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POPULATION MODELS AT STOCHASTIC TIMES

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Abstract

In this paper we consider time-changed models of population evolution $\mathcal{X}^f(t) = \mathcal{X}(H^f(t))$, where $\mathcal{X}$ is a counting process and H^f is a subordinator with Laplace exponent f. In the case where $\mathcal{X}$ is a pure birth process, we study the form of the distribution, the intertimes between successive jumps, and the condition of explosion (also in the case of killed subordinators). We also investigate the case where $\mathcal{X}$ represents a death process (linear or sublinear) and study the extinction probabilities as a function of the initial population size n_0. Finally, the subordinated linear birth–death process is considered. Special attention is devoted to the case where birth and death rates coincide; the sojourn times are also analysed.

Keywords: Nonlinear birth process; sublinear death process; linear death process; sojourn time; fractional birth process; random time

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1. Introduction

Birth and death processes can be applied in the modelling of many dynamical systems such as cosmic showers, fragmentation processes, queueing systems, epidemics, population growth, and aftershocks in earthquakes. The time-changed version of such processes has also been analysed since it is useful to describe the dynamics of various systems when the underlying environmental conditions randomly change. For example, the fractional birth and death processes, studied in Orsingher and Polito [10]–[12], [14], are time-changed processes where the distribution of the time is related to the fractional diffusion equations. We refer the reader to Cahoy and Polito [4], [5] for some applications and simulations.

In this paper we consider the case where the random time is a subordinator. Actually, subordinated Markov processes have been extensively studied since the 1950s. The case of birth and death processes merits, however, a further investigation and this is our aim in this paper. We consider here compositions of point processes $\mathcal{X}(t)$, $t > 0$, with an arbitrary subordinator $H^f(t)$ related to the Bernštein functions f (for the theory of subordinators, we refer the reader to [1]). We denote such processes as $\mathcal{X}^f(t) = \mathcal{X}(H^f(t))$. The general form of f is

$$f(x) = \alpha + \beta x + \int_0^\infty (1 - \mathrm{e}^{-xs})\nu(\mathrm{d}s), \quad \alpha \ge 0,\ \beta \ge 0, \tag{1.1}$$

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where ν is the Lévy measure satisfying

$$\int_0^\infty (s \wedge 1)\nu(\mathrm{d}s) < \infty. \tag{1.2}$$

In this paper we refer to the $\alpha = \beta = 0$ case, unless explicitly stated. The structure of the paper is as follows. In Section 2 we treat the subordinated nonlinear birth process, in Section 3 we deal with the subordinated linear and sublinear death processes, and in Section 4 we analyse the linear birth–death process, with particular attention to the case where birth and death rates coincide. In all three cases, we compute directly the state probabilities by means of the composition formula

$$\mathbb{P}\{\mathcal{X}^f(t) = k\} = \int_0^\infty \mathbb{P}\{\mathcal{X}(s) = k\}\mathbb{P}\{H^f(t) \in \mathrm{d}s\}.$$

Despite most of the subordinators not possessing an explicit form for the probability density function, the distribution of $\mathcal{X}(H^f(t))$ always presents a closed form in terms of the Laplace exponent f. We also study the transition probabilities, both for finite and infinitesimal time intervals. We emphasise that the subordinated point processes have a fundamental difference with respect to the classical ones in that they perform upward or downward jumps of arbitrary size. For infinitesimal time intervals, we provide a direct and simple proof of the following fact:

$$\mathbb{P}\{\mathcal{X}^f(t + \mathrm{d}t) = k \mid \mathcal{X}^f(t) = r\} = \mathrm{d}t \int_0^\infty \mathbb{P}\{\mathcal{X}(s) = k \mid \mathcal{X}(0) = r\}\nu(\mathrm{d}s), \tag{1.3}$$

which is related to Bochner subordination (see [15]).

The first case taken into account is that of a nonlinear birth process with birth rates $\lambda_k, k \geq 1$, which is denoted by $\mathcal{N}(t)$. The subordinated process $\mathcal{N}^f(t)$ does not explode if and only if the following condition is fulfilled:

$$\sum_{j=1}^\infty \frac{1}{\lambda_j} = \infty.$$

This is the same condition of nonexplosion holding for the classical case. Such a condition ceases to hold if we consider a Lévy exponent with $\alpha \neq 0$, which is related to the so-called killed subordinator. In this case, indeed, the process $\mathcal{N}^f(t)$ can explode in a finite time, even if $\mathcal{N}(t)$ does not; more precisely

$$\mathbb{P}\{\mathcal{N}^f(t) = \infty\} = 1 - \mathrm{e}^{-\alpha t}.$$

We note that $\mathcal{N}^f(t)$ can be regarded as a process where upward jumps are separated by exponentially distributed time intervals Y_k such that

$$\mathbb{P}\{Y_k > t \mid \mathcal{N}^f(T_{k-1}) = r\} = \mathrm{e}^{-f(\lambda_r)t},$$

where T_{k-1} is the instant of the $(k-1)$th jump.

In Section 3 we study the subordinated linear and sublinear death processes, that we respectively denote by $M^f(t)$ and $\mathbb{M}^f(t)$, with an initial number of components equal to n_0. We emphasise that in the sublinear case the annihilation is initially slower, then accelerates when few survivors remain. So despite $M^f(t)$ and $\mathbb{M}^f(t)$ presenting different state probabilities, we observe that the extinction probabilities coincide and we prove that they decrease for increasing values of n_0.

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In Section 4, the subordinated linear birth–death process $L^f(t)$ is considered. If the birth and death rates coincide and H^f is a stable subordinator, we compute the mean sojourn time in each state and find, in some particular cases, the distribution of the intertimes between successive jumps. We finally study the probability density of the sojourn times by giving a sketch of the derivation of their Laplace transforms.

2. Subordinated nonlinear birth process

In this section we consider the process $\mathcal{N}^f(t) = \mathcal{N}(H^f(t))$, where $\mathcal{N}$ is a nonlinear birth process with one progenitor and rates λ_k, $k \geq 1$, and $H^f(t)$ is a subordinator independent from $\mathcal{N}(t)$. It is well known that the state probabilities of $\mathcal{N}(t)$ are

$$\mathbb{P}\{\mathcal{N}(t) = k \mid \mathcal{N}(0) = 1\} = \begin{cases} \displaystyle\prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{\mathrm{e}^{-\lambda_m t}}{\prod_{l=1,l\neq m}^{k}(\lambda_l - \lambda_m)}, & k > 1, \\ \mathrm{e}^{-t\lambda_1}, & k = 1. \end{cases}$$

The subordinated process $\mathcal{N}^f(t)$ thus possesses the following distribution:

$$\begin{aligned} \mathbb{P}\{\mathcal{N}^f(t) = k \mid \mathcal{N}^f(0) = 1\} &= \int_0^\infty \mathbb{P}\{\mathcal{N}(s) = k \mid \mathcal{N}(0) = 1\}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\ &= \begin{cases} \displaystyle\prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{\mathrm{e}^{-tf(\lambda_m)}}{\prod_{l=1,l\neq m}^{k}(\lambda_l - \lambda_m)}, & k > 1, \\ \mathrm{e}^{-tf(\lambda_1)}, & k = 1. \end{cases} \end{aligned} \tag{2.1}$$

