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Residually finite tubular groups

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A tubular group G is a finite graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. We characterize residually finite tubular groups: G is residually finite if and only if its edge groups are separable. Methods are provided to determine if G is residually finite. When G has a single vertex group an algorithm is given to determine residual finiteness.

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1. Introduction

A f.g. group G is *tubular* if it splits as a finite graph of groups with $\mathbb{Z}$ edge groups and $\mathbb{Z}^2$ vertex groups. A group G is *residually finite* if for each nontrivial $g \in G$, there is a finite quotient of G so that the image of g is nontrivial. The goal of this paper is to determine which tubular groups are residually finite.

The case where G is a single HNN extension was handled by Andreadakis, Raptis and Varsos [1]. However the full complexity of the situation is not apparent for a single HNN extension, as residual finiteness coincides with virtual specialness whereas failure of residual finiteness coincides with a problematic Baumslag–Solitar subgroup. Kim [12] proved that having *isolated cyclic subgroups* is a sufficient condition for residual finiteness. In the language of this paper, isolated cyclic subgroups translate to saying the tubular group is *primitive*.

1.1. Quick survey of results about tubular groups

Tubular groups form a class of seemingly straightforward groups that are increasingly recognized as a surprisingly rich source of diverse behaviour. Burns, Karass, and Solitar gave the first example of a f.g. 3-manifold group that is not subgroup separable, and their example arises as a tubular group [4]. Croke and Kleiner used this same tubular group to show that the boundary of a CAT(0) space is not an invariant of CAT(0) groups [8]. Gersten gave a tubular group as an example of a free-by-cyclic group that does not act properly and semi-simply on a CAT(0) space [10]. Wise gave an example of a tubular group that is CAT(0) but not Hopfian [15]. Brady and Bridson [3] characterized the Dehn functions of *snowflake groups*, a subclass of tubular groups, to show that there are f.p. groups with isoperimetric functions n^d where $d \in D$ is a dense subset of $[2, \infty]$. Gardam and Woodhouse showed that certain Snowflake groups embed as finite index subgroups of one-relator groups [9], and Button observed that many of these groups are not residually finite [6]. Cashen gave a quasi-isometric classification of tubular groups [7]. Wise gave a criterion for a tubular group to be cubulated [16]. Button showed that if a tubular group is free-by-cyclic, then it is cubulated [5]. Woodhouse classified which cubulations are finite dimensional and showed that a tubular group is virtually special if and only if it acts freely on a finite dimensional CAT(0) cube complex [17, 18].

1.2. Statement of main result

A f.g. group G is *tubular* if it splits as a finite graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. A tubular group G is *primitive* if each edge group is a maximal cyclic subgroup of its vertex groups, and hence of G. A nontrivial element $(a, b) \in \mathbb{Z}^2$ is *primitive* if $\gcd(a, b) = 1$, that is (a, b) is not a 'proper power.'

There are two goals to this paper. The first is to characterize which tubular groups are residually finite, and the second is to provide practical means of deciding the question. The following theorem, addressing the first goal, is a special case of a more extensive characterization given in theorem 4.3.

Theorem 1.1. *A tubular group is residually finite if and only if it is virtually primitive.*

Although we are unable to settle the question of decidability of residual finiteness in general, in the motivating case, where G has a single vertex group we are able to provide the following, which is obtained as a consequence of proposition 7.1 and lemma 7.4 in § 7.

Theorem 1.2. *Let G be a tubular group with a single vertex group. Then there is an algorithm that decides in finite time if G is residually finite or not.*

To address tubular groups in general, we introduce the *expansion sequence* for a tubular group, which we motivate in the following subsection.

1.3. Two illustrative examples

The expansion sequence for a tubular group is nontrivial, even in the simple case of a graph of groups with a single vertex group and two edge groups. Given a tubular group $G = G_0$ the expansion sequence is a series of homomorphisms. At the i-th stage of the computation we obtain a tubular group G_i and a homomorphism $G_{i-1} \to G_i$. The sequence $G = G_0 \to G_1 \to G_2 \to \cdots$ is the *expansion sequence*.

We are presented with a dichotomy: either the expansion sequence *terminates* or it continues indefinitely, that is to say it is *nonterminating*. By lemma 5.4, a terminating expansion sequence is equivalent to G being residually finite. Ideally, we would like to determine if an expansion sequence is nonterminating after a finite number of steps. The simplest way to verify this is if the sequence starts repeating itself. We call such sequences *recurrent*. Unfortunately, not all nonterminating expansion sequences are recurrent. See example 5.5. We conjecture however that if a tubular group is not residually finite, then some subtubular group will have an expansion sequence that repeats itself.

We give two examples of such computations to illustrate and motivate what will be happening in this paper.

Example 1.3. The tubular group G below splits over a graph with a single vertex group and two edge groups. The elements $(1,0)$ and $(0,1)$ generate the vertex group $G_v = \mathbb{Z}^2$ and s and t are the stable letters associated to the edge groups.

$$G = \langle \mathbb{Z} \times \mathbb{Z}, s, t \mid (1,0)^s = (2,2),\ (0,1)^t = (1,1)\ \rangle$$

G is not primitive since $(2,2)$ is not primitive in G_v. Note that t conjugates a primitive element to a primitive element. We will construct a homomorphism $G \to G'$ from G to another tubular group G' with the same underlying graph, such that vertex and edge groups map injectively, and such that the stable letter s conjugates a pair of primitive elements in G'. A simple way to do this is to add the element $(1/2, 0)$ to the vertex group and extend the conjugation by s linearly so that $(1/2, 0)$ is conjugated to $(1,1)$. We thus obtain the following new tubular group:

$$G' = \langle \tfrac{1}{2}\mathbb{Z} \times \mathbb{Z},\ s, t \mid (\tfrac{1}{2}, 0)^s = (1,1),\ (0,1)^t = (1,1)\rangle.$$

There is a homomorphism $G \longrightarrow G'$ that maps $(0,1)$, $(1,0)$, s, and t to themselves in G'. This morphism is the *expansion map*. As G' is a primitive tubular group we say that we have found a *primitive target* for G, which implies by Theorem 4.3 that G is residually finite.

