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Invariant rigid geometric structures and expanding maps

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Abstract. In the first part of this paper, we consider several natural problems about locally homogeneous rigid geometric structures. In particular, we formulate a notion of topological completeness which is adapted to the study of global rigidity of chaotic dynamical systems. In the second part of the paper, we prove the following result: let φ be a C^∞ expanding map of a closed manifold. If φ preserves a topologically complete C^∞ rigid geometric structure, then φ is C^∞ conjugate to an expanding infra-nilendomorphism.

1. *Introduction*

Rigid geometric structures are natural generalizations of classical finite-type geometric structures such as Riemannian metrics and linear connections. A classical program initiated by Zimmer and Gromov consists of classifying chaotic differential dynamical systems preserving rigid geometric structures (see [**14**]). The following important result related to this program was obtained by Benoist and Labourie in [**5**] (see also [**4**]).

THEOREM 1.1. *Let ϕ be a C^∞ topologically transitive Anosov diffeomorphism with C^∞ stable and unstable distributions. If ϕ preserves a C^∞ linear connection, then ϕ is C^∞ conjugate to an Anosov infra-nilautomorphism.*

It is true that linear connections are very important cases of rigid geometric structures. However, for certain interesting dynamical systems, linear connections are not the naturally associated rigid geometric structures (see for example [**12, 17**]). Therefore, it is necessary to consider general invariant rigid geometric structures in the context of the Zimmer–Gromov program. The following question is due to Katok and Spatzier (see [**18**]).

Question 1.1. Let ϕ be a C^∞ topologically transitive Anosov diffeomorphism with C^∞ stable and unstable distributions. If ϕ preserves a C^∞ rigid geometric structure, prove that ϕ is C^∞ conjugate to an Anosov infra-nilautomorphism.

In this paper, we consider C^∞ expanding maps which are simpler dynamical systems than Anosov diffeomorphisms. However, we consider general invariant rigid geometric structures. We hope that some of the results obtained in the paper will help to give an answer to Question 1.1. Our central result is as follows.

THEOREM 1.2. *Let φ be a C^∞ expanding map of a closed manifold M. If φ preserves a topologically complete C^∞ rigid geometric structure g, then φ is C^∞ conjugate to an expanding infra-nilendomorphism.*

We should mention that our topological completeness formulated in §5 is a natural generalization of the classical completeness of linear connections. Typical rigid geometric structures preserved by expanding maps are generalized connections in frame bundles, which are natural generalizations of linear connections (see §5.3). It is easy to verify that generalized connections preserved by expanding maps are necessarily topologically complete (see §7). Therefore, we obtain the following corollary of Theorem 1.2.

COROLLARY 1. *Let φ be a C^∞ expanding map of a closed manifold M. If φ preserves a C^∞ generalized connection, then φ is C^∞ conjugate to an expanding infra-nilendomorphism.*

In the special case that φ is defined on the circle $\mathbb{S}^1$ and preserves a C^∞ linear connection, the corollary above was proved by Feres in [**11**]. This elegant one-dimensional result is one of the motivations of the present paper.

In order to obtain Theorem 1.2, we prove several geometric propositions about locally homogeneous rigid geometric structures. For example, the following geometric problem is considered.

Recall firstly that by the celebrated open-dense theorem of Gromov (see §3), any C^∞ rigid geometric structure preserved by a chaotic dynamical system is locally homogeneous on at least an open-dense subset. Now let g be a locally homogeneous C^∞ rigid geometric structure. We consider the problem of extending local isometries of g to global ones. In the case that g is real analytic, several results of extending local analytic Killing fields to global ones are well known, which have very interesting applications to the study of analytic semisimple group actions (see [**1, 15**]). However, analytic rigid geometric structures are not adequate for the study of C^∞ dynamical systems. In addition, the extendibility of local Killing fields is not well adapted to the study of global rigidity of dynamical systems.

In this paper, we prove, under some mild assumptions, the extendibility of local isometries to global ones for locally homogeneous C^∞ rigid geometric structures. More precisely, we obtain the following proposition, which gives a partial generalization of the classical Liouville theorem in conformal geometry.

PROPOSITION 1. *Let g be a locally homogeneous C^∞ rigid geometric structure on a connected manifold M. Let $\widetilde{M}$ be the universal covering space of M and $\widetilde{g}$ the lifted rigid geometric structure of g on $\widetilde{M}$. Take $x \in M$. If g verifies the following two conditions:*

(a) *there exists a local isometry of g fixing x whose differential at x is hyperbolic, i.e. without eigenvalue of modulus one;*

(b) *g is topologically complete,*

then any local isometry of $\widetilde{g}$ defined on a connected open subset of $\widetilde{M}$ can be uniquely extended to a C^∞ global isometry of $\widetilde{g}$ on $\widetilde{M}$. In particular, the isometry group of $\widetilde{g}$ acts transitively on $\widetilde{M}$.

The organization of the paper is as follows: in §2 we recall some definitions and elementary properties of rigid geometric structures. In §3 we recall the open-dense theorem of Gromov. Then in §4 we define and study normal rigid geometric structures. In §5 we define topological completeness and prove Proposition 1, which will be used in §6 as the departing point of the proof of Theorem 1.2. Finally, in §7, we prove Corollary 1 by showing the topological completeness of invariant generalized connections.

2. *Preliminaries*

For any $k \geq 1$ and $n \geq 1$, let $\mathrm{Gl}^{(k)}(n)$ be the set of k-jets at zero of diffeomorphisms of $\mathbb{R}^n$ fixing zero, which is a Lie group with respect to the composition of k-jets. For any $k \geq 1$, let $T_0^{(k)}\mathbb{R}^n$ be the vector space of $(k-1)$-jets at zero of C^∞ vector fields of $\mathbb{R}^n$. The Lie group $\mathrm{Gl}^{(k)}(n)$ admits a natural linear representation ρ on $T_0^{(k)}\mathbb{R}^n$ defined as follows: for any $j_0^k\phi \in \mathrm{Gl}^{(k)}(n)$ and $j_0^{k-1}Y \in T_0^{(k)}\mathbb{R}^n$,

$$\rho(j_0^k\phi)(j_0^{k-1}Y) = j_0^{k-1}(D\phi(Y)).$$

It is easy to see that ρ is injective and $\mathrm{Gl}^{(k)}(n)$ is a real algebraic group with respect to this faithful representation.

Let M be a C^∞ manifold of dimension n. For any $k \geq 1$, let $L^{(k)}(M)$ denote the kth order frame bundle of M. This is the principal fiber bundle over M whose elements are the k-jets at the origin $0 \in \mathbb{R}^n$ of diffeomorphisms from a neighborhood of $0 \in \mathbb{R}^n$ into M. The structural group of $L^{(k)}(M)$ is $\mathrm{Gl}^{(k)}(n)$ with the natural right action given by composition of k-jets. Geometric structures arise as sections of bundles associated to $L^{(k)}(M)$.

Let Z be a smooth real algebraic variety admitting an algebraic action of $\mathrm{Gl}^{(k)}(n)$ on the left. Let $L^{(k)}(M) \rtimes Z$ be the fiber bundle over M associated to $L^{(k)}(M)$ and such an action. A *geometric structure of order k and algebraic type Z* on M is a C^∞ section of the fiber bundle $L^{(k)}(M) \rtimes Z$. It is well known that geometric structures of order k and algebraic type Z are in bijection with $\mathrm{Gl}^{(k)}(n)$-equivariant C^∞ maps from $L^{(k)}(M)$ to Z.

For any $i \geq 0$, let $J_n^i(Z)$ be the space of i-jets of C^∞ maps from $\mathbb{R}^n$ to Z, which is a smooth real algebraic variety admitting a natural algebraic action of $\mathrm{Gl}^{(k+i)}(n)$ (see [**6**]). Let g be a C^∞ geometric structure of order k and algebraic type Z on M, viewed as a $\mathrm{Gl}^{(k)}(n)$-equivariant C^∞ map from $L^{(k)}(M)$ to Z. By differentiating g (see [**6**] for the details), we obtain a $\mathrm{Gl}^{(k+i)}(n)$-equivariant C^∞ map $g^{(i)} : L^{(k+i)}(M) \to J_n^i(Z)$, i.e. a C^∞ geometric structure of order $k+i$ and algebraic type $J_n^i(Z)$, which is said to be the *ith order prolongation* of g.

A C^∞ diffeomorphism f of M induces a bundle diffeomorphism $f_{(k)}$ of $L^{(k)}(M)$ given by composition of k-jets. Through such maps, the group of C^∞ diffeomorphisms of M acts naturally on the associated bundles $L^{(k)}(M) \rtimes Z$ and their smooth sections. If g is such a section, i.e. a geometric structure on M, then the group of diffeomorphisms of M that preserve g is denoted by $I(g)$ and is called the group of isometries of g. Similarly, $I^{\mathrm{loc}}(g)$ denotes the pseudogroup of local diffeomorphisms of M which preserve g.

