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Boundary blow-up elliptic problems with nonlinear gradient terms and singular weights

Zhijun Zhang
School of Mathematics and Informational Science, Yantai University, Yantai, Shandong 264005, People's Republic of China
(zhangzj@ytu.edu.cn)

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By Karamata regular variation theory, a perturbation method and construction of comparison functions, we show the exact asymptotic behaviour of solutions near the boundary to nonlinear elliptic problems $\Delta u \pm |\nabla u|^q = b(x)g(u)$, $u > 0$ in Ω, $u|_{\partial\Omega} = \infty$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $q > 0$, $g \in C^1[0,\infty)$ is increasing on $[0,\infty)$, $g(0) = 0$, g' is regularly varying at infinity with positive index ρ and b is non-negative in Ω and is singular on the boundary.

1. Introduction and the main results

The purpose of this paper is to investigate the exact asymptotic behaviour of solutions near the boundary to the following model problems:

$$\Delta u \pm |\nabla u|^q = b(x)g(u), \quad u > 0 \in \Omega, \quad u|_{\partial\Omega} = \infty, \tag{$P_\pm$}$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0$, and the solution is called 'a large solution' or 'an explosive solution', Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geqslant 1$, $q > 0$, g have the properties that

(g_1) $g \in C^1[0,\infty)$, $g(0) = 0$, g is increasing on $[0,\infty)$,

(g_2) the Keller–Osserman condition

$$\int_t^\infty \frac{\mathrm{d}s}{\sqrt{2G(s)}} < \infty \quad \text{for all } t > 0, \quad G(s) = \int_0^s g(z)\,\mathrm{d}z$$

holds,

and b satisfies the condition

(b_1) $b \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$ is non-negative in Ω and $\lim_{d(x)\to 0} b(x) = \infty$.

The main feature of this paper is the presence of the three terms: the nonlinear term $g(u)$, which is regularly varying at infinity with index $1+\rho$, $\rho > 0$, and includes a large class of functions; the nonlinear gradient term $\pm|\nabla u|^q$; and the weight b, which is singular on the boundary, and also includes a large class of functions.

First, let us review the following model:

$$\Delta u = b(x)g(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = +\infty. \tag{P_0}$$

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The problem (P_0) arises from many branches of mathematics and applied mathematics and has been discussed by many authors and in many contexts (see, for example, [2–14, 18–21, 23–31, 33, 34, 36–38, 41–43, 45, 46]).

Problem (P_0) was studied much earlier for the case in which $b \equiv 1$ on Ω for $g(u) = \mathrm{e}^u$. Bieberbach [3] proved that when $N = 2$ there is one solution $u \in C^2(\Omega)$ satisfying the condition that $|u(x) - \ln(d(x))^{-2}|$ is bounded on Ω. The same result was proved by Rademacher [37] for $N = 3$. For more general non-decreasing functions g, Keller and Osserman [19, 34] first supplied a necessary and sufficient condition (g_2) for the existence of large solutions to problem (P_0). Later, Loewner and Nirenberg [26] showed that if $g(u) = u^{p_0}$ with $p_0 = (N+2)/(N-2)$, $N > 2$, then problem (P_0) has a unique positive solution u satisfying

$$\lim_{d(x)\to 0} u(x)(d(x))^{(N-2)/2} = (N(N-2)/4)^{(N-2)/4}.$$

Then, by analysing the corresponding ordinary differential equation and combining it with the maximum principle, Bandle and Marcus [2] established the following results: if g satisfies (g_1) and

(g_3) there exist $\theta > 0$ and $s_0 \geqslant 1$ such that $g(\xi s) \leqslant \xi^{1+\theta} g(s)$ for all $\xi \in (0,1)$ and $s \geqslant s_0/\xi$,

then, for any solution u of problem (P_0)

$$\frac{u(x)}{\psi_1(d(x))} \to 1 \quad \text{as } d(x) \to 0, \tag{1.1}$$

where ψ_1 satisfies

$$\int_{\psi_1(t)}^{\infty} \frac{\mathrm{d}s}{\sqrt{2G(s)}} = t \quad \text{for all } t > 0 \tag{1.2}$$

and, in addition to the conditions given above, g satisfies the condition that

(g_4) $g(s)/s$ is increasing on $(0, \infty)$,

then problem (P_0) has a unique solution.

Lazer and McKenna [24] showed that if g satisfies (g_1) and

(g_5) there exists $a_1 > 0$ such that $g'(s)$ is non-decreasing for $s \geqslant a_1$, and

$$\lim_{s\to\infty} \frac{g'(s)}{\sqrt{G(s)}} = \infty,$$

then, for any solution u of problem (P_0),

$$u(x) - \psi_1(d(x)) \to 0 \quad \text{as } d(x) \to 0. \tag{1.3}$$

Now we introduce a class of functions.

Let Λ denote the set of all positive monotonic functions $k \in L^1(0,\nu) \cap C^1(0,\nu)$ which satisfy

$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K(t)}{k(t)}\right) = C_k \in [0, \infty), \tag{1.4}$$

where

$$K(t) = \int_0^t k(s)\,\mathrm{d}s. \tag{1.5}$$

We note that, for each $k \in \Lambda$, $\lim_{t\to 0^+} K(t)/k(t) = 0$, $C_k \in [0,1]$ if k is non-decreasing and $C_k \geqslant 1$ if k is non-increasing.

Most recently, applying the regular variation theory, which was first introduced and established by Karamata in 1930 and is a basic tool in the stochastic process, and constructing comparison functions, Cîrstea and co-workers [6–9] showed the uniqueness and exact asymptotic behaviour of solutions near the boundary to problem (P_0). A basic result is that if g satisfies (g_1) and

(g_6) there exists $\rho > 0$ such that $\lim_{s\to\infty} g'(\xi s)/g'(s) = \xi^\rho$ for all $\xi > 0$

and $b \in C^\alpha(\bar{\Omega})$, $b \geqslant 0$ in Ω, and satisfies

(b_2) $\lim_{d(x)\to 0} b(x)/k^2(d(x)) = c_0 > 0$ for some non-decreasing function $k \in \Lambda$,

then any solution u of problem (P_0) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(d(x))} = \xi_0, \tag{1.6}$$

where

$$\xi_0 = \left(\frac{2 + \rho C_k}{c_0(2+\rho)}\right)^{1/\rho}$$

and $\psi \in C^2(0,a)$, $a \in (0,\nu)$, is defined by

$$\int_{\psi(t)}^\infty \frac{\mathrm{d}s}{\sqrt{2G(s)}} = K(t) \quad \text{for all } t \in (0,a). \tag{1.7}$$

Zhang [46] considered problem (P_0) for the case $k(t) = t^{\sigma/2}$ with $\sigma > -2$, and Mohammed [33] generalized the above results to non-increasing function $k \in \Lambda$.

Now let us return to problem $(P_\pm)$.

When $b \equiv 1$ on Ω, Lasry and Lions [22] established the existence, uniqueness and exact asymptotic behaviour of solutions near the boundary to problem (P_-) for $g(u) = \lambda u$ with $\lambda > 0$ and $q > 1$; when $g(u) = u^p$, $p > 0$, by the ordinary differential equation theory and the comparison principle, Bandle and Giarrusso [1] showed the following results:

(i) if $p > 1$ or $q > 1$, then problem (P_+) has one solution in $C^2(\Omega)$, and the same statement is true for problem (P_-) if $p > 1$ or $1 < q \leqslant 2$;

(ii) if $p > 1$ and $0 < q < 2p/(p+1)$, then every solution $u_\pm$ of problem $(P_\pm)$ satisfies (1.1), where $\psi_1(t) = (\sqrt{2(p+1)}/(p-1))^{2/(p-1)} t^{-2/(p-1)}$;

(iii) if $2p/(p+1) < q < p$, then, for any solution u_+ of problem (P_+),

$$\lim_{d(x)\to 0} u_+(x) \left(\frac{p-q}{q} d(x)\right)^{q/(p-q)} = 1; \tag{1.8}$$

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(iv) if $\max\{2p/(p+1), 1\} < q < 2$, then every solution u_- of problem (P_-) satisfies

$$\lim_{d(x)\to 0} u_-(x)(2-q)((q-1)d(x))^{(2-q)/(q-1)} = 1; \tag{1.9}$$

(v) if $q = 2$, then every solution u_- of problem (P_-) satisfies

$$\lim_{d(x)\to 0} u_-(x)/\ln(d(x)) = 1. \tag{1.10}$$

Moreover, Bandle and Giarrusso [1] and Giarrusso [15,16] extended the above results for more general $g(u)$ satisfying

$$\lim_{u\to\infty} \frac{\sqrt{G(u)}}{(g(u))^{1/q}} = c_1 \in [0, \infty].$$

Most recently, Porretta and Véron [35] gave the precise expression of the solution for the case $g(u) = u^p$ with $p > 0$. Zhang [49] generalized the above results in [1, 15, 16] to problem $(P_\pm)$ for the case that $b \in C^\alpha(\bar{\Omega})$, $b \geqslant 0$ in Ω, and considered problem $(P_\pm)$ when $0 < q < 2(\rho + 1)/(\rho + 2)$ and $k(t) = t^{\sigma/2}$ with $\sigma > -2$ [47].