Example 1.4. Consider the following tubular group G having a single vertex group and two edge groups. Note that G is almost identical to the group in example 1.3, with a slight adjustment to the elements conjugated to $(1,0)$ and $(0,1)$.

$$G = \langle \mathbb{Z} \times \mathbb{Z},\ s, t \mid (1,0)^s = (2,4),\ (0,1)^t = (1,2)\rangle.$$

G is not primitive since $(2,4)$ is not primitive in the vertex group G_v. All other images of the edge group generators are primitive. As in example 1.3 we will construct an 'expansion map' by adding the element $(\frac{1}{2}, 0)$ to G_v and extending the conjugation by s linearly so that s conjugates $(\frac{1}{2}, 0)$ to $(1,2)$. We thus obtain the tubular group G' below, and obtain a homomorphism $G \to G'$ mapping $\mathbb{Z} \times \mathbb{Z}$ and s and t identically to themselves.

$$G' = \langle \tfrac{1}{2}\mathbb{Z} \times \mathbb{Z},\ s, t \mid (\tfrac{1}{2}, 0)^s = (1,2),\ (0,1)^t = (1,2)\rangle$$

Unfortunately, G' is not primitive. Indeed, $(1,2)$ is no longer primitive since $(1,2) = 2(1/2, 1)$. We may then construct another expansion map. This time however, in order to extend both conjugations linearly we need to include the elements

$(1/4, 0)$ and $(0, 1/2)$. We thus obtain the tubular group

$$G' = \langle \tfrac{1}{4}\mathbb{Z} \times \tfrac{1}{2}\mathbb{Z},\ s, t \mid (\tfrac{1}{4}, 0)^s = (\tfrac{1}{2}, 1),\ (0, \tfrac{1}{2})^t = (\tfrac{1}{2}, 1)\rangle$$

and the expansion map $G' \to G''$. This time the expansion map has not improved our situation at all since G'' is isomorphic to G'. The isomorphism is given by scaling both $(1/4, 0)$ and $(0, 1/2)$ by 2. Repeating this process yields G''' which is again isomorphic to G' and therefore we will never arrive at a primitive target. This situation is a *recurrent expansion sequence* and by lemma 5.4 it implies that G is not residually finite.

In examples 1.3 and 1.4, the vertex group is of the form $1/n\mathbb{Z} \times 1/m\mathbb{Z}$ at each stage. However, the algorithm generally wanders through groups that are not subdirect products of cyclic groups commensurable with the factors of the initial product decomposition.

1.4. Structure of this paper

In § 2 we define a range of algebraic constructions that we will use to characterize residually finite tubular groups in § 4 and § 3. Section 5 defines the expansion sequence of a tubular group and provides a general framework for understanding residual finiteness of tubular groups. Section 6 applies the techniques of § 5 to the snowflake groups of Brady and Bridson [**3**], to determine their residual finiteness and recover a result of Button. Section 7 shows that residual finiteness is decidable when the tubular group has a single vertex group.

2. Morphisms and primitivity

We now establish the notation used for the splitting of a group G as a graph Γ of groups. We refer to [**2**, **13**, **14**] for full background. For the directed graph Γ, let $\mathcal{E}$ and $\mathcal{V}$ denote its sets of edges and vertices. For an edge $e \in \mathcal{E}$, let $-e$ and $+e$ denote its initial and terminal vertices. A *graph of groups* $\mathcal{G}$ is a graph Γ with a *vertex group* G_v for each $v \in \mathcal{V}$, and an *edge group* G_e for each $e \in \mathcal{E}$, and an accompanying pair of monomorphisms $\varphi_e^{\pm} : G_e \to G_{e\pm}$ of the edge group into the incident vertex groups. Associated to $\mathcal{G}$ is its *fundamental group* G (see [**14**]), obtained by a sequence of amalgamated free products and HNN extensions determined by the graph of groups data. Alternatively, we will say that the group G *splits as a graph of groups over* Γ. Finally, G acts on its associated Bass–Serre tree T, and $G \backslash T = \Gamma$, and the conjugacy classes of vertex and edge stabilizers correspond to the vertex and edge groups.

Definition 2.1. A group G is a *tubular group* if it splits as a finite graph of groups with vertex groups isomorphic to $\mathbb{Z}^2$ and edge groups isomorphic to $\mathbb{Z}$. We say that G is a *primitive* tubular group if the edge groups are embedded by $\varphi_e^{\pm}$ as maximal cyclic subgroups in the vertex groups.

Let G and G' be groups which split over the graphs Γ and Γ' respectively. A *morphism* of graphs of groups is a homomorphism $f : G \to G'$ determined by a morphism $f_* : \Gamma \to \Gamma'$ of undirected graphs, and homomorphisms $f_v : G_v \to$

$G'_{v'}$ and $f_e : G_e \to G'_{e'}$ where $v' = f_*(v)$ and $e' = f_*(e)$ such that the following commutes:

$$\begin{tikzcd}
G_v \arrow[r, "f_v"] & G'_{v'} \\
G_e \arrow[u, "\varphi_e^+"] \arrow[r, "f_e"] & G'_{e'} \arrow[u, "{\varphi'}_e^+"']
\end{tikzcd}$$

(assuming that $v = +e$ and $v' = +e'$). Note that Bass [2] defines a more general notion of morphism between graphs of groups where edge groups may be 'twisted', but that our definition will suffice for the purposes of this paper.

A *rigid morphism* $f : G \to G'$ is a morphism such that f_* is an isomorphism and each f_v and f_e is injective.

A tubular group G has a *primitive target* if there a rigid morphism $\overline{f} : G \to \overline{G}$ between tubular groups such that $\overline{G}$ is primitive. Similarly, a tubular group G has a *primitive domain* if there is a rigid morphism $\underline{f} : \underline{G} \to G$ such that $\underline{G}$ is a primitive tubular group.