The distribution (2.1) can be easily generalised to the case of r progenitors and is given by

$$\mathbb{P}\{\mathcal{N}^f(t) = r + k \mid \mathcal{N}^f(0) = r\} = \begin{cases} \displaystyle\prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{\mathrm{e}^{-tf(\lambda_m)}}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)}, & k > 0, \\ \mathrm{e}^{-tf(\lambda_r)}, & k = 0. \end{cases} \tag{2.2}$$

Note that $\mathcal{N}^f(t)$ is indeed a well-defined process. This holds because (2.2) are well-defined Markovian transition functions, which allow us to write all the finite-dimensional distributions and this ensures the existence of a Markovian process in view of the Kolmogorov theorem. Moreover, $\mathcal{N}^f(t)$ is time-homogeneous and (2.2) permits us to write

$$\mathbb{P}\{\mathcal{N}^f(t + \mathrm{d}t) = r + k \mid \mathcal{N}^f(t) = r\} = \begin{cases} \displaystyle\prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{1 - \mathrm{d}tf(\lambda_m)}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)}, & k > 0, \\ 1 - \mathrm{d}tf(\lambda_r), & k = 0. \end{cases} \tag{2.3}$$

To find an alternative expression for the transition probabilities we need the following lemma.

Lemma 2.1. *For any sequence of $k+1$ distinct positive numbers λ_r, $\lambda_{r+1} \cdots \lambda_{r+k}$ the following relationship holds:*

$$c_{r,k} = \sum_{m=r}^{r+k} \frac{1}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)} = 0. \tag{2.4}$$

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Proof. The proof follows as a consequence of (2.2) by letting $t \to 0$. An alternative proof can be obtained by suitably adapting the calculation of [10, Theorem 2.1]. □

We are now able to state the following theorem.

Theorem 2.1. *For $k > r$ the transition probability takes the form*

$$\mathbb{P}\{\mathcal{N}^f(t + \mathrm{d}t) = k \mid \mathcal{N}^f(t) = r\} = \mathrm{d}t \int_0^\infty \mathbb{P}\{\mathcal{N}(s) = k \mid \mathcal{N}(0) = r\}\nu(\mathrm{d}s). \tag{2.5}$$

Proof. By repeatedly using both (2.4) and representation (1.1) of the Bernštein functions f, we have

$$\begin{aligned}
\mathbb{P}\{\mathcal{N}^f(t + \mathrm{d}t) = k \mid \mathcal{N}^f(t) = r\} &= \prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{1 - \mathrm{d}t f(\lambda_m)}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)} \\
&= -\mathrm{d}t \prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{f(\lambda_m)}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)} \\
&= -\mathrm{d}t \int_0^\infty \prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{1 - \mathrm{e}^{-\lambda_m s}}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)} \nu(\mathrm{d}s) \\
&= \mathrm{d}t \int_0^\infty \prod_{j=r}^{r+k-1} \lambda_j \sum_{m=r}^{r+k} \frac{\mathrm{e}^{-\lambda_m s}}{\prod_{l=r,l\neq m}^{r+k}(\lambda_l - \lambda_m)} \nu(\mathrm{d}s).
\end{aligned} \tag{2.6}$$

In light of (2.4), the integrand in (2.6) is $\mathcal{O}(s)$ for $s \to 0$. Recalling (1.2), this ensures the convergence of (2.6), and the proof is thus complete. □

Remark 2.1. For the sake of completeness, we observe that in the $k = r$ case, we have

$$\begin{aligned}
\mathbb{P}\{\mathcal{N}^f(t + \mathrm{d}t) = r \mid \mathcal{N}^f(t) = r\} &= 1 - \mathrm{d}t f(\lambda_r) \\
&= 1 - \mathrm{d}t \int_0^\infty (1 - \mathrm{e}^{-\lambda_r s})\nu(\mathrm{d}s) \\
&= 1 - \mathrm{d}t \int_0^\infty (1 - \mathbb{P}\{\mathcal{N}(s) = r \mid \mathcal{N}(0) = r\})\nu(\mathrm{d}s).
\end{aligned}$$

Remark 2.2. The subordinated nonlinear birth process performs jumps of arbitrary height as does the subordinated Poisson process; see, for example, [13]. Thus, in view of the Markovian property, we can write the governing equations for the state probabilities

$$p_k^f(t) = \mathbb{P}\{\mathcal{N}^f(t) = k \mid \mathcal{N}^f(0) = 1\}.$$

For $k > 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k^f(t) = -f(\lambda_k)p_k^f(t) + \sum_{r=1}^{k-1} p_r^f(t) \int_0^\infty \prod_{j=r}^{k-1} \lambda_j \sum_{m=r}^{k} \frac{\mathrm{e}^{-\lambda_m s}}{\prod_{l=r,l\neq m}^{k}(\lambda_l - \lambda_m)} \nu(\mathrm{d}s),$$

while for $k = 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} p_1^f(t) = -f(\lambda_1)p_1^f(t).$$

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Remark 2.3. The process $\mathcal{N}(H^f(t))$ presents positive and integer-valued jumps occurring at random times $T_1, T_2, \dots, T_n$. The interarrival times $Y_1, Y_2, \dots, Y_n$ are defined as

$$Y_k = T_k - T_{k-1}.$$

It is easy to prove that

$$\mathbb{P}\{Y_k > t \mid \mathcal{N}^f(T_{k-1}) = r\} = \mathrm{e}^{-f(\lambda_r)t}.$$

This can be justified by considering that in the time intervals $[T_{k-1}, T_{k-1}+t]$, no new offspring appears in the population and thus, by (2.3), we have

$$\mathbb{P}\{Y_k > t \mid \mathcal{N}^f(T_{k-1}) = r\} = \mathbb{P}\{\mathcal{N}^f(t + T_{k-1}) = r \mid \mathcal{N}^f(T_{k-1}) = r\} = \mathrm{e}^{-f(\lambda_r)t}.$$

2.1. Condition of explosion for the subordinated nonlinear birth process

We note that the explosion of the process $\mathcal{N}^f(t)$, $t > 0$, in a finite time is avoided if and only if

$$T_\infty = Y_1 + Y_2 + \cdots + Y_\infty = \infty,$$

where Y_j, $j \geq 1$, are the intertimes between successive jumps; see [7, p. 252]. For the nonlinear classical process, we have

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{-T_\infty} &= \mathbb{E}\mathrm{e}^{-\sum_{j=1}^\infty Y_j} \\
&= \lim_{n\to\infty} \prod_{j=1}^n \mathbb{E}\mathrm{e}^{-Y_j} \\
&= \lim_{n\to\infty} \prod_{j=1}^n \frac{\lambda_j}{1+\lambda_j} \\
&= \prod_{j=1}^\infty \frac{1}{1+1/\lambda_j} \\
&= \frac{1}{1+\sum_{j=1}^\infty 1/\lambda_j + \cdots}.
\end{aligned}$$