Closely related to local isometries, isometric jets are algebraic and infinitesimal objects, which can be defined as follows: for any $i \geq 0$ and $x, y \in M$, let $D_{x,y}^{(k+i)}(M)$ be the space of $(k+i)$-jets of local diffeomorphisms from a neighborhood of x into M and which

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send x to y. We define

$$I_{x,y}^{k+i} = \{j_x^{k+i} f \in D_{x,y}^{(k+i)}(M) \mid (f_* g^i)(y) = g^i(y)\},$$

where $f_* g^i$ denotes the image of $g^{(i)}$ under the natural action of f. The elements of $I_{x,y}^{k+i}$ are said to be the $(k+i)$*th order isometric jets* of g from x to y. In the case that $x = y$, we denote $D_{x,x}^{(k+i)}(M)$ by $D_x^{(k+i)}(M)$ and $I_{x,x}^{k+i}$ by I_x^{k+i}. Similar to $\mathrm{Gl}^{(k)}(n)$, it is easy to see that $D_x^{(k+i)}(M)$ is a real algebraic group. Since the action of $\mathrm{Gl}^{(k+i)}(n)$ on $J_n^i(Z)$ is algebraic, I_x^{k+i} is a real algebraic subgroup of $D_x^{k+i}(M)$ (see [**22**]). For any $s \geq t \geq 1$, the natural projection $\pi_t^s : I_x^{k+s} \to I_x^{k+t}$ is defined as $\pi_t^s(j_x^{k+s} f) = j_x^{k+t} f$.

Definition 2.1. Under the notation above, a C^∞ geometric structure g is said to be *rigid* (or more precisely $(k+i)$*-rigid*) if there exists $i \geq 0$ such that for any $x \in M$ the natural projection $\pi_{k+i}^{k+i+1} : I_x^{k+i+1} \to I_x^{k+i}$ is injective.

It is well known that classical geometric structures of finite type are rigid, such as pseudo-Riemannian metrics, complete parallelisms and linear connections (see [**6**]). To illustrate the abstract definitions above, let us verify by a straightforward calculation that Riemannian metrics are rigid. We mention that similar calculations show equally the rigidity of pseudo-Riemannian metrics.

Let g be a C^∞ Riemannian metric on a manifold M of dimension n. For any $x \in M$, let us prove that $\pi_1^2 : I_x^2 \to I_x^1$ is injective. Take a normal coordinate system in an open neighborhood of x. The local isometries fixing x are determined by the following equations:

$$g_{ij}(\phi) \cdot \frac{\partial \phi_i}{\partial x_k} \cdot \frac{\partial \phi_j}{\partial x_l} = g_{kl} \quad \text{for all } 1 \leq k, l \leq n.$$

By differentiating these equations, we get for any $1 \leq s, k, l \leq n$,

$$\partial_r g_{ij}(\phi) \frac{\partial \phi_r}{\partial x_s} \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_j}{\partial x_l} + g_{ij}(\phi) \frac{\partial^2 \phi_i}{\partial x_k \partial x_s} \frac{\partial \phi_j}{\partial x_l} + g_{ij}(\phi) \frac{\partial \phi_i}{\partial x_k} \frac{\partial^2 \phi_j}{\partial x_l \partial x_s} = \partial_s g_{kl}.$$

So, I_x^2 is determined by the following algebraic equations:

$$g_{ij}(x) \cdot \frac{\partial \phi_i}{\partial x_k} \cdot \frac{\partial \phi_j}{\partial x_l} = g_{kl}(x)$$

and

$$\partial_r g_{ij}(x) \frac{\partial \phi_r}{\partial x_s} \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_j}{\partial x_l} + g_{ij}(x) \frac{\partial^2 \phi_i}{\partial x_k \partial x_s} \frac{\partial \phi_j}{\partial x_l} + g_{ij}(x) \frac{\partial \phi_i}{\partial x_k} \frac{\partial^2 \phi_j}{\partial x_l \partial x_s} = \partial_s g_{kl}(x).$$

Since the coordinate system at x is normal,

$$g_{ij}(x) = \delta_{ij}, \quad \partial_k g_{ij}(x) = 0 \quad \text{for all } 1 \leq i, j, k \leq n.$$

Suppose that $j_x^2 \phi \in I_x^2$ and $j_x^1 \phi = j_x^1 \,\mathrm{Id}$. Then we deduce easily from the algebraic equations above that

$$\frac{\partial^2 \phi_l}{\partial x_k \partial x_s} = -\frac{\partial^2 \phi_k}{\partial x_l \partial x_s} \quad \text{for all } 1 \leq s, k, l \leq n.$$

So,

$$\frac{\partial^2 \phi_l}{\partial x_k \partial x_s} = -\frac{\partial^2 \phi_k}{\partial x_l \partial x_s} = \frac{\partial^2 \phi_s}{\partial x_l \partial x_k} = -\frac{\partial^2 \phi_l}{\partial x_s \partial x_k}.$$

Thus, $\partial^2 \phi_l / \partial x_k \partial x_s \equiv 0$ and $j_x^2 \phi = j_x^2 \,\mathrm{Id}$. Therefore, π_1^2 is injective and g is rigid.

3. *Open-dense theorem*

The most striking result about rigid geometric structures is the open-dense theorem due to Gromov. Let us illustrate firstly this important theorem by the simple proposition below.

PROPOSITION 2. *Let A be a C^∞ complete parallelism on a manifold M. If its pseudogroup of local isometries I^{loc} admits a dense orbit, then I^{loc} acts transitively on M.*

Proof. Suppose that M is of dimension n. Recall that a C^∞ complete parallelism on M is a C^∞ section of the frame bundle $\pi : L^{(1)}(M) \to M$. Therefore, A is given by n vector fields $\{X_1, \ldots, X_n\}$ such that for any $x \in M$, $\{X_1(x), \ldots, X_n(x)\}$ are independent. We obtain $n^3 C^\infty$ functions $\{f^k_{i,j}\}_{1\le i,j,k\le n}$ such that

$$[X_i, X_j] = \sum_{1\le k\le n} f^k_{i,j} \cdot X_k.$$

Since the pseudogroup of local isometries of A admits a dense orbit, each function $f^k_{i,j}$ is constant on a dense subset of M. So, they must be all constant. Therefore, $\{X_1, \ldots, X_n\}$ generates a Lie algebra denoted by $\mathfrak{g}$.

Denote by G the simply connected Lie group with $\mathfrak{g}$ as its right-invariant Lie algebra. It is well known that A is induced by a local G-action on M. Denote by $\{\bar{X}_1, \ldots, \bar{X}_n\}$ the right-invariant vector fields of G inducing the complete parallelism A.

For any $x \in M$, we define $\alpha_x : G \to M$ such that $\alpha_x(g) = gx$. It is clear that in a neighborhood of the unit element $e \in G$, α_x is a C^∞ local diffeomorphism sending $\{\bar{X}_1, \ldots, \bar{X}_n\}$ to $\{X_1, \ldots, X_n\}$. Therefore, A is locally homogeneous, i.e. I^{loc} acts transitively on M. □

The following open-dense theorem gives a substantial generalization of the above proposition (see [**2, 9, 13, 15, 23**] for the proof).

THEOREM 3.1. (Gromov) *Let M be a C^∞ manifold and g a C^∞ rigid geometric structure of order k and algebraic type Z on M. If its pseudogroup of local isometries $I^{\mathrm{loc}}(g)$ admits a dense orbit in M, then $I^{\mathrm{loc}}(g)$ admits a unique open-dense orbit in M, which is denoted by Ω.*

Moreover, for any r large enough, g verifies the following condition: for any $x, y \in \Omega$ and any $g \in I^r_{x,y}$, there exists a unique germ of local isometry integrating g, which depends smoothly on g.

Remark 1. Suppose that g is $(k + i)$-rigid and that the pseudogroup of local isometries of g admits a dense orbit. We define the *stable degree* of g as follows:

$$d(g) = \inf\{k + s \mid \dim(I^{k+l}_x) = \dim(I^{k+j}_x), \forall x \in \Omega, \forall l \ge j \ge s\},$$

where Ω is the unique open-dense orbit of I^{loc}. It is well known that for any $x \in M$ and any $l \ge j \ge i$, the natural projection $\pi^{k+l}_{k+j} : I^{k+l}_x \to I^{k+j}_x$ is injective (see [**6**, Corollary 5.5]). Therefore, $d(g)$ is finite. We can show that for any $r \ge d(g)$, the second part of the theorem above is true (see [**9**]).

4. *Normal locally homogeneous structures*

Throughout this section, we denote by g a C^∞ rigid geometric structure of order k and algebraic type Z on M. We suppose that its pseudogroup of local isometries $I^{\mathrm{loc}}(g)$ admits

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a dense orbit. By the open-dense theorem, $I^{\text{loc}}(g)$ admits a unique open-dense orbit Ω in M. Inspired by the ideas in [**4**], we define in this section a normality condition for g, which ensures the existence of a (G, X)-structure on Ω.

4.1. *Definition of normality.* A C^∞ vector field Y on M is said to be *Killing* if the local flow of Y preserves the rigid geometric structure g. Take $x \in \Omega$ and denote by $\mathfrak{g}$ the space of germs at x of local C^∞ Killing fields of g. Let $\mathfrak{h}$ be the subset of $\mathfrak{g}$ of local Killing fields vanishing at x.

