ON A PROBLEM IN GEOMETRICAL PROBABILITY

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We consider the following problem. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix of non-negative numbers with $a_{ii} = 0$ for all i, and let n points $x_1, x_2, \ldots, x_n$ be chosen at random from the interval $[0, L]$. What is the probability $P = P(n, A, L)$ that for all i and j, $|x_i - x_j| \geq a_{ij}$?

Let G be the symmetric group on the numbers $1, 2, \ldots, n$; for $\sigma \in G$ we write $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$. Our result is

THEOREM 1. Let

$$(1) \qquad a_{ij} + a_{jk} \geq a_{ik}, \qquad 1 \leq i, j, k \leq n;$$

then

$$(2) \quad P(n,A,L) = (1/n!) \Sigma_{\sigma \in G} [\max (0, 1 - (1/L) \Sigma_{j=1}^{n-1} a_{\sigma(j)\sigma(j+1)})]^n .$$

Proof. In the n-dimensional Euclidean space E_n let H be the n-dimensional hypercube

$$H = \{(x_1, x_2, \ldots, x_n): 0 \leq x_i \leq L, \quad i = 1, 2, \ldots, n\}.$$

We then have the decomposition

$$H = \cup_{\sigma \in G} T_\sigma$$

where T_σ is the simplex given by

$$T_\sigma = \{(x_1, x_2, \ldots, x_n): 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)} \leq L\},$$

and the volume $V(T_\sigma) = L^n/n!$. We choose a particular $\sigma \in G$, and impose the conditions

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(3) $$| x_i - x_j | \geq a_{ij}, \qquad 1 \leq i, j \leq n;$$

this means that

(4) $$x_{\sigma(i)} \leq x_{\sigma(j)} - a_{\sigma(i)\sigma(j)}, \qquad 1 \leq i < j \leq n.$$

Suppose that the conditions (1) hold. Then the $n(n-1)/2$ conditions (4) are implied by the $n - 1$ conditions

$$x_{\sigma(k)} \leq x_{\sigma(k+1)} - a_{\sigma(k)\sigma(k+1)}, \qquad k = 1, 2, \ldots, n-1.$$

Hence the subset U_σ of T_σ consisting of those points $(x_1, x_2, \ldots, x_n)$ in T_σ for which the conditions (3) are satisfied is given by

$$U_\sigma = \{(x_1, x_2, \ldots, x_n): 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} - a_{\sigma(1)\sigma(2)} \leq x_{\sigma(3)} - a_{\sigma(1)\sigma(2)} - a_{\sigma(2)\sigma(3)} \leq \cdots \leq x_{\sigma(n)} - \Sigma_{j=1}^{n-1} a_{\sigma(j)\sigma(j+1)} \leq L - \Sigma_{j=1}^{n-1} a_{\sigma(j)\sigma(j+1)}\}.$$

U_σ, if it is not the empty set or a single point, is a simplex similar to T_σ and its volume $V(U_\sigma)$ is given by

(5) $$V(U_\sigma) = \frac{1}{n!}[\max(0, L - \Sigma_{j=1}^{n-1} a_{\sigma(j)\sigma(j+1)})]^n.$$

For the desired probability $P(n, A, L)$ we now obtain the expression

(6) $$P(n, A, L) = \frac{1}{L^n} \Sigma_{\sigma \in G} V(U_\sigma).$$

Substituting (5) into (6) yields (2) and the proof is complete.

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