For other existence results of large solutions to elliptic problems with nonlinear gradient terms, see [44, 48, 50] and the references therein.

In this paper, by Karamata regular variation theory, a perturbation method and constructing comparison functions, we reveal that how the singular weights b affect the exact asymptotic behaviour of solutions near the boundary to problems $(P_\pm)$.

First let us recall some basic definitions and the properties to Karamata regular variation theory [32, 39, 40].

Definition 1.1. A positive measurable function f defined on $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity with index ρ, written $f \in RV_\rho$, if, for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^\rho. \tag{1.11}$$

The real number ρ is called the index of regular variation.

In particular, when $\rho = 0$, we have the following.

Definition 1.2. A positive measurable function L defined on $[a, \infty)$, for some $a > 0$, is called slowly varying at infinity, if, for each $\xi > 0$,

$$\lim_{t\to\infty} \frac{L(\xi t)}{L(t)} = 1. \tag{1.12}$$

It follows by definitions 1.1 and 1.2 that if $f \in RV_\rho$ it can be represented in the form

$$f(t) = t^\rho L(t). \tag{1.13}$$

Lemma 1.3 (uniform convergence theorem). *If $f \in RV_\rho$, then (1.11) (and thus (1.12)) holds uniformly for $\xi \in [a, b]$ with $0 < a < b$.*

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LEMMA 1.4 (representation theorem). *The function L is slowly varying at infinity if and only if it may be written in the form*

$$L(t) = c(t) \exp\left(\int_a^t \frac{y(s)}{s} \, ds \right), \quad u \geqslant a, \tag{1.14}$$

for some $a > 0$, where $c(t)$ and $y(t)$ are measurable and $t \to \infty$, $y(t) \to 0$ and $c(t) \to c$, with $c > 0$.

As the corresponding to definitions 1.1 and 1.2, we have the following.

DEFINITION 1.5. A positive measurable function h defined on $(0, a)$, for some $a > 0$, is called regularly varying at zero with index σ, written $h \in RVZ_\sigma$, if, for each $\xi > 0$ and some $\sigma \in \mathbb{R}$,

$$\lim_{t \to 0^+} \frac{h(\xi t)}{h(t)} = \xi^\sigma. \tag{1.15}$$

In particular, when $\sigma = 0$, we have the following.

DEFINITION 1.6. A positive measurable function H defined on $(0, a)$, for some $a > 0$, is called slowly varying at zero, if for each $\xi > 0$

$$\lim_{t \to 0^+} \frac{H(\xi t)}{H(t)} = 1. \tag{1.16}$$

It follows by definitions 1.5 and 1.6 that if $h \in RVZ_\sigma$, then it can be represented in the form

$$h(t) = t^\sigma H(t). \tag{1.17}$$

We note that definition 1.1 is equivalent to saying that $f(1/t)$ is regularly varying at zero with index $-\rho$.

Our main results are the following.

THEOREM 1.7. *Let g satisfy (g_1), $g' \in RV_\rho$ with $\rho > 1$; let b satisfy (b_1) and let*

(b_3) $\lim_{d(x) \to 0} b(x)/k^q(d(x)) = c_q > 0$ *for some non-increasing function $k \in \Lambda$.*

If $2 \leqslant q < \rho + 1$, then every solution $u_+ \in C^2(\Omega)$ to problem (P_+) satisfies

$$\lim_{d(x) \to 0} \frac{u_+(x)}{\varphi(K(d(x)))} = c_q^{-1/(\rho - q + 1)}, \tag{1.18}$$

where $\varphi \in C^2(0, \infty)$ is uniquely determined by

$$\int_{\varphi(t)}^\infty \frac{ds}{(g(s))^{1/q}} = t \quad \textit{for all } t > 0. \tag{1.19}$$

Moreover, $\varphi \in RVZ_{-q/(\rho+1-q)}$ and there exists $H \in RVZ_0$ such that

$$\varphi(t) = H(t) t^{-q/(\rho+1-q)}. \tag{1.20}$$

THEOREM 1.8. *Let g satisfy (g_1), $g' \in RV_\rho$ with $\rho > 0$, let*

$$\frac{2(\rho+1)}{\rho+2} < q < \min\{2, \rho+1\},$$

and let b satisfy (b_1). Suppose that there exists some non-increasing function $k \in C^1(0,\nu)$, $k' \in RVZ_{-\sigma-1}$ with $\sigma \in (0,1)$ and a positive constant c_q such that (b_3) holds. If

$$\frac{\sigma(2-q)}{1-\sigma} < \frac{q(2+\rho)-2(1+\rho)}{\rho+1-q} \quad \text{and} \quad \frac{\sigma+1+q\sigma}{1-\sigma} < \frac{(q-1)(1+\rho)}{\rho+1-q}, \tag{1.21}$$

then the conclusion of theorem 1.7 holds.

THEOREM 1.9. *Let g satisfy (g_1), $g' \in RV_\rho$ with $\rho > 0$, let b* satisfy *(b_1) and*

(b_{21}) $\lim_{d(x)\to 0} b(x)/k^2(d(x)) = c_0 > 0$ *for some non-increasing $k \in \Lambda$.*

If either $0 < q < 2(1+\rho)/(2+\rho)$, or $g(u) = u^{1+\rho}$ with $q = 2(1+\rho)/(2+\rho)$, then every solution $u_\pm \in C^2(\Omega)$ to problem $(P_\pm)$ satisfies

$$\lim_{d(x)\to 0} \frac{u_\pm(x)}{\psi_1(K(d(x)))} = \left(\frac{2+\rho+\rho(C_k-1)}{c_0(2+\rho)}\right)^{1/\rho}, \tag{1.22}$$

where $\psi_1 \in C^2(0,a]$ is uniquely determined by (1.2). Moreover, $\psi_1 \in RVZ_{-2/\rho}$, and there exists $H \in RVZ_0$ such that $\psi_1(t) = H(t)t^{-2/\rho}$.

THEOREM 1.10. *Let $q > 0$, let g satisfy (g_1), (g_2) and $\lim_{u\to\infty} G(u)/(g(u))^{2/q} \in [0,\infty)$, or g satisfy (g_1), where $(g(u))^{2/q}/u$ is increasing and*

$$\int_t^\infty \frac{\mathrm{d}s}{(g(s))^{1/q}} < \infty \quad \text{for all } t > 0, \qquad \lim_{u\to\infty} \frac{G(u)}{(g(u))^{2/q}} \in (0,\infty].$$

If $b \in C^\alpha(\Omega)$ is positive in Ω and

(b_4) *the linear Poisson's problem*

$$-\Delta v = b(x), \quad v > 0, \quad x \in \Omega, \quad v|_{\partial\Omega} = 0 \tag{1.23}$$

has a unique solution $\bar{v} \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$, then problem (P_+) has at least one solution $u_+ \in C^2(\Omega)$.

REMARK 1.11. By (1.2), we see that the asymptotic behaviour (1.22) of $u_\pm$ is independent of $\pm|\nabla u_\pm|^q$ when $0 < q < 2(1+\rho)/(2+\rho)$, and so is that when $g(u) = u^{1+\rho}$ with $\rho > 0$ and $q = 2(1+\rho)/(2+\rho)$.