PROPOSITION 3. *Under the notation above, for any $Y, Z \in \mathfrak{g}$, $Y = Z$ as germs of vector fields at x if and only if $j_x^r Y = j_x^r Z$, where r is any integer greater than or equal to $d(g)$. In particular, $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. For any $Y, Z \in \mathfrak{g}$, their bracket is defined as the germ of the local vector field $[Y, Z]$ at x. If X is a vector field on M, its local flow lifts naturally to a local flow on the principal fiber bundle $L^{(k)}(M)$, which induces a vector field $X_{(k)}$ on $L^{(k)}(M)$.

The following relations are well known (see [**6**, Lemma 4.4]):

$$(Y + Z)_{(k)} = Y_{(k)} + Z_{(k)}, \quad (aY)_{(k)} = a \cdot Y_{(k)}$$

and

$$[Y_{(k)}, Z_{(k)}] = [Y, Z]_{(k)}.$$

Recall that g is of order k and algebraic type Z, which can be viewed as a $\mathrm{Gl}^{(k)}(n)$-equivariant C^∞ map from $L^{(k)}(M)$ to Z. It is easy to see that a vector field X is Killing if and only if $Dg(X_{(k)}) \equiv 0$. Therefore, we deduce from the formulas above that $Y + Z$, aY and $[Y, Z]$ are all local Killing fields. So, $\mathfrak{g}$ is a Lie algebra, and obviously $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

Let $r \geq d(g)$. Suppose that $Y \in \mathfrak{g}$ is such that $j_x^r Y = 0$ and denote by ϕ_t^Y the local flow of Y. By [**6**], we have the following relation:

$$\exp(t \cdot j_x^r Y) = j_x^r \phi_t^Y \quad \text{for all } |t| \ll 1.$$

Since $j_x^r Y = 0$, $j_x^r \phi_t^Y \equiv j_x^r \,\mathrm{Id}$ for any $|t| \ll 1$. Therefore, by the second part of the open-dense theorem, there exists an open neighborhood U of x such that ϕ_t^Y is defined on U and $\phi_t^Y|_U = \mathrm{Id}|_U$ for any $|t| \ll 1$. We deduce that $Y = 0$ in $\mathfrak{g}$. The proof is complete. □

Definition 4.1. Under the notation above, let $\bar{G}$ be the connected and simply connected Lie group whose Lie algebra of right-invariant fields is $\mathfrak{g}$. Let $\bar{H}$ be the connected Lie subgroup of $\bar{G}$ integrating the Lie subalgebra $\mathfrak{h}$. The C^∞ rigid geometric structure g is said to be *normal* if $\bar{H}$ is closed in $\bar{G}$.

Since $g|_\Omega$ is locally homogeneous, the normality of g is independent of the base point x in Ω. The proof of the following proposition is straightforward and can be found in [**9**, Ch. II].

PROPOSITION 4. *Let g be a C^∞ rigid geometric structure whose pseudogroup of local isometries admits a unique open-dense orbit Ω. If g is normal, then there exists a $\bar{G}$-invariant C^∞ rigid geometric structure $\bar{g}$ on $\bar{G}/\bar{H}$ locally isometric to $g|_\Omega$. Moreover,*

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by taking the local isometries from $(\Omega, g|_\Omega)$ *into* $(\bar{G}/\bar{H}, \bar{g})$ *as charts, we obtain a* $(I(\bar{g}), \bar{G}/\bar{H})$*-structure on* Ω*, where* $I(\bar{g})$ *denotes the isometry group of* $\bar{g}$*.*

Remark 2. One of the important points in the proof of Proposition 4 (see [**9**]) is to show that any local isometry of $\bar{g}$ defined on a connected open subset of $\bar{G}/\bar{H}$ can be extended to a global isometry of $\bar{g}$.

4.2. *A criterion for normality.* In this subsection, we prove the following proposition.

PROPOSITION 5. *Under the notation above, if there exists a local isometry of* g *fixing* x *whose differential at* x *has no eigenvalue of modulus one, then* g *is normal.*

LEMMA 4.2. *Under the notation above, if there exists a local isometry of* g *fixing* x *whose differential at* x *is hyperbolic, then the center of* $\mathfrak{g}$ *is trivial.*

Proof. Let I_x^{loc} be the group of local isometries of g fixing x. By the open-dense theorem, I_x^{loc} is identified naturally to I_x^r for any $r \geq d(g)$. Therefore, I_x^{loc} is a real algebraic group. By hypothesis, there exists a local isometry ϕ of g fixing x whose differential at x is hyperbolic. Let ϕ_h be the hyperbolic part in the real Jordan decomposition of ϕ (see [**22**]). Since I_x^{loc} is a real algebraic group, ϕ_h is also contained in I_x^{loc}.

With respect to the local action of I_x^{loc} on M, each right-invariant vector field of I_x^{loc} induces naturally a germ of a local Killing field vanishing at x. We can identify in this way the right-invariant Lie algebra of I_x^{loc} with $\mathfrak{h}$. Since ϕ_h is hyperbolic in the sense that ϕ_h is diagonalizable over $\mathbb{R}$ and with positive eigenvalues, its logarithm in $\mathfrak{h}$ is well defined, which is denoted by X_h. So, we have $\exp^{X_h} = \phi_h$. Moreover, since the differential of ϕ at x has no eigenvalue of modulus one, one is not the eigenvalue of the differential of ϕ_h at x. We deduce that zero is not the eigenvalue of the 1-jet $j_x^1 X_h$, which acts naturally on $T_x M$ as follows:

$$(j_x^1 X_h)(W) = j_x^0([X_h, \bar{W}]) \quad \text{for all } W \in T_x M,$$

where $\bar{W}$ denotes an arbitrary C^∞ vector field extension of W.

Now let Y be an element in the center of $\mathfrak{g}$, i.e. for any $X \in \mathfrak{g}$, $[Y, X] = 0$. In particular, we have $[Y, X_h] = 0$. Suppose that in a local coordinate system, $Y = Y^i(\partial/\partial x_i)$ and $X_h = X_h^i(\partial/\partial x_i)$. Since $[Y, X_h] = 0$, the 0-jet $j_x^0[Y, X_h] = 0$. Moreover, since $X_h \in \mathfrak{h}$, $j_x^0 X_h = 0$. So, we obtain

$$j_x^0[Y, X_h] = \left(Y^i \frac{\partial X_h^j}{\partial x_i}\right)(x) \frac{\partial}{\partial x_j} = 0.$$

However, since zero is not the eigenvalue of $j_x^1 X_h$ acting on $T_x M$, we deduce that $Y(x) = 0$, i.e. $j_x^0 Y = 0$.

Since g is locally homogeneous on Ω, for any $u \in T_x M$ there exists $X \in \mathfrak{g}$ such that $j_x^0 X = u$. Since Y is contained in the center of $\mathfrak{g}$, we have $[Y, X] = 0$. In particular, $j_x^0[Y, X] = 0$. Moreover, since $j_x^0 Y = 0$ by the argument above, for any $u \in T_x M$

$$j_x^0[Y, X] = -\left(X^i \frac{\partial Y^j}{\partial x_i}\right)(x) \frac{\partial}{\partial x_j} = -u^i \frac{\partial Y^j}{\partial x_i}(x) \frac{\partial}{\partial x_j} = 0.$$

We deduce that $j_x^1 Y = 0$.

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Since $[Y, X] = 0$, $j_x^1[Y, X] = 0$. We have by the argument above that $j_x^1 Y = 0$, which implies that the order 0 terms of $j_x^1[Y, X]$ are all zero. In a local coordinate system, the order 1 terms of $j_x^1[Y, X]$ are equally zero and given by

$$\frac{\partial}{\partial x_k}\left(Y^i \frac{\partial X^j}{\partial x_i} - X^i \frac{\partial Y^j}{\partial x_i}\right) = \left(\frac{\partial Y^i}{\partial x_k}\frac{\partial X^j}{\partial x_i} + Y^i \frac{\partial^2 X^j}{\partial x_i \partial x_k} - \frac{\partial X^i}{\partial x_k}\frac{\partial Y^j}{\partial x_i} - X^i \frac{\partial^2 Y^j}{\partial x_i \partial x_k}\right)(x)$$

$$= -\left(X^i \frac{\partial^2 Y^j}{\partial x_i \partial x_k}\right)(x) = -u^i \frac{\partial^2 Y^j}{\partial x_i \partial x_k}(x) = 0.$$

We deduce that for any i, j, k, $(\partial^2 Y^j / \partial x_i \partial x_k)(x) = 0$. Therefore, $j_x^2 Y = 0$.