The following holds by the definitions:

Lemma 2.2. *Let G and G' be tubular groups that split over the same underlying graph Γ. Suppose that $G'_v \leqslant G_v$ and $G'_e \leqslant G_e$ and that the edge maps of G' are restrictions of the edge maps of G. Then there is a rigid morphism $\phi : G' \to G$ induced by the inclusion maps on the vertex and edge groups.*

Definition 2.3 (Local Quotients). Let $f : G' \to G$ be a rigid morphism of tubular groups with underlying graph Γ. Suppose that for the edge inclusions $\varphi'_e : G'_e \to G'_v$ and $\varphi_e : G_e \to G_v$ we have $f \circ \varphi'_e(G'_e) = \varphi_e(G_e) \cap f(G'_v)$. Note that this equality always holds when G' is primitive. Define a group $G /\!\!/ G'$ that splits over Γ as follows:

(1) $(G/\!\!/G')_v = G_v/f(G'_v)$,

(2) $(G/\!\!/G')_e = G_e/f(G'_e)$,

(3) Attaching maps $(G/\!\!/G')_e \to (G/\!\!/G')_v$ are projections of $G_e \to G_v$,

(4) There is a morphism $q : G \to G/\!\!/G'$ that is induced by the quotient maps $G_v \to (G/\!\!/G')_v$ and $G_e \to (G/\!\!/G')_e$.

Each map $G_e/G'_e \to G_v/G'_v$ is injective, since if $g \in G_e$ maps to the identity in G_v/G'_v then the image of g in G_v lies in $f(G'_v)$. But then $g \in \varphi_e(G_e) \cap f(G'_v)$ so $f(g) \in G'_e$ by hypothesis. Hence g represents the identity in G_e/G'_e.

Having verified the injectivity of attaching maps of edge groups of $G/\!\!/G'$ we see that the data for $G/\!\!/G'$ actually yields a splitting over Γ. The induced morphism $q : G \to G/\!\!/G'$ is the *local quotient* of f. Since the vertex and edge groups of $G/\!\!/G'$ are finite groups, the local quotient of f is a virtually free group by [11].

3. Regulating $\mathcal{E}$-tuples

Let G be a tubular group. Let $\underline{k} = (k_e)_{e\in\mathcal{E}}$ be an $\mathcal{E}$-tuple of integers, one for each edge of G. For each edge group G_e let $G_e^{(\underline{k})} = k_e G_e$. For each vertex group G_v let $G_v^{(\underline{k})} \leqslant G_v$ be the subgroup generated by the inclusions of the $G_e^{(\underline{k})}$ under the attaching maps. An $\mathcal{E}$-tuple $\underline{k}$ is *regulating* if for each edge e and generator $g_e \in G_e$, the element $\varphi_e^{\pm}(k_e g_e)$ is primitive in $G_{\pm e}^{(\underline{k})}$.

Remark 3.1. Let $\underline{k}$ be an $\mathcal{E}$-tuple. Then, for any positive integer n, we have that $\underline{k}$ is regulating if and only if $n\underline{k} = (nk_e)_{e\in\mathcal{E}}$ is regulating. So, in searching for regulating $\mathcal{E}$-tuples, it suffices to consider those $\underline{k} = (k_e)_{e\in\mathcal{E}}$ having $\gcd(k_e : e \in \mathcal{E}) = 1$.

Lemma 3.2. *Let G be a tubular group. Then G has a primitive domain if and only if G has a regulating $\mathcal{E}$-tuple.*

Proof. Suppose G has a regulating $\mathcal{E}$-tuple $\underline{k}$. Extend each $G_v^{(\underline{k})}$ to a rank 2 subgroup $\bar{G}_v^{(\underline{k})}$ of G_v such that $G_v^{(\underline{k})}$ is a maximal subgroup of its rank in $\bar{G}_v^{(\underline{k})}$. The inclusions $G_e^{(\underline{k})} \hookrightarrow G_e$ and $\bar{G}_v^{(\underline{k})} \hookrightarrow G_v$ induce a rigid morphism $G^{(\underline{k})} \to G$. Each edge group $G_e^{(\underline{k})}$ is generated by $k_e g_e$ where g_e is a generator of G_e. Since $\varphi_e^{\pm}(k_e g_e)$ is primitive in $G_{\pm e}^{(\underline{k})}$ and so in $\bar{G}_{\pm e}^{(\underline{k})}$, the image $\varphi_e^{\pm}(G_e^{(\underline{k})})$ is a maximal cyclic subgroup of $\bar{G}_{\pm e}^{(\underline{k})}$. Hence $G^{(\underline{k})} \to G$ is a primitive domain.

Suppose G has a primitive domain $G' \to G$. Let $\underline{k} = (k_e)_{e\in\mathcal{E}}$ be an $\mathcal{E}$-tuple where $k_e = [G_e : G'_e]$. Then $\varphi_e^{\pm}(G_e^{(\underline{k})}) = \varphi_e^{\pm}(k_e G_e) = \varphi_e^{\pm}(G'_e)$ is a maximal cyclic subgroup of $G'_{\pm e} < G_{\pm e}$ and so $\varphi_e^{\pm}(G_e^{(\underline{k})})$ is a maximal cyclic subgroup of $G_{\pm e}^{(\underline{k})} < G'_{\pm e}$. Then if g_e generates G_e then $\varphi_e^{\pm}(k_e g_e)$ generates $\varphi_e^{\pm}(G_e^{(\underline{k})})$ and so $\varphi_e^{\pm}(k_e g_e)$ is primitive in $G_{\pm e}^{(\underline{k})}$. ☐

4. Scaling morphisms, naive morphisms and primitivity

Given $H \cong \mathbb{Z}^n$ and a nonzero rational number $\alpha \in \mathbb{Q}^*$, it is natural to define the group αH, and likewise to define αh when $h \in H$. This is justified by noting that there is a unique inclusion $H \hookrightarrow \mathbb{Q}^n$ up to conjugation by $GL_n(\mathbb{Q})$.

Let G be a tubular group with underlying graph Γ. Let $G_e = \langle g_e \rangle$ and $G_v = \langle a_v, b_v \rangle$. For $\alpha \in \mathbb{Q}^*$ we define the tubular group αG with underlying graph Γ as follows: The vertex and edge groups of αG are

$$\alpha G_v = \langle \alpha a_v, \alpha b_v \rangle \quad \text{and} \quad \alpha G_e = \langle \alpha g_e \rangle.$$

Its edge inclusions are determined by linear extension: $\phi_e^{\pm}(\alpha g_e) = \alpha \phi_e^{\pm}(g_e)$.

Note that αG is primitive when G is primitive. The *scaling morphism* is a rigid isomorphism $G \to \alpha G$ induced by $g \mapsto \alpha g$ for each g in a vertex or edge group.

