Math. Struct. in Comp. Science (2017), *vol.* 27, *pp.* 1032–1067.
doi:10.1017/S0960129515000547 First published online 28 December 2015

Isomorphism theorems between models of mixed choice

JEAN GOUBAULT-LARRECQ

LSV, ENS Cachan, CNRS, INRIA Saclay, 61, avenue du président-Wilson, 94230 Cachan, France
Email: goubault@lsv.ens-cachan.fr

Received July 2012; revised November 2015

We relate the so-called powercone models of mixed non-deterministic and probabilistic choice proposed by Tix, Keimel, Plotkin, Mislove, Ouaknine, Worrell, Morgan and McIver, to our own models of previsions. Under suitable topological assumptions, we show that they are isomorphic. We rely on Keimel's cone-theoretic variants of the classical Hahn–Banach separation theorems, using functional analytic methods, and on the Schröder–Simpson Theorem.

1. Introduction

Consider the question of giving semantics to a programming language with *mixed choice*, i.e., with two interacting forms of choice, non-deterministic and probabilistic. For example, probabilistic choice can be the result of random coin flips, while non-deterministic choice can be the result of interaction with an environment, which decides which option to take in our place.

We shall be especially interested in domain-theoretic models, where the space of all choices over a directed complete poset (dcpo) of values is itself a dcpo.

Models of only one form of choice have been known for a long time. For non-deterministic choice, the best known models are the Hoare powerdomain of angelic non-determinism, the Smyth powerdomain of demonic non-determinism and the Plotkin powerdomain of erratic non-determinism, see Abramsky and Jung (1994, Section 6.2) or Gierz et al. (2003, Section IV.8): there, choices are represented as certain sets of values, from which the environment may choose. For probabilistic choice, a natural model is the Jones–Plotkin model of continuous valuations (Jones 1990), where choices are represented as objects akin to (sub)probability distributions over the set of possible values, called continuous valuations.

Several denotational models of mixed choice were proposed in the past. Let us list some of them:

1. In the *powercone models*, choices are modelled as certain sets E of continuous valuations. These sets are, in particular, convex, meaning that $a\nu + (1-a)\nu'$ is in E for all $\nu, \nu' \in E$ and $a \in [0, 1]$. Choice proceeds by picking a valuation ν from the set E, then drawing a value v at random according to ν. The probability that v *may* fall in some set U is $\sup_{\nu \in E} \nu(U)$, while the probability that it *must* fall in U is $\inf_{\nu \in E} \nu(U)$. Such models

were proposed and studied by Mislove (2000), by Tix (1999) and Tix et al. (2009) and by McIver and Morgan (2001).

2. In the *prevision models*, choices are modeled as certain second-order functionals F, called previsions. They represent directly the probability that a value v may fall in U, or must fall in U, depending on the kind of prevision, as $F(\chi_U)$, where χ_U is the characteristic function of U. These were proposed by the author in Goubault-Larrecq (2007), and also underlie the (generalized) predicate transformer semantics of Keimel and Plotkin (2009) or of Morgan et al. (1996).
3. The models of indexed valuations (Varacca 2003, 2002) are alternate models for mere probabilistic choice. They combine well with angelic non-determinism, yielding models akin to the powercone models, except that we do not require the sets of indexed valuations/continuous random variables to be convex. Categorically, the combination is obtained via a distributivity law between monads.
4. The monad coproduct models of Lüth (1997), once instantiated to the monads of non-blocking non-deterministic and probabilistic choice (Ghani and Uustalu 2004), also yield a model of mixed choice, which the latter authors argue is close to Varacca's.

Our goal is to relate the first two, and to show that, under mild assumptions, they are isomorphic. We have already announced this result in Goubault-Larrecq (2008). The proof was complex and was limited to continuous dcpos. Similar isomorphisms were proved, with the general aim of giving generalized predicate transformer semantics to powercone models, by Keimel and Plotkin (2009), again for continuous dcpos, and for unbounded continuous valuations, not (sub)probability valuations[†]. We generalize these results to much larger classes of topological spaces, using more streamlined and more general arguments than in Goubault-Larrecq (2008).

The basic idea is simple. There is a map r from the powercone model, which maps any set E of continuous valuations to $F = r(E)$, defined by $F(h) = \sup_{v \in E} \int_x h(x)dv$ (in the angelic case; replace sup by inf in the demonic case). One needs to show that it is continuous, and that it has a continuous inverse.

Our approach is typical of convex analysis. By a form of the Riesz representation theorem, which we shall make explicit below, there is an isomorphism between continuous valuations v and *linear* previsions, i.e., previsions F such that $F(a.h+(1-a).h') = aF(h)+(1-a)F(h')$ for all $a \in [0,1]$: define $F(h)$ as $\int_x h(x)dv$. It is useful to imagine such linear previsions by drawing the curve $y = F(h)$, where h serves as x-coordinate, and convincing oneself that such curves should be straight lines. Modulo this isomorphism, r maps a set E of linear previsions (the straight lines in Figure 1) to its pointwise sup, shown as a fat curve. This fat curve is always convex (meaning that $F(a.h+(1-a).h') \leqslant aF(h)+(1-a)F(h')$ for all $a \in [0,1]$), and will be our (angelic) prevision $r(E)$. Conversely, given any convex, fat curve F, the set of straight lines below it form a convex set of linear previsions $E = s(F)$.

[†] While we were preparing the final version of this document, Klaus Keimel informed me of a newer paper by the same authors (Keimel and Plotkin 2015), which, among other things, also deals with probability and subprobability valuations, relying on a novel notion of Kegelspitze.

https://doi.org/10.1017/S0960129515000547 Published online by Cambridge University Press

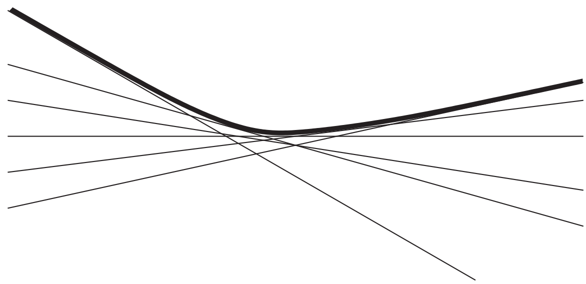

Fig. 1. Convex hulls.

Showing that s is a right inverse to r, i.e., that $r(s(F)) = F$ is essentially the Hahn–Banach theorem (or the corresponding variant in our setting). The difficult part will be to show that s takes its values in the right space, and that it is continuous.

1.1. *Outline*

We present all needed domain-theoretic, topological and cone-theoretic notions in Section 2. This is a fairly long section, and our only excuse is that we have tried to make the paper as self-contained as possible. Section 3 exhibits our prevision models as retracts of spaces of certain sets (closed, compact or lenses) of certain continuous valuations (unbounded, subprobability, probability). The only difference that these sets exhibit compared to powercones is that we are not requiring them to be convex. The existence of these retractions has a certain number of interesting consequences, which we also list there. Finally, using the Schröder–Simpson Theorem, we establish the desired isomorphisms in Section 4. We conclude in Section 5.

2. Preliminaries

We refer the reader to Gierz et al. (2003), Abramsky and Jung (1994) and Mislove (1998) for background on domain theory and topology. We shall write $x \in X \mapsto f(x)$ for the function that maps every element x of some space X to the value $f(x)$, sometimes omitting mention of the space X.

2.1. *Domain theory*

A set with a partial ordering $\leqslant$ is a *poset*. We write $\uparrow E$ for $\{y \in X \mid \exists x \in E \cdot x \leqslant y\}$, $\downarrow E = \{y \in X \mid \exists x \in E \cdot y \leqslant x\}$. A *dcpo* is a poset in which every directed family $(x_i)_{i \in I}$ has a least upper bound (a.k.a., supremum or *sup*) $\sup_{i \in I} x_i$. Symmetrically, we call *inf* (or infimum) any greatest lower bound. A family $(x_i)_{i \in I}$ is *directed* iff it is non-empty, and any two elements have an upper bound in the family. Any poset can be equipped with the *Scott topology*, whose opens are the upward closed sets U such that whenever $(x_i)_{i \in I}$ is a directed family that has a least upper bound in U, then some x_i is in U already. A dcpo X is *pointed* iff it has a least element, which we shall always write $\bot$.

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Given a poset X, we shall write X_σ for X seen as a topological space, equipped with its Scott topology.

We shall always consider $\mathbb{R}^+$, or $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{+\infty\}$, as posets, and implicit endow them with the Scott topology of their ordering $\leqslant$. We shall still write $\mathbb{R}^+_\sigma$ or $\overline{\mathbb{R}^+}_\sigma$ to make this fact clear. The opens of $\mathbb{R}^+$ in its Scott topology are the intervals $(r, +\infty)$, $r \in \mathbb{R}^+$, together with $\mathbb{R}^+$ and $\emptyset$. Those of $[0, 1]$ are $\emptyset$, $[0, 1]$, and $(r, 1]$, $r \in [0, 1)$. Those of $\overline{\mathbb{R}^+}$ are $\emptyset$, $\overline{\mathbb{R}^+}$, and $(r, +\infty]$, $r \in \mathbb{R}^+$.

Given two dcpos X and Y, a map $f : X \to Y$ is *Scott-continuous* iff it is monotonic and $f(\sup_{i\in I} x_i) = \sup_{i\in I} f(x_i)$ for every directed family $(x_i)_{i\in I}$ in X.

We write $[X \to Y]$ for the space of all Scott-continuous maps from X to Y, for two dcpos X and Y. This is again a dcpo, with the pointwise ordering.

The *way-below* relation $\ll$ on a poset X is defined by $x \ll y$ iff, for every directed family $(z_i)_{i\in I}$ that has a least upper bound z such that $y \leqslant z$, then $x \leqslant z_i$ for some $i \in I$ already. We also say that x *approximates* y. Note that $x \ll y$ implies $x \leqslant y$, and that $x' \leqslant x \ll y \leqslant y'$ implies $x' \ll y'$. However, $\ll$ is not reflexive or irreflexive in general. Write $\twoheaduparrow E = \{y \in X \mid \exists x \in E \cdot x \ll y\}$, $\twoheaddownarrow E = \{y \in X \mid \exists x \in E \cdot y \ll x\}$. X is *continuous* iff, for every $x \in X$, $\twoheaddownarrow x$ is a directed family, and has x as least upper bound. A *basis* is a subset B of X such that any element $x \in X$ is the least upper bound of a directed family of elements way-below x *in* B. Then, X is continuous if and only if it has a basis, and in this case X itself is the largest basis.

In a continuous poset with basis B, *interpolation holds*: if $x_1, \ldots, x_n$ are finitely many elements way-below x, then there is a $b \in B$ such that $x_1, \ldots, x_n$ are way-below b, and $b \ll x$. (See for example (Mislove 1998, Section 4.2).) In this case, the Scott opens are exactly the unions of sets of the form $\twoheaduparrow b$, $b \in B$.

2.2. *Topology*

A topology $\mathcal{O}$ on a set X is a collection of subsets of X, called the *opens*, such that any union and any finite intersection of opens is open. The *interior* of a subset A of X is the largest open included in A. A *closed* subset is the complement of an open subset. The *closure* $cl(A)$ of A is the smallest closed subset containing A. An *open neighbourhood* U of a point x is merely an open subset that contains x.

A topology $\mathcal{O}_1$ is *finer* than $\mathcal{O}_2$ if and only if it contains all opens of $\mathcal{O}_2$. We also say that $\mathcal{O}_2$ is *coarser* than $\mathcal{O}_1$.

A *base* $\mathcal{B}$ (not a basis) of $\mathcal{O}$ is a collection of opens such that every open is a union of elements of the base. Equivalently, a family $\mathcal{B}$ of opens is a base iff for every $x \in X$, for every open U containing x, there is a $V \in \mathcal{B}$ such that $x \in V \subseteq U$. A *subbase* of $\mathcal{O}$ is a collection of opens such that the finite intersections of elements of the subbase form a base; equivalently, the coarsest topology containing the elements of the subbase is $\mathcal{O}$, and then we say that $\mathcal{O}$ is *generated* by the subbase.

The specialization preorder of a space X is defined by $x \leqslant y$ if and only if for every open subset U of X that contains x, U also contains y. For every subbase $\mathcal{B}$ of the topology of X, it is equivalent to say that $x \leqslant y$ if and only if every $U \in \mathcal{B}$ that contains x also contains y. The specialization preorder of a dcpo X, with ordering $\leqslant$, in its Scott

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topology, is $\leqslant$. A topological space is T_0 if and only if $\leqslant$ is a partial ordering, not just a preorder. A subset A of X is *saturated* if and only if it is upward-closed in the specialization preorder $\leqslant$.

A map f from X to Y is *continuous* if and only if $f^{-1}(V)$ is open in X for every open subset V of Y. The *characteristic* function $\chi_A : X \to \overline{\mathbb{R}^+}_\sigma$ of a subset A of X, defined as mapping every $x \in A$ to 1 and every $x \notin A$ to 0, is continuous if and only if A is open in X. When X and Y are posets in their Scott topology, $f : X \to Y$ is continuous if and only if it is Scott-continuous.

A subset K of a topological space X is *compact* if and only if every open cover of K has a finite subcover. The image $f[K]$ of any compact subset of X by any continuous map $f : X \to Y$ is compact in Y. A *homeomorphism* is a bijective continuous map whose inverse is also continuous.

The product of two topological spaces X, Y is the set $X \times Y$ with the *product topology*, which is the coarsest that makes the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ continuous. Equivalently, a base of this topology is given by the *open rectangles* $U \times V$, where U is open in X and V is open in Y.

If X and Y are posets, viewed as topological spaces in their Scott topology, then there is some ambiguity about the notation $X \times Y$. We might indeed see the latter as a topological product, with the just mentioned product topology, or as a product of two posets, with the Scott topology of the product ordering. In general, the Scott topology on the poset product is strictly finer than the product topology. This difficulty vanishes when X and Y are continuous posets: indeed, in this case the product poset is continuous as well, and a base of the Scott topology in the product is given by the subsets of the form $\Uparrow(x \times y)$, which are also open in the product topology, as they coincide with $\Uparrow x \times \Uparrow y$.

A topological space X is *locally compact* if and only if for every $x \in X$, for every open subset U of X containing X, there is a compact subset $K \subseteq U$ whose interior contains x. Then, we can require K to be saturated as well, replacing K by its upward closure. In a locally compact space, every open subset U is the union of the directed family of all open subsets V such that $V \subseteq Q \subseteq U$ for some compact saturated subset Q. A refinement of this is the notion of a *core-compact* space, which is by definition a space whose lattice of open subsets is a continuous dcpo. We shall agree to write $\Subset$ for the way-below relation on the lattice of open sets of a space. Every locally compact space is core-compact, in which $V \Subset U$ if and only if $V \subseteq Q \subseteq U$ for some compact saturated subset Q.

Every continuous poset is locally compact, since whenever $x \in U$, U open, there is an $y \in U$ such that $y \ll x$, so that we can take $K = \uparrow y$, and $V = \Uparrow y$.

A topological space X is *well-filtered* if and only if for every filtered family $(Q_i)_{i \in I}$ of compact saturated subsets of X whose intersection is contained in some open subset U, $Q_i \subseteq U$ for some $i \in I$ already. (A family of subsets is *filtered* if and only if it is directed in the reverse inclusion ordering $\supseteq$.) Every sober space is well filtered (Gierz et al. 2003, Theorem II-1.21), and every continuous dcpo is sober in its Scott topology (Gierz et al. 2003, Corollary II-1.12). (We will not define sober spaces, but see the paragraph on Stone duality below.)

There are several topologies one can put on the space $[X \to Y]$ of continuous maps from X to Y, and more generally on any space Z of continuous maps from X to Y.

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Looking at Z, resp., $[X \to Y]$, as a subspace of the product Y^X (i.e., the space of *all* maps from X to Y), we obtain the topology of *pointwise convergence*. This is also the coarsest topology that makes all the maps $f \mapsto f(x)$ continuous, for each $x \in X$. We write $[X \to Y]_{\mathsf{p}}$ for $[X \to Y]$ with the topology of pointwise convergence. Note that if $Z \subseteq [X \to Y]$, then the topology of pointwise convergence on Z is also the subspace topology from $[X \to Y]_{\mathsf{p}}$.

When Y is $\overline{\mathbb{R}^+}_\sigma$ or $\mathbb{R}^+_\sigma$, a subbasis of open sets of $[X \to Y]_{\mathsf{p}}$ is given by the subsets $[x > r] = \{f \in [X \to Y] \mid f(x) > r\}$, $x \in X$, $r \in \mathbb{R}^+$. Note that the latter are Scott open. In particular, the Scott topology on $[X \to \overline{\mathbb{R}^+}_\sigma]$ is always finer than the topology of pointwise convergence.

2.3. *Coherence*

A space X is *coherent* if and only if the intersection of any two compact saturated subsets is again compact.

A related notion is the following. Say the way-below relation $\ll$ on a lattice is *multiplicative* (see Abramsky and Jung (1994, Definition 7.2.18) or Gierz et al. (2003, Proposition I.4.7)) if and only if for all x, y_1, y_2 with $x \ll y_1$ and $x \ll y_2$, we have $x \ll \inf(y_1, y_2)$. We have called *core-coherent* those spaces X such that the way-below relation $\Subset$ was multiplicative on the lattice of open subsets of X (Goubault-Larrecq 2010, Comment before Definition 5.8).

It is easy to see that every locally compact, coherent space is (core-compact and) core-coherent. Indeed, assume $V \Subset U_1$, $V \Subset U_2$. By local compactness, there are compact saturated subsets Q_1, Q_2 such that $V \subseteq Q_1 \subseteq U_1$, $V \subseteq Q_2 \subseteq U_2$. Then, $V \subseteq Q_1 \cap Q_2 \subseteq U_1 \cap U_2$. Since X is coherent, $Q_1 \cap Q_2$ is compact. So $V \Subset U_1 \cap U_2$. In fact, one can check that the sober, core-compact and core-coherent spaces are exactly the sober, locally compact and coherent spaces (Abramsky and Jung 1994, Theorem 7.2.19).

2.4. *Stable compactness*

A space that is sober, locally compact and coherent is called *stably locally compact*. It is *stably compact* iff stably locally compact and compact. An equivalent definition of a stably locally compact space is a space that is T_0, locally compact, well-filtered and coherent.

Stably compact spaces were first studied in Nachbin (1965), see also Gierz et al. (2003, Section VI-6), or Alvarez-Manilla et al. (2004, Section 2). They enjoy the following property, called de Groot duality. Let X^{d} be X, topologized by taking as opens the complements of compact saturated subsets of X. Then, X^{d} is stably compact again, and $X^{\mathsf{dd}} = X$. Moreover, the specialization ordering of X^{d} is $\geqslant$, the opposite of the specialization ordering of X.