It is clear that by successive derivations of $[Y, X]$ at x as above, we get $j_x^s Y = 0$ for any $s \in \mathbb{N}$. By Proposition 3, we know that for any $r \geq d(g)$, each local Killing field of g is determined by its r-jet at x. Therefore, $Y = 0$ in $\mathfrak{g}$. The proof is complete. □

LEMMA 4.3. *Under the notation above, if the center of* $\mathfrak{g}$ *is trivial, then* g *is normal.*

Proof. Let I_x^{loc} be the group of local isometries of g fixing x. Recall that by the open-dense theorem, I_x^{loc} is identified naturally to I_x^r for any $r \geq d(g)$. Therefore, I_x^{loc} is a real algebraic group. Define $\rho : I_x^{\mathrm{loc}} \to \mathrm{Aut}(\mathfrak{g})$ such that

$$\rho(h)(Y) = Dh(Y),$$

where $\mathrm{Aut}(\mathfrak{g})$ denotes the Lie group of automorphisms of $\mathfrak{g}$. Since ρ is a morphism of real algebraic groups, $\rho(I_x^{\mathrm{loc}})$ is a closed subgroup of $\mathrm{Aut}(\mathfrak{g})$ (see [**22**]).

On the other hand, we have the adjoint representation $\mathrm{Ad} : \bar{G} \to \mathrm{Aut}(\mathfrak{g})$. It is easy to see that $\mathrm{Ad}(\bar{H})$ and $\rho(I_x^{\mathrm{loc}})$ have the same Lie algebra in $\mathrm{Aut}(\mathfrak{g})$. So, we have

$$\mathrm{Ad}(\bar{H}) = (\rho(I_x^{\mathrm{loc}}))_e,$$

the identity component of $\rho(I_x^{\mathrm{loc}})$ which is closed in $\mathrm{Aut}(\mathfrak{g})$. Since the center of $\mathfrak{g}$ is supposed to be trivial, $\bar{H} = (\mathrm{Ad}^{-1}((\rho(I_x^{\mathrm{loc}}))_e))_e$. Therefore, $\bar{H}$ is closed in $\bar{G}$, i.e. g is normal. □

The two lemmas above certainly give the proof of Proposition 5. We deduce from Propositions 4 and 5 the following corollary.

COROLLARY 2. *Under the notation above, if there exists a local isometry of* g *fixing* x *whose differential at* x *is hyperbolic, then there exists a* $\bar{G}$*-invariant* C^∞ *rigid geometric structure* $\bar{g}$ *on* $\bar{G}/\bar{H}$ *locally isometric to* $g|_\Omega$. *Moreover, by taking local isometries as charts, we obtain a* $(I(\bar{g}), \bar{G}/\bar{H})$*-structure on* Ω, *where* $I(\bar{g})$ *denotes the isometry group of* $\bar{g}$.

5. *Topological completeness*

Throughout this section, we suppose that g is a locally homogeneous and normal C^∞ rigid geometric structure on a manifold M. Therefore, by Proposition 4, there exists a $\bar{G}$-invariant C^∞ geometric structure $\bar{g}$ on $\bar{G}/\bar{H}$ locally isometric to g. Moreover, by taking the local isometries from (M, g) into $(\bar{G}/\bar{H}, \bar{g})$ as charts, we obtain a $(I(\bar{g}), \bar{G}/\bar{H})$-structure on M, where $I(\bar{g})$ denotes the isometry group of $\bar{g}$. In this section, we formulate a notion of completeness for (M, g) which ensures that this $(I(\bar{g}), \bar{G}/\bar{H})$-structure on M is complete, i.e. its developing map is a surjective covering map.

5.1. *Geodesic structure.* There are several well-known notions of completeness for different types of rigid geometric structures. But most of them (if not all) are purely geometric, which are not adapted to the study of global rigidity of differential dynamical systems. Here we give an axiomatic approach to completeness, which is intuitive and adequate for the study of locally homogeneous rigid geometric structures.

Let us begin with the following simple example to illustrate the idea: let A be a C^∞ complete parallelism on M given by pointwise independent C^∞ vector fields $\{X_1, \ldots, X_n\}$. The main feature of A is the presence of a family of C^∞ curves in M: the orbits of the vector fields aX_i, where $a \in \mathbb{R}$ and $i = 1, \ldots, n$. Let us denote by $\mathcal{C}$ this family of C^∞ curves. It is clear that local isometries of A preserve $\mathcal{C}$. In addition, $\mathcal{C}$ verifies the following condition: for any $x, y \in M$, there exist a continuous curve $c : [a, b] \to M$ and a partition $a_0 = a < a_1 < \cdots < a_m = b$ such that $c(a) = x$, $c(b) = y$ and $c|_{[a_j, a_{j+1}]} \in \mathcal{C}$ for any $j = 0, \ldots, m-1$. It is natural to define A to be complete if each curve in $\mathcal{C}$ can be extended to a curve in $\mathcal{C}$ defined over $\mathbb{R}$. The following definition is motivated by this example.

Definition 5.1. Let g be a C^∞ rigid geometric structure on a manifold M. A *geodesic structure* of (M, g) is given by a family of C^∞ curves $\mathcal{C}$ in M, defined on connected intervals, verifying the following conditions:

(a) local isometries of (M, g) preserve the curves in the family $\mathcal{C}$ as parametric curves;

(b) for any $x \in M$, the constant curve at x is contained in $\mathcal{C}$, denoted by c_x;

(c) the restriction of a curve in $\mathcal{C}$ to a connected subinterval is still contained in $\mathcal{C}$;

(d) let $c_1, c_2 : [a, b] \to M$ be two curves in $\mathcal{C}$. If $c_1(a)$ is close to $c_2(a)$, then the curves c_1 and c_2 are close to each other;

(e) let c_1 and c_2 be two curves in $\mathcal{C}$. If c_1 and c_2 coincide on a non-empty open subinterval, then they coincide on their common connected interval of definition.

Furthermore, let $\bar{\mathcal{C}}$ be the family of continuous curves composed by pieces of curves in $\mathcal{C}$. For any $c \in \bar{\mathcal{C}}$, if c is not differentiable at a, then a is said to be a singular point of c.

(f) For any $x \in M$ and any open neighborhood V of x, there exists an open neighborhood U of x contained in V such that any two points in U can be joined by at least a curve in $\bar{\mathcal{C}}$ contained in U and defined on $[0, 1]$. In addition, let $\bar{\mathcal{C}}_U$ be the space of such curves and π_U be the projection of $\bar{\mathcal{C}}_U$ onto $U \times U$ sending each curve in $\bar{\mathcal{C}}_U$ to its extremities; then there exists a continuous section $\sigma_U : U \times U \to \bar{\mathcal{C}}_U$ of π_U such that the numbers of singular points of the curves in the image of σ_U are bounded above and $\sigma_U(x, x) = c_x$.

5.2. *Existence of geodesic structure.* Let us begin with the case of complete parallelism: let A be a complete parallelism on M given by pointwise independent C^∞ vector fields $\{X_1, \ldots, X_n\}$. Let $\mathcal{C}$ be the set of integral curves of the vector fields aX_i, where $a \in \mathbb{R}$ and $i = 1, \ldots, n$. Since for any $x \in M$, $\{X_1(x), \ldots, X_n(x)\}$ is a basis of T_xM, it is easy to verify that $\mathcal{C}$ is a geodesic structure of A.

Now let us consider the case of linear connection. Let ∇ be a C^∞ linear connection on an n-dimensional manifold M. It is well known that ∇ defines a horizontal distribution H on the frame bundle $L^{(1)}(M)$, which is invariant under the right action of the general linear group $\mathrm{Gl}^{(1)}(n)$. For the sake of completeness, let us recall the classical construction of

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complete parallelisms associated to ∇: for any u contained in the Lie algebra of $\mathrm{Gl}^{(1)}(n)$, the *fundamental vector field* corresponding to u is defined as the vector field on $L^{(1)}(M)$ induced by the right action of the one-parameter subgroup $\exp(tu)$, which is denoted by u^*. For any $\xi \in \mathbb{R}^n$, the *standard horizontal vector field* $B(\xi)$ corresponding to ξ is defined as $D\pi((B(\xi)_\alpha)) = \alpha \cdot \xi$ for any $\alpha \in L^{(1)}(M)$. Now let $\{u_1, \ldots, u_{n^2}\}$ be a basis of the Lie algebra of $\mathrm{Gl}^{(1)}(n)$ and $\{\xi_1, \ldots, \xi_n\}$ a basis of $\mathbb{R}^n$. We obtain the following complete parallelism of $L^{(1)}(M)$:

$$A = (u_1^*, \ldots, u_{n^2}^*, B(\xi_1), \ldots, B(\xi_n)).$$

Let f be a local diffeomorphism of M. It is well known that f is a local isometry of ∇ if and only if the induced action of f on $L^{(1)}(M)$ preserves this complete parallelism (see [**19**]). Now let $\mathcal{C}$ be the projections in M of the integral curves of vector fields defining A. It is evident that the projections of the integral curves of fundamental vector fields are points in M, while the projections of those of standard horizontal vector fields are classical geodesics of ∇. Therefore, $\mathcal{C}$ is a geodesic structure of ∇.

The general case can be treated similarly due to the following interesting result of Candel and Quiroga-Barranco [**7**]: let g be a C^∞ geometric structure of order k on M. If g is $(k+i)$-rigid, then there exists a C^∞ complete parallelism A on $L^{(k+i+1)}(M)$ verifying the following condition: for any local isometry f of g, its induced local diffeomorphism of $L^{(k+i+1)}(M)$ preserves A.