We will also employ the following two rigid morphisms that arise when $\alpha = n \in \mathbb{N}$ and $\alpha = 1/n$ respectively: They map each vertex group and edge group to the obvious copies of itself within the target.

The *first naive morphism* $f : nG \to G$ is defined since $nG_v \leqslant G_v$ and $nG_e \leqslant G_e$ for all vertices $v \in \mathcal{V}$ and $e \in \mathcal{E}$. The inclusions of the vertex and edge groups extend to a rigid morphism by lemma 2.2.

The *second naive morphism* $g : G \to 1/nG$ is defined since $G_v \leqslant 1/nG_v$ and $G_e \leqslant 1/nG_e$ for all vertices $v \in \mathcal{V}$ and $e \in \mathcal{E}$. The inclusions of the vertex and edge groups extend to a rigid morphism since edge maps are extended linearly so lemma 2.2 applies.

We emphasize that the scaling morphism $G \to 1/nG$ and the second naive morphism $G \to 1/nG$ are different for $n > 1$ even though they have the same domain and target. The scaling morphism is an isomorphism and restricts to a scaling isomorphism on each vertex and edge group. The second naive morphism restricts to an inclusion of a subgroup on each vertex and edge group. There is likewise a $1/n$ scaling isomorphism $nG \to G$ which differs from the first naive morphism.

Lemma 4.1. *G has a primitive target if and only if G has a primitive domain.*

Proof. If G has a primitive target then there is a morphism $\overline{f} : G \to \overline{G}$. Let $n_e = [\overline{G}_e : G_e]$ and $n_v = [\overline{G}_v : G_v]$. Let $n = \operatorname{lcm}\{n_e, n_v \mid e \in \mathcal{E}, v \in \mathcal{V}\}$. Then $n\overline{G}$ is also primitive, and there is the naive morphism $F : n\overline{G} \to \overline{G}$. It follows from our choice of n that $n\overline{G}_v \leqslant G_v \leqslant \overline{G}_v$ for $v \in \mathcal{V}$, and $n\overline{G}_e \leqslant G_e \leqslant \overline{G}_e$ for $e \in \mathcal{E}$. Therefore, by lemma 2.2 there is a morphism $\underline{f} : n\overline{G} \to G$ induced by inclusion of the edge groups such that $\underline{f} \circ \overline{f}$ gives the inclusion of $n\overline{G}_v$ into $\overline{G}_v$ for all $v \in \mathcal{V}$, and similarly for all edge groups. Hence G has a primitive domain.

If G has a primitive domain then there is a tubular group $\underline{G}$ and a morphism $\underline{f} : \underline{G} \to G$. Let $m_e = [G_e : \underline{G}_e]$ and $m_v = [G_v : \underline{G}_v]$. Let $m = \operatorname{lcm}\{n_e, n_v \mid e \in \mathcal{E}, v \in \mathcal{V}\}$. Then $1/m\underline{G}$ is a primitive tubular group, and there is the naive morphism $F : \underline{G} \to 1/m\underline{G}$. It follows from our choice of m that $\underline{G}_v \leqslant G_v \leqslant 1/m\underline{G}_v$ for $v \in \mathcal{V}$, and $\underline{G}_e \leqslant G_e \leqslant 1/m\underline{G}_e$ for $e \in \mathcal{E}$. Therefore, by lemma 2.2, there is a morphism $\overline{f} : G \to 1/m\underline{G}$ induced by the inclusions of edge groups such that $\underline{f} \circ \overline{f}$ gives the inclusion of $\underline{G}_v$ into $1/m\underline{G}_v$ for all $v \in \mathcal{V}$, and similarly for all edge groups. Hence G has a primitive target. ☐

A subgroup $H \subset G$ is *separable* if H is the intersection of finite index subgroups of G. The following is well-known:

Lemma 4.2.

(1) *The intersection of separable subgroups of G is separable.*

(2) *A maximal abelian subgroup $A \leqslant G$ of a residually finite group is separable.*

Proof. Statement (4.2) follows from the definition. Statement (4.2) holds as follows: If $g \notin A$, then there exists $a \in A$ such that $k = gag^{-1}a^{-1} \neq 1$. By residual finiteness, there is a finite quotient $\phi : G \to G'$ such that $\phi(k) \neq 1$. Let $A' \leqslant G'$ be a maximal abelian subgroup containing $\phi(A)$, and note that $\phi(g) \notin A'$. Then A lies in the finite index subgroup $\phi^{-1}(A')$, but $g \notin \phi^{-1}(A')$. ☐

Theorem 4.3. *The following are equivalent:*

(1) *G is residually finite.*

(2) *G has a primitive domain.*

(3) *G has a primitive target.*

(4) *G is virtually primitive.*

(5) *G has separable edge groups.*

(6) *G has a regulating $\mathcal{E}$-tuple.*

Proof. (4.3 ⇒ 4.3) Each conjugate of a vertex group is a maximal abelian subgroup, and hence separable by lemma 4.2.(4.2). Each edge group is the intersection of conjugates of incident vertex groups. Hence the edge group is separable by lemma 4.2.(4.2).

(4.3 ⇒ 4.3) By the separability of each edge group G_e, there is a finite index subgroup $J^e \leqslant G$ such that $G_e \leqslant J^e$ and G_e is a direct factor of each of its vertex groups in J^e. Let $G' = \cap_e J^e$. Then G' is primitive.

(4.3 ⇒ 4.3) Since being virtually residually finite is equivalent to being residually finite, we will just show that primitive implies residually finite. Let G be a primitive tubular group. For each n, consider the morphism $nG \to G$ and its associated local quotient $q_n : G \to G /\!\!/ nG$. As $G /\!\!/ nG$ is a graph of finite groups, it is virtually free and hence residually finite. Therefore it suffices to show that for each nontrivial $g \in G$ there exists n such that $q_n(g)$ is nontrivial.

Either g is elliptic or g is hyperbolic with respect to the action on the associated Bass–Serre tree. If g is elliptic we can assume, after conjugation, that $g \in G_v$ regarded as $(p, q) \in \mathbb{Z}^2$. Choose $n > \max\{|p|, |q|\}$. Then $q_n(g)$ is nontrivial. If g is hyperbolic, then it has a normal form without any backtrack. We will explain how to choose n such that q_n also has a normal form without any backtrack. Each potential backtrack is of the form $t^{\pm 1} h t^{\mp 1}$ for some stable letter t and $h \in G_v$. Let G_e be the edge group associated to t, and note that $h \notin G_e$. By primitivity, $G_v = G_e \times \mathbb{Z} \cong \mathbb{Z}^2$ with $(1, 0)$ the generator of G_e. Since $h \notin G_e$, we have $h = (p, q)$ with $q \neq 0$. Hence, this potential backtrack is not a backtrack whenever $n > |q|$. Choosing n to satisfy this condition for each potential backtrack guarantees that $q_n(g)$ is nontrivial.