The *patch topology* on a stably compact space A is the coarsest topology finer than those of A and A^{d}, i.e., it is generated by the opens of A and the complements of compact saturated subsets of A. Write A^{patch} for A with its patch topology: this is a compact T_2 space, and the specialization ordering $\leqslant$ on A has a closed graph in $A^{\mathsf{patch}} \times A^{\mathsf{patch}}$, making $(A^{\mathsf{patch}}, \leqslant)$ a compact pospace Nachbin (1965). We shall be specially interested in $A = \overline{\mathbb{R}^+}_\sigma$;

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then, A^{patch} is merely $\overline{\mathbb{R}^+}$ with its ordinary T_2 topology generated by the intervals (a,b), $[0,b)$, and $(a,+\infty]$. We shall also use the fact that any product of stably compact spaces is stably compact, and the $\_^{\mathsf{patch}}$ operation commutes with product (Alvarez-Manilla et al. 2004, Proposition 14). So $\overline{\mathbb{R}^+}^C_\sigma$ is stably compact, and $(\overline{\mathbb{R}^+}^C_\sigma)^{\mathsf{patch}}$ is the compact T_2 space $\overline{\mathbb{R}^+}^C$.

We agree to prefix with 'patch-' any concept relative to patch topologies. For example, a patch-closed subset of A is a closed subset of A^{patch}, and a patch-continuous map from A to B is a continuous map from A^{patch} to B^{patch}.

2.5. *Stone duality*

There is a functor $\mathcal{O}$ from the category of topological spaces to the opposite of the category of frames (certain complete lattices), defined by the following: for every topological space X, $\mathcal{O}(X)$ is the frame of its open subsets, and for every continuous map $f : X \to Y$, $\mathcal{O}(f)$ is the frame homomorphism that maps every $V \in \mathcal{O}(Y)$ to $f^{-1}(V) \in \mathcal{O}(X)$. This functor has a right adjoint pt, which maps every frame to its set of completely prime filters, with the so-called hull-kernel topology. For details, see Abramsky and Jung (1994, Section 7) or Gierz et al. (2003, Section V-5).

This adjunction establishes a correspondence between properties of spaces X and properties of frames L. For example, X is core-compact if and only if $\mathcal{O}(X)$ is continuous, X is core-coherent if and only if the way-below relation on $\mathcal{O}(X)$ is multiplicative and X is compact if and only if the top element of $\mathcal{O}(X)$ is finite, i.e., way-below itself.

Going the other way around, $\mathsf{pt}(L)$ is always a sober space. (In fact, it is legitimate to call sober any topological space that is homeomorphic to one of this form.) For every topological space X, the fact that pt is right adjoint to $\mathcal{O}$ entails that $\mathsf{pt}(\mathcal{O}(X))$ obeys the following universal property: there is a continuous map η_X from X to $\mathsf{pt}(\mathcal{O}(X))$ (namely, the unit of the adjunction), and every continuous map f from X to any given sober space Y extends to a unique continuous map $f^!$ from $\mathsf{pt}(\mathcal{O}(X))$ to Y, in the sense that $f^! \circ \eta_X = f$. A space obeying that universal property is called a *sobrification* $\mathcal{S}(X)$ of X. Sobrification $\mathcal{S}$ is left adjoint to the forgetful functor from sober spaces to topological spaces; in other words, $\mathcal{S}(X)$ is a free sober space above X. In particular, all the sobrifications of X are naturally isomorphic.

If L is a continuous frame, then $\mathsf{pt}(L)$ is a locally compact, sober space (Abramsky and Jung 1994, Theorem 7.2.16). Equivalently, it is locally compact, T_0, and well filtered (Gierz et al. 2003, Theorem II-1.21). It follows that if X is core-compact, then its sobrification is locally compact, T_0 and well filtered.

If L is arithmetic, i.e., is a continuous frame with a multiplicative way-below relation, then $\mathsf{pt}(L)$ is stably locally compact (Abramsky and Jung 1994, Theorem 7.2.19). So the sobrification of a core-compact, core-coherent space is stably locally compact. Similarly, the sobrification of a compact, core-compact and core-coherent space is stably compact.

For any topological space X, X and its sobrification $\mathcal{S}(X) \cong \mathsf{pt}(\mathcal{O}(X))$ have isomorphic lattices of open subsets. This informally states that any construction, any property that can be expressed in terms of opens will apply to both.

One easy consequence, which we shall use in Section 3.3, is the following.

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Lemma 2.1. The function ! that maps $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$ to $h^! \in [\mathcal{S}(X) \to \overline{\mathbb{R}^+}_\sigma]$ is an order-isomorphism, a homeomorphism when the function spaces are equipped with their Scott topologies, and is natural in X.

Proof. $\overline{\mathbb{R}^+}_\sigma$ is sober, hence $h^!$ is well defined. Since $\mathcal{S}$ is left adjoint to the forgetful functor U from sober spaces to topological spaces, the correspondence between morphisms $h: X \to U(\overline{\mathbb{R}^+}_\sigma)$ and morphisms $h^!: \mathcal{S}(X) \to \overline{\mathbb{R}^+}_\sigma$ is bijective, and is natural in X. Both ! and its inverse $h' \mapsto h' \circ \eta_X$ are monotonic, and therefore define an order-isomorphism, hence an isomorphism of spaces with their Scott topologies. (To show that ! is monotonic, consider $f \leqslant g$, and realize that $\sup(f^!, g^!)$ is a continuous map that extends $\sup(f, g) = g$, hence must coincide with $g^!$ by uniqueness of extensions.) □

2.6. *Cones*

A cone C is an additive commutative monoid with a scalar multiplication by elements of $\mathbb{R}^+$, satisfying laws similar to those of vector spaces. Precisely, a *cone* C is endowed with an addition $+ : C \times C \to C$, a zero element $0 \in C$, and a scalar multiplication $\cdot : \mathbb{R}^+ \times C \to C$ such that

$$(x + y) + z = x + (y + z) \quad x + y = y + x \quad x + 0 = x$$
$$(rs) \cdot x = r \cdot (s \cdot x) \quad 1 \cdot x = x \quad 0 \cdot x = 0$$
$$r \cdot (x + y) = r \cdot x + r \cdot y \quad (r + s) \cdot x = r \cdot x + s \cdot x.$$

An *ordered cone* is a cone with a partial ordering that makes $+$ and $\cdot$ monotonic. Similarly, a *topological cone* is a cone equipped with a T_0 topology that makes $+$ and $\cdot$ continuous, where $\mathbb{R}^+$ is equipped with its Scott topology. In a *semitopological cone*, we only require $+$ and $\cdot$ to be separately continuous, not jointly continuous.

An important example of semitopological cone is given by the ordered cones in which $+$ and $\cdot$ are Scott-continuous (the *s-cones*), in particular by Tix, Keimel and Plotkin's *d-cones* (Tix et al. 2009), which are additionally required to be dcpos. As noticed by Keimel (2008, Remark before Proposition 6.3), an s-cone C may fail to be a topological cone, unless C is a *continuous cone*, i.e., an ordered cone that is continuous as a poset, and where $+$ and $\cdot$ are Scott-continuous. In that case, the product of the Scott topologies is the Scott topology of the product ordering, and separate continuity implies joint continuity.

The most important cone we shall deal with is the ordered cone $[X \to \overline{\mathbb{R}^+}_\sigma]$ of all continuous maps from X to $\overline{\mathbb{R}^+}_\sigma$. With its Scott topology, it is a semitopological cone. It is a topological cone if X is core-compact, since it is a continuous dcpo in that case, as a special case of Gierz et al. (2003, Theorem II-4.7). In previous work, we were using the subcone $\langle X \to \mathbb{R}^+_\sigma \rangle$ of all *bounded* continuous maps from X to $\mathbb{R}^+_\sigma$. This is again a semitopological cone that happens to be topological when X is core-compact.

A subset Z of a topological cone C is *convex* iff $r \cdot x + (1 - r) \cdot y$ is in Z whenever $x, y \in Z$ and $0 \leqslant r \leqslant 1$. C is itself *locally convex* iff every point has a basis of convex open neighbourhoods, i.e., whenever $x \in U$, U open in C, then there is a convex open V such that $x \in V \subseteq U$.

This is the notion of local convexity used by Keimel (2008). Beware that there are others, such as Heckmann's (Heckmann 1996), which are inequivalent.

Every continuous cone C is locally convex; this is a special case of Keimel (2008, Lemma 6.12). The argument, due to J. Lawson, is as follows: let x be a point in some open subset U of C, then there is a point $x_1 \ll x$ that is also in U, hence also an $x_2 \ll x_1$ that is again in U, and continuing this way we have a chain $\ldots \ll x_n \ll \ldots \ll x_2 \ll x_1 \ll x$ of points of U. Now let $V = \bigcup_{n \in \mathbb{N}} \twoheaduparrow x_n$: V is clearly open, $x \in V \subseteq U$, and V is also convex because $V = \bigcup_{n \in \mathbb{N}} \uparrow x_n$, as one can easily check.

We have already noticed that, given a core-compact space X, $[X \to \overline{\mathbb{R}^+}_\sigma]$ is a continuous d-cone. Therefore, $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is a locally convex topological cone in that case.

A map $f : C \to \overline{\mathbb{R}^+}$ is *positively homogeneous* iff $f(r \cdot x) = rf(x)$ for all $x \in C$ and $r \in \mathbb{R}^+$. It is *additive* (resp. *superadditive*, resp. *subadditive*) iff $f(x+y) = f(x) + f(y)$ (resp. $f(x+y) \geqslant f(x) + f(y)$, resp. $f(x+y) \leqslant f(x) + f(y)$) for all $x, y \in C$. It is *linear* (resp. *superlinear*, resp. *sublinear*) iff it is both positively homogeneous and additive (resp. superadditive, resp. subadditive). Every pointwise supremum of sublinear maps is sublinear, and every pointwise infimum of superlinear maps is superlinear.

One of the key results in the theory of cones is Keimel's Sandwich Theorem (Keimel 2008, Theorem 8.2), an analogue of the Hahn–Banach Theorem: in a semitopological cone C, given a continuous superlinear map $q : C \to \overline{\mathbb{R}^+}$ and a sublinear map $p : C \to \overline{\mathbb{R}^+}$ such that $q \leqslant p$, there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}^+}$ such that $q \leqslant \Lambda \leqslant p$. Note that p need not be continuous or even monotonic for this to hold. This is a feature we shall make use of in the proof of Lemma 3.16, allowing us to dispense with an assumption of coherence.

Another construction we shall use is the *lower Minkowski functional* M_A of a non-empty closed subset A of a topological cone C (this was called F_A in Keimel (2008, Section 7), but this would conflict with some of our own notations). This is defined by

$$M_A(x) = \inf\{b > 0 \mid (1/b) \cdot x \in A\}, \quad (1)$$

where we agree that the infimum is equal to $+\infty$ if no b exists such that $(1/b) \cdot x \in A$. M_A is continuous (Keimel 2008, Proposition 7.3 (a)), superlinear if and only if $C \smallsetminus A$ is convex and sublinear if and only if A is convex (Keimel 2008, Lemma 7.5).

Using this, Keimel's Sandwich Theorem immediately implies the following Separation Theorem (Keimel 2008, Theorem 9.1): in a semitopological cone C, for every convex non-empty subset A and every convex open subset U such that $A \cap U = \emptyset$, there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}^+}_\sigma$ such that $\Lambda(x) \leqslant 1$ for every $x \in A$ and $\Lambda(x) > 1$ for every $x \in U$. We shall also use the following Strict Separation Theorem (Keimel 2008, Theorem 10.5): in a locally convex semitopological cone C, for every compact convex subset Q and every non-empty closed convex subset A such that $Q \cap A = \emptyset$, there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}^+}_\sigma$ and a real number $r > 1$ such that $\Lambda(x) \geqslant r$ for every $x \in Q$, and $\Lambda(x) \leqslant 1$ for every $x \in A$.

2.7. *Valuations, previsions, forks*

A *valuation* on a topological space X is a map ν from the lattice $\mathcal{O}(X)$ of open subsets of X to $\overline{\mathbb{R}^+}$ that is *strict* ($\nu(\emptyset) = 0$), *monotonic* (if $U \subseteq V$ then $\nu(U) \leqslant \nu(V)$), and *modular* ($\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$). A valuation is *continuous* if and only if it is

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Scott-continuous, i.e., for every directed family $(U_i)_{i \in I}$ of opens, $\nu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$. It is *subnormalized* iff $\nu(X) \leqslant 1$, and *normalized* iff $\nu(X) = 1$. A normalized continuous valuation is a continuous *probability* valuation.

Given any continuous map $h : X \to \overline{\mathbb{R}^+}$, one can define the *integral* $\int_{x \in X} h(x) d\nu$ of h with respect to ν in various equivalent ways. One is by using a Choquet-type formula (Tix 1995, Section 4.1): $\int_{x \in X} h(x) d\nu$ is defined as $\int_0^{+\infty} \nu(h^{-1}(t, +\infty]) dt$, where the latter is a Riemann integral, which is well defined since the integrated function is non-increasing. Letting $F(h) = \int_{x \in X} h(x) d\nu$, one realizes that F is linear and Scott-continuous on the cone $[X \to \overline{\mathbb{R}^+}_\sigma]$ (Tix 1995, Lemma 4.2).

In particular, F also defines a linear Scott-continuous map from the cone $\langle X \to \mathbb{R}^+_\sigma \rangle$ to $\overline{\mathbb{R}^+}$. Such functionals were called (continuous) *linear previsions* in Goubault-Larrecq (2007), except that they were not allowed to take the value $+\infty$. Conversely, any linear Scott-continuous map F from $\langle X \to \mathbb{R}^+_\sigma \rangle$ to $\overline{\mathbb{R}^+}$ extends to a unique linear Scott-continuous map from $[X \to \overline{\mathbb{R}^+}_\sigma]$ to $\overline{\mathbb{R}^+}$, by $F(h) = \sup_{a \in \mathbb{R}^+} F(\min(h, a))$ for example.

In general, call *prevision* on X any positively homogeneous, Scott-continuous map F from $[X \to \overline{\mathbb{R}^+}_\sigma]$ to $\overline{\mathbb{R}^+}$. A prevision is *Hoare* (or *lower*) if and only if it is sublinear, and is *Smyth* (or *upper*) if and only if it is superlinear. We say that F is *subnormalized* (resp., *normalized*) iff $F(1 + h) \leqslant 1 + F(h)$ (resp., $=$) for every $h \in [X \to \mathbb{R}^+_\sigma]$. These conditions simplify to $F(1) \leqslant 1$ (resp., $F(1) = 1$) in the case of linear previsions, but we shall need the more general form when F is not linear.

It is easy to see that the posets $\mathbf{V}(X)$ of all continuous valuations on X (ordered by $\nu \leqslant \nu'$ iff $\nu(U) \leqslant \nu'(U)$ for every open U) and $\mathbb{P}_{\mathtt{P}}(X)$ of all linear previsions on X (ordered by $F \leqslant G$ iff $F(h) \leqslant G(h)$ for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$) are isomorphic. This is the variant of the Riesz representation theorem that we mentioned in the introduction. In one direction, we obtain F from ν by $F(h) = \int_{x \in X} h(x) d\nu$, and conversely we obtain ν from F by letting $\nu(U) = F(\chi_U)$. (For example, this is a special case of Tix (1995, Satz 4.16).) This isomorphism maps (sub)normalized continuous valuations to (sub)normalized linear previsions and conversely.

To handle the mixture of probabilistic and *erratic* non-determinism, we rely on *forks* (Goubault-Larrecq 2007). A fork on X is by definition a pair (F^-, F^+) of a Smyth prevision F^- and a Hoare prevision F^+ satisfying *Walley's condition*:

$$F^-(h + h') \leqslant F^-(h) + F^+(h') \leqslant F^+(h + h')$$

for all $h, h' \in [X \to \overline{\mathbb{R}^+}_\sigma]$. (This condition was independently discovered by Keimel and Plotkin (2009).) By taking $h' = 0$, or $h = 0$, this in particular implies that $F^- \leqslant F^+$. A fork (F^-, F^+) is (sub)normalized if and only if both F^- and F^+ are.

Using some notation that we introduced in Goubault-Larrecq (2015b), write $\mathbb{P}_{\mathtt{AP}}(X)$ for the set of all Hoare previsions on X ($\mathtt{A}$ for angelic non-determinism, $\mathtt{P}$ for probabilistic choice), $\mathbb{P}_{\mathtt{DP}}(X)$ for the set of all Smyth previsions on X. Write $\mathbb{P}^1_{\mathtt{AP}}(X)$ for the set of normalized Hoare previsions, $\mathbb{P}^{\leqslant 1}_{\mathtt{AP}}(X)$ for the set of subnormalized Hoare previsions, and similarly for Smyth (subscript $\mathtt{DP}$) and linear (subscript $\mathtt{P}$) previsions. In any case, the *weak topology* on any of these spaces Y is generated by subbasic open sets, which we write uniformly as $[h > r]$, and are defined as those $F \in Y$ such that $F(h) > r$, where h ranges over $[X \to \overline{\mathbb{R}^+}_\sigma]$ and r over $\mathbb{R}^+$. The specialization ordering of the weak topology

https://doi.org/10.1017/S0960129515000547 Published online by Cambridge University Press

is pointwise: $F \leqslant F'$ iff $F(h) \leqslant F(h')$ for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$. We use a *wk* subscript, e.g., $\mathbb{P}^{\leqslant 1}_{\mathtt{AP}\ wk}(X)$, to refer to a space in its weak topology. Since $[h > r] = [1/r.h > 1]$ when $r \neq 0$, and $[h > 0] = \bigcup_{r>0}[h > r]$, note that the subsets of the form $[h > 1]$, $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$ form a smaller subbase.

Similarly, write $\mathbb{P}_{\mathtt{ADP}}(X)$ for the set of all forks on X (A for angelic, D for demonic, and P for probabilistic), and $\mathbb{P}^{\leqslant 1}_{\mathtt{ADP}}(X)$, $\mathbb{P}^{1}_{\mathtt{ADP}}(X)$ for their subsets of subnormalized, resp. normalized, forks. On each, define the *weak topology* as the subspace topology induced from the larger space $\mathbb{P}_{\mathtt{DP}\ wk}(X) \times \mathbb{P}_{\mathtt{AP}\ wk}(X)$. It is easy to see that a subbase of the weak topology is composed of two kinds of open subsets: $[h > b]^-$, defined as $\{(F^-, F^+) \mid F^-(h) > b\}$, and $[h > b]^+$, defined as $\{(F^-, F^+) \mid F^+(h) > b\}$, where $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, $b \in \mathbb{R}^+$. The specialization ordering of spaces of forks is the product ordering $\leqslant \times \leqslant$, where $\leqslant$ denotes the pointwise ordering on previsions. As usual, we adjoin a subscript '*wk*' to denote spaces of forks with the weak topology.

