1}{\nu^\omega(U)}\bigg|\int_Y \mathbf{1}_U\mathbf{1}_V \circ T_\omega^N h_\omega\, dm - \nu^\omega(U)\int_Y \mathbf{1}_V \circ T_\omega^N h_\omega\, dm\bigg| \\
&\quad = \frac{1}{\nu^\omega(U)}\bigg|\int_Y \mathbf{1}_V P_\omega^N([\mathbf{1}_U - \nu^\omega(U)]h_\omega)\, dm\bigg| \\
&\quad \le \frac{1}{\nu^\omega(U)}\|P_\omega^N([\mathbf{1}_U - \nu^\omega(U)]h_\omega)\|_{L^1(m)}. \qquad \square
\end{aligned}$$

8.1. *Exponential law: proof of Theorem 6.1.* We can now prove the exponential law for $A_n = \phi^{-1}(b_n J)$, $J \in \mathcal{J}$.

Proof of Theorem 6.1. Due to rounding errors when taking the integer parts, we have

$$\begin{aligned}
&|\nu^{\sigma^{\lfloor ns\rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns\rfloor}\omega) > \lfloor n(t-s)\rfloor) - \nu^{\sigma^{\lfloor ns\rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns\rfloor}\omega) > \lfloor nt\rfloor - \lfloor ns\rfloor)| \\
&\quad \le \nu^{\sigma^{\lfloor nt\rfloor}\omega}(A_n) \le Cm(A_n) \to 0,
\end{aligned}$$

and it is thus enough to prove the convergence of $\nu^{\sigma^{\lfloor ns\rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns\rfloor}\omega) > \lfloor nt\rfloor - \lfloor ns\rfloor)$.

By Lemmas 8.4 and 8.5, for all $N \ge 1$, we have

$$\bigg|\nu^{\sigma^{\lfloor ns\rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns\rfloor}\omega) > \lfloor nt\rfloor - \lfloor ns\rfloor) - \prod_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(1 - \nu^{\sigma^j\omega}(A_n))\bigg| \le \text{(I)} + \text{(II)} + \text{(III)}, \tag{8.4}$$

with

$$\text{(I)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n \cap \{R_{A_n}(\sigma^j\omega) \le N\}),$$

$$\text{(II)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n)\nu^{\sigma^j\omega}(R_{A_n}(\sigma^j\omega) \le N),$$

and

$$\text{(III)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \|P_{\sigma^j\omega}^N([\mathbf{1}_{A_n} - \nu^{\sigma^j\omega}(A_n)]h_{\sigma^j\omega})\|_{L^1(m)}.$$

To estimate (I), we choose $\varepsilon > 0$ such that $J \subset \{|x| > \varepsilon\}$ and we introduce $V_n = \{|\phi| > \varepsilon b_n\}$. For a measurable subset $V \subset Y$, we also define the shortest return to V by

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$$r_\omega(V) = \inf_{x \in V} R_V(\omega)(x),$$

and we set

$$r(V) = \inf_{\omega \in \Omega} r_\omega(V).$$

We have

$$\begin{aligned}\nu^{\sigma^j\omega}(A_n \cap \{R_{A_n}(\sigma^j\omega) \le N\}) &\le \nu^{\sigma^j\omega}(V_n \cap \{R_{V_n}(\sigma^j\omega) \le N\}) \\ &\le \sum_{i=r_{\sigma^j\omega}(V_n)}^{N} \nu^{\sigma^j\omega}(V_n \cap (T^i_{\sigma^j\omega})^{-1}(V_n)) \\ &\le \sum_{i=r_{\sigma^j\omega}(V_n)}^{N} \int_Y \mathbf{1}_{V_n} P^i_{\sigma^j\omega}(\mathbf{1}_{V_n} h_{\sigma^j\omega})\, dm.\end{aligned}$$

It follows from condition (Dec) that

$$\begin{aligned}&\bigg| \int_Y \mathbf{1}_{V_n} P^i_{\sigma^j\omega}(\mathbf{1}_{V_n} h_{\sigma^j\omega})\, dm - \nu^{\sigma^j\omega}(V_n)\nu^{\sigma^{i+j}\omega}(V_n)\bigg| \\ &\quad\le \|\mathbf{1}_{V_n}\|_{L^1_m} \|P^i_{\sigma^j\omega}([\mathbf{1}_{V_n} - \nu^{\sigma^j\omega}(V_n)]h_{\sigma^j\omega})\|_{L^\infty_m} \\ &\quad\le C\theta^i m(V_n)\|[\mathbf{1}_{V_n} - \nu^{\sigma^j\omega}(V_n)]h_{\sigma^j\omega}\|_{\mathrm{BV}} \\ &\quad\le C\theta^i m(V_n),\end{aligned}$$

as BV is a Banach algebra, and both $\|\mathbf{1}_{V_n}\|_{\mathrm{BV}}$ and $\|h_{\sigma^j\omega}\|_{\mathrm{BV}}$ are uniformly bounded. (Recall that, from the definition of ϕ, it follows that V_n is an open interval, and thus $\mathbf{1}_{V_n}$ has a uniformly bounded BV norm.)

Consequently,

$$\begin{aligned}\text{(I)} &\le \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \sum_{i=r_{\sigma^j\omega}(V_n)}^{N} [\nu^{\sigma^j\omega}(V_n)\nu^{\sigma^{i+j}\omega}(V_n) + \mathcal{O}(\theta^i m(V_n))] \\ &\le C(m(V_n)^2 nN + m(V_n)n\theta^{r(V_n)}).\end{aligned}$$

However, we have by equation (8.2),

$$\begin{aligned}\text{(II)} &\le \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n) \sum_{i=1}^{N} \nu^{\sigma^{i+j}\omega}(A_n) \\ &\le CnNm(A_n)^2,\end{aligned}$$

and it follows from condition (Dec) that

$$\begin{aligned}\text{(III)} &\le C\theta^N \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \|[\mathbf{1}_{A_n} - \nu^{\sigma^j\omega}(A_n)]h_{\sigma^j\omega}\|_{\mathrm{BV}} \\ &\le Cn\theta^N,\end{aligned}$$

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since $\{h_\omega\}_{\omega\in\Omega}$ is a bounded family in BV, A_n is the union of at most two intervals, and thus $\|\mathbf{1}_{A_n}\|_{\mathrm{BV}}$ is uniformly bounded. We can thus bound equation (8.4) by

$$C(m(V_n)^2 nN + m(V_n)n\theta^{r(V_n)} + m(A_n)^2 nN + n\theta^N) \le C(n^{-1}N + \theta^{r(V_n)} + n\theta^N),$$

and, assuming for the moment that $r(V_n) \to +\infty$, we obtain the conclusion by choosing $N = N(n) = 2\log n$ and letting $n \to \infty$.

It thus remains to show that $r(V_n) \to +\infty$. Recall that V_n is the ball of center x_0 and radius $b^{-1}\varepsilon^{-\alpha}n^{-1}$. Let $R \ge 1$ be a positive integer. Since x_0 is assumed to be non-recurrent, and that the collection of maps T_ω^j for $\omega \in \Omega$ and $0 \le j < R$ is finite, we have that

$$\delta_R := \inf_{\omega\in\Omega} \inf_{0\le j<R} |T_\omega^j(x_0) - x_0| > 0$$

is positive. Since all the maps T_ω^j are continuous at x_0 by assumption, there exists $n_R \ge 1$ such that for all $n \ge n_R$, $j < R$, and $\omega \in \Omega$,

$$x \in V_n \Longrightarrow |T_\omega^j(x) - T_\omega^j(x_0)| < \frac{\delta_R}{2}.$$

Increasing n_R if necessary, we can assume that $b^{-1}\varepsilon^{-\alpha}n^{-1} < (\delta_R/2)$ for all $n \ge n_R$.

Then, for all $n \ge n_R$, $\omega \in \Omega$, $j < R$, and $x \in V_n$, we have

$$|T_\omega^j(x) - x_0| \ge |T_\omega^j(x_0) - x_0| - |T_\omega^j(x) - T_\omega^j(x_0)| > \frac{\delta_R}{2} > b^{-1}\varepsilon^{-\alpha}n^{-1},$$

and thus $T_\omega^j(x) \notin V_n$.

This implies that $r(V_n) > R$ for all $n \ge n_R$, which concludes the proof as R is arbitrary. □

Remark 8.6. A quenched exponential law for random piecewise expanding maps of the interval is proved in [**HRY20**, Theorem 7.1]. Our proof follows the same standard approach. We are able to specify that Theorem 6.1 holds for non-recurrent x_0, since our assumptions imply decay of correlations against L^1 observables, which is known to be necessary for this purpose, see [**AFV15**, §3.1]. Our proof is shorter, as we consider the simpler setting of finitely many maps, which are all uniformly expanding. In addition, we use the exponential law in the intermittent case of [**HRY20**, Theorem 7.2] to establish the short returns condition of Lemma 8.7 below.

8.2. *Exponential law: proof of Theorem 6.2.* To prove the exponential law in the intermittent setting, Theorem 6.2, we need a genericity condition on the point x_0 in the definition (2.1) of ϕ_{x_0}.

LEMMA 8.7. *If $\gamma_{\max} < \frac{1}{3}$, for m-a.e. x_0 and for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\lim_{n\to\infty} \sum_{j=\lfloor sn\rfloor+1}^{\lfloor tn\rfloor} m(B_{cn^{-1}}(x_0) \cap \{R^{\sigma^j\omega}_{B_{cn^{-1}}(x_0)} \le \lfloor n(\log n)^{-1}\rfloor\}) = 0$$

for all $c > 0$ and all $0 \le s < t$.