(4.3 ⇔ 4.3) This is lemma 4.1.

(4.3 ⇒ 4.3) Let $\underline{f} : \underline{G} \to G$ be the primitive domain for G. Let $G /\!\!/ \underline{G}$ be the associated local quotient. If G has an edge group generator g_e which has a proper root $1/k g_e \in G$, then $1/k g_e$ maps to a torsion element in $G /\!\!/ \underline{G}$. Note that $G /\!\!/ \underline{G}$ is virtually free as a graph of finite groups [**11**]. Let $F \leqslant G /\!\!/ \underline{G}$ be a finite index-free subgroup, and let $G' \leqslant G$ be the preimage of F in G. As G' is a finite index subgroup of G, it will also split as a tubular group over a finite graph Γ'. Finally, observe that G' is primitive as any proper root of an edge generator in G' would map to a torsion element in $G /\!\!/ \underline{G}$.

(4.3 ⇒ 4.3) Since finite index subgroups of primitive tubular groups are primitive, there also exists a finite index normal subgroup $G' \leqslant G$ such that G' is primitive. The induced splitting of G' shows that G' is also tubular, so inclusion of $G' \hookrightarrow G$ is a morphism of tubular groups. Let $p : \Gamma' \to \Gamma$ be the morphism of graphs associated to the inclusion. Let $v \in \mathcal{V}$. If $u', v' \in p^{-1}(v)$, then as G' is a normal subgroup the vertex groups $G'_{u'}$ and $G'_{v'}$ have identical images inside G_v. The analogous statement holds for each edge $e \in \mathcal{E}$.

We construct $\underline{G}$ from G' as follows: The vertex group $\underline{G}_v$ is the image of $G'_{v'}$ in G_v for some and hence any choice $v' \in p^{-1}(v)$. The edge group $\underline{G}_e$ is the image of $G'_{e'}$ in G_e for some and hence any choice $e' \in p^{-1}(e)$. The edge group inclusions of G' determine the edge group inclusions of $\underline{G}$. By lemma 2.2 we get a rigid morphism $\underline{F} : \underline{G} \to G$ determined by the inclusions of the vertex and edge groups. As G' is a primitive tubular group, $\underline{G}$ is also a primitive tubular group.

(4.3 ⇔ 4.3) This is lemma 3.2. □

5. The expansion sequences

Let G be a tubular group with underlying graph Γ. For each edge $e \in \mathcal{E}$ fix a choice of generator g_e of G_e. The *degree* $d_e^{\pm}$ of an attaching map $\varphi_e^{\pm}$ is the order of the torsion factor in $G_{\pm e}/\phi_e^{\pm}(G_e)$. Let $d_e = \operatorname{lcm}\{d_e^+, d_e^-\}$. We refer to the tuple $(d_e)_{e\in\mathcal{E}}$ as the *edge degrees.*

Define a tubular group G' with underlying graph Γ as follows: The edge group $G'_e = 1/d_e G_e$ and the vertex group $G'_v = \langle G_v, H_v\rangle$, where

$$H_v = \left\{ \frac{1}{d_e}\phi_e^+(g_e) \mid e \in \mathcal{E}, +e = v \right\} \cup \left\{ \frac{1}{d_e}\phi_e^-(g_e) \mid e \in \mathcal{E}, -e = v \right\}.$$

As $1/d_e\phi_e^+(g_e) \in G'_v$, for all $e \in \mathcal{E}$ such that $+e = v$, we obtain the edge map $\phi_e'^+ : G'_e \to G'_v$ by extending ϕ_e^+ linearly. The inclusions $p_v : G_v \to G'_v$ and $p_e : G_e \to G'_e$ determine a rigid morphism $p : G \to G'$ called the *expansion morphism.* An expansion is *trivial* if it is the identity map. This occurs precisely when G is primitive.

The following lemma shows that there is a bound on the complexity of the tubular group produced by the expansion morphism.

Lemma 5.1. *Let G be a tubular group and $(d_e)_{e\in\mathcal{E}}$ the edge degrees. Let $\ell = \operatorname{lcm}\{d_e \mid e \in \mathcal{E}\}$. Let $G \to G'$ be the expansion morphism, and $(d'_e)_{e\in\mathcal{E}}$ be the edge degrees of G'. Then d'_e divides ℓ for all $e \in \mathcal{E}$.*

Proof. Let $v = +e$. Let $K \leqslant G_v$ be the maximal cyclic subgroup of G_v containing $\phi_e^+(G_e)$. Then d_e^+ is the order of the quotient $K/\phi_e^+(G_e)$. Let $K' \leqslant G'_v$ be the maximal cyclic subgroup of G'_v containing ${\phi'}_e^+(G'_e)$. The claim follows by showing that ℓ is divided by the order of $K'/{\phi'}_e^+(G'_e)$.

First note that $K = \langle 1/d_e^+\phi_e^+(g_e)\rangle \leqslant \langle 1/d_e\phi_e^+(g_e)\rangle = {\phi'}_e^+(G'_e)$. Second note that $G'_v = \langle G_v, 1/d_e\phi_e^+(g_e), \ldots\rangle \leqslant 1/\ell G_v$, so $K' \leqslant 1/\ell K$. Together this implies that $K \leqslant {\phi'}_e^+(G'_e) \leqslant K' \leqslant 1/\ell K$ so the order of $K'/{\phi'}_e^+(G'_e)$ is a factor of ℓ. □

Lemma 5.2. *Let $p : G \to G'$ be the expansion map. If G has a primitive target $\overline{f} : G \to \overline{G}$, then $\overline{f}$ factors as $\overline{f} = \overline{p} \circ p$ for some rigid morphism $\overline{p} : G' \to \overline{G}$.*