Throughout, we shall use the convention of writing $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$, $\mathbb{P}^{\bullet}_{\mathtt{DP}\ wk}(X)$, etc. $\bullet$ is either the empty superscript, '$\leqslant 1$,' or '1.' This will allow us to factor some notation. Later, we shall also use this scheme for maps $s^{\bullet}_{\mathtt{AP}}$, $s^{\bullet}_{\mathtt{DP}}$, and so forth.

3. Retracting powercones onto previsions

Given a topological space X, let

— the *Hoare powerdomain* $\mathcal{H}(X)$ be the set of all closed, non-empty subsets of X;

— the *Smyth powerdomain* $\mathcal{Q}(X)$ be the set of all compact saturated, non-empty subsets of X;

— the *Plotkin powerdomain* $\mathcal{P}\ell(X)$ be the set of all lenses of X; a *lens* L is a non-empty subset of X that can be written as the intersection of a compact saturated subset Q and a closed subset F of X. A canonical form is obtained by taking $Q = \uparrow L$ and $F = cl(L)$.

$\mathcal{H}(X)$ is the traditional model for so-called *angelic* non-determinism. We write $\mathcal{H}_{\mathcal{V}}(X)$ for $\mathcal{H}(X)$ with its *lower Vietoris topology*, generated by subbasic open sets $\Diamond V = \{C \in \mathcal{H}(X) \mid C \cap V \neq \emptyset\}$, where V ranges over the open subsets of X. Its specialization ordering is inclusion $\subseteq$, and we shall also consider $\mathcal{H}(X)$ as a dcpo in this ordering. The Scott topology is always finer than the lower Vietoris topology, and coincides with it when X is a continuous dcpo in its Scott topology (Schalk 1993, Section 6.3.3).

$\mathcal{Q}(X)$ is the traditional model for so-called *demonic* non-determinism. We write $\mathcal{Q}_{\mathcal{V}}(X)$ for $\mathcal{Q}(X)$ with its *upper Vietoris topology*, generated by basic open sets $\Box V = \{Q \in \mathcal{Q}(X) \mid Q \subseteq V\}$, where V ranges over the open subsets of X. Its specialization ordering is reverse inclusion $\supseteq$. When X is T_0, well filtered, and locally compact, $\mathcal{Q}(X)$ is a continuous dcpo, and the Scott and upper Vietoris topologies coincide (Schalk 1993, Section 7.3.4).

$\mathcal{P}\ell(X)$ is the traditional model for *erratic* non-determinism. Write $\mathcal{P}\ell_{\mathcal{V}}(X)$ for $\mathcal{P}\ell(X)$ with its *Vietoris topology*, generated by subbasic open sets $\Box V = \{L \in \mathcal{P}\ell(X) \mid L \subseteq V\}$ and $\Diamond V = \{L \in \mathcal{P}\ell(X) \mid L \cap V \neq \emptyset\}$. Its specialization ordering is the topological Egli-Milner ordering $\sqsubseteq_{\mathrm{EM}}$, defined by $L \sqsubseteq_{\mathrm{EM}} L'$ iff $\uparrow L \supseteq \uparrow L'$ and $cl(L) \subseteq cl(L')$.

When X is a semitopological cone C, it makes sense to consider the subsets $\mathcal{H}^{cvx}(C)$, $\mathcal{Q}^{cvx}(C)$, $\mathcal{P}\ell^{cvx}(C)$ of those elements of $\mathcal{H}(C)$, $\mathcal{Q}(C)$, $\mathcal{P}\ell(C)$ respectively that are convex. We again equip them with their respective (lower, upper, plain) Vietoris topologies, yielding spaces which we write with a $\mathcal{V}$ subscript, and which happen to be subspaces of $\mathcal{H}_{\mathcal{V}}(C)$, $\mathcal{Q}_{\mathcal{V}}(C)$, $\mathcal{P}\ell_{\mathcal{V}}(C)$, respectively. Their specialization orderings are as for their non-convex variants, and again give rise to Scott topologies. But beware that the Scott topologies on the latter may fail to be subspace topologies.

We now define formally the maps that we named r and s in the introduction. They come in three flavours, angelic, demonic and erratic, but we shall ignore the erratic forms for now. Since continuous valuations are isomorphic to spaces of linear of previsions, we reason with the latter; this is what we shall really need in proofs.

Definition 3.1. Let X be a topological space. For every non-empty set E of linear previsions on X, let $r_{\mathtt{AP}}(E) \colon [X \to \overline{\mathbb{R}^+}_\sigma] \to \overline{\mathbb{R}^+}$ (resp., $r_{\mathtt{DP}}(E)$) map h to $\sup_{G \in E} G(h)$ (resp., inf).

Conversely, for every Hoare prevision (resp., subnormalized, normalized) F on X, let $s_{\mathtt{AP}}(F)$ (resp., $s_{\mathtt{AP}}^{\leqslant 1}(F)$, $s_{\mathtt{AP}}^{1}(F)$) be the set of all linear previsions (resp., subnormalized, normalized) G such that $G \leqslant F$. For every Smyth prevision (resp., subnormalized, normalized) F on X, let $s_{\mathtt{DP}}(F)$ (resp., $s_{\mathtt{DP}}^{\leqslant 1}(F)$, $s_{\mathtt{DP}}^{1}(F)$) be the set of all linear previsions (resp., subnormalized, normalized) G such that $F \leqslant G$.

Our aim in this section is to show that the various matching pairs of maps r and s form retractions onto the adequate spaces. A *retraction* of X onto Y is a pair of two continuous maps $r \colon X \to Y$ (also called, somewhat ambiguously, a retraction) and $s \colon Y \to X$ (called the associated section) such that $r \circ s = \mathrm{id}_Y$.

Some of our retractions will have the extra property that $s \circ r \leqslant \mathrm{id}_X$, meaning that s is left-adjoint to r. We shall call such retraction *embedding-projection pair*, following a domain-theoretic tradition (Abramsky and Jung 1994, Definition 3.1.15); r is the projection, and s is the embedding. Accordingly, we shall say that X *projects onto* Y through r in that situation. There is also a dual situation where $\mathrm{id}_X \leqslant s \circ r$ instead, meaning that s is right-adjoint to r. In that case, we shall say that X *coprojects onto* Y through r, that the latter is a *coprojection*, and that s is the associated *coembedding*.

We start with the angelic case. We shall deal with the demonic case in Section 3.2, and with the erratic case in Section 3.3.

3.1. *The retraction in the angelic case*

Our aim in this section is to prove that if X is a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex, then $r_{\mathtt{AP}}^\bullet$ and $s_{\mathtt{AP}}^\bullet$ form a retraction. Note that the assumption holds as soon as X is locally compact, or, more generally, core-compact.

We proceed through a series of lemmata. The first one is clear.

Lemma 3.2. Let X be a topological space. For every $C \in \mathcal{H}_{\mathcal{V}}(\mathbb{P}_{\mathtt{P}\,wk}^\bullet(X))$, $r_{\mathtt{AP}}(C) = (h \in [X \to \overline{\mathbb{R}^+}_\sigma] \mapsto \sup_{G \in C} G(h))$ is an element of $\mathbb{P}_{\mathtt{AP}}^\bullet(X)$.

For clarity, write $[h > b]_{\mathtt{AP}}$ or $[h > b]_{\mathtt{P}}$ for the subbasic open subset $[h > b]$, depending whether in a space of Hoare previsions or of linear previsions.

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Lemma 3.3. Let X be a topological space. The map $r_{\mathtt{AP}}$ is continuous from $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$ to $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$.

Proof. The inverse image of the subbasic open $[h > b]_{\mathtt{AP}} \subseteq \mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$ by $r_{\mathtt{AP}}$ is $\Diamond[h > b]_{\mathtt{P}}$, hence is open in $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}wk}(X))$. □

The following Lemma is very similar to Lemma 5.8 of Keimel and Plotkin (2009), and has a similar proof.

Lemma 3.4. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex, and F be an element of $\mathbb{P}^{\bullet}_{\mathtt{AP}}(X)$. For every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, and every real number $r > 0$ such that $F(h) > r$, there is a $G \in \mathbb{P}^{\bullet}_{\mathtt{P}}(X)$ with $G \leqslant F$ and $G(h) > r$.

Proof. Since F is continuous, $F^{-1}(r, +\infty]$ is open, and contains h. Using local convexity, there is a convex open subset U containing h such that $F(h') > r$ for every $h' \in U$. Let $q : C \to \overline{\mathbb{R}^+}$ be defined by $q(h') = rM_A(h')$ where $A = C \smallsetminus U$. M_A is the lower Minkowski functional of A, see equation (1) in the Cones subsection of Section 2. Since A is non-empty, closed and its complement is convex, q is continuous and superlinear. Moreover, $q \leqslant F$: for every $h' \in C$, we observe that for every $b > 0$ such that $b < M_A(h')$, $(1/b) \cdot h'$ cannot be in A by the definition of M_A, hence is in U; so $F((1/b) \cdot h') > r$, whence $F(h') > br$; taking sups over b, $F(h') \geqslant rM_A(h') = q(h')$.

So we can apply Keimel's Sandwich Theorem and conclude that there is a continuous linear map G such that $q \leqslant G \leqslant F$.

We claim that $G(h) = F(h)$, which will certainly imply $G(h) > r$. The inequality $G(h) \leqslant F(h)$ is clear. Conversely, $G(h) \geqslant q(h) = rM_A(h) = r\ \inf\{b > 0 \mid F((1/b) \cdot h) \leqslant r\} = F(h)$. This holds even if $F(h_0) = +\infty$, the important thing being that r is real and non-zero.

When $\bullet$ is '$\leqslant 1$,' then $G(1) \leqslant F(1) \leqslant 1$, so G is in $\mathbb{P}^{\leqslant 1}_{\mathtt{P}}(X)$. When $\bullet$ is '1,' additionally, $q(1) = r\ \inf\{b > 0 \mid F(1/b) \leqslant r\} = 1$, whence $G(1) = 1$. In any case, G is in $\mathbb{P}^{\bullet}_{\mathtt{P}}(X)$. □

Lemma 3.5. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex. Then, $r_{\mathtt{AP}} \circ s^{\bullet}_{\mathtt{AP}}$ is the identity map on $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$.

Proof. We must show that for every $F \in \mathbb{P}_{\mathtt{AP}}(X)$, for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, $F(h) = \sup_{G \in \mathbb{P}^{\bullet}_{\mathtt{P}}(X), G \leqslant F} G(h)$. The inequality $F(h) \leqslant \sup_{G \in \mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X), G \leqslant F} G(h)$ follows directly from Lemma 3.4, while the converse inequality is obvious. □

Lemma 3.6. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex. For every $F \in \mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$, $s^{\bullet}_{\mathtt{AP}}(F) = \{G \in \mathbb{P}^{\bullet}_{\mathtt{P}}(X) \mid G \leqslant F\}$ is a closed subset of $\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X)$.

Proof. Consider any $G \notin s^{\bullet}_{\mathtt{AP}}(F)$. There is an $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$ such that $G(h) > F(h)$. Then, $[h > r]$ (where $r = F(h)$) is an open neighbourhood of G that does not meet $s^{\bullet}_{\mathtt{AP}}(F)$. So the complement of $s^{\bullet}_{\mathtt{AP}}(F)$ is open. □

Lemma 3.7. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex. For every $F \in \mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$, $s^{\bullet}_{\mathtt{AP}}(F) = \{G \in \mathbb{P}^{\bullet}_{\mathtt{P}}(X) \mid G \leqslant F\}$ is an element of $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$.

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Proof. It is closed by Lemma 3.6. When $\bullet$ is the empty superscript or '$\leqslant 1$,' $s^\bullet_{\mathtt{AP}}(F)$ is non-empty since it contains the zero prevision. When $\bullet$ is '1,' non-emptiness follows from Lemma 3.4 with $h = 1$, and taking $r = 0$ for example. □

The fact that $s^\bullet_{\mathtt{AP}}$ is continuous is the most complicated result of this section. One of the needed ingredients is von Neumann's original minimax theorem (von Neumann 1928). That theorem was vastly generalized since then, and in numerous ways, but the original form will be sufficient to us:

Lemma 3.8 (Von Neumann's minimax). For each $n \in \mathbb{N}$, let Δ_n be the set of n-tuples of non-negative real numbers $(a_1, a_2, \ldots, a_n)$ such that $\sum_{i=1}^n a_i = 1$. For every $n \times m$ matrix M with real entries,

$$\min_{\vec{\alpha}\in\Delta_m} \max_{\vec{\beta}\in\Delta_n} \vec{\beta}^t M \vec{\alpha} = \max_{\vec{\beta}\in\Delta_n} \min_{\vec{\alpha}\in\Delta_m} \vec{\beta}^t M \vec{\alpha}.$$

Note that an implicit fact in that lemma, hidden in the notation min, max, is that the suprema over $\vec{\beta}$ and the infima over $\vec{\alpha}$ are attained. This is a consequence of the fact that Δ_m and Δ_n are compact, and that multiplication by M is continuous. For that, it is important that M has real-valued entries, and in particular, not $+\infty$.

Lemma 3.9. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex. Then, $s^\bullet_{\mathtt{AP}}$ is continuous from $\mathbb{P}^\bullet_{\mathtt{AP}\ wk}(X)$ to $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$.

Proof. Among the subbases of the weak topology on $\mathbb{P}^\bullet_{\mathtt{P}}(X)$, one is given by subsets of the form $[h > b]_{\mathtt{P}}$, $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, $b \in \mathbb{R}^+$. Since $[h > 0]_{\mathtt{P}} = \bigcup_{r>0}[h > r]_{\mathtt{P}}$, we may restrict to $b \neq 0$. When $b \neq 0$, $[h > b]_{\mathtt{P}} = [h/b > 1]_{\mathtt{P}}$, so another subbase is given by the subsets of the form $[h > 1]_{\mathtt{P}}$. Finally, since h is the directed supremum of the maps $\min(h, r)$, $r \in \mathbb{R}^+$, we can even restrict h to be bounded.

Since $\Diamond$ commutes with unions, a subbase of the topology of $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$ is then given by the subsets of the form $\Diamond W$, where W is a finite intersection $\bigcap_{i=1}^m [h_i > 1]_{\mathtt{P}}$, where each h_i is continuous and bounded. Moreover, we may require $m > 0$.

We must show that the inverse image of $\Diamond W$ by $s^\bullet_{\mathtt{AP}}$, where W is as above, is open in $\mathbb{P}^\bullet_{\mathtt{AP}\ wk}(X)$. Let F be an arbitrary element of ${s^\bullet_{\mathtt{AP}}}^{-1}(\Diamond W)$. By definition, there is an element $G \in \mathbb{P}^\bullet_{\mathtt{P}\ wk}(X)$ such that $G \leqslant F$ and $G \in \bigcap_{i=1}^m [h_i > 1]_{\mathtt{P}}$.

Fix a positive real number ϵ such that $G(h_i) > 1 + \epsilon$ for every i, $1 \leqslant i \leqslant m$. We claim that we can find a finite set A of m-tuples of non-negative real numbers $\vec{a} = (a_1, a_2, \ldots, a_m)$ such that $\bigcap_{\vec{a}\in A}[\sum_{i=1}^m a_i h_i > (1+\epsilon)\sum_{i=1}^m a_i]_{\mathtt{AP}}$ is an open neighbourhood of F included in ${s^\bullet_{\mathtt{AP}}}^{-1}(\Diamond W)$.

To this end, we shall define A as the set of m-tuples of non-negative real numbers $\vec{a} = (a_1, a_2, \ldots, a_m)$ such that $0 < \sum_{i=1}^m a_i \leqslant 1$ and each a_i is an integer multiple of $1/N$, for some fixed, large enough natural number N. Taking N so that $\frac{m}{N} < \frac{\epsilon}{1+\epsilon}$ will be enough for our purposes.

The fact that F is in $\bigcap_{\vec{a}\in A}[\sum_{i=1}^m a_i h_i > (1+\epsilon)\sum_{i=1}^m a_i]_{\mathtt{AP}}$ is obvious. For every $\vec{a} \in A$, $F(\sum_{i=1}^m a_i h_i) \geqslant G(\sum_{i=1}^m a_i h_i) = \sum_{i=1}^m a_i G(h_i)$, and this is larger than or equal to $(1 + \epsilon)\sum_{i=1}^m a_i$. To show that it is strictly larger, recall that some a_i is non-zero, since $0 < \sum_{i=1}^m a_i$.

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To show that $\bigcap_{\vec{a}\in A}[\sum_{i=1}^m a_i h_i > (1+\epsilon)\sum_{i=1}^m a_i]_{\mathtt{AP}}$ is included in ${s^\bullet_{\mathtt{AP}}}^{-1}(\Diamond W)$ is more technical. The main observation is the following:

Fact A. For every positively homogeneous map $\Phi : [X \to \overline{\mathbb{R}^+}_\sigma] \to \overline{\mathbb{R}^+}_\sigma$ such that $\Phi(\sum_{i=1}^m a_i h_i) > (1+\epsilon)\sum_{i=1}^m a_i$ for every $\vec{a} \in A$, it holds that $\Phi(\sum_{i=1}^m \alpha_i h_i) > 1$ for every $\vec{\alpha} \in \Delta_m$.

This fact is proved by elementary computation. Fix $\vec{\alpha} \in \Delta_m$, and let $a_i = \frac{1}{N}\lfloor N\alpha_i \rfloor$ be the largest multiple of $1/N$ below α_i, for each i. Notice that $\frac{1}{1+\epsilon} < \sum_{i=1}^m a_i \leqslant 1$; the inequality on the left follows from the consideration that $a_i > \alpha_i - 1/N$, and $\frac{m}{N} < 1 - \frac{1}{1+\epsilon} = \frac{\epsilon}{1+\epsilon}$, remembering that $\sum_{i=1}^m \alpha_i = 1$. In particular, $\vec{a} = (a_1, a_2, \dots, a_m)$ is in A, so $\Phi(\sum_{i=1}^m a_i h_i) > (1+\epsilon)\sum_{i=1}^m a_i$. Since $\alpha_i \geqslant a_i$ for each i, $\Phi(\sum_{i=1}^m \alpha_i h_i) \geqslant \Phi(\sum_{i=1}^m a_i h_i) > (1+\epsilon)\sum_{i=1}^m a_i$, and we have just seen that the latter is strictly greater than 1.