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Proof. Let $N = \lfloor n(\log n)^{-1} \rfloor$ an $V_n = B_{cn^{-1}}(x_0)$. First, we remark that for m-a.e. x_0 and $\mathbb{P}$-a.e. ω,

$$m(V_n \cap \{R_{V_n}(\omega) \le N\}) = o(n^{-1}). \tag{8.5}$$

This is a consequence of [HRY20, Theorem 7.2]. Their result is stated for two intermittent LSV maps both with $\gamma < \frac{1}{3}$ but generalizes immediately to a finite collection of maps with a uniform bound of $\gamma_{\max} < \frac{1}{3}$. The exponential law for return times to nested balls implies that for a fixed t, for m-a.e x_0, and $\mathbb{P}$-a.e. ω,

$$\lim_{n\to\infty} \frac{1}{\nu^\omega(V_n)} \nu^\omega(V_n \cap \{R_{V_n}(\omega) \le nt\}) = 1 - e^{-t},$$

which shows in particular, since $\{R_{V_n}(\omega) \le N\} \subset \{R_{V_n}(\omega) \le nt\}$ for all n large enough, that for all $t > 0$, m-a.e x_0, and $\mathbb{P}$-a.e. ω,

$$\limsup_{n\to\infty} \frac{1}{\nu^\omega(V_n)} \nu^\omega(V_n \cap \{R_{V_n}(\omega) \le N\}) \le 1 - e^{-t}. \tag{8.6}$$

Using equation (5.12), taking the limit $t \to 0$ proves equation (8.5). Note that, even though the set of full measure of x_0 and ω such that equation (8.6) holds may depend on t, it is enough to consider only a sequence $t_k \to 0$.

Now, for $k \ge 0$ and $n_0 \ge 1$, we introduce the set

$$\Omega_k^{n_0} = \left\{\omega \in \Omega : m(V_n \cap \{R_{V_n}(\omega) \le N\}) \le \frac{2^{-k}}{n} \text{ for all } n \ge n_0\right\}.$$

According to equation (8.5), we have for all $k \ge 0$,

$$\lim_{n_0\to\infty} \mathbb{P}(\Omega_k^{n_0}) = \mathbb{P}\bigg(\bigcup_{n_0\ge 1} \Omega_k^{n_0}\bigg) = 1.$$

By the Birkhoff ergodic theorem, for all $k \ge 0$, $n_0 \ge 1$, and $\mathbb{P}$-a.e. ω,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{\Omega_k^{n_0}}(\sigma^j \omega) = \mathbb{P}(\Omega_k^{n_0}),$$

which implies that for all $0 \le s < t$,

$$\lim_{n\to\infty} \frac{1}{(\lfloor nt \rfloor - \lfloor ns \rfloor)} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbf{1}_{\Omega_k^{n_0}}(\sigma^j \omega) = \mathbb{P}(\Omega_k^{n_0}).$$

Let $n_0 = n_0(\omega, k)$ such that $\mathbb{P}(\Omega_k^{n_0}) \ge 1 - 2^{-k}$, and for all $n \ge n_0$,

$$\frac{1}{(\lfloor nt \rfloor - \lfloor ns \rfloor)} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbf{1}_{\Omega_k^{n_0}}(\sigma^j \omega) \ge \mathbb{P}(\Omega_k^{n_0}) - 2^{-k}.$$

Then, for all $n \ge n_0(\omega, k)$, we have

$$\frac{1}{(\lfloor nt \rfloor - \lfloor ns \rfloor)} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbf{1}_{(\Omega_k^{n_0})^c}(\sigma^j \omega) \le 2^{-(k-1)}.$$

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Consequently,

$$\sum_{\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} m(V_n \cap \{R_{V_n}(\omega) \le N\}) \le (\lfloor nt\rfloor - \lfloor ns\rfloor)\frac{2^{-k}}{n} + (\lfloor nt\rfloor - \lfloor ns\rfloor)2^{-(k-1)}m(V_n).$$

This proves that

$$\limsup_{n\to\infty} \sum_{\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} m(V_n \cap \{R_{V_n}(\omega) \le N\}) \le C\, 2^{-k}$$

and the result follows by taking the limit $k \to \infty$.

Note that the set of x_0 and ω for which the lemma holds depends *a priori* on $c > 0$, but it is enough to consider a countable and dense set of c, since for $c < c'$,

$$\{B_{cn^{-1}}(x_0) \cap \{R^{\omega}_{B_{cn^{-1}}(x_0)} \le N\}\} \subset \{B_{c'n^{-1}}(x_0) \cap \{R^{\omega}_{B_{c'n^{-1}}(x_0)} \le N\}\}. \qquad \square$$

The exponential law for random intermittent maps follows from Lemma 8.7.

Proof of Theorem 6.2. We consider the three terms in equation (8.4) with $N = \lfloor n(\log n)^{-1}\rfloor$.

Let $V_n = \{|\phi| > \varepsilon b_n\}$, where $\varepsilon > 0$ is such that $A_n \subset V_n$ for all $n \ge 1$. Since V_n is a ball of center x_0 and radius $b^{-1}\varepsilon^{-\alpha}n^{-1}$, and since $V_n \subset [\delta, 1]$, the term

$$\text{(I)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n \cap \{R_{A_n}(\sigma^j\omega) \le N\}) \le C \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} m(V_n \cap \{R_{V_n}(\sigma^j\omega) \le N\})$$

tends to zero by Lemma 8.7 for m-a.e x_0.

The term

$$\text{(II)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n)\nu^{\sigma^j\omega}(R_{A_n}(\sigma^j\omega) \le N) \le CnNm(A_n)^2$$

also tends to zero since $N = o(n)$. Lastly, we consider

$$\text{(III)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \|P^N_{\sigma^j\omega}([\mathbf{1}_{A_n} - \nu^{\sigma^j\omega}(A_n)]h_{\sigma^j\omega})\|_{L^1(m)}.$$

We approximate $\mathbf{1}_{A_n}$ by a C^1 function g such that $\|g\|_{C^1} \le n^{\tau}$, $g = \mathbf{1}_{A_n}$ on A_n, and $\|g - \mathbf{1}_{A_n}\|_{L^1} \le n^{-\tau}$ (recall A_n is two intervals of length roughly $1/n$ so a simple smoothing at the endpoints of the intervals allows us to find such a function g). Later we will specify $\tau > 1$ as needed. By [**NPT21**, Lemma 3.4] with $h = h_\omega$ and $\varphi = g - m(gh_\omega)$, for all ω,

$$\begin{aligned}\|P^N_\omega([g - m(gh_\omega)]h_\omega)\|_{L^1} &\le Cn^{\tau}N^{1-1/\gamma_{\max}}(\log N)^{1/\gamma_{\max}}\\ &\le Cn^{\tau+1-1/\gamma_{\max}}(\log n)^{(2/\gamma_{\max})-1}.\end{aligned}$$

Using the decomposition $\mathbf{1}_{A_n} - \nu^\omega(A_n) = (\mathbf{1}_{A_n} - g) - (\nu^\omega(A_n) - m(gh_\omega)) + (g - m(gh_\omega))$, we estimate, leaving out the log term,

$$(\mathrm{III}) \le C[n^{1-\tau} + n^{\tau+2-1/\gamma_{\max}}],$$

where the value of C may change line to line. Taking $\gamma_{\max} < \frac{1}{3}$ and $1 < \tau < (1/\gamma_{\max}) - 2$ suffices. □

9. *Point process results*

We now proceed to the proof of the Poisson convergence. In §11, we will consider an annealed version of our results.

9.1. *Uniformly expanding maps: proof of Theorem 6.3.*

Recall Theorem 6.3: under the conditions of §2.1, in particular conditions (LY), (Min), and (Dec), if $x_0 \notin \mathcal{D}$ is non-recurrent, then for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}$$

under the probability measure ν^ω.

Our proof of Theorem 6.3 uses the existence of a spectral gap for the associated transfer operators P_ω^n, and breaks down in the setting of intermittent maps. The use of the spectral gap is encapsulated in the following lemma.

LEMMA 9.1. *Assume condition* (LY). *Then there exists* $C > 0$ *such that for all* $\omega \in \Omega$, *all* $f, f_n \in \mathrm{BV}$ *with*

$$\sup_{j\ge 1} \|f_j\|_{L^\infty(m)} \le 1 \quad \text{and} \quad \sup_{j\ge 1} \|f_j\|_{\mathrm{BV}} < \infty,$$

we have

$$\sup_{n\ge 0} \bigg\| P_\omega^n\bigg(f \cdot \prod_{j=1}^n f_j \circ T_\omega^j\bigg)\bigg\|_{\mathrm{BV}} \le C\|f\|_{\mathrm{BV}}\Big(\sup_{j\ge 1}\|f_j\|_{\mathrm{BV}}\Big).$$

Proof. We proceed in four steps.