Now let $\mathcal{C}$ be the projections in M of the integral curves of vector fields composing this complete parallelism A. By the construction of A in [**7**], it is straightforward to verify that $\mathcal{C}$ gives a geodesic structure of g. Therefore, we have the following proposition.

PROPOSITION 6. *Let g be a C^∞ rigid geometric structure on M. There exists at least one geodesic structure of (M, g). Moreover, if the rigid structure g is preserved by a group action, then the geodesic structure constructed above is also invariant.*

5.3. *Non-uniqueness of geodesic structure.* Given a C^∞ rigid geometric structure g, there exist usually several geodesic structures. For example, let $g = (\nabla, E^+, E^-)$ on M, where ∇ is a C^∞ linear connection and E^+ and E^- are complementary C^∞ ∇-parallel subbundles of TM. It is clear that g is a rigid geometric structure on M. A natural geodesic structure $\mathcal{C}_1$ of (M, g) is simply given by the geodesics of ∇. Another more interesting geodesic structure $\mathcal{C}_2$ is given by ∇-geodesics tangent either to E^+ or to E^- (see [**4, 5**] for applications of $\mathcal{C}_2$ to Anosov systems).

More examples about non-uniqueness of geodesic structures are given by generalized connections. Let M be a C^∞ manifold of dimension n and $k \in \mathbb{N}$. Let H be a connection in the principal fiber bundle $L^{(k)}(M)$, which means that H is a C^∞ n-dimensional distribution on $L^{(k)}(M)$ transverse to the fibers and invariant under the natural right action of $\mathrm{Gl}^{(k)}(n)$. For $k = 1$, H is nothing but the usual linear connection on M. Therefore, connections in frame bundles are said to be *generalized connections*. It is easy to verify that H is a rigid geometric structure of order $k+1$.

A geodesic structure of H is given by Proposition 6 via the general construction in [**7**]. Another more interesting geodesic structure is given by the following observation: let π be the natural projection from $L^{(k)}(M)$ onto $L^{(1)}(M)$. It is easy to verify that $D\pi(H)$ is

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well defined and gives a horizontal distribution on $L^{(1)}(M)$ invariant under the right action of $\mathrm{Gl}^{(1)}(n)$. Therefore, $D\pi(H)$ is a linear connection on M and the geodesics of $D\pi(H)$ define a geodesic structure of H.

In a concrete dynamical context, the essential point is to find the best geodesic structure adapted to the situation.

5.4. *Definition of topological completeness.* Given a rigid geometric structure g and a geodesic structure $\mathcal{C}$ of (M, g), curves in the family $\mathcal{C}$ are said to be *geodesics* of g. Curves in $\bar{\mathcal{C}}$ are said to be *piecewise geodesics* of g. For any $c \in \mathcal{C}$, there exists by condition (e) above a unique geodesic $\bar{c}$ containing c and defined on a maximal interval. Such a geodesic $\bar{c}$ is said to be a *maximal geodesic*.

Definition 5.2. Let g be a C^∞ rigid geometric structure on a manifold M. A geodesic structure of (M, g) is said to be *topologically complete* if and only if all of its maximal geodesics are defined over $\mathbb{R}$. The rigid geometric structure g is said to be *topologically complete* if and only if (M, g) admits at least one topologically complete geodesic structure.

Let g be a generic rigid geometric structure on M. Generally speaking, g does not have any local isometry. So, it is clear that by considering the geodesics of an arbitrary complete Riemannian metric on M, g is topologically complete in our sense. Therefore, our topological completeness is useless for generic rigid geometric structures. However, in the case that g admits plenty of local isometries, the topological completeness becomes useful and induces an interesting topological property to be proved below.

5.5. *Topological completeness and the developing map.*

PROPOSITION 7. *Let g_1, g_2 be locally homogeneous C^∞ rigid geometric structures on connected manifolds M_1 and M_2. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a C^∞ local diffeomorphism sending g_1 to g_2. If g_1 is topologically complete, then f is a surjective covering map.*

Proof. Let $\mathcal{C}_1$ be a topologically complete geodesic structure of (M_1, g_1). Since g_1 and g_2 are both locally homogeneous, and f is a local isometry sending g_1 to g_2, it is easy to see that f pushes forward $\mathcal{C}_1$ to a well-defined geodesic structure of (M_2, g_2), denoted by $\mathcal{C}_2$. It is clear that f sends geodesics in $\mathcal{C}_1$ to geodesics in $\mathcal{C}_2$. For any $x, y \in M_2$, because of condition (f) in Definition 5.1, there exists a piecewise geodesic γ in $\bar{\mathcal{C}}_2$ joining y to x. Since $\mathcal{C}_1$ is topologically complete, we can lift γ to a piecewise geodesic $\hat{\gamma}$ in $\bar{\mathcal{C}}_1$. Therefore, f is a surjective map.

For any $x \in M_2$, take a small open neighborhood U of x verifying condition (f) of Definition 5.1. The set $f^{-1}(U)$ is decomposed as the disjoint union of its connected components

$$f^{-1}(U) = \bigcup U_i.$$

By lifting piecewise geodesics in $\bar{\mathcal{C}}_2$, it is easy to see that $f(U_i) = U$ for any i. So, in order to prove that f is a covering map, it is enough to show that for any i, $f|_{U_i} : U_i \to U$ is injective.

Take $y, z \in U_i$ such that $f(y) = f(z)$. Take a C^∞ curve γ in U_i such that $\gamma(0) = y$ and $\gamma(1) = z$. Define $\bar{\gamma} = f \circ \gamma$ and $\bar{y} = \bar{\gamma}(0)$. Since $f(y) = f(z)$, $\bar{\gamma}$ is a closed curve. By condition (f) in Definition 5.1, there exists a continuous family of piecewise geodesics $c_t(s)$ in $\bar{\mathcal{C}}_2$, defined on [0, 1], and verifying the following conditions:

(1) for any $t \in [0, 1]$, $c_t(0) = \bar{y}$ and $c_t(1) = \bar{\gamma}(t)$;

(2) $c_0(s) \equiv \bar{y}$ and $c_1(s) \equiv \bar{y}$.

Then, by conditions (d), (e) and (f) in Definition 5.1 and the topological completeness of $\mathcal{C}_1$, we can lift this family of piecewise geodesics to a continuous family of piecewise geodesics $\hat{c}_t(s)$ in $\bar{\mathcal{C}}_1$ such that $\hat{c}_t(0) \equiv y$ for any $t \in [0, 1]$. Therefore, we get a continuous curve $\hat{\gamma} = \hat{c}_t(1)$ in M_1 such that $f \circ \hat{\gamma} = \bar{\gamma}$. It is clear that $\hat{\gamma}(0) = \hat{\gamma}(1) = y$. Moreover, we can see as follows that $\hat{\gamma} = \gamma$.

Define $\Lambda = \{t \in [0, 1] \mid \hat{\gamma}(t) = \gamma(t)\}$, which is not empty and closed in [0, 1]. Since f is a local diffeomorphism and

$$f \circ \hat{\gamma} = \bar{\gamma} = f \circ \gamma,$$

Λ is also open in [0, 1]. We deduce that $\Lambda = [0, 1]$, i.e. $\hat{\gamma} = \gamma$.

Therefore, $y = \hat{\gamma}(1) = \gamma(1) = z$, which implies that $f|_{U_i}$ is injective. The proof is complete. □

Now let us prove Proposition 1: let $\mathcal{C}$ be a topologically complete geodesic structure of (M, g) which satisfies the conditions of Proposition 1. It is clear that we can pull back $\mathcal{C}$ to a topologically complete geodesic structure $\widetilde{\mathcal{C}}$ of $\widetilde{g}$ on $\widetilde{M}$. Therefore, $\widetilde{g}$ is also topologically complete. Since g is supposed to be locally homogeneous, by Corollary 1 there exists a $\bar{G}$-invariant C^∞ rigid geometric structure $\bar{g}$ on $\bar{G}/\bar{H}$ locally isometric to $\widetilde{g}$. Moreover, by taking local isometries as charts, we obtain a $(I(\bar{g}), \bar{G}/\bar{H})$-structure on $\widetilde{M}$, where $I(\bar{g})$ denotes the isometry group of $\bar{g}$.

Since $\widetilde{g}$ is topologically complete, by Proposition 6 the developing map of this $(I(\bar{g}), \bar{G}/\bar{H})$-structure, $D : \widetilde{M} \to \bar{G}/\bar{H}$, is a surjective covering map. Since $\bar{G}/\bar{H}$ is simply connected, the developing map D is a C^∞ diffeomorphism sending $\widetilde{g}$ to $\bar{g}$. Therefore, any local isometry of $\widetilde{g}$ defined on a connected open subset of $\widetilde{M}$ can be extended to a C^∞ global isometry of $\widetilde{g}$ on $\widetilde{M}$ (see Remark 2). Since local isometries are determined by finite-order isometric jets, the global extension of any local isometry of $\widetilde{g}$ is unique. The proof of Proposition 1 is complete.