Now consider any element F' of $\bigcap_{\vec{a}\in A}[\sum_{i=1}^m a_i h_i > (1+\epsilon)\sum_{i=1}^m a_i]_{\mathtt{AP}}$. By Lemma 3.5, F' is a pointwise supremum of elements of $\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X)$. Write this supremum as a directed supremum of finite suprema, and observe that $\bigcap_{\vec{a}\in A}[\sum_{i=1}^m a_i h_i > (1+\epsilon)\sum_{i=1}^m a_i]_{\mathtt{AP}}$ is Scott open. As a consequence, it contains one of the finite suprema, viz., there are finitely many elements $G'_1, G'_2, \dots, G'_n$ of $\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X)$ below F' such that, for every $\vec{a} \in A$, there is a j, $1 \leqslant j \leqslant n$, such that $\sum_{i=1}^m a_i G'_j(h_i) > (1+\epsilon)\sum_{i=1}^m a_i$. We can even take G'_j bounded in the sense that $G'_j(1) < +\infty$: this is clear when $\bullet$ is '$\leqslant 1$' or '1,' otherwise we use (Heckmann 1996, Theorem 4.2), which says that every linear prevision on X is a directed supremum of bounded linear previsions, allowing us to replace each G'_j by a bounded linear prevision whose value on h_i is close enough to $G'_j(h_i)$.

For every $\vec{a} \in A$, there is a j such that $\sum_{i=1}^m a_i G'_j(h_i) > (1+\epsilon)\sum_{i=1}^m a_i$, so trivially, there is a $\vec{\beta} \in \Delta_n$ such that $\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} a_i \beta_j G'_j(h_i) > (1+\epsilon)\sum_{i=1}^m a_i$. Applying Fact A to $\Phi = \sup_{\vec{\beta}\in\Delta_n}\sum_{j=1}^n \beta_j G'_j$, we obtain that for every $\vec{\alpha} \in \Delta_m$, $\sup_{\vec{\beta}\in\Delta_n}\sum_{j=1}^n \beta_j G'_j(\sum_{i=1}^m \alpha_i h_i) > 1$, so $\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \alpha_i \beta_j G'_j(h_i) > 1$ for some $\vec{\beta} \in \Delta_n$. Since G'_j is bounded, and h_i is bounded too, $G'_j(h_i) < +\infty$, so $M = (G'_j(h_i))_{\substack{1\leqslant j\leqslant n\\ 1\leqslant i\leqslant m}}$ is a matrix of real numbers. Rephrasing what we have just obtained, for every $\vec{\alpha} \in \Delta_m$, there is a $\vec{\beta} \in \Delta_n$ such that $\vec{\beta}^t M \vec{\alpha} > 1$. In particular (and using the fact that infima are attained) $\min_{\vec{\alpha}\in\Delta_m}\max_{\vec{\beta}\in\Delta_n}\vec{\beta}^t M\vec{\alpha} > 1$. By von Neumann's minimax theorem (Lemma 3.8), $\max_{\vec{\beta}\in\Delta_n}\min_{\vec{\alpha}\in\Delta_m}\vec{\beta}^t M\vec{\alpha} > 1$. Therefore, there is a tuple $\vec{\beta} \in \Delta_n$ such that, for every $\vec{\alpha} \in \Delta_m$, $\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \alpha_i \beta_j G'_j(h_i) > 1$. Take $G' = \sum_{j=1}^n \beta_j G'_j$. Since $\sum_{j=1}^n \beta_j = 1$, G' is in $\mathbb{P}^\bullet_{\mathtt{P}}(X)$, and $G' \leqslant F'$. Also, we have just proved that $\sum_{i=1}^m \alpha_i G'(h_i) > 1$ for every $\vec{\alpha} \in \Delta_m$, in particular, G' is in $\bigcap_{i=1}^m [h_i > 1]_{\mathtt{P}}$. Therefore, F' is, indeed, in ${s^\bullet_{\mathtt{AP}}}^{-1}(\Diamond W)$. ☐

Lemma 3.10. Let X be a topological space. For every $C \in \mathcal{H}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$, $C \subseteq s^\bullet_{\mathtt{AP}}(r^\bullet_{\mathtt{AP}}(C))$.

Proof. That amounts to checking that for every $G \in C$, $G(h) \leqslant \sup_{G'\in C} G'(h)$, which is obvious. ☐

We sum up these results in the following proposition.

Proposition 3.11. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1,'

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Let X be a topological space. Then, $r_{\mathtt{AP}}$ is a continuous map from $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$ to $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$, and $s^{\bullet}_{\mathtt{AP}} \circ r^{\bullet}_{\mathtt{AP}}$ is above the identity.

If $[X \to \overline{\mathbb{R}^{+}}_{\sigma}]_{\sigma}$ is locally convex, then $s^{\bullet}_{\mathtt{AP}}$ is a continuous map from $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$ to $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$ such that $r_{\mathtt{AP}} \circ s^{\bullet}_{\mathtt{AP}}$ equals the identity.

Recall that a coembedding-coprojection pair is a pair of continuous maps s, r, such that $r \circ s = \mathrm{id}$ and $\mathrm{id} \leqslant s \circ r$.

Corollary 3.12 ($\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$ coprojects onto $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$). Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a topological space such that $[X \to \overline{\mathbb{R}^{+}}_{\sigma}]_{\sigma}$ is locally convex, for example, a locally compact space, or more generally, a core-compact space.

Then, $s^{\bullet}_{\mathtt{AP}}$ and $r_{\mathtt{AP}}$ together define an coembedding-coprojection pair of $\mathcal{H}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$ onto $\mathbb{P}^{\bullet}_{\mathtt{AP}\ wk}(X)$.

3.2. *The retraction in the demonic case*

In the demonic cases, we have defined $r_{\mathtt{DP}}(Q)(h)$ as $\inf_{G \in Q} G(h)$. We shall be interested in cases where Q is in a Smyth powerdomain. We can then say more:

Lemma 3.13. The map $r_{\mathtt{DP}}$, once restricted to the Smyth powerdomain of some space of linear previsions, is defined by $r_{\mathtt{DP}}(Q)(h) = \min_{G \in Q} G(h)$.

Proof. The evaluation map $G \mapsto G(h)$ is continuous from the given space of linear previsions to $\overline{\mathbb{R}^{+}}_{\sigma}$, by the very definition of the weak topology. Such continuous maps are called lower semicontinuous real maps in the mathematical literature, and it is standard that lower semicontinuous real maps attain their infimum on every compact set. □

We wish to show that $r_{\mathtt{DP}}$, $s_{\mathtt{DP}}$ forms a retraction. We again progress through a series of lemmata. In each, $\bullet$ may be the empty subscript, '$\leqslant 1$,' or '1.'

We use a similar proof as sketched in Goubault-Larrecq (2008) to show that $r_{\mathtt{DP}} \circ s^{\bullet}_{\mathtt{DP}}$ is the identity map (for whichever superscripts $\bullet$). An additional trick allows us to dispense with an assumption of stable compactness, in Lemma 3.16 below.

Lemma 3.14. Let X be a topological space. Then, $r_{\mathtt{DP}}$ is a map from $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$ to $\mathbb{P}^{\bullet}_{\mathtt{DP}\ wk}(X)$.

Proof. Let $Q \in \mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathtt{P}\ wk}(X))$. Writing $F(h)$ for $r_{\mathtt{DP}}(Q)(h) = \min_{G \in Q} G(h)$, we must show that F is a Smyth prevision. Clearly, F is positively homogeneous, monotonic and superlinear. It is also subnormalized in case $\bullet$ is '$\leqslant 1$' and normalized when $\bullet$ is '1.' It remains to show that F is Scott-continuous. This follows from the fact that the pointwise infimum of a compact family of lower semicontinuous functions is lower semicontinuous Keimel (1984). For completeness, here is a short argument. For any directed family $(h_i)_{i \in I}$ with supremum h, we must show that $F(h) \leqslant \sup_{i \in I} F(h_i)$, since the other inequality stems from monotonicity. If that were not the case, let $b = \sup_{i \in I} F(h_i)$, so that $F(h) > b$. The open subsets $[h_i > b]$, $i \in I$, form a directed open cover of Q, since for every $G \in Q$, $\sup_{i \in I} G(h_i) = G(h) \geqslant F(h) > b$. By compactness, $Q \subseteq [h_i > b]$ for some i, whence $F(h_i) = \min_{G \in Q} G(h_i) > b$, contradiction. □

Lemma 3.15. Let X be a topological space. Then, r_{DP} is a continuous map from $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$ to $\mathbb{P}^{\bullet}_{\mathrm{DP}\ wk}(X)$.

Proof. For clarity, write $[h > b]_{\mathrm{DP}}$ or $[h > b]_{\mathrm{P}}$ for the subbasic open subset $[h > b]$, depending whether in a space of Smyth previsions or of linear previsions.

The inverse image of the subbasic open $[h > b]_{\mathrm{DP}} \subseteq \mathbb{P}^{\bullet}_{\mathrm{DP}\ wk}(X)$ by r_{DP} is $\{Q \in \mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)) \mid \min_{G \in Q} G(h) > b\} = \{Q \in \mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)) \mid \forall G \in Q \cdot G(h) > b\}$ (that this is a min, and not just an inf, is important here) $= \Box[h > b]_{\mathrm{P}}$. This is open in $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$. Therefore, r_{DP} is continuous. □

To show that $s^{\bullet}_{\mathrm{DP}}$ maps Smyth previsions to compact saturated subsets of linear previsions, we shall use the following piece of logic.

Let A be a fixed stably compact space. (This will be $\overline{\mathbb{R}^+}_\sigma$ in our case.) Let T be a set. A *patch-continuous inequality* on T, A is any formula of the form:

$$f(\_(t_1), \dots, \_(t_m)) \dot{\leqslant} g(\_(t'_1), \dots, \_(t'_n)),$$

where f and g are patch-continuous maps from A^m to A and from A^n to A respectively, and $t_1, \dots, t_m, t'_1, \dots, t'_n$ are $m + n$ fixed elements of T. E *holds* at $\alpha : T \to A$ iff $f(\alpha(t_1), \dots, \alpha(t_m)) \leqslant g(\alpha(t'_1), \dots, \alpha(t'_n))$, where $\leqslant$ is the specialization quasi-ordering of A. A *patch-continuous system* Σ on T, A is a set of patch-continuous inequalities on T, A, and Σ holds at $\alpha : T \to A$ if every element of Σ holds at α. By convention, we shall allow equations $a \doteq b$ in such systems, and agree that they stand for the pair of inequalities $a \dot{\leqslant} b$ and $b \dot{\leqslant} a$. Then, the set $[\Sigma]$ of maps $\alpha : T \to A$ such that Σ holds at α is patch-closed in A^T, and in particular stably compact. This is (Goubault-Larrecq 2010, Proposition 5.5), but also a fairly simple exercise.

Lemma 3.16. Let X be a topological space. For every $F \in \mathbb{P}^{\bullet}_{\mathrm{DP}\ wk}(X)$, $s^{\bullet}_{\mathrm{DP}}(F)$ is a compact saturated subset of $\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$.

Proof. Recall that $s^{\bullet}_{\mathrm{DP}}$ maps every $F \in \mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$ to the set of all $G \in \mathbb{P}^{\bullet}_{\mathrm{P}}(X)$ such that $G \geqslant F$.

Let $Y^{\bullet}$ be the space of all linear, monotonic maps from $C = [X \to \overline{\mathbb{R}^+}_\sigma]$ to $\overline{\mathbb{R}^+}$ that are subnormalized if $\bullet$ is '$\leqslant 1$,' and normalized if $\bullet$ is '1.' Compared to $\mathbb{P}^{\bullet}_{\mathrm{P}}(X)$, we are no longer requiring Scott-continuity. Equip $Y^{\bullet}$ with the weak topology, which is again generated by subbasic open sets $[h > b]_Y$, defined as $\{\alpha \in Y \mid \alpha(h) > b\}$. $Y^{\bullet}$ is then a subspace of the space $\overline{\mathbb{R}^+}_\sigma^C$ with its product topology, and $\mathbb{P}^{\bullet}_{\mathrm{P}}(X)$ is a subspace of $Y^{\bullet}$.

Now $Y^{\bullet}$ is $[\Sigma^{\bullet}]$, where $\Sigma^{\bullet}$ is the set of (polynomial, hence patch-continuous) inequalities:

— $\_(ah) \doteq a\ \_(h)$, for all $a \in \mathbb{R}^+$, $h \in C$ (positive homogeneity);
— $\_(h + h') \doteq \_(h) + \_(h')$ for all $h, h' \in C$ (additivity);
— $\_(h) \dot{\leqslant} \_(h')$ for all $h, h' \in C$ with $h \leqslant h'$ (monotonicity);
— if $\bullet$ is '$\leqslant 1$,' $\_(a + h) \dot{\leqslant} a + \_(h)$ for all $a \in \mathbb{R}^+$, $h \in C$;
— if $\bullet$ is '1,' $\_(a + h) \doteq a + \_(h)$ for all $a \in \mathbb{R}^+$, $h \in C$.

So $Y^{\bullet}$ is a patch-closed subset of $\overline{\mathbb{R}^+}_\sigma^C$, and a stably compact space.

Consider the set $s(F)$ of all $\alpha \in Y^{\bullet}$ such that $\alpha \geqslant F$, meaning $\alpha(h) \geqslant F(h)$ for every $h \in C$. This is again patch-closed in $\overline{\mathbb{R}^+}_\sigma^C$ (consider $\Sigma^{\bullet}$ plus the inequalities $F(h) \dot{\leqslant} \_(h)$),

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hence also in $Y^\bullet$. Note that $s(F)$ is almost $s^\bullet_{\mathrm{DP}}(F)$: the latter is the subset of those elements of $s(F)$ that are Scott-continuous maps.

At this point, the natural next move would be to show that $s^\bullet_{\mathrm{DP}}(F)$ arises as a retract of $s(F)$, and conclude that $s(F)$ is (stably) compact, using the fact that every retract of a stably compact space is stably compact. This idea was pioneered by Jung (2004), and taken again in Goubault-Larrecq (2010). But this requires X to be stably compact, and does not use the fact that F is superlinear and Scott-continuous. We use a different argument.

Note that $s(F)$ is upward closed. But the patch-closed upward closed subsets of a stably compact space are exactly its compact saturated subsets (Gierz et al. 2003, Theorem VI.6.18 (3)), so $s(F)$ is compact saturated in $Y^\bullet$.

To show that $s^\bullet_{\mathrm{DP}}(F)$ is compact saturated in $\mathbb{P}^\bullet_{\mathrm{DP}}(X)$, we shall appeal to Alexander's Subbase Lemma (Kelley 1955, Theorem 5.6), which states that in a space X with subbase $\mathcal{A}$, a subset K is compact if and only if one can extract a finite subcover from every cover of K consisting of elements of $\mathcal{A}$. In our case, assume $s^\bullet_{\mathrm{DP}}(F)$ is included in a union of open subsets $\bigcup_{i \in I}[h_i > b_i]$. We wish to show that there is a finite subset J of I such that $s^\bullet_{\mathrm{DP}}(F) \subseteq \bigcup_{i \in J}[h_i > b_i]$. We do it by contraposition: we assume that $s^\bullet_{\mathrm{DP}}(F) \not\subseteq \bigcup_{i \in J}[h_i > b_i]$ for any J, namely, we assume that for every finite subset J of I, there is a $G_J \in s^\bullet_{\mathrm{DP}}(F)$ such that $G_J(h_i) \leqslant b_i$ for every $i \in J$; and we build an element G of $s^\bullet_{\mathrm{DP}}(F)$ such that $G(h_i) \leqslant b_i$ for every $i \in I$. Note that, for every finite J, $F \leqslant G_J$, so G_J is in $s(F)$.

We claim that there is an $\alpha \in s(F)$ such that $\alpha(h_i) \leqslant b_i$ for every $i \in I$. Otherwise, every $\alpha \in s(F)$ would be in some $[h_i > b_i]_Y$, $i \in I$. Since $s(F)$ is compact in $Y^\bullet$, there would be a finite subset J of I such that $s(F) \subseteq \bigcup_{i \in J}[h_i > b_i]_Y$. In particular, $G_J \in \bigcup_{i \in J}[h_i > b_i]_Y$, contradicting the fact that $G_J(h_i) \leqslant b_i$ for every $i \in J$.

Since α is linear, it is in particular sublinear. Since $\alpha \in s(F)$, we have $F \leqslant \alpha$. So we can apply Keimel's Sandwich Theorem on the semitopological cone $C = [X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$: there is a *continuous* linear map G such that $F \leqslant G \leqslant \alpha$. (Yes, we are using Keimel's Sandwich Theorem only to buy continuity. It is crucial that the larger map, α here, need not be continuous to apply this theorem.) When $\bullet$ is '$\leqslant 1$,' $\alpha(1) \leqslant 1$ implies that G is subnormalized, and when $\bullet$ is '1,' this together with $F(1) = 1$ implies that G is normalized. So G is in $\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X)$. Also, since $F \leqslant G$, G is in $s^\bullet_{\mathrm{DP}}(F)$. Finally, $G(h_i) \leqslant b_i$ for every $i \in I$, because $G \leqslant \alpha$. □

The most difficult part now is the following lemma, which will be used in particular to show that $s^\bullet_{\mathrm{DP}}$ is well defined.

Lemma 3.17. Let X be a topological space, $F \in \mathbb{P}^\bullet_{\mathrm{DP}}(X)$, and $h_0 \in [X \to \overline{\mathbb{R}^+}_\sigma]$. Then there is a $G \in \mathbb{P}^\bullet_{\mathrm{P}\ wk}(X)$ such that $F \leqslant G$ and $F(h_0) = G(h_0)$.

Proof. If F is the constant 0 prevision, this is clear. Indeed, first, $\bullet$ cannot be '1' in this case (since we must have $F(1) = 1$ for example). Then, we can take G to be the constant 0 prevision again. So assume F is not constantly 0.

Write C for $[X \to \overline{\mathbb{R}^+}_\sigma]$. Let c be the smallest non-negative real number such that $F(a + h) \leqslant ac + F(h)$ for every $a \in \mathbb{R}^+$ and $h \in C$, if one exists, and $+\infty$ otherwise. Notice that if $\bullet$ is '$\leqslant 1$,' then $c \leqslant 1$. If $\bullet$ is '1,' then $c = 1$, as one sees by taking

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$a = 1$, $h = 0$. We also note that $c > 0$, in any case. Indeed, if $c = 0$, then we would have $F(a+h) \leqslant F(h)$, hence $F(a+h) = F(h)$ for all a, h, in particular $F(a) = 0$ for every $a \in \mathbb{R}^+$ (taking $h = 0$). Since F is Scott-continuous, for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, $F(h) = \sup_{a \in \mathbb{R}^+} F(\min(h, a)) \leqslant \sup_{a \in \mathbb{R}^+} F(a) = 0$, contradicting the fact that F is not constantly 0.