Step 1. We define

$$g_\omega^n = \prod_{j=0}^n f_j \circ T_\omega^j,$$

where we have set $f_0 = \mathbf{1}$. We observe that for all $n \ge 0$, there exists $C_n > 0$ such that for all $\omega \in \Omega$,

$$\|g_\omega^n\|_{L^\infty(m)} \le \Big(\sup_{j\ge 1}\|f_j\|_{L^\infty(m)}\Big)^{n+1} \le 1 \ \text{ and } \ \|g_\omega^n\|_{\mathrm{BV}} \le C_n\Big(\sup_{j\ge 1}\|f_j\|_{\mathrm{BV}}\Big). \tag{9.1}$$

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The first estimate is immediate, and the second follows, because

$$\begin{aligned}
\mathrm{Var}(g_\omega^{n+1}) &\le \mathrm{Var}(g_\omega^n)\|f_{n+1}\circ T_\omega^{n+1}\|_{L^\infty(m)} + \|g_\omega^n\|_{L^\infty(m)}\mathrm{Var}(f_{n+1}\circ T_\omega^{n+1}) \\
&\le \mathrm{Var}(g_\omega^n) + \mathrm{Var}(f_{n+1}\circ T_\omega^{n+1}) \\
&= \mathrm{Var}(g_\omega^n) + \sum_{I\in\mathcal{A}_\omega^{n+1}} \mathrm{Var}_I(f_{n+1}\circ T_\omega^{n+1}) \\
&= \mathrm{Var}(g_\omega^n) + \sum_{I\in\mathcal{A}_\omega^{n+1}} \mathrm{Var}_{T_\omega^{n+1}(I)}(f_{n+1}) \\
&\le \mathrm{Var}(g_\omega^n) + (\#\mathcal{A}_\omega^{n+1})\mathrm{Var}(f_{n+1}),
\end{aligned}$$

and so we can define by induction $C_{n+1} = C_n + \sup_{\omega\in\Omega}\#\mathcal{A}_\omega^{n+1}$ which is finite, as there are only finitely many maps in $\mathcal{S}$.

Step 2. We first prove the lemma in the case where $r = 1$ in the condition (LY). Before, we claim that for $f \in \mathrm{BV}$ and sequences $(f_j) \subset \mathrm{BV}$ as in the statement, we have

$$\begin{aligned}
\mathrm{Var}(P_\omega^n(fg_\omega^n)) \le \sum_{j=0}^{n} \rho^j \|P_\omega^{n-j}(fg_\omega^{n-j-1})\|_{L^\infty(m)}\|f_{n-j}\|_{\mathrm{BV}} \\
+ D\sum_{j=0}^{n-1} \rho^j \|P_\omega^{n-1-j}(fg_\omega^{n-1-j})\|_{L^1(m)}\|f_{n-j}\|_{L^\infty(m)}.
\end{aligned} \tag{9.2}$$

This implies the lemma when $r = 1$, since

$$\|P_\omega^{n-j}(fg_\omega^{n-j-1})\|_{L^\infty(m)} \le \|g_\omega^{n-j-1}\|_{L^\infty(m)}\|P_\omega^{n-j}|f|\|_{L^\infty(m)} \le C\|f\|_{\mathrm{BV}}$$

and

$$\|P_\omega^{n-j}(fg_\omega^{n-j})\|_{L^1(m)} \le \|fg_\omega^{n-j}\|_{L^1(m)} \le \|f\|_{L^\infty(m)}\|g_\omega^{n-j}\|_{L^1(m)} \le \|f\|_{\mathrm{BV}}.$$

We prove the claim by induction on $n \ge 0$. It is immediate for $n = 0$, and for the induction step, we have, using condition (LY),

$$\begin{aligned}
&\mathrm{Var}(P_\omega^{n+1}(fg_\omega^{n+1})) \\
&\quad= \mathrm{Var}(P_\omega^{n+1}(fg_\omega^n f_{n+1}\circ T_\omega^{n+1})) = \mathrm{Var}(P_\omega^{n+1}(fg_\omega^n)f_{n+1}) \\
&\quad\le \mathrm{Var}(P_\omega^{n+1}(fg_\omega^n))\|f_{n+1}\|_{L^\infty(m)} + \|P_\omega^{n+1}(fg_\omega^n)\|_{L^\infty(m)}\mathrm{Var}(f_{n+1}) \\
&\quad\le (\rho\mathrm{Var}(P_\omega^n(fg_\omega^n)) + D\|P_\omega^n(fg_\omega^n)\|_{L^1(m)})\|f_{n+1}\|_{L^\infty(m)} \\
&\qquad+ \|P_\omega^{n+1}(fg_\omega^n)\|_{L^\infty(m)}\mathrm{Var}(f_{n+1}) \\
&\quad\le \rho\mathrm{Var}(P_\omega^n(fg_\omega^n)) + D\|P_\omega^n(fg_\omega^n)\|_{L^1(m)}\|f_{n+1}\|_{L^\infty(m)} + \|P_\omega^{n+1}(fg_\omega^n)\|_{L^\infty(m)}\|f_{n+1}\|_{\mathrm{BV}},
\end{aligned}$$

which proves equation (9.2) for $n + 1$, assuming it holds for n.

Step 3. Now, we consider the general case $r \ge 1$ and we assume that n is of the particular form $n = pr$, with $p \ge 0$. We note that the random system defined with $\mathcal{T} = \{T_\omega^r\}_{\omega\in\Omega}$

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satisfies the condition (LY) with $r = 1$. Consequently, by the second step and equation (9.1), we have

$$\|P_\omega^n(fg_\omega^n)\|_{\mathrm{BV}} = \|P^r_{\sigma^{r-1}\omega} \circ \cdots \circ P^r_\omega\bigg(f \prod_{j=1}^p g^r_{\sigma^{jr}\omega} \circ T_\omega^{jr}\bigg)\|_{\mathrm{BV}}$$
$$\le C\|f\|_{\mathrm{BV}}\Big(\sup_{j\ge 1}\|g^r_{\sigma^{jr}\omega}\|_{\mathrm{BV}}\Big) \le CC_r\|f\|_{\mathrm{BV}}\Big(\sup_{j\ge1}\|f_j\|_{\mathrm{BV}}\Big).$$

Step 4. Finally, if $n = pr + q$, with $p \ge 0$ and $q \in \{0, \ldots, r-1\}$, as an immediate consequence of condition (LY), we obtain

$$\|P_\omega^n(fg_\omega^n)\|_{\mathrm{BV}} = \|P^q_{\sigma^{pr}\omega}P^{pr}_\omega(fg^{pr}_\omega g^q_{\sigma^{pr}\omega} \circ T^{pr}_\omega)\|_{\mathrm{BV}}$$
$$= \|P^q_{\sigma^{pr}\omega}(P^{pr}_\omega(fg^{pr}_\omega)g^q_{\sigma^{pr}\omega})\|_{\mathrm{BV}} \le C\|P^{pr}_\omega(fg^{pr}_\omega)g^q_{\sigma^{pr}\omega}\|_{\mathrm{BV}}.$$

However, from Step 3, we have

$$\|P^{pr}_\omega(fg^{pr}_\omega)g^q_{\sigma^{pr}\omega}\|_{L^1(m)} \le \|g^q_{\sigma^{pr}\omega}\|_{L^\infty(m)}\|P^{pr}_\omega(fg^{pr}_\omega)\|_{L^1(m)}$$
$$\le \|P^{pr}_\omega(fg^{pr}_\omega)\|_{L^1(m)} \le C\|f\|_{\mathrm{BV}}\Big(\sup_{j\ge1}\|f_j\|_{\mathrm{BV}}\Big),$$

and, using equation (9.1),

$$\begin{aligned}\mathrm{Var}(P^{pr}_\omega(fg^{pr}_\omega)g^q_{\sigma^{pr}\omega}) &\le \|P^{pr}_\omega(fg^{pr}_\omega)\|_{L^\infty(m)}\mathrm{Var}(g^q_{\sigma^{pr}\omega}) \\ &\quad + \mathrm{Var}(P^{pr}_\omega(fg^{pr}_\omega))\|g^q_{\sigma^{pr}\omega}\|_{L^\infty(m)} \\ &\le [C_q\|g^{pr}_\omega\|_{L^\infty(m)}\|P^{pr}_\omega|f|\|_{L^\infty(m)} + C\|f\|_{\mathrm{BV}}]\Big(\sup_{j\ge1}\|f_j\|_{\mathrm{BV}}\Big) \\ &\le C\Big(1 + \max_{q=0,\ldots,r-1} C_q\Big)\|f\|_{\mathrm{BV}}\Big(\sup_{j\ge1}\|f_j\|_{\mathrm{BV}}\Big),\end{aligned}$$

which concludes the proof of the lemma. □

Proof of Theorem 6.3. We denote by $\mathcal{R}$ the family of finite unions of rectangles R of the form $R = (s, t] \times J$ with $J \in \mathcal{J}$. By Kallenberg's theorem, see [**Kal76**, Theorem 4.7] or [**Res87**, Proposition 3.22], $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ if for any $R \in \mathcal{R}$,

$$\text{(a)} \quad \lim_{n\to\infty} \nu^\omega(N_n^\omega(R) = 0) = \mathbb{P}(N_{(\alpha)}(R) = 0)$$

and

$$\text{(b)} \quad \lim_{n\to\infty} \mathbb{E}_{\nu^\omega} N_n^\omega(R) = \mathbb{E}N_{(\alpha)}(R).$$

We first prove equation (b). We write

$$R = \bigcup_{i=1}^k R_i,$$

with $R_i = (s_i, t_i] \times J_i$ disjoint.