6. *Applications to expanding maps*

6.1. *Preliminaries.* Let M be a closed C^∞ manifold. A C^∞ map $\varphi : M \to M$ is said to be *expanding* if there exist a Riemannian metric on M and two constants $C > 0$ and $\lambda > 1$ such that

$$\|D\varphi^n(u)\| \geq C \cdot \lambda^n \|u\| \quad \text{for all } u \in TM, \text{ for all } n \in \mathbb{N}.$$

Algebraic examples of expanding maps can be constructed as follows: let N be a connected and simply connected nilpotent Lie group. Denote by Aut(N) the group of automorphisms of N. Let C be a compact subgroup of Aut(N) and Γ a torsion-free uniform lattice of $N \rtimes C$, which is a closed subgroup of the semidirect product $N \rtimes \mathrm{Aut}(N)$. If $\psi \in \mathrm{Aut}(N)$ verifies the condition

$$\psi \circ \Gamma \circ \psi^{-1} \subseteq \Gamma,$$

then there exists a well-defined smooth map $\bar{\psi} : \Gamma\backslash N \to \Gamma\backslash N$ defined as

$$\bar{\psi}(\Gamma \cdot a) = \Gamma \cdot \psi(a) \quad \text{for all } a \in N.$$

If the moduli of the eigenvalues of $D_e\psi$ are all greater than one, then $\bar{\psi}$ is expanding, and is said to be an *expanding infra-nilendomorphism*. In the case that N is the Abelian group $\mathbb{R}^n$ and ψ is given by a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$, the induced map $\bar{A} : \mathbb{T}^n \to \mathbb{T}^n$ is said to be an *expanding linear endomorphism*.

The dynamical and topological study of expanding maps began with the fundamental work of Shub [**20**]. The complete topological classification of expanding maps was obtained by Gromov in [**16**], where it was shown that any expanding map is C^0 conjugate to an expanding infra-nilendomorphism. This means that for any expanding map ϕ, there exist an expanding infra-nilendomorphism ψ and a homeomorphism H such that $\phi \circ H = H \circ \psi$.

A natural problem worthy of consideration is the problem of classifying expanding maps up to C^k conjugacies, for any $k \geq 1$. However, the picture in this context is far from clear. Here are two reasons: firstly, it is clear that a generic C^1 perturbation of an expanding map is not C^1 conjugate to the initial one because of the modulus at periodic points. Secondly, there exist plenty of C^∞ expanding maps on exotic tori (see [**10**]) which are not even C^1 diffeomorphic to any infra-nilmanifold. So, a complete differential classification of expanding maps seems to be totally out of reach. However, in the context of the Zimmer–Gromov program, the following elegant one-dimensional result was proved by Feres in [**11**].

THEOREM 6.1. *Let φ be a C^∞ expanding map of $\mathbb{S}^1$. If φ preserves a C^∞ linear connection, then φ is C^∞ conjugate to an expanding linear endomorphism of $\mathbb{S}^1$.*

Motivated by Theorem 6.1 and inspired by [**5**], we prove in this section the following result.

THEOREM 6.2. *Let φ be a C^∞ expanding map of a closed manifold. If φ preserves a topologically complete C^∞ rigid geometric structure, then φ is C^∞ conjugate to an expanding infra-nilendomorphism.*

It is well known that expanding maps are covering maps. So, in particular, expanding maps are local diffeomorphisms. Now, let g be a C^∞ rigid geometric structure on M. By definition, g is said to be preserved by an expanding map φ of M if and only if for any $x \in M$ there exists an open neighborhood U of x such that $\varphi|_U : U \to \varphi(U)$ is a C^∞ diffeomorphism and φ sends $g|_U$ to $g|_{\varphi(U)}$. The proof of Theorem 6.2 will be given in the following subsections.

6.2. *A global homogeneous structure.* Let φ be a C^∞ expanding map of M. Let g be a topologically complete C^∞ rigid geometric structure on M preserved by φ.

PROPOSITION 8. *Under the notation above, g is locally homogeneous on M.*

Proof. Since φ is expanding, φ admits a dense orbit in M (see [**20**]). Therefore, the pseudogroup of local isometries $I^{\mathrm{loc}}(g)$ of g admits a dense orbit. We deduce by the

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open-dense theorem that $I^{\mathrm{loc}}(g)$ admits a unique open-dense orbit Ω in M. Since φ is expanding, for any integer m large enough, $\varphi^m(\Omega) = M$. Therefore, $\Omega = M$, i.e. g is locally homogeneous. □

Since φ is expanding, φ admits fixed points in M. Let $x \in M$ be a fixed point of φ. The germ of φ at x is a local g-isometry whose differential at x has no eigenvalue of modulus one. Therefore, we deduce from Proposition 1 the following proposition.

PROPOSITION 9. *Under the notation above, let $\widetilde{M}$ be the universal covering space of M and $\widetilde{g}$ the lifted rigid geometric structure of g on $\widetilde{M}$. Then any local isometry of $\widetilde{g}$ defined on a connected open subset of $\widetilde{M}$ can be uniquely extended to a global isometry of $\widetilde{g}$ on $\widetilde{M}$. In particular, the isometry group of $\widetilde{g}$ acts transitively on $\widetilde{M}$.*

Let $x \in M$ be a fixed point of φ. Let $\widetilde{x} \in \widetilde{M}$ be such that $\pi(\widetilde{x}) = x$, where $\pi : \widetilde{M} \to M$ denotes the canonical projection. Since φ is a covering map, there exists uniquely a lifted map $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$ of φ, which is a C^∞ diffeomorphism and satisfies $\widetilde{\varphi}(\widetilde{x}) = \widetilde{x}$. Let G be the isometry group of $\widetilde{g}$ and H the isotropy subgroup of $\widetilde{x}$ in G. Then, by Proposition 8, $\widetilde{M}$ is C^∞ diffeomorphic to G/H. Let Γ be the fundamental group of M, which is a discrete subgroup of G. We have $M \cong \Gamma \backslash G/H$.

By Proposition 8 and the open-dense theorem, H is canonically identified to $I^r_{\widetilde{x}}$ for any r large enough. Therefore, H is a naturally a real algebraic group, which implies that H has finitely many connected components. Since $\widetilde{M} \cong G/H$ is connected and simply connected, H and G have the same number of connected components. Therefore, $H_e = G_e \cap H$, where H_e and G_e denote the identity components of H and G. Since G acts transitively on $\widetilde{M}$, G_e also acts transitively on $\widetilde{M}$. We deduce that $\widetilde{M} \cong G_e/H_e$.

Since G has finitely many connected components, $\Gamma \cap G_e$ is a finite-index subgroup of Γ. So, up to finite covers, we can suppose in the following that $\Gamma \subseteq G_e$. Therefore, up to finite covers, M is identified to $\Gamma \backslash G_e/H_e$.

Since $\widetilde{\varphi} \in H$ and H has finitely many connected components, there exists $m > 0$ such that $\widetilde{\varphi}^m \in H_e$. It is clear that once φ^m is proved to be C^∞ conjugate to an expanding infra-nilendomorphism, we can deduce easily that φ itself is also an expanding infra-nilendomorphism. Therefore, without any loss of generality, we can suppose in the following that $\widetilde{\varphi} \in H_e$.

6.3. *A subgroup of the isometry group.* Let us illustrate firstly the idea of this subsection by a simple example: let $\widetilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $\widetilde{\varphi}(x, y) = (2x, 2y)$. It is clear that $\widetilde{\varphi}$ induces an expanding map of the torus $\mathbb{T}^2$, denoted by φ. Let ∇ be the canonical flat connection on $\mathbb{T}^2$, which is preserved by φ. The isometry group of the lifted connection $\widetilde{\nabla}$ is the group of affine maps $\mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$, which is a real algebraic group. Let $\Gamma = \mathbb{Z}^2$ be the fundamental group of $\mathbb{T}^2$. The Zariski closure of Γ in the isometry group of $\widetilde{\nabla}$ is the subgroup of translations $\mathbb{R}^2$, which also acts transitively on $\mathbb{R}^2$.

In the general case, we shall obtain a similar subgroup of the isometry group $I(\widetilde{g})$ by using the Zariski topology. Let us begin with the following definition.

Definition 6.3. Let V be a finite-dimensional real vector space and $\mathrm{GL}(V)$ the general linear group. A subgroup of $\mathrm{GL}(V)$ is said to be *almost real algebraic* if and only if it is the union of finitely many connected components of a real algebraic group (see [**22**]).

PROPOSITION 10. *Under the notation above, let* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ *be the adjoint representation. The groups* $\mathrm{Ad}(G_e)$ *and* $\mathrm{Ad}(H_e)$ *are both almost real algebraic.*

Proof. Since H is a real algebraic group and $\mathrm{Ad}|_H : H \to \mathrm{Aut}(\mathfrak{g})$ is a morphism of real algebraic groups, by [**22**] $\mathrm{Ad}(H)$ is almost real algebraic in $\mathrm{Aut}(\mathfrak{g})$. Moreover, since by Lemma 4.2 the center of $\mathfrak{g}$ is trivial, $\mathrm{Ad}(H_e)$ is also almost real algebraic in $\mathrm{Aut}(\mathfrak{g})$.