For every real $\lambda > 0$, let A_λ be the closed convex hull of the pair of points $\{\lambda/c, h_0\}$ in C, where we write λ/c for the constant function with value λ/c. By (Keimel 2008, Lemma 4.10 (a)), A_λ is the closure of the convex hull $H_\lambda = \{a\lambda/c + (1-a)h_0 \mid a \in [0,1]\}$ of $\{\lambda/c, h_0\}$. Define $p_\lambda : C \to \overline{\mathbb{R}^+}$ by $p_\lambda(h) = \lambda M_{A_\lambda}(h)$. (Notice that this makes sense only for $\lambda > 0$: this is undefined when $\lambda = 0$ if $M_{A_\lambda}(h) = +\infty$. As before, M_{A_λ} is the lower Minkowski functional, see equation (1) in the Cones subsection of Section 2.) Since A_λ is non-empty, closed and convex, p_λ is continuous and sublinear for every $\lambda > 0$. Let $p(h) = \inf_{\lambda > F(h_0)} p_\lambda(h)$. It may fail to be continuous, but we don't need p to be continuous to apply Keimel's Sandwich Theorem.

We shall see below that p is sublinear. (Beware that an infimum of sublinear maps is not in general sublinear.) Before we prove this, we notice that whenever $0 < \lambda \leqslant \mu$: (*) for every $h \in C$, for every $r > 0$ such that $(1/r) \cdot h \in H_\mu$, there is an $r' > 0$ such that $(1/r') \cdot h \in H_\lambda$ and $r'\lambda \leqslant r\mu$. Indeed, by assumption $(1/r) \cdot h = a\mu/c + (1-a)h_0$ for some $a \in [0,1]$, i.e., $h = ra\mu/c + r(1-a)h_0$. Let $r' = ra\frac{\mu}{\lambda} + r(1-a)$. We have $r'\lambda = ra\mu + r(1-a)\lambda \leqslant r\mu$. Moreover, letting $a' = \frac{ra\mu}{r'\lambda}$, we check that $1 - a' = \frac{r(1-a)}{r'}$, so that $h = r'a'\lambda/c + r'(1-a')h_0$, and in particular $(1/r') \cdot h = a'\lambda/c + (1-a')h_0$ is in H_λ. This finishes the proof of (*). In turn, (*) implies that for every $r > 0$, for every $h \in C$ such that $(1/r) \cdot h \in H_\mu$,

$$\begin{aligned} p_\lambda(h) = \lambda M_{A_\lambda}(h) &= \lambda \inf\{r' > 0 \mid (1/r') \cdot h \in A_\lambda\} \\ &\leqslant \lambda \inf\{r' > 0 \mid (1/r') \cdot h \in H_\lambda\} \quad \text{(since } H_\lambda \subseteq A_\lambda\text{)} \\ &\leqslant r\mu \quad \text{(since we can pick } r' \text{ so that } r'\lambda \leqslant r\mu\text{)}. \end{aligned}$$

So H_μ is included in the set of those maps $(1/r) \cdot h$ such that $p_\lambda(h) \leqslant r\mu$, i.e., in $r \cdot p_\lambda^{-1}([0, r\mu]) = p_\lambda^{-1}[0, \mu]$. Since the latter is closed (p_λ is continuous), A_μ is also included in this set. So, working backwards, we obtain that for every $h \in C$, for every $r > 0$ such that $(1/r) \cdot h \in A_\mu$, $p_\lambda(h) \leqslant r\mu$. Taking infima over r, $p_\lambda(h) \leqslant p_\mu(h)$. We have proved that p_λ was a monotonic function of λ.

It follows that the family $(p_\lambda)_{\lambda > F(h_0)}$ is a chain. In particular, $(p_\lambda)_{\lambda > F(h_0)}$ is filtered, i.e., directed in the opposite ordering $\geqslant$. Since addition commutes with filtered infima, $p(h) + p(h') = \inf_{\lambda > F(h_0)}[p_\lambda(h) + p_\lambda(h')] \geqslant \inf_{\lambda > F(h_0)} p_\lambda(h + h') = p(h + h')$, using the fact that p_λ is sublinear. So p is subadditive, and it follows easily that it is sublinear.

For every $\lambda > 0$, since λ/c is in A_λ, $M_{A_\lambda}(\lambda/c) \leqslant 1$, so $p_\lambda(\lambda/c) \leqslant \lambda$. Since p_λ is positively homogeneous and $\lambda > 0$, $p_\lambda(1) \leqslant c$. It follows that $p(1) \leqslant c$, a fact we shall need later.

Since h_0 is in A_λ, $M_{A_\lambda}(h_0) \leqslant 1$, so $p_\lambda(h_0) \leqslant \lambda$. Taking infs over $\lambda > F(h_0)$, it follows that $p(h_0) \leqslant F(h_0)$, another fact we shall need later.

For every $\lambda > F(h_0)$, for all $r > 0$ and $h \in C$ such that $(1/r) \cdot h \in H_\lambda$, write $(1/r) \cdot h$ as $a\lambda/c + (1-a)h_0$, $a \in [0,1]$. If $c \neq +\infty$, then $F(h) = rF(a\lambda/c + (1-a)h_0) \leqslant r(a\lambda + (1-a)F(h_0))$ (by definition of c) $\leqslant r(a\lambda + (1-a)\lambda) = \lambda r$. If $c = +\infty$, then $F(h) = rF((1-a)h_0) \leqslant$

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$rF(h_0) \leqslant \lambda r$. Taking infs over r, $F(h) \leqslant \lambda M_{A_\lambda}(h) = p_\lambda(h)$. So, taking infs over $\lambda > F(h_0)$, $F(h) \leqslant p(h)$. So $F \leqslant p$.

So we can apply Keimel's Sandwich Theorem: there is a continuous linear map G such that $F \leqslant G \leqslant p$. Also, $G(1) \leqslant p(1)$, and we have seen that $p(1) \leqslant c$, and that $c \leqslant 1$ whenever $\bullet$ is '$\leqslant 1$' or '1': so, in these cases, $G(1) \leqslant 1$, implying that G is subnormalized. If $\bullet$ is '1,' additionally, $F(1) = 1$, so $G(1) = 1$ and G is normalized. In any case, G is in $\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$.

We have also seen that $p(h_0) \leqslant F(h_0)$. Since also $F(h_0) \leqslant G(h_0) \leqslant p(h_0)$, $G(h_0) = F(h_0)$, and we are done. □

Lemma 3.18. Let X be a topological space. Then, $s^{\bullet}_{\mathrm{DP}}$ is a map from $\mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$ to $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$.

Proof. For every $F \in \mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$, $s^{\bullet}_{\mathrm{DP}}(F)$ is compact saturated by Lemma 3.16. Lemma 3.17 implies that it is non-empty. □

Lemma 3.19. Let X be a topological space. Then, $r_{\mathrm{DP}} \circ s^{\bullet}_{\mathrm{DP}}$ is the identity map on $\mathbb{P}^{\bullet}_{\mathrm{DP}\ wk}(X)$.

Proof. We must show that for every $F \in \mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$, for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, $F(h) = \min_{G \in \mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X), F \leqslant G} G(h)$. The difficult part is to show that $F(h) \geqslant \min_{G \in \mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X), F \leqslant G} G(h)$ for every h, and this is a direct consequence of Lemma 3.17, taking $h_0 = h$. □

Lemma 3.20. Let X be a topological space. Then, $s^{\bullet}_{\mathrm{DP}}$ is a continuous map from $\mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$ to $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$.

Proof. As we already mentioned in the proof of Lemma 3.9, every open subset V of $\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$ can be written $\bigcup_{i \in I} \bigcap_{j \in J_i} V_{ij}$, where each V_{ij} is of the form $[h > 1]_{\mathrm{P}}$, $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, and each J_i is finite. We can also write this as the directed union, over all finite subsets I' of I, of $\bigcup_{i \in I'} \bigcap_{j \in J_i} V_{ij}$. Now we can distribute unions over intersections, and write V as a directed union of finite intersections of finite unions of subsets of the form $[h > b]_{\mathrm{P}}$. It is easy to check that $\Box$ distributes over directed unions (using the fact that the elements of a Smyth powerdomain are compact, and that from every open cover of a compact subset K by a directed family, one can extract a single element of the family containing K), and over finite intersections. It follows that a subbase of the topology on $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$ is given by the subsets of the form $\Box W$ with $W = \bigcup_{i=1}^m [h_i > 1]_{\mathrm{P}}$, U_i open in X.

To show that $s^{\bullet}_{\mathrm{DP}}$ is continuous, we fix such an open W, and we claim that ${s^{\bullet}_{\mathrm{DP}}}^{-1}(\Box W)$ is open in the weak topology. It will be easier to show that its complement D is closed. Observe that F is in D if and only if there is a linear prevision G such that $F \leqslant G$, and for every i, $1 \leqslant i \leqslant m$, $G(h_i) \leqslant 1$.

Case 1: $\bullet$ is neither '$\leqslant 1$' nor '1.' We claim that D is the intersection of the complements of the subbasic weak opens $[\sum_{i=1}^m a_i h_i > 1]_{\mathrm{DP}}$ over all m-tuples of non-negative real numbers $a_1, \ldots, a_m$, such that $\sum_{i=1}^m a_i = 1$, which will prove the claim. Clearly D is included in this intersection: if $F \leqslant G$, G is linear, and $G(h_i) \leqslant 1$ for every i, then $F(\sum_{i=1}^m a_i h_i) \leqslant G(\sum_{i=1}^m a_i h_i) \leqslant \sum_{i=1}^m a_i = 1$. Conversely, let $F \in \mathbb{P}_{\mathrm{DP}}(X)$ be such that $F(\sum_{i=1}^m a_i h_i) \leqslant 1$ for all $a_1, \ldots, a_m \in \mathbb{R}^+$ such that $\sum_{i=1}^m a_i = 1$, and let us show that F is

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in D. Define $p(h)$ as $\inf\{\sum_{i=1}^m a_i \mid a_1,\dots,a_m \in \mathbb{R}^+, \sum_{i=1}^m a_i h_i \geqslant h\}$. This is easily seen to be a sublinear map such that $F \leqslant p$. By Keimel's Sandwich Theorem, there is a continuous linear map G between F and p. By taking $a_i = 1$ and $a_j = 0$ for all $j \neq i$, we see that $p(h_i) \leqslant 1$, whence $G(h_i) \leqslant 1$ for every i, $1 \leqslant i \leqslant m$. So F is in D. This is all we need to show that D is closed in the weak topology, hence that $s^\bullet_{\mathrm{DP}}$ is weakly continuous.

Case 2: $\bullet$ is '$\leqslant 1$' or '1.' Underlining the changes, we claim that D is the intersection of the complements of the subbasic weak opens $[\underline{a_0} + \sum_{i=1}^m a_i h_i > 1]_{\mathrm{DP}}$ over all $\underline{(m+1)}$-tuples of non-negative real numbers $\underline{a_0,}$ $a_1, \dots, a_m$ such that $\sum_{i=0}^m a_i = 1$, which will prove the claim. Clearly, D is included in this intersection: if $F \leqslant G$, G is linear <u>and subnormalized</u>, and $G(h_i) \leqslant 1$ for every i, then $F(\underline{a_0+} \sum_{i=1}^m a_i h_i) \leqslant G(\underline{a_0+} \sum_{i=1}^m a_i h_i) \underline{\leqslant a_0+} \sum_{i=1}^m a_i = 1$. Conversely, let $F \in \mathbb{P}_{\mathrm{DP}}(X)$ be such that $F(\underline{a_0+} \sum_{i=1}^m a_i h_i) \leqslant \underline{a_0+} \sum_{i=1}^m a_i$ for all $a_1,\dots,a_m \in \mathbb{R}^+$ such that $\sum_{i=0}^m a_i = 1$, and let us show that F is in D. Define $p(h)$ as $\inf\{\underline{a_0+} \sum_{i=1}^m a_i \mid \underline{a_0,} a_1,\dots,a_m \in \mathbb{R}^+, \underline{a_0+} \sum_{i=1}^m a_i h_i \geqslant h\}$. This is a sublinear map such that $F \leqslant p$. By Keimel's Sandwich Theorem, there is a continuous linear map G between F and p. <u>Additionally, since $G(1) \leqslant p(1)$ and $p(1) \leqslant 1$ (take $a_0 = 1$, $a_i = 0$ for $i \neq 0$), G is subnormalized; if in addition $\bullet$ is '1,' then $F(1) = 1$, so $F(1) \leqslant G(1)$, so G is normalized. In any case, G is in $\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X)$.</u> By taking $a_i = 1$ and $a_j = 0$ for all $j \neq i$, we see that $p(h_i) \leqslant 1$, whence $G(h_i) \leqslant 1$ for every i, $1 \leqslant i \leqslant m$. So F is in D. This is all we need to show that D is closed in the weak topology, hence that $s^\bullet_{\mathrm{DP}}$ is continuous. □

Lemma 3.21. Let X be a topological space. For every $Q \in \mathcal{Q}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X))$, $Q \subseteq s^\bullet_{\mathrm{DP}}(r^\bullet_{\mathrm{DP}}(Q))$.

Proof. For every $G \in Q$, $G(h) \geqslant \inf_{G' \in Q} G'(h)$. □

We sum up the above results as follows. More than a retraction, we now have a projection, by Lemma 3.21: remember that the ordering on Smyth powerdomains is reverse inclusion $\supseteq$, so $s^\bullet_{\mathrm{DP}} \circ r^\bullet_{\mathrm{DP}}$ is below the identity.

Proposition 3.22 ($\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X))$ projects onto $\mathbb{P}^\bullet_{\mathrm{DP}\ wk}(X)$). Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a topological space. Then, r_{DP} defines a projection of $\mathcal{Q}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X))$ onto $\mathbb{P}^\bullet_{\mathrm{DP}\ wk}(X)$, with associated embedding $s^\bullet_{\mathrm{DP}}$.

3.3. *The retraction in the erratic case*

In the erratic cases, recall that a fork (F^-, F^+) is a pair of a Smyth prevision F^- and of a Hoare prevision F^+ satisfying Walley's condition.

Definition 3.23. Let X be a topological space. For every non-empty set E of linear previsions on X, let $r_{\mathrm{ADP}}(E) : ([X \to \overline{\mathbb{R}^+}_\sigma] \to \overline{\mathbb{R}^+})^2$ be $(r_{\mathrm{DP}}(E), r_{\mathrm{AP}}(E))$.

Conversely, for every (subnormalized, normalized) fork (F^-, F^+) on X, let $s_{\mathrm{ADP}}(F^-, F^+)$ (resp., $s^{\leqslant 1}_{\mathrm{ADP}}(F^-, F^+)$, $s^1_{\mathrm{ADP}}(F^-, F^+)$) be the set of all (subnormalized, normalized) linear previsions G such that $F^- \leqslant G \leqslant F^+$.

So $s^\bullet_{\mathrm{ADP}}(F^-, F^+) = s^\bullet_{\mathrm{DP}}(F^-) \cap s^\bullet_{\mathrm{AP}}(F^+)$, whatever the superscript $\bullet$. We shall now prove that r_{ADP}, $s^\bullet_{\mathrm{ADP}}$ form a retraction, once again.

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This will require not only that $C = [X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ be locally convex, as in the Hoare cases, but also that that addition on C be almost open. Following Keimel (2008, Definition 4.6), we say that addition is *almost open* on a semitopological cone if and only if $\uparrow(U+V)$ is open for every pair of open subsets U, V of C.

For every monotonic map q from C to $\overline{\mathbb{R}^+}$, there is a (pointwise) largest continuous map $\check{q}$ less than or equal to q: $\check{q}(h)$ is the least upper bound of all real numbers r such that h is in the interior of $q^{-1}(r, +\infty]$. When C is almost open, and q is superlinear, $\check{q}$ is again superlinear (Keimel 2008, Lemma 5.7). This will be our main new ingredient.

There is a canonical situation in which all the above assumptions are satisfied:

Lemma 3.24. Let X be a core-compact, core-coherent space (for example, a stably locally compact space). Then, $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is a locally convex topological cone in which addition is almost open.

Proof. Since X is core-compact, we already know that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is a locally convex topological cone. $[X \to \overline{\mathbb{R}^+}_\sigma]$ is a continuous d-cone, too, and in that case, the fact that addition is almost open is equivalent to the property that the way-below relation $\ll$ on C is *additive*, namely, $f \ll f'$ and $g \ll g'$ together imply $f + g \ll f' + g'$ (Keimel 2008, Lemma 6.14).

The space $Y = \mathcal{S}(X)$ is stably locally compact, and Proposition 2.28 of Tix et al. (2009) then states that $\ll$ is additive on $[Y \to \overline{\mathbb{R}^+}_\sigma]$. By Lemma 2.1, $\ll$ is also additive on the homeomorphic space $[X \to \overline{\mathbb{R}^+}_\sigma]$, which allows us to conclude. (Note that core-coherence is required here. Using the same sobrification trick, Proposition 2.29 of Tix et al. (2009) says that if X is core-compact, and $\ll$ is additive on $[X \to \overline{\mathbb{R}^+}_\sigma]$, then X must in fact be core-coherent.) □

Lemma 3.25. Let X be a compact space. For every $a \in [0, 1)$, $a\chi_X \ll \chi_X$ in $[X \to \overline{\mathbb{R}^+}_\sigma]$, hence $\Uparrow a\chi_X$ is an open neighbourhood of χ_X in $[X \to \overline{\mathbb{R}^+}_\sigma]$.

Proof. Let $(f_i)_{i \in I}$ be a directed family of continuous maps such that $\sup_{i\in I} f_i \geqslant \chi_X$. So $(\sup_{i\in I} f_i)^{-1}(a, +\infty] = \bigcup_{i \in I} f_i^{-1}(a, +\infty]$ contains the compact set X. Since the union is directed, $X \subseteq f_i^{-1}(a, +\infty]$ for some $i \in I$, proving the claim. □

Lemma 3.26. Let X be a topological space. Then, r_{ADP} is a continuous map from $\mathcal{P}\ell_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathsf{P}\ wk}(X))$ to $\mathbb{P}^\bullet_{\mathsf{ADP}\ wk}(X)$.

Proof. We check that $r_{\mathsf{ADP}}(L)$ satisfies Walley's condition for every lens L. Let $(F^-, F^+) = r_{\mathsf{ADP}}(L)$. Then, $F^-(h + h') = \inf_{G\in L} G(h + h') = \inf_{G\in L}(G(h) + G(h')) \leqslant \inf_{G\in L}(G(h) + \sup_{G\in L} G(h')) = \inf_{G\in L} G(h) + \sup_{G\in L} G(h')$ (since L is non-empty) $= F^-(h) + F^+(h')$. We prove $F^-(h) + F^+(h') \leqslant F^+(h + h')$ similarly. That $r_{\mathsf{ADP}}(L)$ is then a fork, and that r_{ADP} is continuous, then follows from Lemmas 3.3 and 3.15. □

Lemma 3.27. Let X be a topological space. Then, $s^\bullet_{\mathsf{ADP}}$ is a map from $\mathbb{P}^\bullet_{\mathsf{ADP}}(X)$ to $\mathcal{P}\ell_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathsf{P}\ wk}(X))$.