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Then

$$\mathbb{E}N_{(\alpha)}(R) = \sum_{i=1}^{k}(t_i - s_i)\Pi_\alpha(J_i)$$

and

$$\begin{aligned}\mathbb{E}_{\nu^\omega} N_n^\omega(R) = \sum_{i=1}^{k} \mathbb{E}_{\nu^\omega} N_n^\omega((s_i, t_i] \times J_i) &= \sum_{i=1}^{k} \sum_{ns_i < j \le nt_i} \mathbb{E}_{\nu^\omega}(\mathbf{1}_{\phi_{x_0}^{-1}(b_n J_i)} \circ T_\omega^{j-1}) \\ &= \sum_{i=1}^{k} \sum_{ns_i < j \le nt_i} \nu^{\sigma^{j-1}\omega}(\phi_{x_0}^{-1}(b_n J_i)) \\ &= \sum_{i=1}^{k} \sum_{j=\lfloor ns_i \rfloor}^{\lfloor nt_i \rfloor - 1} \nu^{\sigma^j\omega}(\phi_{x_0}^{-1}(b_n J_i)).\end{aligned}$$

By Lemma 8.1, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{k} \sum_{j=\lfloor ns_i \rfloor}^{\lfloor nt_i \rfloor - 1} \nu^{\sigma^j\omega}(\phi_{x_0}^{-1}(b_n J_i)) = (t_i - s_i)\Pi_\alpha(J_i),$$

which proves equation (b).

We next establish equation (a). We will use induction on the number of 'time' intervals $(s_i, t_i] \subset (0, \infty]$. Let $R = (s_1, t_1] \times J_1$, where $J_1 \in \mathcal{J}$. Define

$$A_n = \phi_{x_0}^{-1}(b_n J_1).$$

Since

$$\begin{aligned}\{N_n^\omega(R) = 0\} &= \{x : T_\omega^j(x) \notin A_n, ns_1 < j+1 \le nt_1\} \\ &= \{1_{A_n^c} \circ T_\omega^{\lfloor ns_1 \rfloor} \cdot 1_{A_n^c} \circ T_\omega^{\lfloor ns_1 \rfloor + 1} \cdot \cdots \cdot 1_{A_n^c} \circ T_\omega^{\lfloor nt_1 \rfloor - 1} \neq 0\} \\ &= \left\{x : \left(\prod_{j=0}^{\lfloor nt_1 \rfloor - 1 - \lfloor ns_1 \rfloor} 1_{A_n^c} \circ T_{\sigma^{\lfloor ns_1 \rfloor}\omega}^{j} \right) \circ T_\omega^{\lfloor ns_1 \rfloor}(x) \neq 0 \right\},\end{aligned}$$

we have that

$$\begin{aligned}&|\nu^\omega(N_n^\omega(R) = 0) - \nu^{\sigma^{\lfloor ns_1 \rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns_1 \rfloor}\omega) > \lfloor n(t_1 - s_1) \rfloor)| \\ &\quad \le \nu^{\sigma^{\lfloor ns_1 \rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns_1 \rfloor}\omega) = 0) = \nu^{\sigma^{\lfloor ns_1 \rfloor}\omega}(A_n) \le Cm(A_n) \to 0, \qquad (9.3)\end{aligned}$$

because, due to rounding when taking integer parts, $\lfloor nt_1 \rfloor - \lfloor ns_1 \rfloor - 1$ is either equal to $\lfloor n(t_1 - s_1) \rfloor - 1$ or to $\lfloor n(t_1 - s_1) \rfloor$. By Theorem 6.1,

$$\nu^{\sigma^{\lfloor ns_1 \rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns_1 \rfloor}\omega) > \lfloor n(t_1 - s_1) \rfloor) \to e^{-(t_1 - s_1)\Pi_\alpha(J)}$$

as desired.

Now let $R = \bigcup_{j=1}^{k}(s_i, t_i] \times J_i$ with $0 \le s_1 < t_1 < \cdots < s_k < t_k$ and $J_i \in \mathcal{J}$. Furthermore, define $s_i' = s_i - s_1$ and $t_i' = t_i - s_1$.

Observe that, accounting for the rounding errors when taking integer parts as for equation (9.3), we get

$$\bigg|\nu^{\omega}\bigg(N_n^{\omega}\bigg(\bigcup_{i=1}^{k}(s_i,t_i]\times J_i\bigg)=0\bigg)-\nu^{\sigma^{\lfloor ns_1\rfloor}\omega}\bigg(N_n^{\sigma^{\lfloor ns_1\rfloor}\omega}\bigg(\bigcup_{i=1}^{k}(s_i',t_i']\times J_i\bigg)=0\bigg)\bigg|$$
$$\leq 2C\sum_{i=1}^{k} m(\phi_{x_0}^{-1}(b_nJ_i))\to 0 \tag{9.4}$$

so, after replacing ω by $\sigma^{\lfloor ns_1\rfloor}\omega$, we can assume that $s_1=0$. Let

$$R_1=(0,t_1]\times J_1,$$
$$R_2=\bigcup_{i=2}^{k}(s_i,t_i]\times J_i,$$
$$R_2'=\bigcup_{i=2}^{k}(s_i-s_2,t_i-s_2]\times J_i.$$

Then, with $A_n=\phi_{x_0}^{-1}(b_nJ_1)$,

$$|\nu^{\eta}(N_n^{\eta}(R_1\cup R_2)=0)-\nu^{\eta}[\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}\cap T_{\eta}^{-\lfloor ns_2\rfloor}(N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0)]|\to 0 \tag{9.5}$$

as $n\to\infty$, uniformly in $\eta\in\Omega$, as in equation (9.4). Moreover, as we check below,

$$|\nu^{\eta}[\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}\cap T_{\eta}^{-\lfloor ns_2\rfloor}(N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0)]$$
$$-\nu^{\eta}(R_{A_n}(\eta)>\lfloor nt_1\rfloor)\cdot\nu^{\eta}(N_n^{\eta}(R_2)=0)|\to 0 \tag{9.6}$$

as $n\to\infty$, uniformly in $\eta\in\Omega$. Therefore, setting $\eta=\sigma^{\lfloor ns_2\rfloor}\omega$ in equations (9.5) and (9.6), we have, by Theorem 6.1,

$$\lim_{n\to\infty}|\nu^{\sigma^{\lfloor ns_2\rfloor}\omega}(N_n^{\sigma^{\lfloor ns_2\rfloor}\omega}(R_1\cup R_2)=0)-e^{-t_1\Pi_\alpha(J_1)}\nu^{\sigma^{\lfloor ns_2\rfloor}\omega}(N_n^{\sigma^{\lfloor ns_2\rfloor}\omega}(R_2)=0)|=0,$$

which gives the induction step in the proof of equation (a).

We prove now equation (9.6). Our proof uses the spectral gap for P_ω^n and breaks down for random intermittent maps.

Similarly to equation (9.4),

$$|\nu^{\eta}(N_n^{\eta}(R_2)=0)-\nu^{\eta}(T_{\eta}^{-\lfloor ns_2\rfloor}(N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0))|\to 0\quad\text{as } n\to\infty,\ \text{uniformly in } \eta.$$

We have, using the notation

$$U=\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\},\qquad V=\{N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0\},$$

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that

$$\begin{aligned}
&|\nu^\eta(U \cap T_\eta^{-\lfloor ns_2\rfloor}(V)) - \nu^\eta(U)\nu^\eta(T_\eta^{-\lfloor ns_2\rfloor}(V))| \\
&= \left| \int P_\eta^{\lfloor ns_2\rfloor}((\mathbf{1}_U - \nu^\eta(U))h_\eta)\mathbf{1}_V \, dm \right| \\
&\le C\| P_\eta^{\lfloor ns_2\rfloor}((\mathbf{1}_U - \nu^\eta(U))h_\eta)\|_{\mathrm{BV}} \\
&= \| P_{\sigma^{\lfloor nt_1\rfloor}\eta}^{\lfloor ns_2\rfloor-\lfloor nt_1\rfloor} P_\eta^{\lfloor nt_1\rfloor}((\mathbf{1}_U - \nu^\eta(U))h_\eta)\|_{\mathrm{BV}} \\
&\le C\theta^{\lfloor ns_2\rfloor-\lfloor nt_1\rfloor}\| P_\eta^{\lfloor nt_1\rfloor}((\mathbf{1}_U - \nu^\eta(U))h_\eta)\|_{\mathrm{BV}},
\end{aligned}$$

where the last inequality follows from the decay, uniform in η, of $\{P_\eta^k\}_k$ in BV (condition (Dec)).

However,

$$\sup_\eta \sup_n \| P_\eta^{\lfloor nt_1\rfloor}((\mathbf{1}_{\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}} - \nu^\eta(R_{A_n}(\eta) > \lfloor nt_1\rfloor))h_\eta)\|_{\mathrm{BV}} < \infty, \tag{9.7}$$

which proves equation (9.6). This follows from Lemma 9.1 applied to $f = h_\eta$ and $f_j = \mathbf{1}_{A_n^c}$, because

$$\mathbf{1}_{\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}} = \prod_{j=1}^{\lfloor nt_1\rfloor} \mathbf{1}_{A_n^c} \circ T_\eta^j,$$

and both $\|h_\eta\|_{\mathrm{BV}}$ and $\|\mathbf{1}_{A_n^c}\|_{\mathrm{BV}}$ are uniformly bounded. Note that for the stationary case, the estimate in equation (9.7) is used in the proof of [**TK10b**, Theorem 4.4], which refers to [**ADSZ04**, Proposition 4]. □

9.2. *Intermittent maps: proof of Theorem 6.4.* We prove a weaker form of convergence in the setting of intermittent maps, which suffices to establish stable limit laws but not functional limit laws.