REMARK 1.12. Some basic examples of g which satisfies (g_1) and $g' \in RV_\rho$ with $\rho > 0$ are

(i) $g(u) = u^{\rho+1}$,

(ii) $g(u) = u^{\rho+1}(\ln(u+1))^\beta$, $\beta > 0$,

(iii) $g(u) = u^{\rho+1} \arctan u$,

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(iv)
$$g(u) = c_0 u^{\rho+1} \exp\left(\int_0^u \frac{y(s)}{s}\,\mathrm{d}s\right), \quad u \geqslant 0,$$
where $c_0 > 0$, $y \in C[0,\infty)$ is non-negative such that $\lim_{s\to 0^+} y(s)/s \in [0,\infty)$ and $\lim_{s\to\infty} y(s) = 0$.

REMARK 1.13. Some basic examples of the non-increasing functions in Λ are

(i) $k \equiv C_0 > 0$, $K(t) = C_0 t$, $C_k = 1$,

(ii) $k(t) = t^{-\sigma/2}$ with $\sigma \in (0,2)$,
$$K(t) = \frac{2t^{(2-\sigma)/2}}{2-\sigma}, \quad C_k = \frac{2}{2-\sigma} > 1,$$

(iii) $k(t) = -\ln t$, $K(t) = t(1-\ln t)$, $C_k = 1$,

(iv) $k(t) = -\ln t/t^{\sigma}$ with $\sigma \in (0,1)$,
$$K(t) = \frac{t^{1-\sigma}}{(1-\sigma)^2}(1-(1-\sigma)\ln t), \quad C_k = \frac{1}{1-\sigma} > 1.$$

At the same time, we note that $k(t) = 1/t(-\ln t)^{\sigma}$ with $\sigma > 1$, $k \in L^1(0,\nu) \cap C^1(0,\nu)$, $K(t) = (\ln t)^{1-\sigma}/(\sigma-1)$, $C_k = +\infty$.

REMARK 1.14. When $g(u) = u^{\rho+1}$, $\rho > 0$, $0 < q < \rho + 1$,
$$\varphi(t) = \left(\frac{q}{\rho+1-q}\right)^{q/(\rho+1-q)} t^{-q/(\rho+1-q)}, \quad t > 0.$$

REMARK 1.15. When $g(u) = u^{\rho+1}$, $\rho > 0$,
$$\psi_1(t) = \left(\frac{2(2+\rho)}{\rho^2}\right)^{1/\rho} t^{-2/\rho}, \quad t > 0.$$

REMARK 1.16. If $2(\rho+1)/(\rho+2) < q < 2$, then
$$1 < \frac{(q-1)(1+\rho)}{\rho+1-q}.$$

REMARK 1.17 (Zhang [48, theorem 1.2]). Let g satisfy (g_1) and $g(s) \leqslant C_1 s^{p_1}$ for all $s \in (0,\infty)$ and $g(s) \geqslant C_2 s^{p_2}$ for large enough s, where $p_1 \geqslant p_2 > 1$ and C_1, C_2 are positive constants; let $b \in C^{\alpha}(\Omega)$ and $b > 0$ in Ω, satisfying the following assumption: there exist constants $\gamma_1 \geqslant \gamma_2 > -2$ such that
$$C_2(d(x))^{\gamma_2} \leqslant b(x) \leqslant C_1(d(x))^{\gamma_1} \quad \text{for all } x \in \Omega. \tag{1.24}$$
If
$$1 < q \leqslant \frac{2p_1+\gamma_1}{p_1+\gamma_1+1},$$
then problem (P_-) has at least one solution $u_- \in C^2(\Omega)$.

Remark 1.18. If b satisfies (b_1) and $\sup_{x\in\Omega}(d(x))^{2-\sigma}b(x) < \infty$ with $\sigma \in (0, 2)$, then (b_4) holds (see [17, ch. 4, theorem 4.3, problems 4.3 and 4.6, pp. 70–71]).

The paper is organized as follows. In § 2 we continue to recall Karamata regular variation theory. In § 3 we prove theorems 1.7–1.9. Finally, we show existence of solutions to problem (P_+).

2. Some basic definitions and the properties to Karamata regularly varying theory

Let us continue to recall some basic definitions and the properties to Karamata regular variation theory (see [32, 39, 40]).

Some basic examples of slowly varying functions at infinity are as follows:

(i) every measurable function on $[a, \infty)$ which has a positive limit at infinity;

(ii) $L(t) = \prod_{m=1}^{m=n} (\log_m t)^{\alpha_m}$, $\alpha_m \in \mathbb{R}$;

(iii) $L(t) = \exp\left(\prod_{m=1}^{m=n} (\log_m t)^{\alpha_m}\right)$, $0 < \alpha_m < 1$;

(iv) $L(t) = \dfrac{1}{t}\displaystyle\int_a^t \frac{\mathrm{d}s}{\ln s}$;

(v) $L(t) = \exp((\ln t)^{1/3} \cos((\ln t))^{1/3})$, where

$$\liminf_{t\to\infty} L(t) = 0 \quad \text{and} \quad \limsup_{t\to\infty} L(t) = +\infty;$$

(vi) we have

$$L_1(t) = \frac{1}{t}\int_a^t \frac{L(s)\,\mathrm{d}s}{s},$$

where L is slowly varying at infinity and, in this case,

$$\lim_{t\to\infty} \frac{L_1(t)}{L(t)} = 0.$$

Lemma 2.1. *If the functions L, L_1 are slowly varying at infinity, then*

(i) *L^σ for every $\sigma \in \mathbb{R}$, $L(t) + L_1(t)$, $L(L_1(t))$ (if $L_1(t) \to \infty$ as $t \to \infty$) are also slowly varying at infinity,*

(ii) *for every $\theta > 0$ and $t \to \infty$,*

$$t^\theta L(t) \to \infty, \quad t^{-\theta} L(t) \to 0, \tag{2.1}$$

(iii) *for $t \to \infty$, $\ln(L(t))/\ln t \to 0$.*

Lemma 2.2 (asymptotic behaviour). *If the function L is slowly varying at infinity, then [32, appendix, proposition 2] for $t \to \infty$,*

$$\int_a^t s^\beta L(s)\,\mathrm{d}s \cong (\beta+1)^{-1} t^{1+\beta} L(t) \qquad \text{for } \beta > -1, \tag{2.2}$$

$$\int_t^\infty s^\beta L(s)\,\mathrm{d}s \cong (-\beta-1)^{-1} t^{1+\beta} L(t) \quad \text{for } \beta < -1. \tag{2.3}$$

Let Ψ be non-decreasing in $\mathbb{R}$. We define (as in [39]) the inverse of Ψ by

$$\Psi^{\leftarrow}(t) = \inf\{s : \Psi(s) \geqslant t\}. \tag{2.4}$$

Lemma 2.3 (Resnick [39, proposition 0.8]). *The following hold:*

(i) *if $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, then $f_1 \cdot f_2 \in RV_{\rho_1+\rho_2}$;*

(ii) *if $f_1 \in RV_{\rho_1}$, $f_2 \in RV_{\rho_2}$, where $\lim_{t\to\infty} f_2(t) = \infty$, then $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$;*

(iii) *if Ψ is non-decreasing in $\mathbb{R}$, $\lim_{t\to\infty} \Psi(t) = \infty$, and $\Psi \in RV_\rho$ with $\rho \neq 0$, then $\Psi^{\leftarrow} \in RV_{\rho^{-1}}$.*

By the above lemmas, we can obtain the following results directly.

Lemma 2.4 (representation theorem). *The function H is slowly varying at zero if and only if it may be written in the form*

$$H(t) = c(t) \exp\left(\int_t^a \frac{y(s)}{s}\,\mathrm{d}s\right), \quad 0 < t < a, \tag{2.5}$$

for some $a > 0$, where $c(t)$ is a bounded measurable function, $y \in C[0,a]$ with $y(0) = 0$ and, for $t \to 0^+$, $c(t) \to c$ with $c > 0$.