So if $\sum_{j=1}^\infty 1/\lambda_j = \infty$, we have $\mathrm{e}^{-T_\infty} = 0$ almost surely, i.e. $T_\infty = \infty$. Therefore, for the subordinated nonlinear birth process, we have

$$\begin{aligned}
\mathbb{P}\{\mathcal{N}^f(t) < \infty\} &= \int_0^\infty \sum_{k=1}^\infty \mathbb{P}\{\mathcal{N}(s) = k\}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \int_0^\infty \mathbb{P}\{H^f(t) \in \mathrm{d}s\} = 1 \quad \text{for all } t > 0.
\end{aligned}$$

Instead, if $\sum_{j=1}^\infty 1/\lambda_j < \infty$, we obtain $\sum_{k=1}^\infty \mathbb{P}\{\mathcal{N}(s) = k\} < \infty$, and this implies that $\mathbb{P}\{\mathcal{N}^f(t) < \infty\} < 1$.

We can now consider the case of killed subordinators $\mathcal{H}^g(t)$, defined as

$$\mathcal{H}^g(t) = \begin{cases} H^f(t), & t < T, \\ \infty, & t \geq T, \end{cases}$$

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where $T \sim \mathrm{Exp}(\alpha)$ and $H^f(t)$ is an ordinary subordinator related to the function $f(x) = \int_0^\infty (1 - \mathrm{e}^{-sx})\nu(\mathrm{d}s)$. It is well known that $\mathcal{H}^g(t)$ is related to a Bernštein function

$$g(x) = \alpha + f(x).$$

In this case, even if $\sum_{j=1}^\infty 1/\lambda_j = \infty$, the probability of explosion for $\mathcal{N}^f(t)$ is positive and equal to

$$\mathbb{P}\{\mathcal{N}^f(t) = \infty\} = 1 - \mathrm{e}^{-t\alpha}.$$

This can be proven by observing that

$$\begin{aligned}
\mathbb{P}\{\mathcal{N}^f(t) < \infty\} &= \int_0^\infty \sum_{k=1}^\infty \mathbb{P}\{\mathcal{N}(s) = k\}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \int_0^\infty \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \int_0^\infty \mathrm{e}^{-\mu s}\mathbb{P}\{H^f(t) \in \mathrm{d}s\}|_{\mu=0} \\
&= \mathrm{e}^{-\alpha t - f(\mu)t}|_{\mu=0} \\
&= \mathrm{e}^{-\alpha t}.
\end{aligned}$$

If, instead, $\sum_{j=1}^\infty 1/\lambda_j < \infty$, we have $\sum_{k=1}^\infty \mathbb{P}\{\mathcal{N}(s) = k\} < 1$ and thus, a fortiori, $\mathbb{P}\{\mathcal{N}^f(t) < \infty\} < \mathrm{e}^{-\alpha t}$.

2.2. Subordinated linear birth process

The subordinated Yule–Furry process $N^f(t)$ with one initial progenitor possesses the following distribution:

$$\begin{aligned}
p_k^f(t) &= \int_0^\infty \mathrm{e}^{-\lambda s}(1 - \mathrm{e}^{-\lambda s})^{k-1}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \int_0^\infty \mathrm{e}^{-\lambda s} \sum_{J=0}^{k-1} \binom{k-1}{j} (-1)^j \mathrm{e}^{-\lambda s j}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \int_0^\infty \mathrm{e}^{-s\lambda(1+j)}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \mathrm{e}^{-tf(\lambda(j+1))}.
\end{aligned}$$

Of course, this is obtainable from the distribution $\mathcal{N}^f(t)$ by assuming that $\lambda_j = \lambda j$. For an application of the Yule–Furry process, see, for example, [6]. We now compute the factorial moments of the subordinated linear birth process. The probability generating function is

$$G^f(u,t) = \sum_{k=1}^\infty u^k \int_0^\infty \mathrm{e}^{-\lambda s}(1 - \mathrm{e}^{-\lambda s})^{k-1}\mathbb{P}(H^f(t) \in \mathrm{d}s).$$

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The rth-order factorial moments are

$$\begin{aligned}
&\frac{\partial^r}{\partial u^r} G^f(u,t)|_{u=1} \\
&\quad = \sum_{k=r}^{\infty} k(k-1)\cdots(k-r+1) \int_0^{\infty} \mathrm{e}^{-\lambda s}(1-\mathrm{e}^{-\lambda s})^{k-1} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&\quad = \sum_{k=r}^{\infty} k(k-1)\cdots(k-r+1) \int_0^{\infty} \mathrm{e}^{-\lambda s}(1-\mathrm{e}^{-\lambda s})^{k-r}(1-\mathrm{e}^{-\lambda s})^{r-1} \mathbb{P}\{H^f(t) \in \mathrm{d}s\},
\end{aligned}$$

and since

$$\sum_{k=r}^{\infty} k(k-1)\cdots(k-r+1)(1-p)^{k-r} = (-1)^r \frac{\mathrm{d}^r}{\mathrm{d}p^r} \sum_{k=0}^{\infty}(1-p)^k = (-1)^r \frac{\mathrm{d}^r}{\mathrm{d}p^r}\frac{1}{p} = \frac{r!}{p^{r+1}},$$

we have

$$\begin{aligned}
\frac{\partial^r}{\partial u^r} G(u,t)|_{u=1} &= r! \int_0^{\infty} \mathrm{e}^{\lambda r s}(1-\mathrm{e}^{-\lambda s})^{r-1} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= r! \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^m \int_0^{\infty} \mathrm{e}^{-\lambda s(m-r)} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= r! \sum_{m=0}^{r-1} \binom{r-1}{m} (-1)^m \mathrm{e}^{-tf(\lambda(m-r))}.
\end{aligned}$$

By $f(-x)$, $x > 0$ we mean the extended Bernštein function having representation

$$f(-x) = \int_0^{\infty} (1-\mathrm{e}^{sx})\nu(\mathrm{d}s), \qquad x > 0, \tag{2.7}$$

provided that the integral in (2.7) is convergent. In particular, we infer that

$$\mathbb{E}(\mathcal{N}^f(t)) = \mathrm{e}^{-tf(-\lambda)}, \qquad \operatorname{var}(\mathcal{N}^f(t)) = 2\mathrm{e}^{-tf(-2\lambda)} - \mathrm{e}^{-tf(-\lambda)} - \mathrm{e}^{-2tf(-\lambda)}.$$

For a stable subordinator, i.e. with Lévy measure $\nu(\mathrm{d}s) = (\alpha s^{-\alpha-1}/\Gamma(1-\alpha))\,\mathrm{d}s$, $\alpha \in (0,1)$, all the factorial moments are infinite. Instead, for a tempered stable subordinator where $\nu(\mathrm{d}s) = (\alpha \mathrm{e}^{-\theta s} s^{-\alpha-1}/\Gamma(1-\alpha))\,\mathrm{d}s$, $\alpha \in (0,1)$ and $\theta > 0$, only the factorial moments of order r such that $r < \theta/\lambda$ are finite. If we then consider the gamma subordinator with $\nu(\mathrm{d}s) = (\mathrm{e}^{-\alpha s}/s)\,\mathrm{d}s$, only the factorial moments of order r such that $r < \alpha/\lambda$ are finite.