In the setting of intermittent maps, we will show that for $\mathbb{P}$-a.e. ω,

$$N_n^\omega((0,1]\times\cdot) \xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$$

Proof of Theorem 6.4. We will show that for $\mathbb{P}$-a.e. $\omega \in \Omega$, the assumptions of Kallenberg's theorem [**Kal76**, Theorem 4.7] hold.

Recall that $\mathcal{J}$ denotes the set of all finite unions of intervals of the form $(x, y]$, where $x < y$ and $0 \notin [x, y]$.

By Kallenberg's theorem [**Kal76**, Theorem 4.7], $N_n^\omega[(0,1]\times\cdot) \to^d N_{(\alpha)}((0,1]\times\cdot)$ if for all $J \in \mathcal{J}$,

$$\text{(a)} \lim_{n\to\infty} \nu^\omega(N_n^\omega((0,1]\times J) = 0) = \mathbb{P}(N_{(\alpha)}((0,1]\times J) = 0)$$

and

$$\text{(b)} \lim_{n\to\infty} \mathbb{E}_{\nu^\omega} N_n^\omega((0,1]\times J) = \mathbb{E}[N_{(\alpha)}((0,1]\times J)].$$

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We prove first equation (b) following [**TK10b**, p. 12]. Write

$$J = \bigcup_{i=1}^{k} J_i$$

with $J_i = (x_i, y_i]$ disjoint.

Then

$$\mathbb{E}N_{(\alpha)}((0,1] \times J) = \sum_{i=1}^{k} \Pi_\alpha(J_i) = \Pi_\alpha(J)$$

and

$$\mathbb{E}_{\nu^\omega} N_n^\omega((0,1] \times J) = \sum_{i=1}^{k} \sum_{j=1}^{n} \mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_n J_i))} \circ T_\omega^{j-1}] = \sum_{j=1}^{n} \mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_n J))} \circ T_\omega^{j-1}].$$

We check that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_{\nu^\omega}(\mathbf{1}_{\{\phi_{x_0}^{-1}(b_n J)\}} \circ T_\omega^{j}) = \Pi_\alpha(J)$$

for $J = \bigcup_{i=1}^{k} J_i$.

Write $A_n := \phi_{x_0}^{-1}(b_n J)$. Then

$$\mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_n J))} \circ T_\omega^{j}] = \nu^{\sigma^j \omega}(A_n),$$

hence

$$\lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_n J_i))} \circ T_\omega^{j}(x)] = \Pi_\alpha(J)$$

by Lemma 7.2.

Now we prove equation (a), that is,

$$\lim_{n\to\infty} \nu^\omega(N_n^\omega((0,1] \times J) = 0) = P(N_{(\alpha)}((0,1] \times J) = 0)$$

for all $J \in \mathcal{J}$.

Let $J \in \mathcal{J}$ and denote as above $A_n := \phi_{x_0}^{-1}(b_n J) \subset X = [0,1]$. Then

$$\{N_n^\omega((0,1] \times J) = 0\} = \{x : T_\omega^j(x) \notin A_n, 0 < j+1 \le n\} = \{R_{A_n}(\omega) > n-1\} \cap A_n^c.$$

Hence,

$$|\nu^\omega(N_n^\omega((0,1] \times J) = 0) - \nu^\omega(R_{A_n}(\omega) > n)| \le Cm(A_n) \to 0$$

and by Theorem 6.2, for m-a.e. x_0

$$\nu^\omega(R_{A_n}(\omega) > n) \to e^{-\Pi_\alpha(J)}.$$

This proves equation (a). □

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10. *Stable laws and functional limit laws*

10.1. *Uniformly expanding maps: proof of Theorem 2.4.*

In this section, we prove Theorem 2.4, under the conditions given in §2.1, in particular, conditions (LY), (Dec), and (Min).

For this purpose, we consider first some technical lemmas regarding short returns. For $\omega \in \Omega$, $n \geq 1$, and $\varepsilon > 0$, let

$$\mathcal{E}_n^\omega(\varepsilon) = \{x \in [0,1] : |T_\omega^n(x) - x| \leq \varepsilon\}.$$

LEMMA 10.1. *There exists $C > 0$ such that for all $\omega \in \Omega$, $n \geq 1$, and $\varepsilon > 0$,*

$$m(\mathcal{E}_n^\omega(\varepsilon)) \leq C\varepsilon.$$

Proof. We follow the proof of [**HNT12**, Lemma 3.4], conveniently adapted to our setting of random non-Markov maps. Recall that $\mathcal{A}_\omega^n$ is the partition of monotonicity associated to the map T_ω^n. Consider $I \in \mathcal{A}_\omega^n$. Since $\inf_I |(T_\omega^n)'| \geq \lambda^n > 1$, there exists at most one solution $x_I^\pm \in I$ to the equation

$$T_\omega^n(x_I^\pm) = x_I^\pm \pm \varepsilon, \tag{10.1}$$

and since there is no sign change of $(T_\omega^n)'$ on I, we have

$$\mathcal{E}_n^\omega(\varepsilon) \cap I \subset [x_I^-, x_I^+]. \tag{10.2}$$

We have

$$T_\omega^n(x_I^+) - T_\omega^n(x_I^-) = x_I^+ - x_I^- + 2\varepsilon,$$

and by the mean value theorem,

$$|T_\omega^n(x_I^+) - T_\omega^n(x_I^-)| = |(T_\omega^n)'(c)|\,|x_I^+ - x_I^-| \quad \text{for some } c \in I.$$

Consequently,

$$|x_I^+ - x_I^-| \leq \bigg(\sup_I \frac{1}{|(T_\omega^n)'|}\bigg)[|x_I^+ - x_I^-| + 2\varepsilon] \leq \lambda^{-n}|x_I^+ - x_I^-| + 2\varepsilon \sup_I \frac{1}{|(T_\omega^n)'|}. \tag{10.3}$$

Note that if there is no solutions to equation (10.1), then the estimate in equation (10.3) is actually improved. Rearranging equation (10.3) and summing over $I \in \mathcal{A}_\omega^n$, we obtain, thanks to equation (10.2),

$$m(\mathcal{E}_n^\omega(\varepsilon)) \leq \sum_{I \in \mathcal{A}_\omega^n} |x_I^+ - x_I^-| \leq \frac{2\varepsilon}{1 - \lambda^{-n}} \sum_{I \in \mathcal{A}_\omega^n} \sup_I \frac{1}{|(T_\omega^n)'|} \leq C\varepsilon.$$

The fact that

$$\sum_{I \in \mathcal{A}_\omega^n} \sup_I \frac{1}{|(T_\omega^n)'|} \leq C \tag{10.4}$$

for a constant $C > 0$ independent from ω and n follows from a standard distortion argument for one-dimensional maps that can be found in the proof of [**ANV15**, Lemma 8.5(3)] (see also [**AR16**, Lemma 7]), where finitely many piecewise C^2 uniformly expanding maps

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with finitely many discontinuities are also considered. Since it follows from condition (LY) that $\|P^n_\omega f\|_{\mathrm{BV}} \le C\|f\|_{\mathrm{BV}}$ for some uniform $C > 0$, we do not have to average equation (10.4) over ω as in [**ANV15**], but instead, we can simply have an estimate that holds uniformly in ω. □

Recall that, for a measurable subset U, $R^\omega_U(x) \ge 1$ is the hitting time of (ω, x) to U defined by equation (6.1).

LEMMA 10.2. *Let $a > 0$, $\frac{2}{3} < \psi < 1$, and $0 < \kappa < 3\psi - 2$. Then there exist sequences $(\gamma_1(n))_{n\ge1}$ and $(\gamma_2(n))_{n\ge1}$ with $\gamma_1(n) = \mathcal{O}(n^{-\kappa})$ and $\gamma_2(n) = o(1)$, and for all $\omega \in \Omega$, a sequence of measurable subsets $(A^\omega_n)_{n\ge1}$ of $[0, 1]$ with $m(A^\omega_n) \le \gamma_1(n)$ and such that for all $x_0 \notin A^\omega_n$,*

$$(\log n) \sum_{i=0}^{n-1} m(B_{n^{-\psi}}(x_0) \cap \{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\}) \le \gamma_2(n).$$

Proof. Let

$$E^\omega_n = \{x \in [0, 1] : |T^j_\omega(x) - x| \le 2n^{-\psi} \text{ for some } 0 < j \le \lfloor a \log n \rfloor\}.$$

Since $B_{n^{-\psi}}(x_0) \cap \{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\} \subset B_{n^{-\psi}}(x_0) \cap E^{\sigma^i\omega}_n$, it is enough to consider

$$(\log n) \sum_{i=0}^{n-1} m(B_{n^{-\psi}}(x_0) \cap E^{\sigma^i\omega}_n).$$

According to Lemma 10.1, we have

$$m(E^\omega_n) \le \sum_{j=1}^{\lfloor a \log n \rfloor} m(\mathcal{E}^\omega_j(2n^{-\psi})) \le C \frac{\log n}{n^\psi}.$$