Proof. The vertex and edge groups of G can be viewed as subgroups of the corresponding vertex and edge groups of both $\overline{G}$ and G'. We deduce that $G_e \leqslant G'_e \leqslant \overline{G}_e$ as $\overline{G}$ is primitive, so $1/d_eG_e$ must be a subgroup of $\overline{G}_e$. Similarly, $G_v \leqslant G'_v \leqslant \overline{G}_v$ as the primitivity of $\overline{G}$ implies that $1/d_e\phi_e^{\pm}(g_e)$ must be in $\overline{G}_v$ for $v = \pm e$. Therefore, by lemma 2.2 there exists a rigid morphism $\overline{p} : G' \to \overline{G}$ such that $\overline{p} \circ p = \overline{f}$. □

Lemma 5.3. *Let $p : G \to G'$ be an expansion map. Then $p(G) = G'$.*

Proof. Recall that for each edge $e \in \mathcal{E}$ we fixed a generator g_e. We then let $d_e^{\pm}$ be the degree of the attaching map $\phi_e^{\pm}$, and $d_e = \mathrm{lcm}\{d_e^+, d_e^-\}$. Then $G'_e = 1/d_e G_e$, and $G'_{\pm e}$ were defined to include the element $1/d_e \phi_e^{\pm}(g_e)$, for all incident edges e. Note that $1/d_e^{\pm}\phi_e^{\pm}(g_e)$ was already an element of $G_{\pm e}$, since $d_e^{\pm}$ was the order of the torsion factor in $G_{\pm e}/\phi_e^{\pm}(G_e)$. Therefore $1/d_e^{\pm} g_e$ and thus $1/d_e g_e$ will be in the image of p. It then follows that G'_e is contained in $p(G)$ for all edges e, and therefore G'_v is contained in $p(G)$ for all $v \in \mathcal{V}$. □

An *expansion sequence* is a sequence of nontrivial expansions

$$G \to G_1 \to G_2 \to \cdots \to G_t \to \cdots$$

The following asserts that a finite expansion sequence is equivalent to residual finiteness.

Lemma 5.4. *If G has a primitive target then any expansion sequence starting with G has length bounded by $\sum_e [\overline{G_e} : G_e]$.*

Conversely, if the expansion sequence $G \to \cdots \to G_t$ terminates *in the sense that it cannot be extended, then G_t is primitive, and hence G has a primitive target.*

Proof. Let $\overline{f} : G \to \overline{G}$ be a primitive target for G. By lemma 5.2, $\overline{f}$ factors through the map $G \to G_m$ for each m. Therefore, the sum of the degrees of each edge group G_e in $\overline{G}_e$ provides an upper bound on the length of a sequence of edge expansions.

The composition $G = G_1 \to G_t = \overline{G}$ yields the converse. For if G_t is not primitive then $d_e \neq 1$ for some edge e. Hence there is a nontrivial expansion of G_t. □

The expansion sequence is computable so lemma 5.4 shows that there is an algorithm which can find a primitive target, should one exist. Specifically, the algorithm would perform edge expansions until the expansion sequence terminates. An effective algorithm would also need to identify when G is nonresidually finite. Suppose that $G \to G_1 \to G_2 \to \cdots$ is a nonterminating expansion sequence. Then we say the expansion sequence is *recurrent* if G_i is isomorphic to G_j via some rigid isomorphism, for some $i < j$. Therefore if either a terminating or a recurrent expansion sequence could be found in finite time, the question of residual finiteness would be algorithmically decidable. Unfortunately, in general, there are nonresidually finite tubular groups with nonrecurrent, infinite expansion sequences.

Example 5.5. Let

$$G = G_0 = \langle \mathbb{Z} \times \mathbb{Z}, s, t \mid s(1,0)s^{-1} = (2,0),\ t(0,1)t^{-1} = (1,1)\rangle.$$

There is a single nonprimitive vector $(2,0)$ among the relations so the first edge expansion is given by dividing the first edge group by two to obtain

$$G_1 = \langle \tfrac{1}{2}\mathbb{Z} \times \mathbb{Z}, s, t \mid s(\tfrac{1}{2},0)s^{-1} = (1,0),\ t(0,1)t^{-1} = (1,1)\rangle.$$

Observe that the elements $(0,1)$ and $(1,1)$ remain primitive in $1/2\mathbb{Z} \times \mathbb{Z}$, so the only nonprimitive element in the relations is $(1,0)$. Therefore the n-th term in the expansion sequence is:

$$G_n = \langle \tfrac{1}{2^n}\mathbb{Z} \times \mathbb{Z}, s, t \mid s(\tfrac{1}{2^n}, 0)s^{-1} = (\tfrac{1}{2^{n-1}}, 0), t(0,1)t^{-1} = (1,1)\rangle$$

Thus the expansion sequence does not terminate so G is not residually finite. But $G_n \neq G_m$ for $n \neq m$. Indeed, since all maximal rank 2 free abelian groups in G_n are conjugate to the vertex group $1/2^n\mathbb{Z} \times \mathbb{Z}$, we can assume that an isomorphism $G_n \to G_m$ sends $1/2^n\mathbb{Z} \times \mathbb{Z}$ to $1/2^m\mathbb{Z} \times \mathbb{Z}$. Any conjugate of the vertex group in G_n that nontrivially intersects $1/2^n\mathbb{Z} \times \mathbb{Z}$ does so in a cyclic subgroup $\langle(1/2^n, 0)\rangle$, $\langle(1/2^{n-1}, 0)\rangle$, $\langle(0,1)\rangle$ and $\langle(1,1)\rangle$. Similarly, in G_m nontrivial intersections of conjugates of the vertex group intersect $1/2^m\mathbb{Z} \times \mathbb{Z}$ in the cyclic subgroups $\langle(1/2^m, 0)\rangle$, $\langle(1/2^{m-1}, 0)\rangle$, $\langle(0,1)\rangle$ and $\langle(1,1)\rangle$. By identifying $G_n \cong \mathbb{Z}^2$ we can compute the *unsigned intersection numbers* of these cyclic subgroups. The unsigned intersection number of $\langle(p,q)\rangle$ and $\langle(r,s)\rangle$ is the absolute value of the determinant of the matrix $\big(p \quad rq \quad s\big)$. The unsigned intersection number is invariant up to multiplication by elements of $GL_2(\mathbb{Z})$. So, as any isomorphism $G_n \to G_m$ must send conjugates of vertex groups to conjugates of vertex groups, the unsigned intersection numbers are an invariant of G_n. The largest intersection number of G_n is 2^n and is achieved by the vectors $(0,1)$ and $(1,1)$, which are identified with $(0,1)$ and $(2^n, 1)$ when $(G_n)_v$ is identified with $\mathbb{Z}^2$. Therefore G_n is not isomorphic to G_m if $n \neq m$.