2.3. Fractional subordinated nonlinear birth process

The fractional nonlinear birth process has state probabilities $p_k^{\nu}(t)$ solving the fractional differential equation

$$\frac{\mathrm{d}^{\nu} p_k^{\nu}(t)}{\mathrm{d}t^{\nu}} = -\lambda_k p_k^{\nu}(t) + \lambda_{k-1} p_{k-1}^{\nu}(t), \qquad \nu \in (0,1),\ k \geq 1$$

with initial condition

$$p_k^{\nu}(0) = \begin{cases} 1, & k = 1, \\ 0, & k > 1. \end{cases}$$

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The state probabilities are (see [11])

$$p_k^\nu(t) = \mathbb{P}\{\mathcal{N}^\nu(t) = k \mid \mathcal{N}^\nu(0) = 1\} = \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{l=1,l\neq m}^{k}(\lambda_l - \lambda_m)}, \qquad \nu \in (0,1),$$

where

$$E_{\nu,1}(-\eta t^\nu) = \frac{\sin(\nu\pi)}{\pi} \int_0^\infty \frac{r^{\nu-1}\exp(-r\eta^{1/\nu}t)}{r^{2\nu} + 2r^\nu\cos(\nu\pi) + 1}\,\mathrm{d}r$$

is the Mittag–Leffler function; see Haubold *et al.* [8, Equation (7.3)]. So the subordinated nonlinear fractional birth process has distribution

$$\begin{aligned}
&\mathbb{P}\{\mathcal{N}^\nu(H^f(t)) = k \mid \mathcal{N}^\nu(0) = 1\} \\
&\quad = \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{1}{\prod_{l=1,l\neq m}^{k}(\lambda_l - \lambda_m)} \frac{\sin(\nu\pi)}{\pi} \int_0^\infty \frac{r^{\nu-1}\exp(-tf(r\lambda_m^{1/\nu}))}{r^{2\nu} + 2r^\nu\cos(\nu\pi) + 1}\,\mathrm{d}r.
\end{aligned}$$

For further readings on fractional point processes consult, for example, [9] and [16].

3. Subordinated death processes

We now consider the process $M^f(t) = M(H^f(t))$, where M is a linear death process with n_0 progenitors. The state probabilities are

$$\begin{aligned}
\mathbb{P}\{M^f(t) = k \mid M^f(0) = n_0\} &= \int_0^\infty \binom{n_0}{k} \mathrm{e}^{-\mu k s}(1 - \mathrm{e}^{-\mu s})^{n_0-k}\,\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \binom{n_0}{k} \sum_{j=0}^{n_0-k} \binom{n_0-k}{j} (-1)^j \int_0^\infty \mathrm{e}^{-(\mu k + \mu j)s}\,\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= \binom{n_0}{k} \sum_{j=0}^{n_0-k} \binom{n_0-k}{j} (-1)^j \mathrm{e}^{-tf(\mu k + \mu j)}, \qquad 0 \le k \le n_0.
\end{aligned}$$

In particular, the extinction probability is

$$\mathbb{P}\{M^f(t) = 0 \mid M^f(0) = n_0\} = \sum_{j=0}^{n_0} \binom{n_0}{j} (-1)^j \mathrm{e}^{-tf(\mu j)} = 1 + \sum_{j=1}^{n_0} \binom{n_0}{j} (-1)^j \mathrm{e}^{-tf(\mu j)}$$

and converges to 1 exponentially fast with rate $f(\mu)$.

Remark 3.1. We observe that the extinction probability is a decreasing function of n_0 for any choice of the subordinator $H^f(t)$. This can be shown by observing that

$$\begin{aligned}
&\mathbb{P}\{M^f(t) = 0 \mid M^f(0) = n_0\} - \mathbb{P}\{M^f(t) = 0 \mid M^f(0) = n_0 - 1\} \\
&\quad = \sum_{j=1}^{n_0} \binom{n_0}{j} (-1)^j \mathrm{e}^{-tf(\mu j)} - \sum_{j=1}^{n_0-1} \binom{n_0-1}{j} (-1)^j \mathrm{e}^{-tf(\mu j)} \\
&\quad = \sum_{j=1}^{n_0-1} \binom{n_0-1}{j-1} (-1)^j \mathrm{e}^{-tf(\mu j)} + (-1)^{n_0} \mathrm{e}^{-tf(\mu n_0)}
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{j=1}^{n_0} \binom{n_0-1}{j-1} (-1)^j \mathrm{e}^{-tf(\mu j)} \\
&= -\sum_{j=0}^{n_0-1} \binom{n_0-1}{j} (-1)^j \mathrm{e}^{-tf(\mu(j+1))} \\
&= -\int_0^\infty \sum_{j=0}^{n_0-1} \binom{n_0-1}{j} (-1)^j \mathrm{e}^{-s\mu(j+1)} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&= -\int_0^\infty \mathrm{e}^{-\mu s}(1-\mathrm{e}^{-\mu s})^{n_0-1} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&< 0.
\end{aligned}
$$

This permits us also to establish the following upper bound which is valid for all values of n_0:

$$
\mathbb{P}\{M^f(t) = 0 \mid M^f(0) = n_0\} < \mathbb{P}\{M^f(t) = 0 \mid M^f(0) = 1\} = 1 - \mathrm{e}^{-tf(\mu)}.
$$

We also infer that

$$
\begin{aligned}
\mathbb{P}\{M^f(t) = 0 \mid M^f(0) = n_0\} &= \mathbb{P}\{M^f(t) = 0 \mid M^f(0) = n_0 - 1\} \\
&\quad - \frac{1}{n_0}\mathbb{P}\{M^f(t) = 1 \mid M^f(0) = n_0\} \quad \text{for all } k < n_0.
\end{aligned}
$$

Remark 3.2. The probability generating function of the subordinated linear death process is

$$
G(u, t) = \int_0^\infty (u\mathrm{e}^{-\mu s} + 1 - \mathrm{e}^{-\mu s})^{n_0} \mathbb{P}\{H^f(t) \in \mathrm{d}s\}.
$$