Proof. We check that $s^\bullet_{\mathsf{ADP}}(F^-, F^+)$ is non-empty for every (subnormalized, normalized) fork (F^-, F^+), i.e., that there is a (subnormalized, normalized) linear prevision G such that $F^- \leqslant G \leqslant F^+$: this is a trivial consequence of Keimel's Sandwich Theorem on the semitopological cone $C = [X \to \overline{\mathbb{R}^+}_\sigma]$. Then, $s^\bullet_{\mathtt{ADP}}(F^-, F^+) = s^\bullet_{\mathtt{DP}}(F^-) \cap s^\bullet_{\mathtt{AP}}(F^+)$ is a lens, by Lemmas 3.6 and 3.16. □

Before we continue, we establish two consequences of Walley's condition. Those are akin to the Main Lemma (Lemma 5.1) of Keimel and Plotkin (2009).

Lemma 3.28. Let X be a topological space, and $(F^-, F^+) \in \mathbb{P}^\bullet_{\mathtt{ADP}}(X)$. For every $G' \in \mathbb{P}^\bullet_{\mathtt{P}}(X)$ such that $F^- \leqslant G'$, there is a $G \in \mathbb{P}^\bullet_{\mathtt{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ and $G \leqslant G'$.

Proof. Write C for $[X \to \overline{\mathbb{R}^+}_\sigma]$. Let $p(h) = \inf_{\substack{h_1,h_2 \in C \\ h \leqslant h_1+h_2}} (F^+(h_1)+G'(h_2))$. This is a sublinear map, notably, subadditivity is proved as follows:

$$\begin{aligned} p(h) + p(h') &= \inf_{\substack{h_1,h_2,h'_1,h'_2 \in C \\ h \leqslant h_1+h_2 \\ h' \leqslant h'_1+h'_2}} (F^+(h_1) + F^+(h'_1) + G'(h_2) + G'(h'_2)) \\ &\geqslant \inf_{\substack{h_1,h_2,h'_1,h'_2 \in C \\ h \leqslant h_1+h_2 \\ h' \leqslant h'_1+h'_2}} (F^+(h_1 + h'_1) + G'(h_2 + h'_2)) \\ &\geqslant \inf_{\substack{h''_1,h''_2 \in C \\ h+h' \leqslant h''+h''_2}} (F^+(h''_1) + G'(h''_2)) = p(h + h'). \end{aligned}$$

Note also that $p \leqslant F^+$ (take $h_1 = h$, $h_2 = 0$) and $p \leqslant G'$ (take $h_1 = 0$, $h_2 = h$). We check that $F^- \leqslant p$, i.e., that for all $h_1, h_2 \in C$ such that $h \leqslant h_1 + h_2$, $F^-(h) \leqslant F^+(h_1) + G'(h_2)$. This is by Walley's condition, since $F^-(h) \leqslant F^-(h_1 + h_2) \leqslant F^+(h_1) + F^-(h_2)$, and $F^- \leqslant G'$. By Keimel's Sandwich Theorem, there is a continuous linear map G such that $F^- \leqslant G \leqslant p$. When $\bullet$ is '$\leqslant 1$,' $G(1) \leqslant p(1) \leqslant F^+(1) \leqslant 1$, so G is subnormalized, and when $\bullet$ is '1,' $p(1) \geqslant F^-(1) = 1$, so G is normalized. In any case, G is in $\mathbb{P}^\bullet_{\mathtt{P}}(X)$. Moreover, $F^- \leqslant G \leqslant p \leqslant F^+$, and $G \leqslant p \leqslant G'$. □

Lemma 3.29. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex and has an almost open addition map. Assume that X is also compact in case $\bullet$ is '1.' Let $(F^-, F^+) \in \mathbb{P}^\bullet_{\mathtt{ADP}}(X)$. For every $G' \in \mathbb{P}^\bullet_{\mathtt{P}}(X)$ such that $G' \leqslant F^+$, there is a $G \in \mathbb{P}^\bullet_{\mathtt{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ and $G' \leqslant G$.

Proof. Write again C for $[X \to \overline{\mathbb{R}^+}_\sigma]$. Let $q_1(h)$ be defined as $\sup_{\substack{h_1,h_2 \in C \\ h_1+h_2 \leqslant h}} (F^-(h_1)+G'(h_2))$. By an argument similar to the one we used in the proof of Lemma 3.28 on p, q_1 is superlinear, and $F^- \leqslant q_1$, $G' \leqslant q_1$. Moreover, we check that $q_1 \leqslant F^+$, i.e., that for all $h_1, h_2 \in C$ such that $h_1 + h_2 \leqslant h$, $F^-(h_1) + G'(h_2) \leqslant F^+(h)$: this is by Walley's condition again, since $F^+(h) \geqslant F^+(h_1 + h_2) \geqslant F^-(h_1) + F^+(h_2) \geqslant F^-(h_1) + G'(h_2)$.

However, to apply Keimel's Sandwich Theorem to q_1 and F^+, we would need q_1 to be continuous. Instead, consider $\check{q}_1$. We have already noted that, since addition on $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is almost open, $\check{q}_1$ is not only continuous, but also superlinear (Keimel 2008, Lemma 5.7). Since $\check{q}_1$ is the largest Scott-continuous map below q_1, and F^-, G' are Scott-continuous and below q_1, we also have $F^- \leqslant \check{q}_1$, $G' \leqslant \check{q}_1$. Since $\check{q}_1 \leqslant q_1 \leqslant F^+$, we

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can now apply Keimel's Sandwich Theorem, and obtain a linear prevision G such that $\check{q}_1 \leqslant G \leqslant F^+$.

If $\bullet$ is '$\leqslant 1$,' then $G(1) \leqslant F^+(1) \leqslant 1$, so G is subnormalized. If $\bullet$ is '1,' remember that we have assumed that X was also compact. By Lemma 3.25, $\Uparrow a\chi_X$ is an open neighbourhood of χ_X for every $a \in [0,1)$. We check that is is included in $q_1^{-1}(r,+\infty]$ for every $r < a$: for every $h \in \Uparrow a\chi_X$, $q_1(h) \geqslant q_1(a\chi_X) \geqslant F^-(a\chi_X) = a > r$. In particular, for every $r \in [0,1)$, we have just shown that χ_X is in the interior of $q_1^{-1}(r,+\infty]$ (pick any $a \in (r,1)$). Since $\check{q}_1(1) = \check{q}_1(\chi_X)$ is the least upper bound of all real numbers r with that property, $\check{q}_1(1) \geqslant 1$. Since $G(1) \geqslant \check{q}_1(1)$, G is normalized in the '1' case. To sum up, whatever $\bullet$ is, G is in $\mathbb{P}^\bullet_{\mathrm{P}}(X)$.

Finally, $F^- \leqslant \check{q}_1 \leqslant G \leqslant F^+$, and $G' \leqslant \check{q}_1 \leqslant G$. $\square$

Lemma 3.30. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex and has an almost open addition map. Assume that X is also compact in case $\bullet$ is '1.' Then, $s^\bullet_{\mathrm{ADP}}$ is a continuous map from $\mathbb{P}^\bullet_{\mathrm{ADP}\ wk}(X)$ to $\mathcal{P}\ell_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X))$.

Proof. For every open subset V of $\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X)$, ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Box V)$ is the set of (subnormalized, normalized) forks (F^-,F^+) such that every $G \in \mathbb{P}^\bullet_{\mathrm{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ is in V. We claim that this is exactly $\pi_1^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Box V))$, where π_1 is first projection. The inclusion $\pi_1^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Box V)) \subseteq {s^\bullet_{\mathrm{ADP}}}^{-1}(\Box V)$ is trivial. Conversely, for every element (F^-,F^+) of ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Box V)$, we must show that every $G' \in \mathbb{P}^\bullet_{\mathrm{P}}(X)$ such that $F^- \leqslant G'$ is in V. By Lemma 3.28, one can find $G \in \mathbb{P}^\bullet_{\mathrm{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ and $G \leqslant G'$. In particular, since (F^-,F^+) is in ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Box V)$, G is in V. Since V is upward closed, G' is also in V.

For every open subset V of $\mathbb{P}^\bullet_{\mathrm{P}\ wk}(X)$, ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Diamond V)$ is the set of (subnormalized, normalized) forks (F^-,F^+) such that there is a $G \in \mathbb{P}^\bullet_{\mathrm{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ that is also in V. We claim that this is exactly $\pi_2^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Diamond V))$, where π_2 is second projection. The inclusion ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Diamond V) \subseteq \pi_2^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Diamond V))$ is trivial. Conversely, for every element (F^-,F^+) of $\pi_2^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Diamond V))$, i.e., such that there is a $G' \in V$ with $G' \leqslant F^+$, we must show that there is a $G \in V$ such that $F^- \leqslant G \leqslant F^+$. We use Lemma 3.29 to this end: this yields such a G, since V is upward closed.

To sum up, we have shown that both ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Box V) = \pi_1^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Box V))$ and ${s^\bullet_{\mathrm{ADP}}}^{-1}(\Diamond V) = \pi_2^{-1}({s^\bullet_{\mathrm{DP}}}^{-1}(\Diamond V))$ are open, so $s^\bullet_{\mathrm{ADP}}$ is continuous. $\square$

Lemma 3.31. Let X be topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex and has an almost open addition map. Assume that X is also compact in case $\bullet$ is '1.' Then, $r_{\mathrm{ADP}} \circ s^\bullet_{\mathrm{ADP}}$ is the identity map on $\mathbb{P}^\bullet_{\mathrm{ADP}\ wk}(X)$.

Proof. For every (subnormalized, normalized) fork (F^-,F^+), let $L = s^\bullet_{\mathrm{ADP}}(F^-,F^+)$. One may write L as $Q \cap F$, where $Q = s^\bullet_{\mathrm{DP}}(F^-)$, and $F = s^\bullet_{\mathrm{AP}}(F^+)$, and by Propositions 3.22 and 3.11, $r_{\mathrm{DP}}(Q) = F^-$, $r_{\mathrm{AP}}(F) = F^+$. To show that $r_{\mathrm{ADP}}(s^\bullet_{\mathrm{ADP}}(F^-,F^+)) = (F^-,F^+)$, we must show that $r_{\mathrm{DP}}(L) = F^-$ and $r_{\mathrm{AP}}(L) = F^+$.

Since $L \subseteq Q$, using the definition of r_{DP} (Definition 3.1), $r_{\mathrm{DP}}(L) \geqslant F^-$. Since $r_{\mathrm{DP}}(Q) = F^-$ (and using Lemma 3.13), for every $h \in C$, there is a $G' \in Q$ such that $G'(h) = F^-(h)$. By definition of Q, $F^- \leqslant G'$. By Lemma 3.28, there is a $G \in \mathbb{P}^\bullet_{\mathrm{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ (i.e., such that $G \in L$) and $G \leqslant G'$. So $r_{\mathrm{DP}}(L)(h) \leqslant G(h) \leqslant G'(h) = F^-(h)$. It follows that $r_{\mathrm{DP}}(L) \leqslant F^-$, hence $r_{\mathrm{DP}}(L) = F^-$.

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Since $L \subseteq F$, using the definition of $r_{\mathtt{AP}}$, $r_{\mathtt{AP}}(L) \leqslant F^+$. Since $r_{\mathtt{AP}}(F) = F^+$, for every $h \in C$, and for every real number $r < F^+(h)$, there is a $G' \in F$ such that $G'(h) \geqslant r$. By definition of F, $G' \leqslant F^+$. By Lemma 3.29, there is a $G \in \mathbb{P}^\bullet_{\mathtt{P}}(X)$ such that $F^- \leqslant G \leqslant F^+$ (i.e., such that $G \in L$) and $G' \leqslant G$. So $r_{\mathtt{AP}}(L)(h) \geqslant G(h) \geqslant G'(h) \geqslant r$. Taking sups over r, $r_{\mathtt{AP}}(L)(h) \geqslant F^+(h)$. So $r_{\mathtt{AP}}(L) \geqslant F^+$, whence $r_{\mathtt{AP}}(L) = F^+$, and we are done. □

We sum up these results as follows.

Proposition 3.32 ($\mathcal{P}\ell_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$ retracts onto $\mathbb{P}^\bullet_{\mathtt{ADP}\ wk}(X)$). Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex and has an almost open addition map, for example a stably locally compact space, or more generally, a core-compact, core-coherent space. Assume also that X is compact in case $\bullet$ is '1.'

Then, $r_{\mathtt{ADP}}$ defines a retraction of $\mathcal{P}\ell_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$ onto $\mathbb{P}^\bullet_{\mathtt{ADP}\ wk}(X)$, with associated section $s^\bullet_{\mathtt{ADP}}$.

3.4. *Domain-theoretic consequences*

The focus in domain theory is on Scott topologies rather than on weak topologies. But Scott-continuity quickly follows provided we use the following result (Goubault-Larrecq 2012b, Lemma 3.8): for every quasi-monotone convergence space Z, and every T_0 topological space Z', every continuous map from Z to Z' is continuous from Z_σ to Z'_σ. Here, Z_σ is Z with the Scott topology of its specialization preorder, and a quasi-monotone convergence space is a space where this topology is finer than the original one on Z. The specialization ordering of any space of previsions with the weak topology is the pointwise ordering $\leqslant$, and each of these spaces is quasi-monotone convergence, because $[h > r]$ is Scott open. So, given any space of previsions Z, Z_σ is just the same space with the Scott topology of its ordering. Similarly for spaces of forks. Also, $\mathcal{Q}_{\mathcal{V}}(Y)_\sigma = \mathcal{Q}(Y)$ and $\mathcal{H}_{\mathcal{V}}(Y)_\sigma = \mathcal{H}(Y)$. To stress that we are using Scott-continuity, call a *Scott retraction* any retraction between posets equipped with their Scott topologies.

Applying this reasoning to Proposition 3.11, we obtain the following:

Proposition 3.33. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a core-compact space. Then, $r_{\mathtt{AP}}$ defines a Scott retraction of $\mathcal{H}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$ onto $\mathbb{P}^\bullet_{\mathtt{AP}}(X)$, with associated section $s^\bullet_{\mathtt{AP}}$. Moreover, $r_{\mathtt{AP}} \circ s_{\mathtt{AP}} \geqslant \mathrm{id}$.

Applying this reasoning to Proposition 3.22, we obtain:

Proposition 3.34. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a topological space. Then $r_{\mathtt{DP}}$ defines a Scott retraction of $\mathcal{Q}(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$ onto $\mathbb{P}^\bullet_{\mathtt{DP}}(X)$, with associated section $s^\bullet_{\mathtt{DP}}$. Moreover $r_{\mathtt{DP}} \circ s_{\mathtt{DP}} \leqslant \mathrm{id}$.

Applying it to Proposition 3.32, finally, we obtain the following:

Proposition 3.35. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a core-compact, core-coherent space (and compact if $\bullet$ is '1"). Then, $r_{\mathtt{ADP}}$ defines a Scott retraction of $\mathcal{P}\ell(\mathbb{P}^\bullet_{\mathtt{P}\ wk}(X))$ onto $\mathbb{P}^\bullet_{\mathtt{ADP}}(X)$, with associated section $s^\bullet_{\mathtt{ADP}}$.

There is still a bit of the weak topology lying around in the latter three propositions, in the various spaces $\mathbb{P}^{\bullet}_{\mathrm{P}\,wk}(X)$ of linear previsions involved.

The Scott topology is finer than the weak topology on any space of previsions. Up to the canonical isomorphism between $\mathbb{P}^{\bullet}_{\mathrm{P}}(X)$ and $\mathbf{V}^{\bullet}(X)$ (for whichever superscript $\bullet$), the *Kirch–Tix Theorem* states that, whenever X is a continuous dcpo, the Scott and weak topologies coincide on $\mathbb{P}_{\mathrm{P}}(X)$ (Tix 1995, Satz 4.10), and on $\mathbb{P}^{\leqslant 1}_{\mathrm{P}}(X)$ (Kirch 1993, Satz 8.6).

Given a poset X, let $X_\perp$ be X plus a fresh bottom element $\perp$, below all points of X. By a trick due to Edalat (Edalat 1995, Section 3), the spaces $\mathbf{V}^1(X_\perp)$ of all normalized continuous valuations on $X_\perp$ and $\mathbf{V}^{\leqslant 1}(X)$ of all subnormalized continuous valuations on X are order-isomorphic, and also homeomorphic in their weak topologies. (For $\nu \in \mathbf{V}^1(X_\perp)$, define a subnormalized valuation on X by considering ν restricted to the opens contained in X. Conversely, for $\nu \in \mathbf{V}^{\leqslant 1}(X)$, define $\nu' \in \mathbf{V}^1(X_\perp)$ by $\nu'(U) = \nu(U)$ if U is open in X, and $\nu'(X_\perp) = 1$.) It follows that, on pointed continuous dcpos, that is, on continuous dcpos that we can write as $X_\perp$ for some, necessarily continuous, dcpo X, the Scott and weak topologies also coincide on $\mathbb{P}^1_{\mathrm{P}}(X)$.

Every continuous dcpo is locally compact, and every pointed poset is compact in its Scott topology. We therefore obtain the following purely domain-theoretic statements (all spaces come with their Scott topologies) from Proposition 3.33, Proposition 3.34 and Proposition 3.35 respectively.

Proposition 3.36. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a continuous dcpo (and pointed if $\bullet$ is '1'). Then, r_{AP} defines a Scott retraction (even a coembedding-coprojection pair) of $\mathcal{H}(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$ onto $\mathbb{P}^{\bullet}_{\mathrm{AP}}(X)$, with associated section $s^{\bullet}_{\mathrm{AP}}$.

Proposition 3.37. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a continuous dcpo. Then, r_{DP} defines a Scott retraction (even an embedding-projection pair) of $\mathcal{Q}(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$ onto $\mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$, with associated section $s^{\bullet}_{\mathrm{DP}}$.

Proposition 3.38. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a coherent, continuous dcpo (and pointed if $\bullet$ is '1'). Then, r_{ADP} defines a Scott retraction of $\mathcal{P}\ell(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$ onto $\mathbb{P}^{\bullet}_{\mathrm{ADP}}(X)$, with associated section $s^{\bullet}_{\mathrm{ADP}}$.

Let us draw a few domain-theoretic consequences of the above results. All of these will state that, under suitable conditions, spaces of Smyth, resp. Hoare previsions, and of forks are continuous dcpos, with natural bases; and that the Scott and the weak topologies will coincide on such spaces.

When X is a continuous dcpo, $\mathbf{V}(X)$ is a continuous dcpo, with a basis of *simple valuations*, i.e., valuations of the form $\sum_{i=1}^n a_i\delta_{x_i}$, where $a_i \in \mathbb{R}^+$, $x_i \in X$ (Gierz et al. 2003, Theorem IV.9.16). This is an extension of Jones' original theorem, that $\mathbf{V}^{\leqslant 1}(X)$ is a continuous dcpo, with a basis of subnormalized simple valuations (Jones 1990, Chapter 5). Using Edalat's trick, we obtain a similar result for $\mathbf{V}^1(X)$ and normalized simple valuations, provided X is also pointed. Notice that the isomorphism with linear previsions yields bases of $\mathbb{P}_{\mathrm{P}}(X)$ (resp., $\mathbb{P}^{\leqslant 1}_{\mathrm{P}}(X)$, $\mathbb{P}^1_{\mathrm{P}}(X)$) consisting of *simple linear previsions* (resp., subnormalized, normalized) of the form $h \mapsto \sum_{i=1}^n a_i h(x_i)$.