Note that if we consider the subtubular group

$$G' = G'_0 = \langle \mathbb{Z} \times \mathbb{Z}, s \mid s(1,0)s^{-1} = (2,0)\rangle.$$

Then we can compute that

$$G'_1 = \langle \tfrac{1}{2}\mathbb{Z} \times \mathbb{Z}, s \mid s(\tfrac{1}{2}, 0)s^{-1} = (1,0)\rangle.$$

As there is a rigid isomorphism $G'_1 \to G'$ we deduce that G' is recurrent.

Consideration of examples and computer experiments leads to the following:

PROBLEM 5.6. Does every nonresidually finite tubular group contain a subtubular group with recurrent expansion sequence?

The following example illustrates that even a terminating expansion sequence can be arbitrarily long for a fixed graph Γ.

EXAMPLE 5.7. For each n let $G^{(n)}$ be the tubular group presented by:

$$\langle \mathbb{Z} \times \mathbb{Z}, t \mid t(1,0)t^{-1} = (2, 2^n)\rangle$$

$G^{(n-1)}$ is isomorphic to the expansion of $G^{(n)}$ which has the following presentation:

$$\langle \tfrac{1}{2}\mathbb{Z} \times \mathbb{Z}, t \mid t(\tfrac{1}{2}, 0)t^{-1} = (1, 2^{n-1})\rangle$$

We thus have the terminating expansion sequence $G^{(n)} \to G^{(n-1)} \to \cdots \to G^{(1)} \to G^{(0)}$. So the expansion sequence of $G^{(n)}$ has length $n+1$.

6. The residually finite snowflake groups

Snowflake groups are the following tubular groups for positive integers $p \geqslant q$:

$$G_{pq} = \langle \mathbb{Z}^2, s, t \mid (q, 0)^s = (p, 1), (q, 0)^t = (p, -1) \rangle$$

Brady and Bridson showed that G_{pq} has Dehn function $\simeq n^{2\alpha}$ for $\alpha = \log_2(2p/q)$ in [3]. Gardam and Woodhouse showed that many snowflake groups are finite index subgroups of one-relator groups [9]. This provided examples of nonautomatic one-relator groups that do not contain Baumslag–Solitar subgroups of the form $BS(m, n) = \langle a, t \mid (a^m)^t = a^n \rangle$ with $m \neq \pm n$. Subsequently, Button observed that some of these one-relator groups are CAT(0) but not residually finite and has classified the residually finite snowflake groups [6]. We now reproduce his classification using our method.

Theorem 6.1. *G_{pq} is residually finite if and only if q divides $2p$.*

Proof. If $q = 1$ then G_{pq} is a primitive tubular group and hence residually finite by theorem 4.3. If $q > 1$ then we perform the expansion map $G_{pq} \to G'_{pq}$ where each edge group is divided by q. The vertex group of G'_{pq} is:

$$\left\langle \left(\frac{p}{q}, \frac{1}{q}\right), \left(\frac{p}{q}, -\frac{1}{q}\right), (1, 0), (0, 1) \right\rangle$$

We swap the components of these generators, scale them by q and set them as the rows of a matrix below. We obtain a two-element basis by performing integer row operations to reduce the matrix to Hermite normal form:

$$\begin{bmatrix} 1 & p \\ -1 & p \\ 0 & q \\ q & 0 \end{bmatrix} \to \begin{bmatrix} 1 & p \\ 0 & 2p \\ 0 & q \\ 0 & -qp \end{bmatrix} \to \begin{bmatrix} 1 & p \\ 0 & \gcd(2p, q) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus G'_{pq} has the following presentation:

$$G'_{pq} = \left\langle \left(\frac{\gcd(q, 2p)}{q}, 0\right), \left(\frac{p}{q}, \frac{1}{q}\right), s, t \;\middle|\; (1, 0)^s = \left(\frac{p}{q}, \frac{1}{q}\right), (1, 0)^t = \left(\frac{p}{q}, \frac{-1}{q}\right) \right\rangle.$$

If $q \mid 2p$ then G'_{pq} is primitive and hence G_{pq} is residually finite, by lemma 5.4 and theorem 4.3. Otherwise, G'_{pq} is not primitive and has a nontrivial expansion map where each edge group is divided by the degree of the torsion factor in

$$\left\langle \left(\frac{\gcd(q, 2p)}{q}, 0\right), \left(\frac{p}{q}, \frac{1}{q}\right) \right\rangle \Big/ \langle (1, 0) \rangle.$$

Since the vertex group in G'_{pq} is generated by the elements conjugated by s and t we deduce that the expansion map $G'_{pq} \to G''_{pq}$ is a scaling morphism and therefore an isomorphism. Thus, the expansion sequence is recurrent if $q \nmid 2p$ and so G_{pq} is not residually finite by lemma 5.4 and theorem 4.3. ☐

7. Deciding residual finiteness for single vertex group

Let G be a tubular group with a single vertex group G_v. We will show that the problem of determining the residual finiteness of G is decidable.

Proposition 7.1. *Let G be a tubular group with a single vertex group G_v. Assume that G has at least two edges and that $\langle \varphi_e^+(G_e), \varphi_e^-(G_e)\rangle < G_v$ has rank 2 for every edge e. Let $e_1, \dots, e_n$ be the edges in the underlying graph of G. Let $u_i, v_i \in G_v$ correspond to the generators of the cyclic subgroups of G_v conjugated by the stable letter associated to e_i. Let $t_i \in \mathbb{Q}_{>0}$ be minimal such that $t_i u_i \in \langle u_{i+1}, v_{i+1}\rangle$, where the indices are considered modulo n. Let $\underline{k} = (k_e)_{e\in\mathcal{E}}$ be given by $k_{e_i} = k_i$ and write $\underline{k} = (k_1, k_2, \dots, k_n)$. If $\underline{k}$ is regulating then*

$$\underline{k} = \left(m, m\frac{z_1}{t_1}, \dots, m\frac{z_1 z_2 \cdots z_{n-1}}{t_1 t_2 \cdots t_{n-1}}\right)$$

for some $m, z_1, z_2, \dots, z_n \in \mathbb{Z}$ for which $z_1 \cdots z_n = t_1 \cdots t_n$.