We now compute the factorial moments of order r for the process $M^f(t)$. Thus,

$$
\begin{aligned}
&\mathbb{E}(M^f(t)(M^f(t)-1)(M^f(t)-2)\cdots(M^f(t)-r+1)) \\
&\quad = \int_0^\infty \frac{\partial^r}{\partial u^r}(u\mathrm{e}^{-\mu s} + 1 - \mathrm{e}^{-\mu s})^{n_0}\big|_{u=1} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&\quad = n_0(n_0-1)(n_0-2)\cdots(n_0-r+1) \int_0^\infty \mathrm{e}^{-\mu r s} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\
&\quad = n_0(n_0-1)(n_0-2)\cdots(n_0-r+1)\mathrm{e}^{-tf(\mu r)} \\
&\quad = r!\binom{n_0}{r}\mathrm{e}^{-tf(\mu r)} \quad \text{for } r \le n_0.
\end{aligned}
$$

In particular, we extract the expressions

$$
\mathbb{E}M^f(t) = n_0\mathrm{e}^{-tf(\mu)}, \qquad \operatorname{var} M^f(t) = n_0\mathrm{e}^{-tf(\mu)} - n_0\mathrm{e}^{-tf(2\mu)} + n_0^2\mathrm{e}^{-tf(2\mu)} - n_0^2\mathrm{e}^{-2tf(\mu)}.
$$

The variance can be also be obtained as

$$
\begin{aligned}
\operatorname{var} M^f(t) &= \mathbb{E}\{\operatorname{var}(M(H^f(t)) \mid H^f(t))\} + \operatorname{var}\{\mathbb{E}(M(H^f(t)) \mid H^f(t))\} \\
&= \mathbb{E}(n_0\mathrm{e}^{-\mu H^f(t)}(1-\mathrm{e}^{-\mu H^f(t)})) + \operatorname{var}(n_0\mathrm{e}^{-\mu H^f(t)}) \\
&= n_0\mathrm{e}^{-tf(\mu)} - n_0\mathrm{e}^{-tf(2\mu)} + n_0^2\mathrm{e}^{-tf(2\mu)} - n_0^2\mathrm{e}^{-2tf(\mu)}.
\end{aligned}
$$

Remark 3.3. The transition probabilities

$$\mathbb{P}\{M^f(t_0+t)=k \mid M^f(t_0)=r\} = \binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j \mathrm{e}^{-tf(\mu k+\mu j)}$$

permit us to write, for a small time interval $[t, t+\mathrm{d}t)$,

$$\begin{aligned}
&\mathbb{P}\{M^f(t_0+\mathrm{d}t)=k \mid M^f(t_0)=r\} \\
&\quad = \binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j(1-\mathrm{d}tf(\mu k+\mu j)) \\
&\quad = -\mathrm{d}t\binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j f(\mu k+\mu j) \\
&\quad = -\mathrm{d}t\binom{r}{k}\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j\int_0^\infty(1-\mathrm{e}^{-(\mu k+\mu j)s})\nu(\mathrm{d}s) \\
&\quad = \mathrm{d}t\binom{r}{k}\int_0^\infty\sum_{j=0}^{r-k}\binom{r-k}{j}(-1)^j\mathrm{e}^{-\mu js}\mathrm{e}^{-\mu ks}\nu(\mathrm{d}s) \\
&\quad = \mathrm{d}t\int_0^\infty\binom{r}{k}(1-\mathrm{e}^{-\mu s})^{r-k}\mathrm{e}^{-\mu ks}\nu(\mathrm{d}s) \\
&\quad = \mathrm{d}t\int_0^\infty\mathbb{P}\{M(s)=k \mid M(0)=r\}\nu(\mathrm{d}s), \qquad 0\le k<r\le n_0.
\end{aligned} \tag{3.1}$$

It follows that the subordinated death process decreases with downwards jumps of arbitrary size. Equation (3.1) is a special case of (1.3) for the linear death process.

Remark 3.4. If $M^f(t_0)=r$, the probability that the number of individuals does not change during a time interval of length t is

$$\mathbb{P}\{M^f(t_0+t)=r \mid M^f(t_0)=r\} = \mathrm{e}^{-tf(\mu r)}. \tag{3.2}$$

As a consequence, the random time between two successive jumps has exponential distribution with rate $f(\mu r)$, i.e.

$$T_r \sim \mathrm{Exp}(f(\mu r)).$$

From (3.2), we also have

$$\mathbb{P}\{M^f(t+\mathrm{d}t)=r \mid M^f(t)=r\} = 1-\mathrm{d}tf(\mu r).$$

Remark 3.5. In view of (3.1), we can write the governing equations for the transition probabilities $p_k^f(t)=\mathbb{P}\{M^f(t)=k \mid M^f(0)=n_0\}$ for $0\le k\le n_0$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}p_k^f(t) = -p_k^f(t)f(\mu k)+\sum_{j=k+1}^{n_0}p_j^f(t)\int_0^\infty\binom{j}{k}(1-\mathrm{e}^{-\mu s})^{j-k}\mathrm{e}^{-\mu ks}\nu(\mathrm{d}s).$$

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3.1. The subordinated sublinear death process

In the sublinear death process, we have, for $0 \le k \le n_0$,

$$\mathbb{P}\{\mathbb{M}(t+\mathrm{d}t) = k-1 \mid \mathbb{M}(t) = k, \mathbb{M}(0) = n_0\} = \mu(n_0 - k + 1)\,\mathrm{d}t + o(\mathrm{d}t)$$

so that the probability that a particle disappears in $[t, t+\mathrm{d}t)$ is proportional to the number of deaths occurred in $[0, t)$. It is well known that

$$\mathbb{P}\{\mathbb{M}(t) = k \mid \mathbb{M}(0) = n_0\} = \begin{cases} \mathrm{e}^{-\mu t}(1-\mathrm{e}^{-\mu t})^{n_0-k}, & k = 1, 2, \dots, n_0, \\ (1-\mathrm{e}^{-\mu t})^{n_0}, & k = 0. \end{cases}$$

So the probability law of the subordinated process immediately follows as

$$\mathbb{P}\{\mathbb{M}^f(t) = k \mid \mathbb{M}^f(0) = n_0\} = \begin{cases} \displaystyle\sum_{j=0}^{n_0-k} \binom{n_0-k}{j} (-1)^j \mathrm{e}^{-tf(\mu(j+1))}, & k = 1, \dots, n_0, \\ \displaystyle\sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \mathrm{e}^{-tf(\mu k)}, & k = 0. \end{cases}$$

The extinction probability is a decreasing function of n_0 as in the sublinear death process. Furthermore, we observe that the extinction probabilities for the subordinated linear and sublinear death processes coincide.