We introduce the maximal function

$$M^\omega_n(x_0) = \sup_{t>0} \frac{1}{2t} \int_{x_0-t}^{x_0+t} \left(\sum_{i=0}^{n-1} \mathbf{1}_{E^{\sigma^i\omega}_n}(z) \right) dz = \sup_{t>0} \frac{1}{2t} \sum_{i=0}^{n-1} m(B_t(x_0) \cap E^{\sigma^i\omega}_n).$$

By [**Rud87**, Equation (5), p. 138], for all $\lambda > 0$, we have

$$m(M^\omega_n > \lambda) \le \frac{C}{\lambda} \left\| \sum_{i=0}^{n-1} \mathbf{1}_{E^{\sigma^i\omega}_n} \right\|_{L^1_m} \le \frac{C}{\lambda} \sum_{i=0}^{n-1} m(E^{\sigma^i\omega}_n) \le \frac{C}{\lambda} \frac{\log n}{n^{\psi-1}}. \tag{10.5}$$

Let $\rho > 0$ and $\xi > 0$ to be determined later. We define

$$F^\omega_n = \{x_0 \in [0, 1] : m(B_{n^{-\psi}}(x_0) \cap E^\omega_n) \ge 2n^{-\psi(1+\rho)}\},$$

so that we have

$$\sum_{i=0}^{n-1} m(B_{n^{-\psi}}(x_0) \cap E^{\sigma^i\omega}_n) \ge \left(\sum_{i=0}^{n-1} \mathbf{1}_{F^{\sigma^i\omega}_n}(x_0) \right) 2n^{-\psi(1+\rho)}.$$

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By definition of the maximal function M_n^ω, this implies that

$$M_n^\omega(x_0) \geq n^{-\psi\rho}\bigg(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}}(x_0)\bigg),$$

from which it follows, by equation (10.5) with $\lambda = (\log n)n^{\xi-\psi\rho}$,

$$m(A_n^\omega) \leq m(M_n^\omega > (\log n)n^{\xi-\psi\rho}) \leq Cn^{-(\xi+(1-\rho)\psi-1)} =: \gamma_1(n),$$

where

$$A_n^\omega = \bigg\{\bigg(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}}\bigg) > (\log n)n^{\xi}\bigg\}.$$

If $x_0 \notin A_n^\omega$, then

$$\begin{aligned}
&(\log n)\sum_{i=0}^{n-1} m(B_{n^{-\psi}}(x_0)\cap E_n^{\sigma^i\omega})\\
&\leq (\log n)\bigg(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}}(x_0)\bigg) m(B_{n^{-\psi}}(x_0)) + 2(\log n)n^{1-\psi(1+\rho)}\\
&\leq C(\log n)((\log n)n^{-(\psi-\xi)} + n^{-(\psi(1+\rho)-1)}) =: \gamma_2(n).
\end{aligned}$$

Since $\frac{2}{3} < \psi < 1$ and $0 < \kappa < 3\psi - 2$, it is possible to choose $\rho > 0$ and $\xi > 0$ such that $\kappa = \xi + (1-\rho)\psi - 1$, $\psi > \xi$, and $\psi(1+\rho) > 1$ (for instance, take $\xi = \psi - \delta$ and $\rho = \psi^{-1} - 1 + \delta\psi^{-1}$ with $\delta = (3\psi - 2 - \kappa)/2$), which concludes the proof. □

LEMMA 10.3. *Suppose that $a > 0$ and $\frac{3}{4} < \psi < 1$. Then for m-a.e. $x_0 \in [0, 1]$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ and, we have*

$$\lim_{n\to\infty} (\log n)\sum_{i=0}^{n-1} m(B_{n^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\}) = 0.$$

Proof. Let $0 < \kappa < 3\psi - 2$ to be determined later. Consider the sets $(A_n^\omega)_{n\geq 1}$ given by Lemma 10.2, with $m(A_n^\omega) \leq \gamma_1(n) = \mathcal{O}(n^{-\kappa})$. Since $\kappa < 1$, we need to consider a subsequence $(n_k)_{k\geq 1}$ such that $\sum_{k\geq 1}\gamma_1(n_k) < \infty$. For such a subsequence, by the Borel–Cantelli lemma, for m-a.e. x_0, there exists $K = K(x_0, \omega)$ such that for all $k \geq K$, $x_0 \notin A^\omega_{n_k}$. Since $\lim_{k\to\infty}\gamma_2(n_k) = 0$, this implies

$$\lim_{k\to\infty} (\log n_k)\sum_{i=0}^{n_k-1} m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \leq \lfloor a\log n_k\rfloor\}) = 0.$$

We take $n_k = \lfloor k^\zeta\rfloor$ for some $\zeta > 0$ to be determined later. To have $\sum_{k\geq 1}\gamma_1(n_k) < \infty$, we need to require that $\kappa\zeta > 1$. Set $U_n^\omega(x_0) = B_{n^{-\psi}}(x_0)\cap\{R^\omega_{B_{n^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\}$. To obtain the convergence to 0 of the whole sequence, we need to prove that

$$\lim_{k\to\infty}\sup_{n_k\leq n<n_{k+1}}\bigg|(\log n)\sum_{i=0}^{n-1} m(U_n^{\sigma^i\omega}(x_0)) - (\log n_k)\sum_{i=0}^{n_k-1} m(U_{n_k}^{\sigma^i\omega}(x_0))\bigg| = 0. \quad (10.6)$$

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For this purpose, we estimate

$$\left|(\log n)\sum_{i=0}^{n-1} m(U_n^{\sigma^i\omega}(x_0)) - (\log n_k)\sum_{i=0}^{n_k-1} m(U_{n_k}^{\sigma^i\omega}(x_0))\right|$$
$$\leq \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)},$$

where

$$\text{(I)} = |\log n - \log n_k| \sum_{i=0}^{n-1} m(U_n^{\sigma^i\omega}(x_0)), \quad \text{(II)} = (\log n_k)\sum_{i=n_k}^{n-1} m(U_n^{\sigma^i\omega}(x_0)),$$

$$\text{(III)} = (\log n_k)\sum_{i=0}^{n_k-1} |m(B_{n^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})$$
$$- m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})|,$$

$$\text{(IV)} = (\log n_k)\sum_{i=0}^{n_k-1} |m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})$$
$$- m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})|,$$

$$\text{(V)} = (\log n_k)\sum_{i=0}^{n_k-1} |m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})$$
$$- m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \leq \lfloor a\log n_k\rfloor\})|.$$

Before proceeding to estimate each term, we note that $|n_{k+1}-n_k| = \mathcal{O}(k^{-(1-\zeta)})$, $|n_{k+1}^{-\psi}-n_k^{-\psi}| = \mathcal{O}(k^{-(1+\zeta\psi)})$, $|\log n_{k+1}-\log n_k| = \mathcal{O}(k^{-1})$ and $m(U_n^{\omega}(x_0)) \leq m(B_{n^{-\psi}}(x_0)) = \mathcal{O}(k^{-\zeta\psi})$.

From these observations, it follows

$$\text{(I)} \leq C|\log n_{k+1}-\log n_k| n_{k+1}k^{-\zeta\psi} \leq Ck^{-(1-(1-\psi)\zeta)},$$

$$\text{(II)} \leq C(\log n_k)|n_{k+1}-n_k|k^{-\zeta\psi} \leq C(\log k)k^{-(1-(1-\psi)\zeta)},$$

$$\text{(III)} \leq C(\log n_k)n_k m(B_{n_k^{-\psi}}(x_0)\setminus B_{n^{-\psi}}(x_0))$$
$$\leq C(\log n_k)n_k|n_{k+1}^{-\psi}-n_k^{-\psi}| \leq C(\log k)k^{-(1-(1-\psi)\zeta)},$$

$$\text{(IV)} \leq C(\log n_k)\sum_{i=0}^{n_k-1} m(B_{n_k^{-\psi}}(x_0)\cap\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)\setminus B_{n^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})$$
$$\leq C(\log n_k)\sum_{i=0}^{n_k-1} a(\log n) m(B_{n_k^{-\psi}}(x_0)\setminus B_{n^{-\psi}}(x_0))$$
$$\leq C(\log k)^2 k^{-(1-(1-\psi)\zeta)},$$

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and

$$\begin{aligned}(\mathrm{V}) &\leq C(\log n_k)\sum_{i=0}^{n_k-1} m(B_{n_k^{-\psi}}(x_0)\cap\{\lfloor a\log n_k\rfloor < R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \leq \lfloor a\log n\rfloor\})\\ &\leq C(\log n_k)\sum_{i=0}^{n_k-1} a|\log n_{k+1}-\log n_k| m(B_{n_k^{-\psi}}(x_0))\\ &\leq C(\log k)k^{-(1-(1-\psi)\zeta)}.\end{aligned}$$

To obtain equation (10.6), it is thus sufficient to choose $\kappa > 0$ and $\zeta > 0$ such that $\kappa < 3\psi - 2$, $\kappa\zeta > 1$, and $(1-\psi)\zeta < 1$, which is possible if $\psi > \frac{3}{4}$. □

We can now prove the functional convergence to a Lévy stable process for i.i.d. uniformly expanding maps.