In turn, if Y is a continuous dcpo, with basis B, then $\mathcal{H}(Y)$ is a continuous dcpo, too, and a basis is given by the subsets of the form $\downarrow E$, where E is a finite non-empty subset of

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B. (See Gierz et al. (2003, Corollary IV.8.7), which does not mention the basis explicitly, or Abramsky and Jung (1994, Theorem 6.2.10, Item 1), which does, but in less explicit a form; in the latter case, one should also note that our version of the Hoare powerdomain coincides with theirs, by their own Theorem 6.2.13.) Now, for any retraction $r : D \to E$, where D is a continuous dcpo, E is also a continuous dcpo, and a basis of E is given by the image under r of a basis of D (Abramsky and Jung 1994, Lemma 3.1.3). Using Proposition 3.36, we obtain the following:

Proposition 3.39. Let X be a continuous dcpo. Then, $\mathbb{P}_{\mathtt{AP}}(X)$ (resp., $\mathbb{P}^{\leqslant 1}_{\mathtt{AP}}(X)$) is a continuous dcpo, with basis given by the finite non-empty sups of simple (resp., and subnormalized) linear previsions:

$$h \mapsto \max_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} h(x_{ij}),$$

where $m \geqslant 1$ (resp., and $\sum_{j=1}^{n_i} a_{ij} \leqslant 1$ for every i).

If X is a pointed continuous dcpo, then $\mathbb{P}^{1}_{\mathtt{AP}}(X)$ is a pointed continuous dcpo, with basis given by the finite sups of simple normalized previsions (i.e., $\sum_{j=1}^{n_i} a_{ij} = 1$ for every i). The least element is $h \mapsto h(\bot)$, where $\bot$ is the least element of X.

Recall that, when Y is a continuous dcpo, then the Scott and the lower Vietoris topologies coincide on $\mathcal{H}(Y)$ (Schalk 1993, Section 6.3.3). In particular, under the assumptions of Proposition 3.39, $\mathcal{H}(\mathbb{P}_{\mathtt{P}}(X)) = \mathcal{H}_{\mathcal{V}}(\mathbb{P}_{\mathtt{P}}(X)) = \mathcal{H}_{\mathcal{V}}(\mathbb{P}_{\mathtt{P}\ wk}(X))$ (by the Kirch–Tix Theorem), and similarly in the subnormalized and normalized cases.

Now, every section is a topological embedding. Under the same assumptions as above, Propositions 3.11 and 3.36 imply that $\mathbb{P}_{\mathtt{AP}\ wk}(X)$ and $\mathbb{P}_{\mathtt{AP}}(X)$ both embed into the same space $\mathcal{H}(\mathbb{P}_{\mathtt{P}}(X)) = \mathcal{H}_{\mathcal{V}}(\mathbb{P}_{\mathtt{P}\ wk}(X))$. Hence, they have the same topology:

Proposition 3.40. Let X be a continuous dcpo. The Scott topology coincides with the weak topology on $\mathbb{P}_{\mathtt{AP}}(X)$, on $\mathbb{P}^{\leqslant 1}_{\mathtt{AP}}(X)$; also on $\mathbb{P}^{1}_{\mathtt{AP}}(X)$ if X is additionally assumed to be pointed.

Similarly, if Y is a continuous dcpo with basis B, then $\mathcal{Q}(Y)$ is a continuous dcpo, with basis given by the subsets of the form $\uparrow E$, E a finite and non-empty subset of B (Abramsky and Jung 1994, Theorem 6.2.10, Item 2) (and our Smyth powerdomain is the same as theirs, by their Theorem 6.2.14).

Proposition 3.41. Let X be a continuous dcpo. Then, $\mathbb{P}_{\mathtt{DP}}(X)$ (resp., $\mathbb{P}^{\leqslant 1}_{\mathtt{DP}}(X)$) is a continuous dcpo, with basis given by the finite non-empty infs of simple (resp., and subnormalized) linear previsions:

$$h \mapsto \min_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} h(x_{ij}),$$

where $m \geqslant 1$ (resp., and $\sum_{j=1}^{n_i} a_{ij} \leqslant 1$ for every i).

If X is a pointed continuous dcpo, then $\mathbb{P}^{1}_{\mathtt{DP}}(X)$ is a pointed continuous dcpo, with basis given by the finite mins of simple normalized previsions (i.e., $\sum_{j=1}^{n_i} a_{ij} = 1$ for every i). The least element is $h \mapsto h(\bot)$, where $\bot$ is the least element of X.

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The Scott and the upper Vietoris topologies coincide on $\mathcal{Q}(Y)$ on every T_0, well-filtered, locally compact space (Schalk 1993, Section 7.3.4), in particular on every continuous dcpo Y. By the same argument as for Proposition 3.40:

Proposition 3.42. Let X be a continuous dcpo. The Scott topology coincides with the weak topology on $\mathbb{P}_{\mathtt{DP}}(X)$, on $\mathbb{P}^{\leqslant 1}_{\mathtt{DP}}(X)$; also on $\mathbb{P}^{1}_{\mathtt{DP}}(X)$ is X is additionally assumed to be pointed.

When Y is a continuous, coherent dcpo with basis B, then $\mathcal{P}\ell(Y)$ is again a continuous dcpo, with basis given by the subsets of the form $\uparrow E \cap \downarrow E$, E a finite and non-empty subset of B. This is Theorem 6.2.3 of Abramsky and Jung (1994), together with Theorem 6.2.22, which states that our Plotkin powerdomain is the same as theirs.

Proposition 3.43. Let X be a continuous, coherent dcpo. Then $\mathbb{P}_{\mathtt{ADP}}(X)$ (resp., $\mathbb{P}^{\leqslant 1}_{\mathtt{ADP}}(X)$) is a continuous dcpo, with basis given by the simple (resp., and subnormalized) forks of the form:

$$\left(h \mapsto \min_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} h(x_{ij}), h \mapsto \max_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} h(x_{ij}) \right),$$

where $m \geqslant 1$ (resp., and $\sum_{j=1}^{n_i} a_{ij} \leqslant 1$ for every i).

If X is a pointed continuous, coherent dcpo, then $\mathbb{P}^{1}_{\mathtt{ADP}}(X)$ is a pointed continuous dcpo, with basis the simple normalized forks (i.e., $\sum_{j=1}^{n_i} a_{ij} = 1$ for every i). The least element is $(h \mapsto h(\bot), h \mapsto h(\bot))$, where $\bot$ is the least element of X.

We conclude this section with a result similar to Propositions 3.40 and 3.42. This time, we use the fact that the way-below relation $\ll_{\mathcal{P}\ell}$ on $\mathcal{P}\ell(Y)$, when Y is a continuous, coherent dcpo, if given by $L \ll_{\mathcal{P}\ell} L'$ iff $\uparrow L \ll_{\mathcal{Q}} \uparrow L'$ and $cl(L) \ll_{\mathcal{H}} cl(L')$ (Abramsky and Jung 1994, Section 6.2.1); so our subbasic Scott open subsets are $\{L' \in \mathcal{P}\ell(Y) \mid L' \subseteq \Uparrow E \text{ and } L' \cap \Downarrow E' \neq \emptyset\} = \Box \Uparrow E \cap \bigcap_{y \in E'} \Diamond \Downarrow y$, where E and E' are finite and non-empty. Since they are all open in the Vietoris topology, the Scott and Vietoris topologies coincide on $\mathcal{P}\ell(Y)$. As before, we use the fact that our spaces of forks with the weak and the Scott topologies are subspaces of the same space to conclude:

Proposition 3.44. Let X be a continuous, coherent dcpo. The Scott topology coincides with the weak topology on $\mathbb{P}_{\mathtt{ADP}}(X)$, on $\mathbb{P}^{\leqslant 1}_{\mathtt{ADP}}(X)$; also on $\mathbb{P}^{1}_{\mathtt{ADP}}(X)$ is X is additionally assumed to be pointed.

4. The isomorphisms

Let $\bullet$ be the empty superscript, or '$\leqslant 1$,' or '1,' depending on the case. When $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex, we know that $r_{\mathtt{AP}} \circ s^\bullet_{\mathtt{AP}}$ is the identity map, where $r_{\mathtt{AP}} : \mathcal{H}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\,wk}(X)) \to \mathbb{P}^\bullet_{\mathtt{AP}\,wk}(X)$ and $s^\bullet_{\mathtt{AP}} : \mathbb{P}^\bullet_{\mathtt{AP}\,wk}(X) \to \mathcal{H}_{\mathcal{V}}(\mathbb{P}^\bullet_{\mathtt{P}\,wk}(X))$ (Corollary 3.12).

Now write $\mathcal{H}^{cvx}_{\mathcal{V}}(D)$ for the subspace of $\mathcal{H}_{\mathcal{V}}(D)$ consisting of *convex* closed, non-empty, subsets of $D = \mathbb{P}^\bullet_{\mathtt{AP}\,wk}(X)$, and similarly $\mathcal{H}^{cvx}(D)$ is the underlying poset, with the inclusion ordering. Note that although $\mathbb{P}^\bullet_{\mathtt{AP}\,wk}(X)$ is not a cone (when $\bullet$ is '$\leqslant 1$' or '1'), convexity makes sense. We also write $\mathcal{Q}^{cvx}_{\mathcal{V}}(D)$, $\mathcal{Q}^{cvx}(D)$, $\mathcal{P}\ell^{cvx}(D)$, $\mathcal{P}\ell^{cvx}_{\mathcal{V}}(D)$ with the obvious meaning.

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Clearly, for every $F \in \mathbb{P}^{\bullet}_{\mathrm{AP}\,wk}(X)$, $s^{\bullet}_{\mathrm{AP}}(F) = \{G \in D \mid G \leqslant F\}$ is convex, and similarly for $s^{\bullet}_{\mathrm{DP}}(F) = \{G \in D \mid F \leqslant G\}$ and $s^{\bullet}_{\mathrm{ADP}}(F^-, F^+) = \{G \in D \mid F^- \leqslant G \leqslant F^+\}$. So $s^{\bullet}_{\mathrm{AP}}$ corestricts to a continuous map from $\mathbb{P}^{\bullet}_{\mathrm{AP}\,wk}(X)$ to $\mathcal{H}^{cvx}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\,wk}(X))$, and similarly for $s^{\bullet}_{\mathrm{DP}}$ and $s^{\bullet}_{\mathrm{ADP}}$. We wish to show that this is a homeomorphism, with inverse the corresponding restriction of r_{AP} to $\mathcal{H}^{cvx}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\,wk}(X))$, resp., of r_{DP} to $\mathcal{Q}^{cvx}_{\mathcal{V}}((\mathbb{P}^{\bullet}_{\mathrm{P}\,wk}(X))$, resp., of r_{ADP} to $\mathcal{P}\ell^{cvx}_{\mathcal{V}}((\mathbb{P}^{\bullet}_{\mathrm{P}\,wk}(X))$.

4.1. *The case of unbounded previsions*

The case where $\bullet$ is the empty superscript, i.e., where general previsions are considered, was already dealt with in Keimel and Plotkin (2009, Section 6.1). We improve on their result, which required X to be a continuous dcpo. The general plan of the proof is the same.

The key to the generalization is the following *Schröder–Simpson Theorem.*

Theorem 4.1 (Schröder–Simpson). Let X be a topological space. For every continuous linear map ψ from $\mathbf{V}(X)$ to $\overline{\mathbb{R}^+}_\sigma$ (resp., from $\mathbb{P}_{\mathrm{P}\,wk}(X)$ to $\overline{\mathbb{R}^+}_\sigma$), there is a unique continuous map $h\colon X \to \overline{\mathbb{R}^+}_\sigma$ such that, for every $\nu \in \mathbf{V}(X)$, $\psi(\nu) = \int_{x\in X} h(x)d\nu$ (resp., for every $G \in \mathbb{P}_{\mathrm{P}\,wk}(X)$, $\psi(G) = G(h)$).

The converse direction, that given a unique continuous map $h\colon X \to \overline{\mathbb{R}^+}_\sigma$, the map $G \mapsto G(h)$ is continuous and linear from $\mathbb{P}_{\mathrm{P}\,wk}(X)$ to $\overline{\mathbb{R}^+}_\sigma$, is obvious.

Theorem 4.1 is due to Schröder and Simpson. It was announced at the end of the presentation (Schröder and Simpson 2006), and a full proof was given in another talk (Schröder and Simpson 2009). Another proof was discovered by Keimel, who stresses the role of Hahn–Banach-like extension theorems in quasi-uniform cones (Keimel 2012). A short, elementary proof of this theorem can be found in Goubault-Larrecq (2015a).

Lemma 4.2. Let X be a topological space. For all $A, B \in \mathcal{H}^{cvx}(\mathbb{P}_{\mathrm{P}\,wk}(X))$, if $r_{\mathrm{AP}}(A) \leqslant r_{\mathrm{AP}}(B)$ then $A \subseteq B$.

Proof. Assume $A \not\subseteq B$, so there is a $G \in A$ that is not in B. $\mathbb{P}_{\mathrm{P}\,wk}(X)$ is a locally convex topological cone, since every subbasic open set $[h > b]$ is convex. Therefore, there is a convex open subset U containing G that does not intersect B. By the Separation Theorem (Keimel 2008, Theorem 9.1), there is a continuous linear map $\Lambda\colon C \to \overline{\mathbb{R}^+}$ such that $\Lambda(G') \leqslant 1$ for every $G' \in B$, and $\Lambda(G') > 1$ for every $G' \in U$; in particular, $\Lambda(G) > 1$. By the Schröder–Simpson Theorem, Λ is the map $G' \mapsto G'(h)$ for some $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$. So $\Lambda(G') = G'(h) \leqslant 1$ for every $G' \in B$, which implies that $r_{\mathrm{AP}}(B) \leqslant 1$. And $\Lambda(G) = G(h) > 1$, which implies that $r_{\mathrm{AP}}(A) > 1$, contradiction. □

Proposition 4.3. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex, for example, a core-compact space. Then, r_{AP} defines a homeomorphism between $\mathcal{H}^{cvx}_{\mathcal{V}}(\mathbb{P}_{\mathrm{P}\,wk}(X))$ and $\mathbb{P}_{\mathrm{AP}\,wk}(X)$ and an order-isomorphism between $\mathcal{H}^{cvx}(\mathbb{P}_{\mathrm{P}\,wk}(X))$ and $\mathbb{P}_{\mathrm{AP}}(X)$.

Proof. By Proposition 3.11, r_{AP} is continuous hence monotonic, and $r_{\mathrm{AP}} \circ s_{\mathrm{AP}}$ equals the identity. In particular, r_{AP} is surjective, and Lemma 4.2 implies in particular that it is injective. So r_{AP} is a bijection, with inverse s_{AP}. Both are continuous, by Proposition 3.11,

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hence define a homeomorphism. The final part follows from the fact that every homeomorphism induces an order-isomorphism with respect to the underlying specialization preorders. □

Lemma 4.4. Let X be a topological space. For all $Q, Q' \in \mathcal{Q}^{cvx}(\mathbb{P}_{\mathrm{P}\,wk}(X))$, if $r_{\mathrm{DP}}(Q) \leqslant r_{\mathrm{DP}}(Q')$ then $Q \supseteq Q'$.

Proof. Assume $Q \not\supseteq Q'$, so there is a $G \in Q'$ that is not in Q. $\mathbb{P}_{\mathrm{P}\,wk}(X)$ is a locally convex topological cone, and $A = \downarrow G$ is a closed convex set disjoint from Q. By the Strict Separation Theorem (Keimel 2008, Theorem 10.5), there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}^+}_\sigma$ and a real number $r > 1$ such that $\Lambda(G') \geqslant r$ for every $G' \in Q$, and $\Lambda(G') \leqslant 1$ for every $G' \in A$. In particular, $\Lambda(G) \leqslant 1$. By the Schröder–Simpson Theorem, $\Lambda = \Phi(h)$ for some $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, so $G(h) \leqslant 1$, which implies $r_{\mathrm{DP}}(Q')(h) \leqslant 1$. Also, $r_{\mathrm{DP}}(Q)(h) = \min_{G' \in Q} G'(h) = \min_{G' \in Q} \Lambda(G') \geqslant r > 1$, contradiction. □

Proposition 4.5. Let X be a topological space. Then, r_{DP} defines a homeomorphism between $\mathcal{Q}^{cvx}_{\mathcal{V}}(\mathbb{P}_{\mathrm{P}\,wk}(X))$ and $\mathbb{P}_{\mathrm{DP}\,wk}(X)$, and an order-isomorphism between $\mathcal{Q}^{cvx}(\mathbb{P}_{\mathrm{P}\,wk}(X))$ and $\mathbb{P}_{\mathrm{DP}}(X)$.

Proof. Same argument as in Proposition 4.3, using Proposition 3.22 and Lemma 4.4. □

Lemma 4.6. Let X be a topological space. For every $L \in \mathcal{P}\ell(\mathbb{P}_{\mathrm{P}\,wk}(X))$, $r_{\mathrm{AP}}(L) = r_{\mathrm{AP}}(cl(L))$, and $r_{\mathrm{DP}}(L) = r_{\mathrm{DP}}(\uparrow L)$.

Proof. Since $L \subseteq cl(L)$, $r_{\mathrm{AP}}(L)(h) = \sup_{G \in L} G(h) \leqslant r_{\mathrm{AP}}(cl(L))(h)$ for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$. Assume the inequality were strict: for some $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, there would be a $G \in cl(L)$ such that $r_{\mathrm{AP}}(L)(h) < G(h)$. Let $b = r_{\mathrm{AP}}(L)(h)$. Then, G is in the open subset $[h > b]$, and since $G \in cl(L)$, $[h > b]$ must meet L. So there is a $G' \in L$ such that $G'(h) > b$, which implies $r_{\mathrm{AP}}(L)(h) > b$, contradiction. The second claim is obvious. □

Lemma 4.7. Let X be a topological space. For all $L, L' \in \mathcal{P}\ell^{cvx}(\mathbb{P}_{\mathrm{P}\,wk}(X))$, if $r_{\mathrm{ADP}}(L) \leqslant r_{\mathrm{DP}}(L')$ then $L \sqsubseteq_{\mathrm{EM}} L'$.