4. Subordinated linear birth–death processes

In this section we consider the linear birth and death process $L(t)$ with one progenitor at the time $H^f(t)$. We recall that, for $k \ge 1$ (see Bailey [2, p. 90]),

$$\mathbb{P}\{L(t) = k \mid L(0) = 1\} = \begin{cases} \dfrac{(\lambda-\mu)^2\mathrm{e}^{-(\lambda-\mu)t}(\lambda(1-\mathrm{e}^{-(\lambda-\mu)t}))^{k-1}}{(\lambda-\mu\mathrm{e}^{-(\lambda-\mu)t})^{k+1}}, & \lambda > \mu, \\ \dfrac{(\mu-\lambda)^2\mathrm{e}^{-(\mu-\lambda)t}(\lambda(1-\mathrm{e}^{-(\mu-\lambda)t}))^{k-1}}{(\mu-\lambda\mathrm{e}^{-(\mu-\lambda)t})^{k+1}}, & \lambda < \mu, \\ \dfrac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}}, & \lambda = \mu, \end{cases}$$

while the extinction probabilities have the form

$$\mathbb{P}\{L(t) = 0 \mid L(0) = 1\} = \begin{cases} \dfrac{\mu-\mu\mathrm{e}^{-t(\lambda-\mu)}}{\lambda-\mu\mathrm{e}^{-t(\lambda-\mu)}}, & \lambda > \mu, \\ \dfrac{\mu-\mu\mathrm{e}^{-t(\mu-\lambda)}}{\lambda-\mu\mathrm{e}^{-t(\mu-\lambda)}}, & \mu > \lambda, \\ \dfrac{\lambda t}{1+\lambda t}, & \lambda = \mu. \end{cases}$$

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We now study the subordinated process $L^f(t) = L(H^f(t))$. When $\lambda \neq \mu$, after a series expansion, we easily obtain

$$\mathbb{P}\{L^f(t) = k \mid L^f(0) = 1\}$$
$$= \begin{cases} \left(\dfrac{\lambda-\mu}{\lambda}\right)^2 \displaystyle\sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{\mu}{\lambda}\right)^l \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \mathrm{e}^{-tf((\lambda-\mu)(l+r+1))}, & \lambda > \mu, \\ \left(\dfrac{\mu-\lambda}{\mu}\right)^2 \left(\dfrac{\lambda}{\mu}\right)^{k-1} \displaystyle\sum_{l=0}^{\infty} \binom{l+k}{l} \left(\frac{\lambda}{\mu}\right)^l & \\ \qquad \times \displaystyle\sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \mathrm{e}^{-tf((\mu-\lambda)(l+r+1))}, & \lambda < \mu, \end{cases}$$

provided that $k \geq 1$. Moreover, the extinction probabilities have the following form:

$$\mathbb{P}\{L^f(t) = 0\} = \begin{cases} \dfrac{\mu-\lambda}{\lambda} \left(\displaystyle\sum_{m=1}^{\infty} \left(\frac{\mu}{\lambda}\right)^m \mathrm{e}^{-tf((\lambda-\mu)m)} \right) + \dfrac{\mu}{\lambda}, & \lambda > \mu, \\ 1 - \left(\dfrac{\mu-\lambda}{\lambda}\right) \displaystyle\sum_{m=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^m \mathrm{e}^{-tf((\mu-\lambda)m)}, & \lambda < \mu. \end{cases}$$

Similarly to the classical process, we have

$$\lim_{t\to\infty} \mathbb{P}\{L^f(t) = 0\} = \begin{cases} \dfrac{\mu}{\lambda}, & \lambda > \mu, \\ 1, & \lambda < \mu. \end{cases}$$

4.1. Processes with equal birth and death rates

We now concentrate on the $\lambda = \mu$ case, which leads to some interesting results. The extinction probability is given by

$$\begin{aligned} \mathbb{P}\{L^f(t) = 0 \mid L^f(0) = 1\} &= \int_0^\infty \frac{\lambda s}{1+\lambda s} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\ &= 1 - \int_0^\infty \frac{1}{1+\lambda s} \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\ &= 1 - \int_0^\infty \mathbb{P}\{H^f(t) \in \mathrm{d}s\} \int_0^\infty \mathrm{d}w \mathrm{e}^{-w\lambda s} \mathrm{e}^{-w} \\ &= 1 - \int_0^\infty \mathrm{d}w \mathrm{e}^{-w} \mathrm{e}^{-tf(\lambda w)}. \end{aligned} \tag{4.1}$$

We note that $\lim_{t\to\infty} \mathbb{P}\{L^f(t) = 0 \mid L^f(0) = 1\} = 1$ as in the classical case. From (4.1) we infer that the distribution of the extinction time $T_0^f = \inf\{t \geq 0 \colon L^f(t) = 0\}$ has the following form:

$$\frac{\mathbb{P}\{T_0^f \in \mathrm{d}t\}}{\mathrm{d}t} = \int_0^\infty \mathrm{e}^{-w} f(\lambda w) \mathrm{e}^{-tf(\lambda w)} \,\mathrm{d}w.$$

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We now observe that all the state probabilities of the process $L(t)$ depend on the extinction probability (see [11])

$$\begin{aligned}\mathbb{P}\{L(t)=k \mid L(0)=1\} &= \frac{(\lambda t)^{k-1}}{(1+\lambda t)^{k+1}} \quad (k \geq 1)\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg(\frac{\lambda}{1+\lambda t}\bigg)\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(\lambda(1-\mathbb{P}\{L(t)=0\})). \end{aligned} \tag{4.2}$$

Hence, the state probabilities of $L^f(t)$ can be written as, for $k \geq 1$,

$$\begin{aligned}\mathbb{P}\{L^f(t)=k \mid L^f(0)=1\} &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg[\lambda\int_0^\infty(1-\mathbb{P}\{L(s)=0\})\mathbb{P}\{H^f(t)\in \mathrm{d}s\}\bigg]\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}[\lambda(1-\mathbb{P}\{L^f(t)=0\})]\\ &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg[\lambda\int_0^\infty \mathrm{d}w\,\mathrm{e}^{-w}\mathrm{e}^{-tf(\lambda w)}\bigg]. \end{aligned} \tag{4.3}$$

4.2. Transition probabilities

To compute the transition probabilities of $L^f(t)$, we recall that the linear birth–death process with r progenitors has the following probability law (see [2, p. 94, Equation (8.47)]):

$$\mathbb{P}\{L(t)=n \mid L(0)=r\} = \sum_{j=0}^{\min(r,n)}\binom{r}{j}\binom{r+n-j-1}{r-1}\alpha^{r-j}\beta^{n-j}(1-\alpha-\beta)^j,$$

where $n \geq 0$ and

$$\alpha = \frac{\mu(\mathrm{e}^{(\lambda-\mu)t}-1)}{\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu}, \qquad \beta = \frac{\lambda(\mathrm{e}^{(\lambda-\mu)t}-1)}{\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu}.$$

In the $\lambda = \mu$ case, we have

$$\lim_{\mu\to\lambda}\alpha = \lim_{\mu\to\lambda}\beta = \frac{\lambda t}{1+\lambda t}$$

so that

$$\begin{aligned}\mathbb{P}\{L(t)=n \mid L(0)=r\} &= \sum_{j=0}^{\min(r,n)}\binom{r}{j}\binom{r+n-j-1}{r-1}\bigg(\frac{\lambda t}{1+\lambda t}\bigg)^{r+n-2j}\bigg(1-2\frac{\lambda t}{1+\lambda t}\bigg)^j\\ &= \sum_{j=0}^{\min(r,n)}\sum_{k=0}^{j}\binom{r}{j}\binom{r+n-j-1}{r-1}\binom{j}{k}(-2)^k\bigg(\frac{\lambda t}{1+\lambda t}\bigg)^{r+n-2j+k}. \end{aligned} \tag{4.4}$$

One can check that for $r = 1$ the last equation reduces to

$$\mathbb{P}\{L(t)=n \mid L(0)=1\} = \frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}}.$$

The transition probabilities related to the subordinated process $L^f(t)$ can be written in an elegant form, as shown in the following theorem.