Proof of Theorem 2.4. We apply Theorem 7.3. By Theorem 6.3, we have $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ under the probability ν^ω for $\mathbb{P}$-a.e. $\omega\in\Omega$. It thus remains to check that equation (7.2) holds for m-a.e. x_0 when $\alpha\in[1,2)$ to complete the proof. For this purpose, we will use a reverse martingale argument from [**NTV18**] (see also [**AR16**, Proposition 13]). Because of equation (5.8), it is enough to work on the probability space $([0,1],\nu^\omega)$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. Let $\mathcal{B}$ denote the σ-algebra of Borel sets on $[0,1]$ and

$$\mathcal{B}_{\omega,k} = (T_\omega^k)^{-1}(\mathcal{B}).$$

To simplify notation a bit, let

$$f_{\omega,j,n}(x) = \phi_{x_0}(x)\mathbf{1}_{\{|\phi_{x_0}|\leq\varepsilon b_n\}}(x) - \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\leq\varepsilon b_n\}}).$$

From equation (5.8), it follows that $\mathbb{E}_m(|f_{\omega,j,n}|)\leq C\varepsilon b_n$, and from the explicit definition of ϕ, we can estimate the total variation of $f_{\omega,j,n}$ and obtain the existence of $C>0$, independent of ω, ε, n, and j, such that

$$\|f_{\omega,j,n}\|_{\mathrm{BV}} \leq C\varepsilon b_n. \tag{10.7}$$

We define

$$S_{\omega,k,n} := \sum_{j=0}^{k-1} f_{\omega,j,n}\circ T_\omega^j$$

and

$$H_{\omega,k,n}\circ T_\omega^n := \mathbb{E}_{\nu^\omega}(S_{\omega,k,n}|\mathcal{B}_{\omega,k}). \tag{10.8}$$

Hence, $H_{\omega,1,n}=0$ and an explicit formula for $H_{\omega,k,n}$ is

$$H_{\omega,k,n} = \frac{1}{h_{\sigma^k\omega}}\sum_{j=0}^{k-1} P_{\sigma^j\omega}^{k-j}(f_{\omega,j,n}h_{\sigma^j\omega}).$$

From the explicit formula, the exponential decay in the BV norm of $P_{\sigma^j\omega}^{n-j}$ from condition (Dec), equations (5.8) and (10.7), we see that $\|H_{\omega,k,n}\|_{\mathrm{BV}}\leq C\varepsilon b_n$, where the constant C may be taken as constant over $\omega\in\Omega$. If we define

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$$M_{\omega,k,n} = S_{\omega,k,n} - H_{\omega,k,n} \circ T_\omega^k,$$

then the sequence $\{M_{\omega,k,n}\}_{k\geq 1}$ is a reverse martingale difference for the decreasing filtration $\mathcal{B}_{\omega,k} = (T_\omega^n)^{-1}(\mathcal{B})$ as

$$\mathbb{E}_{\nu^\omega}(M_{\omega,k,n}|\mathcal{B}_{\omega,k}) = 0.$$

The martingale reverse differences are

$$M_{\omega,k+1,n} - M_{\omega,k,n} = \psi_{\omega,k,n} \circ T_\omega^k,$$

where

$$\psi_{\omega,k,n} := f_{\omega,k,n} + H_{\omega,k,n} - H_{\omega,k+1,n} \circ T_{\sigma^{k+1}\omega}.$$

We see from the L^∞ bounds on $\|H_{\omega,k,n}\|_\infty \leq C b_n \varepsilon$ and the telescoping sum that

$$\left|\sum_{j=0}^{k-1} \psi_{\omega,j,n} \circ T_\omega^j - \sum_{j=0}^{k-1} f_{\omega,j,n} \circ T_\omega^j\right| \leq C\varepsilon b_n. \tag{10.9}$$

By Doob's martingale maximal inequality,

$$\nu^\omega\left\{\max_{1\leq k\leq n}\left|\sum_{j=0}^{k-1} \psi_{\omega,j,n} \circ T_\omega^j\right| \geq b_n\delta\right\} \leq \frac{1}{b_n^2\delta^2}\mathbb{E}_{\nu^\omega}\left|\sum_{j=0}^{n-1} \psi_{\omega,j,n} \circ T_\omega^j\right|^2.$$

Note that

$$\sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[\psi_{\omega,j,n}^2 \circ T_\omega^j] = \mathbb{E}_{\nu^\omega}\left[\sum_{j=0}^{n-1} \psi_{\omega,j,n} \circ T_\omega^j\right]^2$$

by pairwise orthogonality of martingale reverse differences.

As in [**HNTV17**, Lemma 6],

$$\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2] = \sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[\psi_{\omega,j,n}^2 \circ T_\omega^j] + \mathbb{E}_{\nu^\omega}[H_{\omega,1,n}^2] - \mathbb{E}_{\nu^\omega}[H_{\omega,n,n}^2 \circ T_\omega^n].$$

So we see that

$$\nu^\omega\left\{\max_{1\leq k\leq n}\left|\sum_{j=0}^{k-1} \psi_{\omega,j,n} \circ T_\omega^j\right| \geq b_n\delta\right\} \leq \frac{1}{b_n^2\delta^2}\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2] + 2\frac{C^2\varepsilon^2}{\delta^2}, \tag{10.10}$$

where we have used $\|H_{\omega,j,n}^2\|_\infty \leq C^2 b_n^2 \varepsilon^2$.

Now we estimate

$$\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2] \leq \sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}^2 \circ T_\omega^j] + 2\sum_{i=0}^{n-1}\sum_{i<j} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n} \circ T_\omega^j \cdot f_{\omega,i,n} \circ T_\omega^i]. \tag{10.11}$$

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Using the equivariance of the measures $\{\nu^\omega\}_{\omega\in\Omega}$ and equation (5.8), we have

$$\sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[f^2_{\omega,j,n} \circ T^j_\omega] \le Cn\mathbb{E}_\nu(\phi^2_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}}) \sim C\varepsilon^{2-\alpha} b_n^2, \tag{10.12}$$

by Proposition 3.2 and that

$$\lim_{n\to\infty} n\ \nu(|\phi_{x_0}| > \lambda b_n) = \lambda^{-\alpha} \quad \text{for } \lambda > 0,$$

since ϕ_{x_0} is regularly varying.

However, we are going to show that for m-a.e. x_0,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{b_n^2}\sum_{i=0}^{n-1}\sum_{i<j} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] = 0. \tag{10.13}$$

The first observation is that, due to condition (Dec),

$$\mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] \le C\theta^{j-i}\|f_{\omega,i,n}\|_{\mathrm{BV}}\|f_{\omega,j,n}\|_{L^1_m} \le C\varepsilon^2 b_n^2\theta^{j-i},$$

where $\theta < 1$. Hence, there exists $a > 0$ independently of n and ε such that

$$\sum_{j-i>\lfloor a\log n\rfloor} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] \le C\varepsilon^2 n^{-2} b_n^2$$

and it is enough to prove that for $\varepsilon > 0$,

$$\sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] = o(b_n^2) = o(n^{2/\alpha}).$$

By construction, the term $\mathbb{E}_{\nu^\omega}[f_{\omega,i,n}\circ T^i_\omega \cdot f_{\omega,j,n}\circ T^j_\omega]$ is a covariance, and since ϕ is positive, we can bound this quantity by $\mathbb{E}_{\nu^\omega}[f\circ T^i_\omega \cdot f\circ T^j_\omega] = \mathbb{E}_{\nu^{\sigma^i\omega}}[f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}]$, where $f_n = \phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}}$. Then, since the densities are uniformly bounded by equation (5.8), we are left to estimate

$$\sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor} \mathbb{E}_m[f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}]. \tag{10.14}$$

Let $\frac{3}{4} < \psi < 1$ and $U_n = B_{n^{-\psi}}(x_0)$. We bound equation (10.14) by (I) + (II) + (III), where

$$(\mathrm{I}) = \sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor}\int_{U_n\cap(T^{j-i}_{\sigma^i\omega})^{-1}(U_n)} f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}\,dm,$$

$$(\mathrm{II}) = \sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor}\int_{U_n\cap(T^{j-i}_{\sigma^i\omega})^{-1}(U_n^c)} f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}\,dm,$$

and

$$(\mathrm{III}) = \sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor}\int_{U_n^c} f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}\,dm.$$

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Since $\|f_n\|_\infty \le \varepsilon b_n$, it follows that

$$(\mathrm{I}) \le \varepsilon^2 b_n^2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{i+\lfloor a \log n\rfloor} m(U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n))$$
$$\le a\varepsilon^2 b_n^2 (\log n) \sum_{i=0}^{n-1} m(U_n \cap \{R_{U_n}^{\sigma^i\omega} \le a \log n\}),$$

which by Lemma 10.3 is a $o(b_n^2)$ as $n \to \infty$ for m-a.e. x_0.

To estimate terms (II) and (III), we will use Hölder's inequality. We first observe by a direct computation that

$$\int_{U_n^c} \phi_{x_0}^2 \, dm = \mathcal{O}(n^{\psi(2/\alpha-1)}). \tag{10.15}$$

We consider term (III) first. Let $A = U_n^c$. We have

$$\int_{U_n^c} f_n \cdot f_n \circ T_{\sigma^i\omega}^{j-i} \, dm \le \int_A \phi_{x_0} \cdot f_n \circ T_{\sigma^i\omega}^{j-i} \, dm$$
$$\le \bigg(\int_A \phi_{x_0}^2 \, dm\bigg)^{1/2} \bigg(\int f_n^2 \circ T_{\sigma^i\omega}^{j-i} \, dm\bigg)^{1/2} \tag{10.16}$$
$$\le C\bigg(\int_A \phi_{x_0}^2 \, dm\bigg)^{1/2} \bigg(\int f_n^2 \, dm\bigg)^{1/2}. \tag{10.17}$$

By equation (10.15), $(\int_A \phi_{x_0}^2 \, dm)^{1/2} \le Cn^{\psi/2(2/\alpha-1)}$ and by Proposition 3.2, $(\int f_n^2 \, dm)^{1/2} \le Cn^{1/\alpha-1/2}$. Hence, we may bound equation (10.16) by $Cn^{(1+\psi)(1/\alpha-1/2)}$.