Let $\widetilde{\varphi}_h$ be the hyperbolic part in the real Jordan decomposition of $\widetilde{\varphi} \in H$. Let L_h be the logarithm of $\widetilde{\varphi}_h$ in $\mathfrak{h}$. Since $\mathrm{Ad}|_H : H \to \mathrm{Aut}(\mathfrak{g})$ is a morphism of real algebraic groups, by [**22**] $\mathrm{Ad}(\widetilde{\varphi}_h)$ is a hyperbolic element in $\mathrm{Aut}(\mathfrak{g})$. Moreover, since $\mathrm{ad}(L_h)$ is the logarithm of $\mathrm{Ad}(\widetilde{\varphi}_h)$ and φ is expanding, $\mathrm{ad}(L_h)$ is diagonalizable over $\mathbb{R}$ with strictly positive eigenvalues. Therefore, we get the decomposition

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_{\lambda_i},$$

where $\mathfrak{g}_{\lambda_i} = \{Y \in \mathfrak{g} \mid [L_h, Y] = \lambda_i Y\}$. If $\lambda_i \neq 0$, then, for any $Y_i \in \mathfrak{g}_{\lambda_i}$ and any j,

$$\mathrm{ad}(Y_i)(\mathfrak{g}_{\lambda_j}) \subseteq \mathfrak{g}_{\lambda_j + \lambda_i}.$$

Therefore, $\mathrm{ad}(Y_i)$ is a nilpotent element in $\mathrm{GL}(\mathfrak{g})$.

If $[L_h, Y] = 0$, then, as in the proof of Lemma 4.2, we can show that $Y(x) = 0$, i.e. $Y \in \mathfrak{h}$. Therefore, $\mathfrak{g}_0 \subseteq \mathfrak{h}$. We deduce that

$$\mathrm{ad}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{h}) + \sum_{\lambda_i \neq 0} \mathrm{ad}(\mathfrak{g}_{\lambda_i}).$$

Since we have shown above that $\mathrm{Ad}(H_e)$ is almost real algebraic, we can conclude that $\mathrm{Ad}(G_e)$ is also almost real algebraic by using the following result in [**8**].

PROPOSITION 11. *Let V be a finite-dimensional real vector space. A connected Lie subgroup of* $\mathrm{GL}(V)$ *is almost real algebraic if and only if there exists a basis* $\{X_1, \ldots, X_n\}$ *of its Lie algebra such that for any $i = 1, \ldots, n$, either X_i is nilpotent or X_i is semisimple, and their eigenvalues generate a $\mathbb{Q}$-vector space of dimension one.* □

Since $\mathrm{Ad}(G_e)$ is almost real algebraic, then we can consider the Zariski topology of $\mathrm{Ad}(G_e)$. Recall that $\Gamma \subseteq G_e$ by §6.2. Then $\mathrm{Ad}(\Gamma) \subseteq \mathrm{Ad}(G_e)$. Let $\overline{\mathrm{Ad}(\Gamma)}$ be the Zariski closure of $\mathrm{Ad}(\Gamma)$ in $\mathrm{Ad}(G_e)$. We define $U = (\mathrm{Ad}^{-1}(\overline{\mathrm{Ad}(\Gamma)}))_e$, which is a connected closed Lie subgroup of G_e.

6.4. *A transitive nilpotent group action.* In this subsection, we prove the following result.

PROPOSITION 12. *Under the notation above, the connected Lie group U is simply connected and nilpotent, which acts transitively on $\widetilde{M}$.*

We shall prove this proposition via several lemmas. Let $\Gamma_0 = \Gamma$ and, for any $n \geq 1$, let $\Gamma_n = \widetilde{\varphi}^{-n} \circ \Gamma \circ \widetilde{\varphi}^n$. We define $\Lambda = \bigcup_{n \geq 0} \Gamma_n$. Since $\Gamma \subseteq G_e$ and $\widetilde{\varphi} \in G_e$ by §6.2, for any $n \geq 0$, Γ_n is a discrete subgroup of G_e. It is easy to see that $\widetilde{\varphi} \circ \Gamma \circ \widetilde{\varphi}^{-1} \subseteq \Gamma$. Therefore, $\Gamma_0 \subseteq \Gamma_1$. We deduce that

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \cdots \subseteq \Lambda \subseteq G_e.$$

LEMMA 6.4. *Under the notation above, Λ acts minimally on $\widetilde{M}$, i.e. each orbit of Λ is dense in $\widetilde{M}$.*

Proof. Take $y \in \widetilde{M}$. For any $n \geq 0$ and any $z \in \widetilde{M}$, there exists $\gamma \in \Gamma$ such that

$$d(\gamma(\widetilde{\phi}^n(y)), \widetilde{\phi}^n(z)) \leq \operatorname{diam}(M),$$

where d is the distance on $\widetilde{M}$ induced by the lift of an arbitrary Riemannian metric on M. We deduce that

$$d(\widetilde{\phi}^{-n}(\gamma(\widetilde{\phi}^n(y))), z) \leq \frac{1}{c\lambda^n} \operatorname{diam}(M).$$

Therefore, the orbit of y with respect to the Λ-action is dense in $\widetilde{M}$. The proof is complete. □

LEMMA 6.5. *Under the notation above, the Zariski closure $\overline{\mathrm{Ad}(\Gamma)}$ coincides with the Zariski closure $\overline{\mathrm{Ad}(\Lambda)}$.*

Proof. Since, for any $n \geq 0$, $\Gamma_n = \widetilde{\varphi}^{-n} \circ \Gamma \circ \widetilde{\varphi}^n$, we have

$$\overline{\mathrm{Ad}(\Gamma_n)} = \mathrm{Ad}(\widetilde{\varphi})^{-n} \cdot \overline{\mathrm{Ad}(\Gamma)} \cdot \mathrm{Ad}(\widetilde{\varphi})^n.$$

In particular, $\overline{\mathrm{Ad}(\Gamma_n)}$ and $\overline{\mathrm{Ad}(\Gamma)}$ have the same dimension and the same number of connected components. Moreover, we have the evident inclusion $\overline{\mathrm{Ad}(\Gamma)} \subseteq \overline{\mathrm{Ad}(\Gamma_n)}$, since $\Gamma \subseteq \Gamma_n$. We deduce that

$$\overline{\mathrm{Ad}(\Gamma)} = \overline{\mathrm{Ad}(\Gamma_n)} \quad \text{for all } n \geq 0.$$

Therefore,

$$\overline{\mathrm{Ad}(\Lambda)} = \overline{\bigcup_{n\geq 0} \mathrm{Ad}(\Gamma_n)} \subseteq \overline{\bigcup_{n\geq 0} \overline{\mathrm{Ad}(\Gamma_n)}} = \overline{\mathrm{Ad}(\Gamma)} \subseteq \overline{\mathrm{Ad}(\Lambda)},$$

which implies that $\overline{\mathrm{Ad}(\Lambda)} = \overline{\mathrm{Ad}(\Gamma)}$. The proof is complete. □

LEMMA 6.6. *Under the notation above, U is a connected and simply connected nilpotent subgroup of G_e.*

Proof. It is well known that φ has only finitely many fixed points in M, say $\{x_0, x_1, x_2, \ldots, x_p\}$ with $x_0 = x$. For any $i \in \{0, 1, \ldots, p\}$, we take a lift of x_i in $\widetilde{M}$ denoted by $\widetilde{x}_i$. We can suppose that $\widetilde{x}_0 = \widetilde{x}$. Let θ_i denote the lifted map of φ fixing $\widetilde{x}_i$. Therefore, we have $\theta_0 = \widetilde{\varphi}$. Denote by $\langle \Gamma, \widetilde{\varphi} \rangle$ the subgroup of G_e generated by Γ and $\widetilde{\varphi}$. It is easy to see that $\{\theta_0, \theta_1, \ldots, \theta_p\} \subseteq \langle \Gamma, \widetilde{\varphi} \rangle$.

For any $\gamma \in \Gamma$, $\gamma^{-1} \circ \widetilde{\varphi}^{-1}$ is a contraction with respect to the lifted metric of a Lyapunov metric on M. Therefore, $\widetilde{\varphi} \circ \gamma$ admits a unique fixed point in $\widetilde{M}$, say $y \in \widetilde{M}$. So,

$$\pi(y) = \pi(\widetilde{\varphi} \circ \gamma(y)) = \varphi(\pi(y)).$$

Therefore, there exist $i \in \{0, 1, \ldots, p\}$ and $\gamma' \in \Gamma$ such that $y = \gamma'(\widetilde{x}_i)$. We deduce that $(\gamma'^{-1} \circ \widetilde{\varphi} \circ \gamma \circ \gamma')(\widetilde{x}_i) = \widetilde{x}_i$. Therefore,

$$\theta_0 \circ \gamma = \gamma' \circ \theta_i \circ \gamma'^{-1}.$$

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So, we get

$$\begin{aligned}\mathrm{Ad}(\theta_0)\cdot\overline{\mathrm{Ad}(\Gamma)} &\subseteq \overline{\bigcup_{0\le i\le p}\{\mathrm{Ad}(\gamma')\cdot\mathrm{Ad}(\theta_i)\cdot\mathrm{Ad}(\gamma')^{-1}\mid \gamma'\in\Gamma\}}\\ &\subseteq \overline{\bigcup_{0\le i\le p}\{g\cdot\mathrm{Ad}(\theta_i)\cdot g^{-1}\mid g\in\overline{\mathrm{Ad}(\Gamma)}\}}.\end{aligned}$$

Let $\hat{G}=\overline{\langle \mathrm{Ad}(\theta_0), \overline{\mathrm{Ad}(\Gamma)}\rangle}$ be the Zariski closure of the subgroup generated by $\mathrm{Ad}(\theta_0)$ and $\overline{\mathrm{Ad}(\Gamma)}$. Since, for any $i\in\{0,\ldots,p\}$, θ_i is contained in the subgroup generated by $\widetilde{\varphi}$ and Γ, we have $\mathrm{Ad}(\theta_i)\in\hat{G}$ for any $i\in\{0,\ldots,p\}$.