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Theorem 4.1. *In the subordinated linear birth–death process* $L^f(t)$, *when* $\lambda = \mu$, $n \geq 0$, $r \geq 1$, $n \neq r$, *we have*

$$\begin{aligned}\mathbb{P}\{L^f(t+t_0) = n \mid L^f(t_0) = r\} \\ = \sum_{j=0}^{\min(r,n)} \sum_{k=0}^{j} \binom{r}{j}\binom{r+n-j-1}{r-1}\binom{j}{k} 2^k \frac{(-1)^{r+n-1}\lambda^{r+n+k-2j}}{(r+n-2j+k-1)!} \\ \times \frac{\mathrm{d}^{r+n-2j+k-1}}{\mathrm{d}\lambda^{r+n-2j+k-1}}\bigg[\frac{1}{\lambda} - \frac{1}{\lambda}\int_0^\infty \mathrm{d}w\mathrm{e}^{-w}\mathrm{e}^{-tf(\lambda w)}\bigg]. \end{aligned} \tag{4.5}$$

Proof. By subordination, we have

$$\begin{aligned}\mathbb{P}\{L^f(t) = n \mid L^f(0) = r\} &= \int_0^\infty \mathbb{P}\{L(s) = n \mid L(0) = r\}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\ &= \sum_{j=0}^{\min(r,n)} \sum_{k=0}^{j} \binom{r}{j}\binom{r+n-j-1}{r-1}\binom{j}{k}(-2)^k \\ &\quad\times \int_0^\infty \mathbb{P}\{H(t) \in \mathrm{d}s\}\bigg(\frac{\lambda s}{1+\lambda s}\bigg)^{r+n-2j+k}. \end{aligned}$$

To compute the last integral, we preliminarily observe that

$$\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\frac{1}{1+\lambda s} = (-1)^m m!\, s^m \frac{1}{(1+\lambda s)^{m+1}}$$

and, consequently,

$$\bigg(\frac{\lambda s}{1+\lambda s}\bigg)^m = \frac{(-1)^{m-1} s\lambda^m}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}\lambda^{m-1}}\frac{1}{1+\lambda s}. \tag{4.6}$$

So we have

$$\begin{aligned}\mathbb{P}\{L^f(t) = n \mid L^f(0) = r\} \\ = \sum_{j=0}^{\min(r,n)} \sum_{k=0}^{j} \binom{r}{j}\binom{r+n-j-1}{r-1}\binom{j}{k} 2^k \frac{(-1)^{r+n-1}\lambda^{r+n-2j+k}}{(r+n-2j+k-1)!} \\ \times \frac{\mathrm{d}^{r+n-2j+k-1}}{\mathrm{d}\lambda^{r+n-2j+k-1}}\int_0^\infty \frac{s}{1+\lambda s}\mathbb{P}\{H^f(t) \in \mathrm{d}s\}, \end{aligned}$$

where, by using (4.1), we write

$$\begin{aligned}\int_0^\infty \frac{s}{1+\lambda s}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} &= \frac{1}{\lambda}\int_0^\infty \frac{\lambda s}{1+\lambda s}\mathbb{P}\{H^f(t) \in \mathrm{d}s\} \\ &= \frac{1}{\lambda}\bigg[1 - \int_0^\infty \mathrm{d}w\mathrm{e}^{-w}\mathrm{e}^{-tf(\lambda w)}\bigg] \end{aligned}$$

and the desired result immediately follows. □

Remark 4.1. For a small time interval $\mathrm{d}t$, the quantity in square brackets in (4.5) can be written as

$$\frac{1}{\lambda} - \frac{1}{\lambda}\int_0^\infty \mathrm{d}w \mathrm{e}^{-w}(1 - \mathrm{d}t f(\lambda w)) = \mathrm{d}t \frac{1}{\lambda}\int_0^\infty \mathrm{d}w \mathrm{e}^{-w}\int_0^\infty \nu(\mathrm{d}s)(1 - \mathrm{e}^{-\lambda w s})$$
$$= \mathrm{d}t \int_0^\infty \nu(\mathrm{d}s)\frac{s}{1+\lambda s}.$$

Then by using (4.6) and (4.4), (4.5) reduces to

$$\mathbb{P}\{L^f(t_0 + \mathrm{d}t) = n \mid L^f(t_0) = k\} = \mathrm{d}t \int_0^\infty \nu(\mathrm{d}s)\mathbb{P}\{L(s) = n \mid L(0) = k\},$$

thus proving (1.3) for subordinated birth–death processes.

Remark 4.2. If $L^f(0) = 1$, from (4.3) it follows that the probability that the number of individuals does not change during a time interval of length $\mathrm{d}t$ is

$$\mathbb{P}\{L^f(\mathrm{d}t) = 1 \mid L^f(0) = 1\} = 1 - \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg(\lambda \int_0^\infty \mathrm{d}w \mathrm{e}^{-w} f(\lambda w)\bigg).$$

Thus, the waiting time for the first jump, i.e.

$$T_1 = \inf\{t > 0 \colon L^f(t) \neq 1\}$$

has the following distribution:

$$\mathbb{P}\{T_1 > t\} = \exp\bigg(-t \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg(\lambda \int_0^\infty \mathrm{d}w \mathrm{e}^{-w} f(\lambda w)\bigg)\bigg).$$

For example, in the case when $H^f(t)$ is a stable subordinator with index $\alpha \in (0, 1)$, T_1 has an exponential distribution with parameter $\lambda^\alpha \Gamma(\alpha + 2)$.

4.3. Mean sojourn times

Let $V_k(t)$, $k \geq 1$, be the total amount of time that the process $L(t)$ spends in the state k up to time t, i.e.

$$V_k(t) = \int_0^t \mathbf{1}_k(L(s))\,\mathrm{d}s,$$

where $\mathbf{1}_k(\cdot)$ is the indicator function of the state k. The mean sojourn time up to time t is given by

$$\mathbb{E}V_k(t) = \int_0^t \mathbb{P}\{L(s) = k \mid L(0) = 1\}\,\mathrm{d}s.$$

By means of (4.2), we have

$$\mathbb{E}V_k(t) = \int_0^t \mathbb{P}\{L(s) = k \mid L(0) = 1\}\,\mathrm{d}s$$
$$= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg(\lambda\bigg(t - \int_0^t \mathbb{P}\{L(s) = 0\}\,\mathrm{d}s\bigg)\bigg)$$