To bound term (II), let $B = U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)$. Then,

$$\int_{U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)} f_n \cdot f_n \circ T_{\sigma^i\omega}^{j-i} \, dm \le \int_B f_n \cdot \phi_{x_0} \circ T_{\sigma^i\omega}^{j-i} \, dm$$
$$\le \bigg(\int f_n^2 \, dm\bigg)^{1/2} \bigg(\int_B \phi_{x_0}^2 \circ T_{\sigma^i\omega}^{j-i} \, dm\bigg)^{1/2}. \tag{10.18}$$

As before, $(\int f_n^2 dm)^{1/2} \le Cn^{1/\alpha-1/2}$ and

$$\bigg(\int_B \phi_{x_0}^2 \circ T_{\sigma^i\omega}^{j-i} \, dm\bigg)^{1/2} \le \bigg(\int \phi_{x_0}^2 \circ T_{\sigma^i\omega}^{j-i} \mathbf{1}_{(T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)} \, dm\bigg)^{1/2}$$
$$\le C\bigg(\int_{U_n^c} \phi_{x_0}^2 \, dm\bigg)^{1/2} \le Cn^{\psi/2(2/\alpha-1)}$$

by equation (10.15), and so equation (10.18) is bounded by $Cn^{(1+\psi)(1/\alpha-1/2)}$.

It follows that (II) + (III) $\le C(\log n)n^{1+(1+\psi)(1/\alpha-1/2)} = o(n^{2/\alpha})$, since $\psi < 1$. This proves that equation (10.14) is $o(b_n^2)$ and concludes the proof of equation (10.13).

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Finally, from equations (10.11), (10.12), and (10.13), we obtain

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{b_n^2}\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2]=0, \tag{10.19}$$

which gives the result by taking the limit first in n and then in ε in equation (10.10). □

10.2. *Intermittent maps: proof of Theorem 2.6.* We prove convergence to a stable law in the setting of intermittent maps when $\alpha \in (0, 1)$.

Proof of Theorem 2.6. We apply Proposition 5.8. By Theorem 6.4, it remains to prove equation (5.7), since $\alpha \in (0, 1)$. We will need an estimate for $\mathbb{E}_{\nu^\omega}(|\phi_{x_0}|\mathbf{1}_{\{\phi_{x_0}\le\varepsilon b_n\}})$ which is independent of ω. For this purpose, we introduce the absolutely continuous probability measure $\nu_{\max}$ whose density is given by $h_{\max}(x)=\kappa x^{-\gamma_{\max}}$. Since all densities h_ω belong to the cone L, we have that $h_\omega \le (a/\kappa)h_{\max}$ for all ω. Thus,

$$\frac{1}{b_n}\sum_{j=0}^{n-1}\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}})\le\frac{n}{b_n}\frac{a}{\kappa}\mathbb{E}_{\nu_{\max}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}}).$$

We can easily verify that ϕ_{x_0} is regularly varying of index α with respect to $\nu_{\max}$, with scaling sequence equal to $(b_n)_{n\ge1}$ up to a multiplicative constant factor. Consequently, by Proposition 3.2, we have that, for some constant $c > 0$,

$$\mathbb{E}_{\nu_{\max}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}})\sim c\varepsilon^{1-\alpha}n^{1/\alpha-1},$$

which implies equation (5.7). □

11. *The annealed case*

In this section, we consider the annealed counterparts of our results. Even though the annealed versions do not seem to follow immediately from the quenched version, it is easy to obtain them from our proofs in the quenched case. We take $\phi_{x_0}(x)=d(x,x_0)^{-1/\alpha}$ as before we consider the convergence on the measure space $\Omega\times[0,1]$ with respect to $\nu(d\omega,dx)=\mathbb{P}(d\omega)\nu^\omega(dx)$. We give precise annealed results in the case of Theorems 2.4 and 2.6, where we consider

$$X_n^a(\omega,x)(t):=\frac{1}{b_n}\Bigg[\sum_{j=0}^{\lfloor nt\rfloor-1}\phi_{x_0}(T_\omega^j x)-tc_n\Bigg],\ t\ge 0,$$

viewed as a random process defined on the probability space $(\Omega\times[0,1],\nu)$.

THEOREM 11.1. *Under the same assumptions as Theorem 2.4, the random process $X_n^a(t)$ converges in the J_1 topology to the Lévy α-stable process $X_{(\alpha)}(t)$ under the probability measure ν.*

Proof. We apply [TK10b, Theorem 1.2] to the skew-product system $(\Omega\times[0,1],F,\nu)$ and the observable ϕ_{x_0} naturally extended to $\Omega\times[0,1]$. Recall that ν is given by the disintegration $\nu(d\omega,dx)=\mathbb{P}(d\omega)\nu^\omega(dx)$.

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We have to prove that:

(a) $N_n \xrightarrow{d} N_{(\alpha)}$;

(b) if $\alpha \in [1, 2)$, for all $\delta > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \nu((\omega, x) : \max_{1\le k\le n} \bigg|\frac{1}{b_n}\sum_{j=0}^{k-1}[\phi_{x_0}(T_\omega^j x)\mathbf{1}_{\{|\phi_{x_0}\circ T_\omega^j|\le\varepsilon b_n\}}(x) - \mathbb{E}_\nu(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}})]\bigg| \ge \delta) = 0,$$

where

$$N_n(\omega, x)(B) := N_n^\omega(x)(B) = \#\bigg\{j \ge 1 : \bigg(\frac{j}{n}, \frac{\phi_{x_0}(T_\omega^{j-1}(x))}{b_n}\bigg) \in B\bigg\},\ n \ge 1.$$

To prove item (a), we take $f \in C_K^+((0,\infty) \times (\mathbb{R} \setminus \{0\}))$ arbitrary. Then, by Theorem 6.3, we have for $\mathbb{P}$-a.e. ω,

$$\lim_{n\to\infty} \mathbb{E}_{\nu^\omega}(e^{-N_n^\omega(f)}) = \mathbb{E}(e^{-N(f)}).$$

Integrating with respect to $\mathbb{P}$ and using the dominated convergence theorem yields

$$\lim_{n\to\infty} \mathbb{E}_\nu(e^{-N_n(f)}) = \mathbb{E}(e^{-N(f)}),$$

which proves item (a).

To prove item (b), we simply have to integrate with respect to $\mathbb{P}$ in the estimates in the proof of Theorem 2.4, which hold uniformly in $\omega \in \Omega$, and then to take the limits as $n \to \infty$ and $\varepsilon \to 0$. □

Similarly, we have the following theorem.

THEOREM 11.2. *Under the same assumptions as Theorem 2.6, $X_n^a(1) \xrightarrow{d} X_{(\alpha)}(1)$ under the probability measure ν.*

Proof. We can proceed as for Theorem 11.1 to check the assumptions of [**TK10b**, Theorem 1.3] for the skew-product system $(\Omega \times [0, 1], F, \nu)$ and the observable ϕ_{x_0}. □

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A. *Appendix*

The observation that our distributional limit theorems hold for any measures $\mu \ll \nu^\omega$ follows from Zweimüller's work [**Zwe07**, Theorem 1, Corollary 1, and Corollary 3].

Let

$$S_n(x) = \frac{1}{b_n}\bigg[\sum_{j=0}^{n-1} \phi \circ T_\omega^j(x) - a_n\bigg]$$

and suppose

$$S_n \to_{\nu_\omega} Y,$$

where Y is a Lévy random variable.

We consider first the set-up of intermittent maps. We will show that for any measure ν with density h that is $d\nu = h\, dm$ in the cone L, in particular Lebesgue measure m with $h = 1$,

$$S_n \to_\nu Y.$$

We focus on m. According to [**Zwe07**, Theorem 1], it is enough to show that

$$\int \psi(S_n)\, d\nu_\omega - \int \psi(S_n)\, dm \to 0$$

for any $\psi : \mathbb{R} \to \mathbb{R}$ which is bounded and uniformly Lipschitz.

Fix such a ψ and consider

$$\begin{aligned}
&\int \psi\bigg(\frac{1}{b_n}\bigg[\sum_{j=0}^{n-1} \phi \circ T_\omega^j(x) - a_n\bigg]\bigg)(h_\omega - 1)\, dm \\
&\le \int \psi\bigg(\frac{1}{b_n}\bigg[\sum_{j=0}^{n-1} \phi \circ T_{\sigma^k\omega}^j(x) - a_n\bigg]\bigg) P_\omega^k(h_\omega - 1)\, dm \\
&\le \|\psi\|_\infty \|P_\omega^k(h_\omega - 1)\|_{L^1(m)}.
\end{aligned}$$

Since $\|P_\omega^k(h_\omega - 1)\|_{L_m^1} \to 0$ in case of Example 2.2 and maps satisfying conditions (LY), (Dec), and (Min), the assertion is proved. By [**Zwe07**, Corollary 3], the proof for continuous time distributional limits follows immediately.

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