Since $\widetilde{\varphi}$ normalizes Λ, $\mathrm{Ad}(\theta_0)$ normalizes $\overline{\mathrm{Ad}(\Lambda)}$. Moreover, since, by Lemma 6.5, $\overline{\mathrm{Ad}(\Gamma)}=\overline{\mathrm{Ad}(\Lambda)}$, $\overline{\mathrm{Ad}(\Gamma)}$ is a normal subgroup of $\hat{G}$. We shall apply the following group-theoretical result obtained in [**5**, p. 21] to conclude the proof.

PROPOSITION. *Let G be a real algebraic group, $\delta_0\in G$ and G_1 a normal subgroup in G. We suppose that G is the Zariski closure of the group generated by δ_0 and G_1. If, in addition, there exist $\delta_1,\ldots,\delta_p\in G$ such that*

$$\delta_0 G_1\subseteq \overline{\bigcup_{0\le i\le p}\{g\delta_i g^{-1}\mid g\in G_1\}},$$

then $(G_1)_e$ is a unipotent subgroup.

We deduce from this proposition that $(\overline{\mathrm{Ad}(\Gamma)})_e$ is unipotent, i.e. $(\overline{\mathrm{Ad}(\Gamma)})_e$ is conjugate to a subgroup of the group of lower-triangular matrices with 1's on the diagonal. Therefore, $(\overline{\mathrm{Ad}(\Gamma)})_e$ is simply connected and nilpotent. Since $\mathfrak{g}$ has trivial center by Lemma 4.2, $U=(\mathrm{Ad}^{-1}(\overline{\mathrm{Ad}(\Gamma)}))_e$ is also simply connected and nilpotent. □

LEMMA 6.7. *Under the notation above, U acts transitively on $\widetilde{M}$.*

Proof. It is easy to see that the action of $\overline{\mathrm{Ad}(\Gamma)}$ on $\mathrm{Ad}(G_e)/\mathrm{Ad}(H_e)$ is algebraic. Even though $\mathrm{Ad}(G_e)/\mathrm{Ad}(H_e)$ is not necessarily affine algebraic, we can show that each orbit of $\overline{\mathrm{Ad}(\Gamma)}$ is open in its closure with respect to the analytic topology of $\mathrm{Ad}(G_e)/\mathrm{Ad}(H_e)$ (see [**3, 21**]).

Since Λ acts minimally on G_e/H_e by Lemma 6.4, $\mathrm{Ad}(\Lambda)$ also acts minimally on $\mathrm{Ad}(G_e)/\mathrm{Ad}(H_e)$. Moreover, since, by Lemma 6.5, $\overline{\mathrm{Ad}(\Gamma)}=\overline{\mathrm{Ad}(\Lambda)}$, each orbit of $\overline{\mathrm{Ad}(\Gamma)}$ is open dense in $\mathrm{Ad}(G_e)/\mathrm{Ad}(H_e)$. So, $\overline{\mathrm{Ad}(\Gamma)}$ acts transitively on $\mathrm{Ad}(G_e)/\mathrm{Ad}(H_e)$. We deduce that $\mathrm{Ad}^{-1}(\overline{\mathrm{Ad}(\Gamma)})$ acts transitively on $\widetilde{M}\cong G_e/H_e$. Therefore, U also acts transitively on $\widetilde{M}$. □

The proof of Proposition 11 is complete by the lemmas above.

6.5. *The end of the proof of Theorem 6.2.* Denote by $\mathrm{Aff}(U,\widetilde{M})$ the group of diffeomorphisms of $\widetilde{M}$ normalizing the action of U. Since, by Proposition 11, U is simply connected and nilpotent, $\mathrm{Aff}(U,\widetilde{M})$ is naturally a real algebraic group. In addition, the group generated by $\widetilde{\varphi}$ and Γ is included in $\mathrm{Aff}(U,\widetilde{M})$.

Let $N=(\overline{\Gamma})_e$ be the identity component of the Zariski closure of Γ in $\mathrm{Aff}(U,\widetilde{M})$. By the same arguments as in the proof of Lemma 6.6, we can show that N is unipotent.

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Therefore, N is connected, simply connected and nilpotent. Now, by the same group-theoretical arguments as in [**5**, Section 4.3.4], we obtain the following lemma.

LEMMA 6.8. *Under the notation above, N is included in G and acts freely and transitively on $\widetilde{M}$. Moreover, $\Gamma \cap N$ is of finite index in Γ and cocompact in N.*

Therefore, up to finite covers, M is identified to the closed nilmanifold $(\Gamma \cap N)\diagdown N$. Moreover, since it is clear that $\widetilde{\varphi}$ normalizes N (see Lemmas 6.5 and 6.6) and $\widetilde{\varphi} \circ \Gamma \circ \widetilde{\varphi}^{-1} \subseteq \Gamma$, up to finite covers, φ is C^∞ conjugate to the expanding nilendomorphism of $(\Gamma \cap N)\diagdown N$ induced by the group automorphism $\Phi \in \mathrm{Aut}(N)$ defined as

$$\Phi(\xi) = \widetilde{\varphi} \circ \xi \circ \widetilde{\varphi}^{-1} \quad \text{for all } \xi \in N.$$

The proof of Theorem 6.2 is complete. □

7. *Invariant generalized connections*

In §5.3, we have seen that invariant generalized connections project naturally to invariant linear connections. Therefore, to prove Corollary 1, we need only consider the case of invariant linear connections. Moreover, by Theorem 1.2, it is enough to show the completeness of invariant linear connections.

LEMMA 7.1. *Let φ be a C^∞ expanding map of M. If φ preserves a C^∞ linear connection ∇, then ∇ is complete, i.e. maximal geodesics of ∇ are all defined over $\mathbb{R}$.*

Proof. Let $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$ be a lift of φ on the universal covering space. Let g be an arbitrary C^∞ Riemannian metric on M. Since M is closed, there exists $\epsilon > 0$ such that any geodesic γ of ∇ verifying $\|\gamma'(0)\| < 1$ is defined at least over $]-\epsilon, \epsilon[$. Therefore, any geodesic $\bar{\gamma}$ of the lifted connection $\widetilde{\nabla}$ verifying $\|\bar{\gamma}'(0)\| < 1$ is defined at least over $]-\epsilon, \epsilon[$.

For any $v \in T\widetilde{M}$, let $\bar{\gamma}$ be the $\widetilde{\nabla}$-geodesic verifying $\bar{\gamma}'(0) = v$. Since $\widetilde{\varphi}$ is expanding and preserves $\widetilde{\nabla}$, for any integer n large enough, $\widetilde{\varphi}^{-n} \circ \bar{\gamma}$ is a $\widetilde{\nabla}$-geodesic and verifies

$$\|(\widetilde{\varphi}^{-n} \circ \bar{\gamma})'(0)\| = \|D\widetilde{\varphi}^{-n}(v)\| < 1.$$

Therefore, $\widetilde{\varphi}^{-n} \circ \bar{\gamma}$ is defined over $]-\epsilon, \epsilon[$. We deduce that $\bar{\gamma}$ is defined at least on $]-\epsilon, \epsilon[$, which implies that $\widetilde{\nabla}$ is complete. Therefore, ∇ is complete. □

Our Corollary 1 gives a complete characterization of expanding infra-nilendomorphisms because of the following well-known lemma.

LEMMA 7.2. *Any expanding infra-nilendomorphism preserves a C^∞ linear connection.*

Proof. Let $\bar{\psi} : \Gamma\diagdown N \to \Gamma\diagdown N$ be an expanding infra-nilendomorphism, where Γ denotes a uniform lattice of a subgroup $N \rtimes C$ of $N \rtimes \mathrm{Aut}(N)$. Take a basis of the Lie algebra of N, $\{X_1, \ldots, X_n\}$, which are left invariant fields on N. It is clear that there exists a unique C^∞ linear connection ∇ on N verifying

$$\nabla_{X_i} X_j = 0 \quad \text{for all } 1 \le i, j \le n.$$

Since any element of $\mathrm{Aut}(N)$ sends left-invariant fields to left-invariant fields, ∇ is $\mathrm{Aut}(N)$-invariant. Moreover, since ∇ is left invariant, ∇ is $N \rtimes \mathrm{Aut}(N)$-invariant. We deduce that ∇ is ψ-invariant and Γ-invariant. Therefore, the quotient connection of ∇ is well defined on $\Gamma\diagdown N$, which is $\bar{\psi}$-invariant. □

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