Lemma 2.5. *If the function H is slowly varying at zero, then for every $\theta > 0$ and $t \to 0^+$*

$$t^{-\theta} H(t) \to \infty, \qquad t^\theta H(t) \to 0. \tag{2.6}$$

Lemma 2.6 (asymptotic behaviour). *If the function H is slowly varying at zero, then for $t \to 0$ [32, appendix, proposition 2]*

$$\int_0^t s^\beta H(s)\,\mathrm{d}s \cong (\beta+1)^{-1} t^{1+\beta} H(t) \qquad \text{for } \beta > -1, \tag{2.7}$$

$$\int_t^a s^\beta H(s)\,\mathrm{d}s \cong (-\beta-1)^{-1} t^{1+\beta} H(t) \quad \text{for } \beta < -1. \tag{2.8}$$

Lemma 2.7. *If g satisfies (g_1) and $g' \in RV_\rho$ with $\rho > 0$, then*

(i) *$\lim_{u\to\infty} g'(u) = \lim_{u\to\infty} g(u) = \infty$ and $g \in RV_{\rho+1}$, $G \in RV_{\rho+2}$,*

(ii) *g satisfies the Keller–Osserman condition (g_2) and*

$$\int_t^\infty \frac{\mathrm{d}s}{(g(s))^{1/q}} < \infty \quad \text{for all } t > 0$$

provided that $0 < q < \rho + 1$,

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(iii) $\lim_{s\to\infty} \dfrac{g'(s)}{g(s)} = 0, \quad \lim_{s\to\infty} \dfrac{sg'(s)}{g(s)} = \rho + 1, \quad \lim_{s\to\infty} \dfrac{g'(s)G(s)}{g^2(s)} = \dfrac{1+\rho}{2+\rho},$

(iv) *when* $2(\rho+1)/(\rho+2) < q < \rho+1$ *and*

$$\theta \in \left[0, \frac{q(\rho+2) - 2(\rho+1)}{\rho+1-q}\right), \quad \theta_1 \in \left[0, \frac{(q-1)(\rho+1)}{\rho+1-q}\right),$$

$$\lim_{u\to\infty} \frac{g'(u)/(g(u))^{2(q-1)/q}}{(\int_u^\infty \mathrm{d}s/(g(s))^{1/q})^{\theta}} = 0 \quad \textit{and} \quad \lim_{u\to\infty} \frac{1/(g(u))^{(q-1)/q}}{(\int_u^\infty \mathrm{d}s/(g(s))^{1/q})^{\theta_1}} = 0,$$

(v) *when* $0 < q < 2(1+\rho)/(2+\rho)$,

$$\lim_{u\to\infty} \frac{(\int_0^u g(s)\,\mathrm{d}s)^{q/2}}{g(u)} = 0,$$

and when $g(u) = u^{1+\rho}$ *with* $q = 2(1+\rho)/(2+\rho)$,

$$\lim_{u\to\infty} \frac{(G(u))^{q/2}}{g(u)} = (2+\rho)^{-(1+\rho)/(2+\rho)}.$$

Proof. Since g satisfies (g_1) and $g' \in RV_\rho$ with $\rho > 0$, there exists a function L which is slowly varying at infinity such that $g'(t) = t^\rho L(t)$.

(i) This follows by definition 1.1, l'Hôpital's rule and lemma 2.1.

(ii) By $G \in RV_{\rho+2}$ with $\rho > 0$, we see that there exists a function L which is slowly varying at infinity such that $G(t) = t^{2+\rho}L(t)$. Since $\rho > 0$, let $\rho_1 \in (0, \frac{1}{2}\rho)$. We see by lemma 2.1 that

$$\lim_{t\to\infty} \frac{G(t)}{t^{2(1+\rho_1)}} = \lim_{t\to\infty} t^{\rho-2\rho_1} L(t) = \infty,$$

i.e. there exists $T_0 > 0$ such that

$$\frac{G(t)}{t^{2(1+\rho_1)}} > 1, \quad \sqrt{G(t)} > t^{1+\rho_1}, \quad t > T_0.$$

This implies that g satisfies (g_2). In the same way, we can show that

$$\int_t^\infty \frac{\mathrm{d}s}{(g(s))^{1/q}} < \infty \quad \text{for all } t > 0,$$

provided that $0 < q < \rho + 1$.

(iii) By $g'(s) = s^\rho L(s)$ with $\rho > 0$ and lemma 2.2, we see that, for $t \to 0$,

$$g(t) = \int_0^t g'(s)\,\mathrm{d}s = \int_0^t s^\rho L(s)\,\mathrm{d}s \cong (\rho+1)^{-1} t^{1+\rho} L(t),$$

$$G(t) = \int_0^t g(s)\,\mathrm{d}s = (\rho+1)^{-1} \int_0^t s^{\rho+1} L(s)\,\mathrm{d}s \cong (\rho+1)^{-1}(\rho+2)^{-1} t^{2+\rho} L(t).$$

So

$$\lim_{s\to\infty}\frac{g'(s)}{g(s)}=0,\ \lim_{s\to\infty}\frac{sg'(s)}{g(s)}=\rho+1,\ \lim_{s\to\infty}\frac{G(s)}{sg(s)}=(2+\rho)^{-1}$$

and

$$\lim_{s\to\infty}\frac{g'(s)G(s)}{g^2(s)}=\lim_{s\to\infty}\frac{sg'(s)}{g(s)}\lim_{s\to\infty}\frac{G(s)}{sg(s)}=\frac{1+\rho}{2+\rho}.$$

(iv) Since

$$\begin{aligned}q(\rho+2)-2(\rho+1)-\theta(\rho+1-q)&>0,\\(q-1)(\rho+1)-\theta_1(\rho+1-q)&>0,\end{aligned}$$

we see by lemma 2.1 and the proof of (i)–(iii) that

$$\begin{aligned}&\lim_{u\to\infty}\frac{g'(u)/(g(u))^{2(q-1)/q}}{(\int_u^\infty \mathrm{d}s/(g(s))^{1/q})^{\theta}}\\&\quad=\frac{(1+\rho)^{2(q-1)/q}}{q}\frac{(1+\rho-q)^{\theta}}{q^{\theta}}\lim_{u\to\infty}\frac{(u^{(q(\rho+2)-2(\rho+1))/q}(L(u))^{(q-2)/q})^{-1}}{u^{-\theta(\rho+1-q)/q}(L_1(u))^{\theta}}\\&\quad=\frac{(1+\rho-q)^{\theta}(1+\rho)^{2(q-1)/q}}{q^{1+\theta}}\\&\qquad\lim_{u\to\infty}(u^{(q(\rho+2)-2(\rho+1)-\theta(\rho+1-q))/q}(L(u))^{(q-2)/q}(L_1(u))^{\theta})^{-1}\\&\quad=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{u\to\infty}\frac{1/(g(u))^{(q-1)/q}}{(\int_u^\infty \mathrm{d}s/(g(s))^{1/q})^{\theta_1}}\\&\quad=\frac{(1+\rho-q)^{\theta_1}}{q^{\theta_1}}\lim_{u\to\infty}\frac{(u^{(q-1)(\rho+1)/q}(L(u))^{(q-1)/q})^{-1}}{u^{-\theta_1(\rho+1-q)/q}(L_1(u))^{\theta_1}}\\&\quad=\frac{(1+\rho-q)^{\theta_1}}{q^{\theta_1}}\lim_{u\to\infty}(u^{((q-1)(\rho+1)-\theta_1(\rho+1-q))/q}(L(u))^{(q-1)/q}(L_1(u))^{\theta_1})^{-1}\\&\quad=0.\end{aligned}$$

(v) This follows in the same way. □

LEMMA 2.8. *Let b, k be in theorem 1.7. Then*

(i) $\lim_{t\to0^+}k(t)=\infty$, $\lim_{t\to0^+}K(t)=0$,

(ii)

$$\lim_{t\to0^+}\frac{K(t)}{k(t)}=0,\qquad \lim_{t\to0^+}\frac{k'(t)K(t)}{k^2(t)}=1-C_k.$$

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Proof. Since $k \in \Lambda$ is non-increasing, b satisfies (b_1) and (b_3), we have

$$\lim_{t\to 0} k(t) = +\infty, \qquad \lim_{t\to 0^+} K(t) = 0,$$
$$\lim_{t\to 0^+} \frac{k'(t)K(t)}{k^2(t)} = 1 - \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{K(t)}{k(t)}\right) = 1 - C_k.$$

□

LEMMA 2.9. *Let* $2(\rho+1)/(\rho+2) < q < \min\{2, \rho+1\}$, *and* b, k *be in theorem 1.8. If* $\theta > \sigma(2-q)/(1-\sigma)$ *and* $\theta_1 > (\sigma+1+q\sigma)/(1-\sigma)$, *then*

(i) $\lim_{t\to 0^+} k'(t) = -\infty$, $\lim_{t\to 0^+} k(t) = \infty$, $\lim_{t\to 0^+} K(t) = 0$,

(ii) $\lim_{t\to 0^+} (k(t))^{2-q}(K(t))^{\theta} = 0$,

(iii) $\lim_{t\to 0^+} k'(t)(K(t))^{\theta_1}/(k(t))^q = 0$.