Proof. By assumption, $r_{\mathrm{AP}}(L) \leqslant r_{\mathrm{AP}}(L')$, and $r_{\mathrm{DP}}(L) \leqslant r_{\mathrm{DP}}(L')$. Using Lemmas 4.6, 4.2 and 4.4, we obtain $cl(L) \subseteq cl(L')$ and $\uparrow L \supseteq \uparrow L'$. □

Proposition 4.8. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex and has an almost open addition map, for example, a stably locally compact space, or more generally, a core-compact, core-coherent space.

Then, r_{ADP} defines a homeomorphism between $\mathcal{P}\ell^{cvx}_{\mathcal{V}}(\mathbb{P}_{\mathrm{P}\,wk}(X))$ and $\mathbb{P}_{\mathrm{ADP}\,wk}(X)$, and an order-isomorphism between $\mathcal{P}\ell^{cvx}(\mathbb{P}_{\mathrm{P}\,wk}(X))$ and $\mathbb{P}_{\mathrm{ADP}}(X)$.

Proof. Same argument as in Proposition 4.3, using Proposition 3.32 and Lemma 4.7. □

4.2. *Subnormalized and normalized previsions*

The subnormalized and normalized cases reduce to the unbounded case.

Lemma 4.9. Let $\bullet$ be '$\leqslant 1$' or '1.' Let X be a topological space. For every non-empty closed subset C of $\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$, $r_{\mathrm{AP}}(C) = r_{\mathrm{AP}}(cl_{\mathrm{P}}(C))$, where $cl_{\mathrm{P}}(C)$ denotes the closure of C in $\mathbb{P}_{\mathrm{P}\ wk}(X)$.

Proof. Since $C \subseteq cl_{\mathrm{P}}(C)$, for every $h \in [X \to \overline{\mathbb{R}^+}_\sigma]$, $r_{\mathrm{AP}}(C)(h) = \sup_{G \in C} G(h) \leqslant r_{\mathrm{AP}}(cl_{\mathrm{P}}(C))(h)$. Assume by contradiction that $r_{\mathrm{AP}}(C)(h) < r_{\mathrm{AP}}(cl_{\mathrm{P}}(C))(h)$ for some h. Let a be a real number such that $r_{\mathrm{AP}}(C)(h) < a < r_{\mathrm{AP}}(cl_{\mathrm{P}}(C))(h)$. So $cl_{\mathrm{P}}(C)$ is in $r_{\mathrm{AP}}^{-1}[h > a]$, which is open in $\mathcal{H}_{\mathcal{V}}(\mathbb{P}_{\mathrm{P}\ wk}(X))$, since r_{AP} is continuous (Lemma 3.3). Every open subset of $\mathcal{H}_{\mathcal{V}}(\mathbb{P}_{\mathrm{P}\ wk}(X))$ is also Scott open, and $cl_{\mathrm{P}}(C)$ is the least upper bound in $\mathcal{H}(\mathbb{P}_{\mathrm{P}\ wk}(X))$ of the directed family of closed subsets of the form $\downarrow_{\mathrm{P}} E$, where E ranges over the finite subsets of C. (We write $\downarrow_{\mathrm{P}}$ for downward closure in $\mathbb{P}_{\mathrm{P}\ wk}(X)$.) So there is a finite subset E of C such that $\downarrow_{\mathrm{P}} E \in r_{\mathrm{AP}}^{-1}[h > a]$, i.e., such that $\sup_{G \in E} G(h) > a$. It follows that there is a $G \in E$ (hence $G \in C$) such that $G(h) > a$. But $r_{\mathrm{AP}}(C)(h) = \sup_{G \in C} G(h) < a$, contradiction. So $r_{\mathrm{AP}}(C)(h) = r_{\mathrm{AP}}(cl_{\mathrm{P}}(C))(h)$ for every h. □

Lemma 4.10. Let $\bullet$ be '$\leqslant 1$' or '1.' Let X be a topological space. For all $C, C' \in \mathcal{H}^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$, if $r_{\mathrm{AP}}(C) \leqslant r_{\mathrm{AP}}(C')$ then $C \supseteq C'$.

Proof. Assume $r_{\mathrm{AP}}(C) \leqslant r_{\mathrm{AP}}(C')$. By Lemma 4.9, $r_{\mathrm{AP}}(cl_{\mathrm{P}}(C)) \leqslant r_{\mathrm{AP}}(cl_{\mathrm{P}}(C'))$. Now $cl_{\mathrm{P}}(C)$ and $cl_{\mathrm{P}}(C')$ are closed and non-empty. They are convex, because the closure of any convex subset of a semitopological cone is convex again (Keimel 2008, Lemma 4.10 (a)). So Lemma 4.2 applies: $cl_{\mathrm{P}}(C) \subseteq cl_{\mathrm{P}}(C')$. In particular, $C \subseteq cl_{\mathrm{P}}(C')$. Since C' is closed in the subspace $Y = \mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$ of $Z = \mathbb{P}_{\mathrm{P}\ wk}(X)$, one can write it as $Y \cap A$ for some closed subset A of Z. Clearly, $cl_{\mathrm{P}}(C') \subseteq A$, so $C' = Y \cap A \supseteq Y \cap cl_{\mathrm{P}}(C')$. Since $C \subseteq cl_{\mathrm{P}}(C')$ and $C \subseteq Y$, it follows that $C \subseteq C'$. □

Theorem 4.11 (Isomorphism, angelic cases). Let $\bullet$ be the empty superscript, '$\leqslant 1$' or '1.' Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex, for example, a core-compact space.

Then, r_{AP} defines a homeomorphism with inverse $s^{\bullet}_{\mathrm{AP}}$ between $\mathcal{H}^{cvx}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{AP}\ wk}(X)$ and an order-isomorphism between $\mathcal{H}^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{AP}}(X)$.

Proof. When $\bullet$ is the empty superscript, this is Proposition 4.3. Otherwise, we use the same argument as in its proof, replacing Lemma 4.2 by Lemma 4.10. That is, r_{AP} is an injective retraction, hence an isomorphism. □

Corollary 4.12. Let $\bullet$ be the empty superscript, '$\leqslant 1$' or '1.' Let X be a continuous dcpo. Then, r_{AP} define an isomorphism with inverse $s^{\bullet}_{\mathrm{AP}}$ between $\mathcal{H}^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{AP}}(X)$.

The demonic cases are slightly simpler.

Lemma 4.13. Let $\bullet$ be '$\leqslant 1$' or '1.' Let X be a topological space. For every non-empty compact saturated subset Q of $\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$, $r_{\mathrm{DP}}(Q) = r_{\mathrm{DP}}(\uparrow_{\mathrm{P}} Q)$, where $\uparrow_{\mathrm{P}} Q$ denotes the upward closure of Q in $\mathbb{P}_{\mathrm{P}\ wk}(X)$.

Note that $\uparrow_{\mathrm{P}} Q$ is compact saturated in $\mathbb{P}_{\mathrm{P}\ wk}(X)$. Indeed, Q is compact in $\mathbb{P}^{\bullet}_{\mathrm{P}\ wk}(X)$, which is a subspace of $\mathbb{P}_{\mathrm{P}\ wk}(X)$. In particular, Lemma 3.13 applies, and $r_{\mathrm{DP}}(\uparrow_{\mathrm{P}} Q)(h) = \min_{G \in \uparrow_{\mathrm{P}} Q} G(h)$ for every h.

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Proof. Since $Q \subseteq \uparrow_{\mathrm{P}} Q$, $r_{\mathrm{DP}}(\uparrow_{\mathrm{P}} Q)(h) \leqslant r_{\mathrm{DP}}(Q)(h)$. Conversely, for every $G \in \uparrow_{\mathrm{P}} Q$, there is a $G_1 \in Q$ such that $G_1 \leqslant G$, so $r_{\mathrm{DP}}(\uparrow_{\mathrm{P}} Q)(h) = \min_{G \in \mathbb{P}_{\mathrm{P}\;wk}(X), G_1 \in Q, G_1 \leqslant G} G(h) \geqslant \min_{G_1 \in Q} G_1(h) = r_{\mathrm{DP}}(Q)(h)$. □

Lemma 4.14. Let $\bullet$ be '$\leqslant 1$' or '1.' Let X be a topological space. For all $Q, Q' \in \mathcal{Q}^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X))$, if $r_{\mathrm{DP}}(Q) \leqslant r_{\mathrm{DP}}(Q')$ then $Q \supseteq Q'$.

Proof. Assume $r_{\mathrm{DP}}(Q) \leqslant r_{\mathrm{DP}}(Q')$. By Lemma 4.13, $r_{\mathrm{DP}}(\uparrow_{\mathrm{P}} Q) \leqslant r_{\mathrm{DP}}(\uparrow_{\mathrm{P}} Q')$. Now, $\uparrow_{\mathrm{P}} Q$ is compact saturated, non-empty, and it is easy to see that it is convex, since Q is convex. We can therefore apply Lemma 4.4 and conclude that $\uparrow_{\mathrm{P}} Q \supseteq \uparrow_{\mathrm{P}} Q'$. In particular, $\uparrow_{\mathrm{P}} Q \supseteq Q'$. So, for every element G' of Q', there is a $G \in Q$ such that $G \leqslant G'$. But G' is in $\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X)$ since in Q', and Q is upward closed in $\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X)$, so G' itself is in Q. It follows that $Q \supseteq Q'$. □

Theorem 4.15 (Isomorphism, demonic cases). Let $\bullet$ be the empty superscript, '$\leqslant 1$' or '1.' Let X be a topological space. Then, r_{DP} defines a homeomorphism with inverse $s^{\bullet}_{\mathrm{DP}}$ between $\mathcal{Q}^{cvx}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{DP}\;wk}(X)$, and an order-isomorphism between $\mathcal{Q}^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$.

Proof. When $\bullet$ is the empty superscript, this is Proposition 4.5. Otherwise, we use the same argument as in its proof, replacing Lemma 4.4 by Lemma 4.14. That is, r_{DP} is an injective retraction, hence an homeomorphism. □

Corollary 4.16. Let $\bullet$ be the empty superscript, '$\leqslant 1$' or '1.' Let X be a continuous dcpo. Then, r_{DP} defines an isomorphism with inverse $s^{\bullet}_{\mathrm{DP}}$ between $\mathcal{Q}^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{DP}}(X)$.

There is nothing left to do to conclude in the erratic cases.

Theorem 4.17 (Isomorphism, erratic cases). Let $\bullet$ be '$\leqslant 1$' or '1.' Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_{\sigma}]_{\sigma}$ is locally convex and has an almost open addition map, for example a stably locally compact space, or more generally, a core-compact, core-coherent space. Assume also that X is compact in case $\bullet$ is '1.'

Then, r_{ADP} defines an homeomorphism with inverse $s^{\bullet}_{\mathrm{ADP}}$ between $\mathcal{P}\ell^{cvx}_{\mathcal{V}}(\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{ADP}\;wk}(X)$, and an order-isomorphism between $\mathcal{P}\ell^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{ADP}}(X)$.

Proof. The only thing left to prove is that $s^{\bullet}_{\mathrm{ADP}}(r_{\mathrm{ADP}}(L)) = L$ for every $L \in \mathcal{P}\ell^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}\;wk}(X))$. But $s^{\bullet}_{\mathrm{ADP}}(r_{\mathrm{ADP}}(L)) = s^{\bullet}_{\mathrm{ADP}}(r_{\mathrm{DP}}(L), r_{\mathrm{AP}}(L)) = s^{\bullet}_{\mathrm{ADP}}(r_{\mathrm{DP}}(\uparrow L), r_{\mathrm{AP}}(cl(L)))$ (by Lemma 4.6) $= s^{\bullet}_{\mathrm{DP}}(r_{\mathrm{DP}}(\uparrow L)) \cap s^{\bullet}_{\mathrm{AP}}(r_{\mathrm{AP}}(cl(L))) = \uparrow L \cap cl(L)$ (by Theorems 4.15 and 4.11) $= L$. □

Corollary 4.18. Let $\bullet$ be the empty superscript, '$\leqslant 1$' or '1.' Let X be a coherent, continuous dcpo (and pointed if $\bullet$ is '1'). Then, r_{ADP} defines an isomorphism with inverse $s^{\bullet}_{\mathrm{ADP}}$ between $\mathcal{P}\ell^{cvx}(\mathbb{P}^{\bullet}_{\mathrm{P}}(X))$ and $\mathbb{P}^{\bullet}_{\mathrm{ADP}}(X)$.

4.3. *Hulls*

When A is not convex, $s^{\bullet}_{\mathrm{AP}}(r_{\mathrm{AP}}(A))$ cannot be equal to A, since the former is always convex. Similarly in the demonic cases. It is a natural question to ask what the compositions $s^{\bullet}_{\mathrm{AP}} \circ r_{\mathrm{AP}}$, and similar compositions, compute.

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The *convex hull* $conv(A)$ of a set A in a topological cone (or in a convex subspace) is the smallest convex set that contains A. This is also the set of linear combinations of the form $ax + (1 - a)y$, where $x, y \in A$, $a \in [0, 1]$.

The *closed convex hull* of a set A in a semitopological cone is the smallest closed and convex set containing A. This is the closure of the convex hull of A (Keimel 2008, Lemma 4.10 (a)).

Proposition 4.19. Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex, for example, a core-compact space. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.'

For every non-empty closed subset A of $\mathbb{P}^\bullet_{\mathrm{P}\,wk}(X)$, $s^\bullet_{\mathrm{AP}}(r_{\mathrm{AP}}(A))$ is the closed convex hull $cl(conv(A))$ of A in $\mathbb{P}^\bullet_{\mathrm{P}\,wk}(X)$.

Proof. By Proposition 3.11, $s^\bullet_{\mathrm{AP}}(r_{\mathrm{AP}}(A))$ is non-empty and closed. It is convex, and clearly contains A. Conversely, for every closed convex subset A' that contains A, $s^\bullet_{\mathrm{AP}}(r_{\mathrm{AP}}(A)) \subseteq s^\bullet_{\mathrm{AP}}(r_{\mathrm{AP}}(A')) = A'$, where the last equality follows from Theorem 4.11. Therefore, $s^\bullet_{\mathrm{AP}}(r_{\mathrm{AP}}(A))$ is the smallest closed convex set containing A, namely $cl(conv(A))$. □

Symmetrically, one can define the *compact saturated convex hull* of a set A as the smallest compact saturated, convex subset containing A, if it exists. The following shows that this exists if A is any non-empty compact saturated subset of $\mathbb{P}^\bullet_{\mathrm{P}\,wk}(X)$, in particular.

Proposition 4.20. Let X be a topological space. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.'

For every non-empty compact saturated subset Q of $\mathbb{P}^\bullet_{\mathrm{P}\,wk}(X)$, $s^\bullet_{\mathrm{DP}}(r_{\mathrm{DP}}(Q))$ is the compact saturated convex hull of Q in $\mathbb{P}^\bullet_{\mathrm{P}\,wk}(X)$.

Proof. By Proposition 3.22, $s^\bullet_{\mathrm{DP}}(r_{\mathrm{DP}}(Q))$ is compact saturated. It is convex, and clearly contains Q. If Q' is any convex, compact saturated superset of Q, then $s^\bullet_{\mathrm{DP}}(r_{\mathrm{DP}}(Q'))$ contains $s^\bullet_{\mathrm{DP}}(r_{\mathrm{DP}}(Q))$. However, $s^\bullet_{\mathrm{DP}}(r_{\mathrm{DP}}(Q')) = Q'$ since r_{DP} is an isomorphism with inverse $s^\bullet_{\mathrm{DP}}$ on non-empty convex compact saturated sets, by Theorem 4.15; so Q' contains $s^\bullet_{\mathrm{DP}}(r_{\mathrm{DP}}(Q))$. □

Proposition 4.21. Let $\bullet$ be the empty superscript, '$\leqslant 1$,' or '1.' Let X be a topological space such that $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is a locally convex semitopological cone, for example, a core-compact space.

For every lens L of $\mathbb{P}^\bullet_{\mathrm{P}\,wk}(X)$, $s^\bullet_{\mathrm{ADP}}(r_{\mathrm{ADP}}(L))$ is a lens, and is the smallest convex lens that contains L.

Proof. We know that $s^\bullet_{\mathrm{ADP}}(r_{\mathrm{ADP}}(L)) = s^\bullet_{\mathrm{AP}}(F^+) \cap s^\bullet_{\mathrm{DP}}(F^-)$ where $(F^+, F^-) = r_{\mathrm{ADP}}(L)$. By Lemma 4.6, $F^+ = r_{\mathrm{AP}}(cl(L))$ and $F^- = r_{\mathrm{DP}}(\uparrow L)$. By Proposition 4.19, $s^\bullet_{\mathrm{AP}}(F^+)$ is the closed convex hull of $cl(L)$, hence also the smallest closed convex subset that contains L. By Proposition 4.20, $s^\bullet_{\mathrm{DP}}(F^-)$ is the compact saturated convex hull of $\uparrow L$, hence also the smallest compact saturated convex subset that contains L. As a consequence, their intersection is a convex lens.

For the final part of the Proposition, if L' is any convex lens that contains L, then $cl(L')$ contains L, is closed, and is convex (Keimel 2008, Lemma 4.10 (a)), so $cl(L')$ contains $s^\bullet_{\mathrm{AP}}(F^+)$; also, $\uparrow L'$ contains L, is compact saturated and convex, hence contains $s^\bullet_{\mathrm{DP}}(F^-)$. It follows that $L' = cl(L') \cap \uparrow L'$ contains $s^\bullet_{\mathrm{AP}}(F^+) \cap s^\bullet_{\mathrm{DP}}(F^-) = s^\bullet_{\mathrm{ADP}}(r_{\mathrm{ADP}}(L))$. □

https://doi.org/10.1017/S0960129515000547 Published online by Cambridge University Press

5. Conclusion

We have proved that, under some natural conditions, the powercone models and the prevision models of mixed non-deterministic and probabilistic choice coincide. This involved functional analytic methods that heavily rely on Keimel's cone-theoretic variants of the classical Hahn–Banach separation theorems, plus the Schröder–Simpson Theorem.

The demonic cases are the nicest, and require absolutely no assumption on the underlying topological space X. As should be expected, the erratic cases demand stronger assumptions. An intermediate case is given by the angelic cases, which require $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ to be locally convex. We know that this is the case when X is locally compact, and more generally, core-compact, and we do not know of an example of a space X for which $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ would fail to be locally convex. Can we characterize those spaces X for which $[X \to \overline{\mathbb{R}^+}_\sigma]_\sigma$ is locally convex?

I thank the anonymous referees for many useful comments. More than anything, I must thank them for their urging me to try to do away with an assumption of coherence that I had made in some results concerning the Hoare cases, which I had found surprising, but could not do away with initially. That assumption has now disappeared, after some more hard work. I also thank them, as well as the editor, Prakash Panangaden, for their patience. I did not have time to work on this paper for more than two years, between March 2013 and June 2015, and they are to be commended for having waited for so long: my sincere apologies to them. This work was partially supported by the INRIA ARC ProNoBiS and the ANR programme blanc project CPP.

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