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$$
\begin{aligned}
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg(\lambda\bigg(t-\int_0^t \frac{\lambda s}{1+\lambda s}\,\mathrm{d}s\bigg)\bigg) \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\log(1+\lambda t) \\
&= \frac{1}{\lambda k}\bigg(\frac{\lambda t}{1+\lambda t}\bigg)^k
\end{aligned}
$$

and the mean asymptotic sojourn time is therefore given by

$$
\mathbb{E}V_k(\infty) = \frac{1}{\lambda k}. \tag{4.7}
$$

In view of (4.3), for the sojourn time $V_k^f(t)$ of the subordinated process $L^f(t)$, we have

$$
\begin{aligned}
\mathbb{E}V_k^f(t) &= \int_0^t \mathbb{P}\{L^f(s)=k \mid L^f(0)=1\}\,\mathrm{d}s \\
&= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg[\lambda\int_0^\infty \mathrm{d}w\mathrm{e}^{-w}\frac{1}{f(\lambda w)}(1-\mathrm{e}^{-tf(\lambda w)})\bigg]
\end{aligned}
$$

and the mean asymptotic sojourn time is given by

$$
\mathbb{E}V_k^f(\infty) = \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg[\lambda\int_0^\infty \mathrm{d}w\mathrm{e}^{-w}\frac{1}{f(\lambda w)}\bigg].
$$

It is possible to obtain an explicit expression for $\mathbb{E}V_k^f(\infty)$ in the case of a stable subordinator, when $f(x)=x^\alpha$, $\alpha \in (0,1)$, i.e.

$$
\begin{aligned}
\mathbb{E}V_k^f(\infty) &= \frac{(-1)^{k-1}\lambda^{k-1}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\bigg[\lambda\int_0^\infty \mathrm{d}w\mathrm{e}^{-w}\frac{1}{\lambda^\alpha w^\alpha}\bigg] \\
&= \frac{(-1)^{k-1}\lambda^{k-1}\Gamma(1-\alpha)}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\lambda^{1-\alpha} \\
&= \frac{(-1)^{k-1}\lambda^{k-1}\Gamma(1-\alpha)}{k!}(1-\alpha)(-\alpha)(-\alpha-1)\cdots(-\alpha-k+1)\lambda^{-\alpha-k+1} \\
&= \frac{\Gamma(1-\alpha)\Gamma(\alpha+k)}{k!\,\Gamma(\alpha)\lambda^\alpha} \\
&= \frac{B(1-\alpha,k+\alpha)}{\Gamma(\alpha)\lambda^\alpha} \quad \text{for } k\geq 1.
\end{aligned} \tag{4.8}
$$

In the $\alpha=\frac{1}{2}$ case, by using the duplication equation for the gamma function and the Stirling formula, the quantity in (4.8) can be estimated for large values of k in the following way:

$$
\mathbb{E}V_k^f(\infty) = \frac{\Gamma(1/2+k)}{k!\sqrt{\lambda}} = \frac{\Gamma(1/2)2^{1-2k}\Gamma(2k)}{k!\sqrt{\lambda}\Gamma(k)} \simeq \frac{1}{\sqrt{\lambda k}}
$$

which is somehow related to (4.7). We finally note that

$$
\frac{1}{(\alpha+k)\Gamma(\alpha)\lambda^\alpha} < \mathbb{E}V_k^f(\infty) < \frac{1}{(1-\alpha)\Gamma(\alpha)\lambda^\alpha} \quad \text{for all } k\geq 1,
$$

since

$$
\frac{1}{(\alpha+k)} < B(1-\alpha,k+\alpha) < \frac{1}{1-\alpha}.
$$

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4.4. On the distribution of the sojourn times

Let $L_k^f(t)$ be a linear birth–death process with k progenitors. We now study the distribution of the sojourn time

$$V_k(t) = \int_0^t \mathbf{1}_k(L_k^f(s))\,\mathrm{d}s,$$

which represents the total amount of time that the process spends in the state k up to time t. We now define the Laplace transform

$$r_k(\gamma) = \int_0^\infty \mathrm{e}^{-\gamma t}\mathbb{P}\{L_k^f(t) = k\}\,\mathrm{d}t.$$

The hitting time

$$V_k^{-1}(t) = \inf\{w > 0 \colon V_k(w) > t\}$$

is such that

$$\mathbb{E}\int_0^\infty \mathrm{e}^{-\gamma V_k^{-1}(t)}\,\mathrm{d}t = \mathbb{E}\int_0^\infty \mathrm{e}^{-\gamma t}\,\mathrm{d}V_k(t) = \mathbb{E}\int_0^\infty \mathrm{e}^{-\gamma t}\mathbf{1}_k(L_k^f(t))\,\mathrm{d}t = r_k(\gamma).$$

By [3, Proposition 3.17, Chapter V], we have

$$\mathbb{E}\exp(-\gamma V_k^{-1}(t)) = \exp\biggl(-t\frac{1}{r_k(\gamma)}\biggr). \tag{4.9}$$

Now we resort to the fact that

$$\mathbb{P}\{V_k(t) > x\} = \mathbb{P}\{V_k^{-1}(x) < t\}$$

and, thus, we can write

$$\frac{\mathbb{P}\{V_k(t) \in \mathrm{d}x\}}{\mathrm{d}x} = -\frac{\partial}{\partial x}\int_0^t \mathbb{P}\{V_k^{-1}(x) \in \mathrm{d}w\}.$$

Therefore, we have

$$\begin{aligned}
\frac{1}{\mathrm{d}x}\int_0^\infty \mathrm{e}^{-\gamma t}\mathbb{P}\{V_k(t) \in \mathrm{d}x\}\,\mathrm{d}t &= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}t\mathrm{e}^{-\gamma t}\int_0^t \mathbb{P}\{V_k^{-1}(x) \in \mathrm{d}w\}\\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}w\int_w^\infty \mathrm{d}t\mathrm{e}^{-\gamma t}\mathbb{P}\{V_k^{-1}(x) \in \mathrm{d}w\}\\
&= -\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}w\mathrm{e}^{-\gamma w}\mathbb{P}\{V_k^{-1}(x) \in \mathrm{d}w\}\\
&= -\frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}x}\exp\biggl(-x\frac{1}{r_k(\gamma)}\biggr)\\
&= \frac{1}{\gamma r_k(\gamma)}\exp\biggl(-x\frac{1}{r_k(\gamma)}\biggr).
\end{aligned}$$

If $r_k(0) < \infty$, from (4.9) it emerges that $\mathbb{P}\{V_k^{-1}(t) < \infty\} < 1$; so the sample paths of $V_k(t)$ become constant after a random time with positive probability. This is related to the fact that the subordinated birth and death process extinguishes with probability 1 in a finite time when $\lambda = \mu$.

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We finally observe that in the $k = 1$ case, by (4.3) we have

$$r_1(\gamma) = \int_0^\infty e^{-\gamma t}\mathbb{P}\{L^f(t) = k\}\,dt = \frac{d}{d\lambda}\bigg[\lambda\int_0^\infty dw e^{-w}\frac{1}{\gamma + f(\lambda w)}\bigg],$$

provided that the Fubini theorem holds.

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