Proof. (i) Since $k' \in RVZ_{-\sigma-1}$ with $\sigma \in (0,1)$, there exists a function H which is slowly varying at zero such that $k'(t) = -t^{-\sigma-1}H(t)$. It follows by lemmas 2.5 and 2.6 that

$$\lim_{t\to 0^+} k'(t) = -\infty, \qquad k(t) = -\int_t^a k'(s)\,\mathrm{d}s + k(a) \cong \sigma^{-1}t^{-\sigma}H(t) \quad \text{as } t \to 0^+,$$

and

$$K(t) = \int_0^t k(s)\,\mathrm{d}s \cong (\sigma(1-\sigma))^{-1}t^{1-\sigma}H(t) \quad \text{as } t \to 0^+.$$

So

$$\lim_{t\to 0^+} k(t) = \infty, \qquad \lim_{t\to 0^+} K(t) = 0.$$

(ii) Since $\theta(1-\sigma) - \sigma(2-q) > 0$, we see by lemma 2.5 that

$$\lim_{t\to 0^+} (k(t))^{2-q}(K(t))^{\theta} = \lim_{t\to 0^+} \frac{t^{\theta(1-\sigma)-\sigma(2-q)}(L(t))^{2-q+\theta}}{\sigma^{2-q+\theta}(1-\sigma)^{\theta}} = 0.$$

(iii) We see by $\theta_1(1-\sigma) + q\sigma - \sigma - 1 > 0$ and the proof of (ii) that

$$\lim_{t\to 0^+} \frac{k'(t)(K(t))^{\theta_1}}{(k(t))^q} = \lim_{t\to 0^+} \frac{t^{\theta_1(1-\sigma)+q\sigma-\sigma-1}(L(t))^{1+\theta_1-q}}{\sigma^{\theta_1-q}(1-\sigma)^{\theta_1}} = 0.$$

□

By the proof of lemma 2.3, we can show the following results.

LEMMA 2.10. *If* $h_1 \in RVZ_{\rho_1}$, $h_2 \in RVZ_{\rho_2}$ *and* $\lim_{t\to 0^+} h_2(t) = 0$, *then* $h_1 \circ h_2 \in RVZ_{\rho_1\rho_2}$.

LEMMA 2.11. *Let* g_1 *and* g_2 *be positive continuous on* $(0,\infty)$, *let* $g_1 \in RV_{1+\rho}$ *with* $\rho > 0$ *and let [49, lemma 2.4]*

$$\int_t^{\infty} \frac{\mathrm{d}s}{g_1(s)} < \infty \quad \textit{for all } t > 0.$$

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If

$$\lim_{s\to\infty}\frac{g_1(s)}{g_2(s)}=1 \quad \textit{and} \quad \int_{\varphi_1(t)}^{\infty}\frac{\mathrm{d}s}{g_1(s)}=\int_{\varphi_2(t)}^{\infty}\frac{\mathrm{d}s}{g_2(s)}=t \quad \textit{for all } t>0,$$

then

$$\lim_{t\to\infty}\frac{\varphi_1(t)}{\varphi_2(t)}=1.$$

LEMMA 2.12. *Under the assumptions in theorems 1.7 and 1.9,*

$$\varphi\in RVZ_{-q/(\rho+1-q)} \quad \textit{and} \quad \psi_1\in RVZ_{-2/\rho}.$$

Proof. Let

$$f_1(u)=\int_u^{\infty}\frac{\mathrm{d}s}{(g(s))^{1/q}},\quad f_{11}(u)=\int_u^{\infty}\frac{\mathrm{d}s}{\sqrt{2G(s)}},\quad u>0.$$

By l'Hôpital's rule, we can easily see that $f_1\in RV_{-(\rho+1-q)/q}$ and $f_{11}\in RV_{-\rho/2}$. It follows by lemma 2.3 that $\varphi=f_1^{-1}\in RVZ_{-q/(\rho+1-q)}$ and $\psi_1={f_{11}}^{-1}\in RVZ_{-2/\rho}$. □

3. The exact asymptotic behaviour

In this section we prove theorems 1.7–1.9. We need the following preliminary considerations.

LEMMA 3.1. *Let $2(\rho+1)/\rho+2<q<\rho+1$, g and φ be as in theorem 1.7. Then*

(i) $\lim_{t\to0^+}\varphi(t)=\infty$, $\lim_{t\to0^+}\varphi'(t)=-\infty$,

(ii) $\displaystyle\lim_{t\to0^+}\frac{\varphi''(t)}{(-\varphi'(t))^q}=0$ *and* $\displaystyle\lim_{t\to0^+}\frac{\varphi'(t)}{t(-\varphi'(t))^q}=0.$

Proof. (i) By the definition of φ in (1.19) and a direct calculation, we see that

$$\lim_{t\to0^+}\varphi(t)=\infty,\qquad \varphi'(t)=-(g(\varphi(t)))^{1/q},$$
$$\varphi''(t)=\frac{1}{q}g'(\varphi(t))(g(\varphi(t)))^{(2-q)/q},\quad t>0.$$

It follows by lemma 2.7 that $\lim_{t\to0^+}\varphi'(t)=-\infty$.

(ii) Since $2(\rho+1)/(\rho+2)<q$, we see by lemma 2.7 that

$$\lim_{t\to0^+}\frac{\varphi''(t)}{(-\varphi'(t))^q}=\frac{1}{q}\lim_{t\to0^+}\frac{g'(\varphi(t))(g(\varphi(t)))^{(2-q)/q}}{g(\varphi(t))}=\frac{1}{q}\lim_{u\to\infty}\frac{g'(u)}{(g(u))^{2(q-1)/q}}=0$$

and

$$\lim_{t\to0^+}\frac{\varphi'(t)}{t(\varphi'(t))^q}=\lim_{t\to0^+}\frac{(g(\varphi(t)))^{(1-q)/q}}{\int_{\varphi(t)}^{\infty}\mathrm{d}s/(g(s))^{1/q}}=\lim_{u\to\infty}\frac{(g(u))^{(1-q)/q}}{\int_u^{\infty}\mathrm{d}s/(g(s))^{1/q}}$$
$$=\frac{q-1}{q}\lim_{u\to\infty}\frac{g'(u)}{(g(u))^{2(q-1)/q}}=0.$$

□

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LEMMA 3.2. *Let $2(\rho+1)/(\rho+2) < q < \rho+1$, g and φ be as in theorem 1.8, and let*

$$\theta \in \left(\frac{\sigma(2-q)}{1-\sigma}, \frac{q(\rho+2)-2(\rho+1)}{\rho+1-q}\right), \qquad \theta_1 \in \left(\frac{\sigma+1+q\sigma}{1-\sigma}, \frac{(q-1)(\rho+1)}{\rho+1-q}\right).$$

Then

$$\lim_{t\to 0^+} \frac{\varphi''(t)}{t^\theta(-\varphi'(t))^q} = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{\varphi'(t)}{t^{\theta_1}(-\varphi'(t))^q} = 0.$$

Proof. By lemma 2.7, we see that

$$\begin{aligned}\lim_{t\to 0^+} \frac{\varphi''(t)}{t^\theta(-\varphi'(t))^q} &= \frac{1}{q}\lim_{t\to 0^+} \frac{g'(\varphi(t))/(g(\varphi(t)))^{2(q-1)/q}}{(\int_{\varphi(t)}^\infty \mathrm{d}s/(g(s))^{1/q})^\theta} \\ &= \frac{1}{q}\lim_{u\to\infty} \frac{g'(u)/(g(u))^{2(2-q)/q}}{(\int_u^\infty \mathrm{d}s/(g(s))^{1/q})^\theta} = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0^+} \frac{\varphi'(t)}{t^{\theta_1}(-\varphi'(t))^q} &= -\lim_{t\to 0^+} \frac{1/(g(\varphi(t)))^{(q-1)/q}}{(\int_{\varphi(t)}^\infty \mathrm{d}s/(g(s))^{1/q})^{\theta_1}} \\ &= \lim_{u\to\infty} \frac{1/(g(u))^{(q-1)/q}}{(\int_u^\infty \mathrm{d}s/(g(s))^{1/q})^{\theta_1}} = 0.\end{aligned}$$