Proof. Suppose $\underline{k}$ is regulating. Then

$$\frac{k_{i+1}t_i}{k_i} k_i u_i = k_{i+1} t_i u_i \in k_{i+1}\langle u_{i+1}, v_{i+1}\rangle \leqslant G_v^{(\underline{k})}.$$

Since $k_i u_i$ is primitive in $G_v^{(\underline{k})}$ by the definition of regulating, we deduce that $k_{i+1}t_i/k_i = z_i$ for some integer $z_i \in \mathbb{Z}$. Hence $k_{i+1}/k_i = z_i/t_i$ and so

$$\frac{z_1 \cdots z_n}{t_1 \cdots t_n} = \frac{k_2}{k_1}\frac{k_3}{k_2} \cdots \frac{k_n}{k_{n-1}}\frac{k_1}{k_n} = 1.$$

Setting $m = k_1$ we recover the claim. □

We apply proposition 7.1 in the following example.

Example 7.2. Let G be the tubular group with the following presentation.

$$\langle \mathbb{Z}\times\mathbb{Z}, s, t \mid (2,-4)^s = (-1,-2),\ (-6,-6)^t = (2,2)\rangle$$

Following proposition 7.1, let $u_1 = (2,-4)$, $v_1 = (-1,-2)$, $u_2 = (-6,6)$, $v_2 = (2,2)$, and compute that $t_1 = 2$ and $t_2 = 4/3$. Since $t_1 t_2$ is not an integer, there do not exist integers z_1 and z_2 such that $z_1 z_2 = t_1 t_2$. Hence, proposition 7.1 implies that G has no regulating $\mathcal{E}$-tuple. Hence G is not residually finite, by theorem 4.3.

Example 7.3. The snowflake group G_{pq} is the tubular group presented by

$$\left\langle \mathbb{Z}^2, s, t \mid (q,0)^s = (p,1), (q,0)^t = (p,-1)\right\rangle$$

for positive integers $p \geqslant q$. Following proposition 7.1, let $u_1 = u_2 = (q,0)$, $v_1 = (p,1)$, $v_2 = (p,-1)$, and compute that $t_1 = t_2 = 1$. Then, by proposition 7.1 and remark 3.1, there is a regulating $\mathcal{E}$-tuple for G_{pq} if and only if $(k_1,k_2) = (1,1)$ is a regulating $\mathcal{E}$-tuple. That is, if and only if $(q, 0)$, $(p, 1)$ and $(p, -1)$ are primitive in the subgroup $H = \langle (q, 0), (p, 1), (p, -1)\rangle$ that they generate. If

$$r(p, \pm 1) = a(q, 0) + b(p, \mp 1)$$

for some $r \in \mathbb{Q}$ and $a, b \in \mathbb{Z}$ then $r = -b \in \mathbb{Z}$ and so $(p, \pm 1)$ is always primitive in H. On the other hand

$$r(q, 0) = a(p, 1) + b(p, -1)$$

holds for some $r \in \mathbb{Q}$ and $a, b \in \mathbb{Z}$ if and only if $a = b$ and $r = \frac{2p}{q}a$. Hence $(q, 0)$ is primitive in H if and only if $q|2p$. Thus we see that G_{pq} has a regulating $\mathcal{E}$-tuple if and only if $q|2p$.

Theorem 1.2 follows from theorem 4.3 and the following lemma.

Lemma 7.4. *Let G be a tubular group with a single vertex group G_v. There is an algorithm which determines if G has a regulating $\mathcal{E}$-tuple.*

Proof. The algorithm first checks to see if the images $\varphi_e^{\pm}(G_e)$ of any edge group G_e are commensurable but distinct in G_v. In such a case we have $\varphi_e^{+}(k_e g_e) = q\varphi_e^{-}(k_e g_e)$ for some $q \in \mathbb{Q} - \{1\}$ where g_e is a generator of G_e. Then the $\varphi_e^{\pm}(k_e g_e)$ cannot both be primitive in any subgroup of G_v so no $\mathcal{E}$-tuple is regulating and the algorithm may return a 'no' answer.

Henceforth we assume that if $\varphi_e^{\pm}(G_e)$ are commensurable for some $e \in \mathcal{E}$ then they are equal. Let G' be the subtubular group obtained from G by removing an edge e for which the $\varphi_e^{\pm}(G_e)$ are equal. Given a regulating $\mathcal{E}'$-tuple $\underline{k}'$ for G' we may obtain a regulating $\mathcal{E}$-tuple $\underline{k}$ for G as follows. If $G_v^{(\underline{k}')} \cap \varphi_e^{\pm}(G_e)$ is trivial then we obtain $\underline{k}$ by extending $\underline{k}'$ with any $k_e \in \mathbb{Z} - \{0\}$. Otherwise, let $q \in \mathbb{Q}_{>0}$ be minimal such that $q\varphi_e^{\pm}(G_e) < G_v^{(\underline{k}')}$ and choose $m \in \mathbb{Z} - \{0\}$ such that $mq \in \mathbb{Z}$. We obtain $\underline{k}$ by extending $m\underline{k}' = (mk_e')_{e\in\mathcal{E}'}$ with $k_e = mq$.

Thus the algorithm discards all edges e for which the $\varphi_e^{\pm}(G_e)$. If G has a single edge group G_e then any $k_e \in \mathbb{Z} - \{0\}$ gives a regulating $\underline{k}$ and so the algorithm returns a 'yes' answer in this case.

At this point in the algorithm G has at least two edges and for each edge e the $\varphi_e^{\pm}(G_e)$ are not commensurable. By proposition 7.1 and remark 3.1, we need to only consider finitely many integers $z_1, \ldots, z_n$ and m to check if G has a regulating $\mathcal{E}$-tuple. For each $z_1, \ldots, z_n$ and m we compute the corresponding $G_v^{(\underline{k})}$ and determine whether the $k_i u_i$ and $k_i v_i$ are primitive in $G_v^{(\underline{k})}$. □

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