□

LEMMA 3.3. *Let g, k and ψ_1 be as in theorem 1.9.*

(i) *If $0 < q < 2(1+\rho)/(2+\rho)$, then*

$$\lim_{t\to 0^+} \frac{(G(\psi(t)))^{q/2}}{g(\psi(t))} = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{\sqrt{2G(\psi(t))}}{tg(\psi(t))} = \frac{\rho}{2+\rho}.$$

(ii) *If $g(u) = u^{1+\rho}$, $q = 2(1+\rho)/(2+\rho)$, then*

$$\lim_{t\to 0^+} \frac{(G(\psi(t)))^{q/2}}{g(\psi(t))} = (2+\rho)^{-(1+\rho)/(2+\rho)}.$$

Proof. (i) By lemma 2.7, we see that

$$\lim_{t\to 0^+} \frac{(G(\psi(t)))^{q/2}}{g(\psi(t))} = \lim_{u\to\infty} \frac{(\int_0^u g(s)\,\mathrm{d}s)^{q/2}}{g(u)} = 0$$

and

$$\begin{aligned}\lim_{t\to 0^+} \frac{\sqrt{2G(\psi(t))}}{tg(\psi(t))} &= \lim_{u\to\infty} \frac{\sqrt{2G(u)}/g(u)}{\int_u^\infty \mathrm{d}s/\sqrt{2G(s)}} \\ &= \lim_{u\to\infty}\left(\frac{2g'(u)G(u)}{g^2(u)} - 1\right) = \frac{2(1+\rho)}{2+\rho} - 1 = \frac{\rho}{2+\rho}.\end{aligned}$$

(ii) This follows by direct calculation. The proof is finished. □

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LEMMA 3.4 (the comparison principle). *Let $\Psi(x,s,\xi)$ satisfy the following two conditions [17, theorem 10.1]:*

(D_1) *Ψ is non-increasing in s for each $(x,\xi)\in(\Omega\times\mathbb{R}^N)$;*

(D_2) *Ψ is continuously differentiable with respect to the variable ξ in $\Omega\times(0,\infty)\times\mathbb{R}^N$.*

If $u,v\in C(\bar\Omega)\cap C^2(\Omega)$ satisfy $\Delta u+\Psi(x,u,\nabla u)\geqslant\Delta v+\Psi(x,v,\nabla v)$ in Ω and $u\leqslant v$ on $\partial\Omega$, then $u\leqslant v$ in Ω.

Proof of theorem 1.7. Given an arbitrary $\varepsilon\in(0,\frac{1}{4}c_q)$. Let

$$\xi_0=\left(\frac{1}{c_q}\right)^{\rho-q+1},\qquad \xi_{2\varepsilon}=(c_q-2\varepsilon)^{-1/(\rho+1-q)},\qquad \xi_{1\varepsilon}=(c_q+2\varepsilon)^{-1/(\rho+1-q)}.$$

It follows that

$$\left(\frac{2}{3}\right)^{1/(\rho+1-q)}\xi_0<\xi_{1\varepsilon}<\xi_{2\varepsilon}<2^{1/(\rho+1-q)}\xi_0.$$

For any $\delta>0$, we define

$$\Omega_\delta=\{x\in\Omega:0<d(x)<\delta\}.$$

Since $\partial\Omega\in C^2$, there exists a constant $\delta\in(0,\frac{1}{2}\nu)$ which depends only on Ω such that $d(x)\in C^2(\bar\Omega_{2\delta})$ and $|\nabla d|\equiv1$ on $\Omega_{2\delta}$.

Recalling that

$$\lim_{t\to0^+}\frac{k^2(t)}{k^q(t)}=\begin{cases}0 & \text{for } q>2,\\ 1 & \text{for } q=2,\end{cases}$$

and

$$\lim_{t\to0^+}\frac{k'(t)K(t)}{k^q(t)}=\lim_{t\to0^+}\frac{k'(t)K(t)}{k^2(t)}\lim_{t\to0^+}\frac{1}{k^{q-2}(t)}=\begin{cases}0 & \text{for } q>2,\\ 1-C_k & \text{for } q=2,\end{cases}$$

and since $k'(t)<0$, $\varphi(t)<0$ and $Z(x)<K(d(x))$, we see that

$$\begin{aligned}\left|\frac{k'(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q}\right|&=\frac{k'(d(x))K(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{Z(x)(-\varphi'(Z(x)))^q}\frac{Z(x)}{K(d(x))}\\&<\frac{k'(d(x))K(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{Z(x)(-\varphi'(Z(x)))^q}.\end{aligned}$$

It follows by (b_3) and lemma 3.1 that

$$\begin{aligned}\lim_{(d(x),\beta)\to(0,0^+)}\biggl(\frac{k^2(d(x))}{k^q(d(x))}\frac{\varphi''(Z(x))}{(-\varphi'(Z(x)))^q}+\frac{k'(d(x))K(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{Z(x)(-\varphi'(Z(x)))^q}\\+\frac{k(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q}\Delta d(x)\biggr)=0\end{aligned}$$

and

$$\lim_{(d(x),\beta)\to(0,0^+)}\frac{b(x)}{k^q(d(x))}\frac{g(\xi_{2\varepsilon}(\varphi(Z(x))))}{\xi_{2\varepsilon}g(\varphi(Z(x)))}=c_q\xi_{2\varepsilon}^\rho.$$

Note that

$$\xi_{2\varepsilon}^{q-1} - c_q \xi_{2\varepsilon}^{\rho} = \xi_{2\varepsilon}^{q-1}\left(1 - \frac{c_q}{c_q - 2\varepsilon}\right) < 0.$$

Thus, we see that, corresponding to ε, there exists $\delta_\varepsilon \in (0, \delta)$ sufficiently small and, letting $\beta \in (0, \delta_\varepsilon)$ be arbitrary, we define

$$\bar{u}_\beta = \xi_{2\varepsilon}\varphi(Z(x)), \quad Z(x) = K(d(x)) - K(\beta), \quad x \in D_\beta^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\beta,$$
$$\underline{u}_\beta = \xi_{1\varepsilon}\varphi(Y(x)), \quad Y(x) = K(d(x)) + K(\beta), \quad x \in D_\beta^+ = \Omega_{2\delta_\varepsilon - \beta},$$

such that, for $(x, \beta) \in D_\beta^- \times (0, \delta_\varepsilon)$,

$$\begin{aligned}
&\Delta\bar{u}_\beta(x) + |\nabla\bar{u}_\beta(x)|^q - b(x)g(\bar{u}_\beta(x)) \\
&\quad = \xi_{2\varepsilon}k^2(d(x))\varphi''((Z(x))) + \xi_{2\varepsilon}k'(d(x))\varphi'(Z(x)) \\
&\qquad + \xi_{2\varepsilon}k(d(x))\varphi'(Z(x))\Delta d(x) \\
&\qquad + \xi_{2\varepsilon}^q k^q(d(x))(-\varphi'(Z(x)))^q - b(x)g(\xi_{2\varepsilon}(\varphi(Z(x)))) \\
&\quad = \xi_{2\varepsilon}k^q(d(x))g(\varphi(Z(x)))\biggl[\frac{k^2(d(x))}{k^q(d(x))}\frac{\varphi''(Z(x))}{(-\varphi'(Z(x)))^q} + \frac{k'(d(x)}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q} \\
&\qquad + \frac{k(d(x)}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q}\Delta d(x) \\
&\qquad + \xi_{2\varepsilon}^{q-1} - \frac{b(x)}{k^q(d(x))}\frac{g(\xi_{2\varepsilon}(\varphi(Z(x))))}{\xi_{2\varepsilon}g(\varphi(Z(x)))}\biggr] \\
&\quad \leqslant 0.
\end{aligned}$$

Note that $Y(x) > K(d(x))$ and

$$\xi_{1\varepsilon}^{q-1} - c_q \xi_{1\varepsilon}^{\rho} = \xi_{1\varepsilon}^{q-1}\left(1 - \frac{c_q}{c_q + 2\varepsilon}\right) > 0.$$

Thus, in the same way, we can show that, for $(x, \beta) \in D_\beta^+ \times (0, \delta_\varepsilon)$,

$$\begin{aligned}
&\Delta\underline{u}_\beta(x) + |\nabla\underline{u}_\beta(x)|^q - b(x)g(\underline{u}_\beta(x)) \\
&\quad = \xi_{2\varepsilon}k^2(d(x))\varphi''((Y(x))) + \xi_{2\varepsilon}k'(d(x))\varphi'(Y(x)) \\
&\qquad + \xi_{2\varepsilon}k(d(x))\varphi'(Y(x))\Delta d(x) \\
&\qquad + \xi_{2\varepsilon}^q k^q(d(x))(-\varphi'(Y(x)))^q - b(x)g(\xi_{2\varepsilon}(\varphi(Y(x)))) \\
&\quad = \xi_{2\varepsilon}k^q(d(x))g(\varphi(Y(x)))\biggl[\frac{k^2(d(x))}{k^q(d(x))}\frac{\varphi''(Y(x))}{(-\varphi'(Y(x)))^q} + \frac{k'(d(x))}{k^q(d(x))}\frac{\varphi'(Y(x))}{(-\varphi'(Y(x)))^q} \\
&\qquad + \frac{k(d(x))}{k^q(d(x))}\frac{\varphi'(Y(x))}{(-\varphi'(Y(x)))^q}\Delta d(x) \\
&\qquad + \xi_{2\varepsilon}^{q-1} - \frac{b(x)}{k^q(d(x))}\frac{g(\xi_{2\varepsilon}(\varphi(Y(x))))}{\xi_{2\varepsilon}g(\varphi(Y(x)))}\biggr] \\
&\quad \geqslant 0.
\end{aligned}$$

Now let u_+ be an arbitrary solution of problem (P_+) and

$$M_{u+}(2\delta_\varepsilon) = \max_{d(x) \geqslant 2\delta_\varepsilon} u_+(x).$$

We see that

$$u_+ \leqslant M_u(2\delta_\varepsilon) + \bar{u}_\beta \quad \text{on } \partial D_\beta^-.$$

Since φ is decreasing and k is non-increasing, it follows that $\underline{u}_\beta \leqslant \xi_{1\varepsilon}\varphi(K(2\delta_\varepsilon))$ whenever $d(x) = 2\delta_{2\varepsilon} - \beta$. Let $M_{\underline{u}}(2\delta_\varepsilon) = \xi_{1\varepsilon}\varphi(K(2\delta_\varepsilon))$. We see that

$$\underline{u}_\beta \leqslant u_+ + M_{\underline{u}}(2\delta_\varepsilon) \quad \text{on } \partial D_\beta^+.$$

It follows by (g_1) and lemma 3.4 that

$$u_+ \leqslant M_u(2\delta_\varepsilon) + \bar{u}_\beta, x \in D_\beta^- \quad \text{and} \quad \underline{u}_\beta \leqslant u_+ + \xi_{1\varepsilon}\varphi(K(2\delta_\varepsilon)), x \in D_\beta^+.$$

Hence, for $x \in D_\beta^- \cap D_\beta^+$ and letting $\beta \to 0$, we have

$$\xi_{1\varepsilon} - \frac{M_{\underline{u}}(2\delta_\varepsilon)}{\varphi(K(d(x)))} \leqslant \frac{u_+(x)}{\varphi(K(d(x)))} \leqslant \xi_{2\varepsilon} + \frac{M_{u+}(2\delta_\varepsilon)}{\varphi(K(d(x)))}.$$

Recalling that $\lim_{t\to 0^+}\varphi(t) = \infty$, we see that

$$\xi_{1\varepsilon} \leqslant \lim_{d(x)\to 0}\inf \frac{u_+(x)}{\varphi(K(d(x)))} \leqslant \lim_{d(x)\to 0}\sup \frac{u_+(x)}{\varphi(K(d(x)))} \leqslant \xi_{2\varepsilon}.$$

Letting $\varepsilon \to 0$, and looking at the definitions of $\xi_{1\varepsilon}^+$ and $\xi_{2\varepsilon}^+$, we have

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\varphi(K(d(x)))} = \xi_0.$$

By lemma 2.12, the proof is finished. □

Proof of theorem 1.8. By (b_3) and lemma 3.2, we see that

$$\begin{aligned}
&\lim_{(d(x),\beta)\to(0,0^+)} \frac{k^2(d(x))}{k^q(d(x))}\frac{\varphi''(Z(x))}{(-\varphi'(Z(x)))^q} \\
&\qquad = \lim_{(d(x),\beta)\to(0,0^+)} k^{2-q}(d(x))K^\theta(d(x))\frac{\varphi'(Z(x))}{Z^\theta(x)(-\varphi'(Z(x)))^q} \\
&\qquad = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{(d(x),\beta)\to(0,0^+)} \frac{k'(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q} \\
&\qquad = \lim_{(d(x),\beta)\to(0,0^+)} \frac{k'(d(x))(K(d(x)))^{\theta_1}}{k^q(d(x))}\frac{\varphi'(Z(x))}{(Z(x))^{\theta_1}(-\varphi'(Z(x)))^q} \\
&\qquad = 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{(d(x),\beta)\to(0,0^+)} \biggl(&\frac{k^2(d(x))}{k^q(d(x))}\frac{\varphi''(Z(x))}{(-\varphi'(Z(x)))^q} + \frac{k'(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q} \\
&+ \frac{k(d(x))}{k^q(d(x))}\frac{\varphi'(Z(x))}{(-\varphi'(Z(x)))^q}\Delta d(x)\biggr) = 0.
\end{aligned}$$

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By the proof of theorem 1.7, we can obtain

$$\lim_{d(x)\to 0} \frac{u_+(x)}{\varphi(K(d(x)))} = \xi_0.$$

□

Proof of theorem 1.9. Let

$$\tau_0 = \frac{\rho(C_k - 1)}{(2+\rho)}, \qquad \xi_0 = ((1+\tau_0)/c_0)^{1/\rho},$$

$$\xi_{2\varepsilon} = \left(\xi_0^\rho + \frac{2\varepsilon}{c_0}\right)^{1/\rho}, \quad \xi_{1\varepsilon} = \left(\xi_0^\rho - \frac{2\varepsilon}{c_0}\right)^{1/\rho},$$

where $\varepsilon \in (0, \frac{1}{4}c_0\xi_0^\rho)$ is arbitrary. One can easily see that

$$\frac{\xi_0}{2^{1/\rho}} < \xi_{1\varepsilon} < \xi_0 < \xi_{2\varepsilon} < \left(\frac{3}{2}\right)^{1/\rho} \xi_0.$$

We define

$$\bar{u}_\beta = \xi_{2\varepsilon}\psi_1(Z(x)), \qquad \underline{u}_\beta = \xi_{1\varepsilon}\psi_1(Y(x)).$$

It follows by lemmas 2.8 and 3.3 that

$$\begin{aligned}
&\lim_{(d(x),\beta)\to(0,0^+)} \frac{k'(d(x))\sqrt{2G(\psi_1(Z(x)))}}{k^2(d(x))g(\psi_1(Z(x)))} \\
&\qquad = \lim_{(d(x),\beta)\to(0,0^+)} \frac{k'(d(x))K(d(x))}{k^2(d(x))} \frac{\sqrt{2G(\psi_1(Z(x)))}}{Z(x)g(\psi_1(Z(x)))} = -\frac{\rho(C_k-1)}{2+\rho},
\end{aligned}$$

$$\begin{aligned}
&\lim_{(d(x),\beta)\to(0,0^+)} \frac{\sqrt{2G(\psi_1(Z(x)))}}{k(d(x))g(\psi_1(Z(x)))}\Delta d(x) \\
&\qquad = \lim_{(d(x),\beta)\to(0,0^+)} \frac{K(d(x))}{k(d(x))} \frac{\sqrt{2G(\psi_1(Z(x)))}}{Z(x)g(\psi_1(Z(x)))}\Delta d(x) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{(d(x),\beta)\to(0,0^+)} \frac{k^q(d(x))(2G(\psi_1(Z(x))))^{q/2}}{k^2(d(x))g(\psi_1(Z(x)))} \\
&\qquad = \lim_{d(x)\to 0} \frac{k^q(d(x))}{k^2(d(x))} \lim_{(d(x),\beta)\to(0,0^+)} \frac{(2G(\psi_1(Z(x))))^{q/2}}{g(\psi_1(Z(x)))} = 0.
\end{aligned}$$

By the proof of theorem 1.7, we can obtain that, for $(x,\beta) \in D_\beta^- \times (0,\delta_\varepsilon)$,

$$\begin{aligned}
&\Delta\bar{u}_\beta(x) - b(x)g(\bar{u}_\beta(x)) \pm |\nabla\bar{u}_\beta(x)|^q \\
&\quad = \xi_{2\varepsilon}k^2(d(x))g(\psi_1(Z(x))) - b(x)g(\xi_{2\varepsilon}(\psi_1(Z(x)))) \\
&\qquad - \xi_{2\varepsilon}k'(d(x))\sqrt{2G(\psi_1(Z(x)))} - \xi_{2\varepsilon}k(d(x))\sqrt{2G(\psi_1(Z(x)))}\Delta d(x) \\
&\qquad \pm \xi_{2\varepsilon}^q k^q(d(x))(2G(\psi_1(Z(x))))^{q/2}
\end{aligned}$$

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$$
\begin{aligned}
&= \xi_{2\varepsilon} k^2(d(x)) g(\psi_1(Z(x))) \bigg[1 - \frac{b(x) g(\xi_{2\varepsilon}(\psi_1(Z(x))))}{\xi_{2\varepsilon} k^2(d(x)) g(\psi_1(Z(x)))} \\
&\qquad - \frac{k'(d(x))\sqrt{2G(\psi_1(Z(x)))}}{k^2(d(x)) g(\psi_1(Z(x)))} \\
&\qquad - \frac{\sqrt{2G(\psi_1(Z(x)))}}{k(d(x)) g(\psi_1(Z(x)))} \Delta d(x) \\
&\qquad \pm \frac{\xi_{2\varepsilon}^{q-1} k^q(d(x))(2G(\psi_1(Z(x)))^{q/2})}{k^2(d(x)) g(\psi_1(Z(x)))} \bigg] \\
&\leqslant 0
\end{aligned}
$$

and, for $(x,\beta) \in D_\beta^+ \times (0,\delta_\varepsilon)$,

$$
\begin{aligned}
&\Delta \underline{u}_\beta(x) - b(x) g(\underline{u}_\beta(x)) \pm |\nabla \underline{u}_\beta(x)|^q \\
&= \xi_{2\varepsilon} k^2(d(x)) g(\psi_1(Y(x))) - b(x) g(\xi_{2\varepsilon}(\psi_1(Y(x)))) \\
&\quad - \xi_{2\varepsilon} k'(d(x))\sqrt{2G(\psi_1(Y(x)))} - \xi_{2\varepsilon} k(d(x))\sqrt{2G(\psi_1(Y(x)))}\Delta d(x) \\
&\quad \pm \xi_{2\varepsilon}^q k^q(d(x))(2G(\psi_1(Y(x))))^{q/2} \\
&= \xi_{2\varepsilon} k^2(d(x)) g(\psi_1(Y(x))) \bigg[(1-\tau_0) - \frac{b(x) g(\xi_{2\varepsilon}(\psi_1(Y(x))))}{\xi_{2\varepsilon} k^2(d(x)) g(\psi_1(Y(x)))} \\
&\qquad - \frac{k'(d(x))\sqrt{2G(\psi_1(Y(x)))}}{k^2(d(x)) g(\psi_1(Y(x)))} - \tau_0 \\
&\qquad - \frac{\sqrt{2G(\psi_1(Y(x)))}}{k(d(x)) g(\psi_1(Y(x)))} \Delta d(x) \\
&\qquad \pm \frac{\xi_{2\varepsilon}^{q-1} k^q(d(x))(2G(\psi_1(Y(x))))^{q/2}}{k^2(d(x)) g(\psi_1(Y(x)))} \bigg] \\
&\geqslant 0.
\end{aligned}
$$

Thus,

$$
\lim_{d(x)\to 0} \frac{u_\pm(x)}{\psi_1(K(d(x)))} = \xi_0.
$$

By lemma 2.12, the proof is finished. □

4. Existence of solutions

In this section, we consider the existence of solutions to problems (P_+). First we need the following lemmas.

Define $H(u) = \int_u^\infty \mathrm{d}s/g(s)$ for $u > 0$. Then $H : (0,\infty) \to (0,\infty)$ is strictly decreasing and $H'(u) = -1/g(u)$ for $u > 0$.

We note by [21, lemma 1] that if g satisfies (g_1), then (g_2) implies that

$$
\int_u^\infty \frac{\mathrm{d}s}{g(s)} < \infty \quad \text{for all } u > 0.
$$

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Lemma 4.1. *Let $u_+ \in C^2(\Omega)$ be an arbitrary solution to problem (P_+) and let $b \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ be non-negative and non-trivial in Ω. If b satisfies (b_4), then*

$$u_+ \geqslant H^{-1}(\bar{v}(x)) > 0 \quad \textit{for all } x \in \Omega, \tag{4.1}$$

where H^{-1} denotes the inverse function of H.

Proof. Let $u_+(x) = H^{-1}(v(x))$, $x \in \Omega$. We see that $v|_{\partial\Omega} = 0$ and

$$-\Delta v + g_1(v)|\nabla v|^2 + g_2(v)|\nabla v|^q = b(x), \quad x \in \Omega, \quad v|_{\partial\Omega} = 0, \tag{4.2}$$

where $g_1(v) = g'(u_+) = g'(H^{-1}(v))$ and $g_2(v) = (g(H^{-1}(v)))^{q-1}$. It follows that $-\Delta v \leqslant b(x)$, $x \in \Omega$. By the maximum principle, we obtain $v \leqslant \bar{v}$ in Ω, i.e. $u_+ \geqslant H^{-1}(\bar{v}(x))$ for all $x \in \Omega$. □

Lemma 4.2. *Let $q > 0$, $b \equiv c_0 > 0$ in Ω (see [1, theorems 5.1 and 5.2] and [16, theorem 4.1]). If g satisfies (g_1), (g_2) and*

$$\lim_{u\to\infty} G(u)/(g(u))^{2/q} \in [0, \infty),$$

or g satisfies (g_1), $(g(u))^{2/q}/u$ is increasing and

$$\int_t^{\infty} \frac{\mathrm{d}s}{(g(s))^{1/q}} < \infty \quad \textit{for all } t > 0, \qquad \lim_{u\to\infty} \frac{G(u)}{(g(u))^{2/q}} \in (0, \infty],$$

then problem (P_+) admits at least one solution $u_+ \in C^2(\Omega)$.

Lemma 4.3 (Lazer and McKenna [23, theorem 4.2]). *Let Ω be a bounded open set of $\mathbb{R}^N$ with smooth boundary. There then exists a sequence $\{\Omega_m\}_1^{\infty}$ of open sets such that $\Omega_m \subset \Omega_{m+1} \subset \Omega$, $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$ and the boundary $\partial\Omega_m$ is a C^{∞} submanifold of $N-1$ dimension for each $m \geqslant 1$.*

Let $\bar{v}_m \in C^{2+\alpha}(\bar{\Omega}_m)$ be the unique solution to the problem

$$-\Delta v = b(x), \quad v > 0, \quad x \in \Omega_m, \quad v|_{\partial\Omega_m} = 0. \tag{4.3}$$

If follows by the maximum principle that

$$\bar{v}_m \leqslant \bar{v}_{m+1} \leqslant \bar{v} \quad \textit{for all } x \in \bar{\Omega}_m. \tag{4.4}$$

Proof of theorem 1.10. Since $b \in C^{\alpha}(\bar{\Omega}_m)$ and is positive on $\bar{\Omega}_m$, it follows by lemmas 4.1 and 4.2 that the problem

$$\Delta u + |\nabla u|^q = b(x)g(u), \quad u > 0, \quad x \in \Omega, \quad u|_{\partial\Omega_m} = \infty, \quad m \in \mathbb{N} \tag{P_{m+}}$$

admits one solution $u_{m+} \in C^2(\Omega_m)$. Moreover,

$$0 < H^{-1}(\bar{v}(x)) \leqslant H^{-1}(\bar{v}_{m+1}(x)) \leqslant u_{(m+1)+}(x) \leqslant u_{m+}(x) \quad \text{for all } x \in \Omega_m. \tag{4.5}$$

Let D be an arbitrary compact subset of Ω. There exists m_0 such that $D \subset \Omega_{m_0}$ and it follows by (4.5) that the sequence $\{u_{m+}(x)\}_{m_0}^{\infty}$ is non-increasing and is bounded from below in D, so $u_+(x) = \lim_{m\to\infty} u_{m+}(x)$ exists for all $x \in \Omega$. By the standard argument (see, for instance, [1,16,44,51]), we see that $u_+ \in C^2(\Omega)$ and is one solution to problem (P_+). The proof is finished. □

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