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Universal homogeneous graph-like structures and domains

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We present explicit constructions of universal homogeneous objects in categories of domains with stable embedding–projection pairs as arrows. These results make use of a representation of such domains through graph-like structures and apply a generalization of Rado's result on the existence of the universal homogeneous countable graph. In particular, we build universal homogeneous objects in the categories of coherence spaces and qualitative domains, introduced by Girard (Girard 1987; Girard 1986), and two categories of hypercoherences recently studied by Ehrhard (Ehrhard 1993). Our constructions rely on basic numerical notions. We also show that a suitable random construction of Rado's graph and its generalizations produces with probability 1 the universal homogeneous structures presented here.

1. Introduction

A leading theme in the mathematical study of the semantics of programming languages is the search for smooth techniques for solving recursive domain equations. The basic method stems from the work of Scott on models of the λ-calculus (Scott 1972), which was further developed in Wand (1979) and Smyth and Plotkin (1982). Basically, it consists of forming, for a (cartesian closed) category of domains $\mathscr{C}$, a new category $\mathscr{C}^e$ having the same objects as $\mathscr{C}$ and whose arrows witness a relation of *approximation* between the objects of $\mathscr{C}$[¶]. These are the *embeddings*, that is, arrows $e : X \to Y$ of $\mathscr{C}$ that appear in pairs

[¶] The issue of what properties a reasonable category of domains should possess, though an important one, will not be pursued here. In the rest of this paper we shall use categorical concepts only relative to categories whose arrows are basically set-theoretic functions.

$$X \underset{p}{\overset{e}{\rightleftarrows}} Y$$

and satisfy the equations $p \circ e = 1_X$, $e \circ p \sqsubseteq_{Y^Y} 1_Y$ where $\sqsubseteq_{Y^Y}$ is a suitable ordering on the exponential object Y^Y. In these formulas, p is the *projection* and it turns out to be uniquely determined by e. The passage from $\mathscr{C}$ to $\mathscr{C}^e$ is motivated by the need to turn the function space constructor into a functor that is covariant in *both* arguments. This allows us to define solutions of recursive domain equations as fixed points of functors over $\mathscr{C}^e$. Given an expression $T(X)$ involving domain constructors, one method for solving an equation of the form

$$X \cong T(X), \tag{1}$$

builds the solution as $\varinjlim(T^n(\mathbf{1}), T^n(!))$, where $\mathbf{1}$ is the initial† object of $\mathscr{C}^e$ with the unique arrow $! : \mathbf{1} \to T(\mathbf{1})$. This makes use of the observation that T can be seen as an endofunctor on $\mathscr{C}^e$ preserving colimits of chains of embeddings of the form

$$X_0 \xrightarrow{e_0} X_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{n-1}} X_n \xrightarrow{e_n} X_{n+1} \xrightarrow{e_{n+1}} \cdots \tag{2}$$

and that $(T^n(\mathbf{1}), T^n(!))$ is such a chain.

A related method, and one we shall be interested in throughout this paper, can be used whenever the category $\mathscr{C}^e$ contains a *universal domain*, which is an object U such that for every object X of $\mathscr{C}$ there is an embedding $e : X \to U$ (with associated projection $p : U \to X$). Then X can be represented as (the range of) a mapping $\pi_X =_{\text{def}} e \circ p : U \to U$ such that $\pi_X = \pi_X \circ \pi_X \sqsubseteq_{U^U} 1_U$ and $\text{im}[\pi_X] \cong X$, which is also called a *projection* (Amadio *et al.* 1986). Projections $\pi : U \to U$ such that $\text{im}[\pi]$ is an object of $\mathscr{C}$ are called *finitary*, and form a cpo $\mathsf{Fp}[U]$ with the remarkable property that the approximation relation on domains is represented faithfully by the partial order relation on the representing projections. Another essential property of projections is that each of the standard domain constructors used in denotational semantics, seen as a functor $T : \mathscr{C}^e \to \mathscr{C}^e$, is *representable* as a continuous functional

$$R_T : \mathsf{Fp}[U] \to \mathsf{Fp}[U]$$

in such a way that $\text{im}[R_T(\pi_X)] \cong T(\text{im}[\pi_X])$. Furthermore, for a chain of embeddings of the form (2), we have $\varinjlim(X_n, e_n) \cong \text{im}[\bigsqcup_{n \in \mathbb{N}} \pi_{X_n}]$, where the least upper bound is taken in the cpo $\mathsf{Fp}[U]$. In particular, the domain $\text{im}[\mathsf{fix}(R_T)]$ solves equation (1).

The first results on universal domains consisted of proving that some familiar structure had the desired property of universality within some category of interest. In his pioneering work, Scott (Scott 1976) observed that the powerset of natural numbers $\mathscr{P}\omega$ is a universal domain in the category of ω-algebraic lattices with continuous closures as arrows, and

† Note that $\mathbf{1}$ is indeed the *terminal* object in the original category $\mathscr{C}$.

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at the same time a universal continuous lattice (Scott 1972) if continuous retractions are taken as arrows. Plotkin (Plotkin 1978a) generalized Scott's work to cpo's by replacing $\mathscr{P}\omega$ with $\mathbf{T}^{\omega}$, the direct product of denumerably many copies of the flat cpo of truth values. Further, Scott (Scott 1982) used properties of boolean algebras to prove the existence of a universal Scott domain (that is, a consistently complete, ω-algebraic cpo).

A different approach was initiated by Gunter (Gunter 1987), who used model-theoretic methods in the construction of domains, forcing them to have the universality property (in the category of profinite domains). His work has been pursued in Gunter and Jung (1990), and especially in Droste and Göbel (1993) which presented a technique for building universal homogeneous domains through the amalgamation property in a wide range of categories of domains, including categories in which arrows are *stable* embeddings (see also Droste (1991), Droste (1992) and Droste and Göbel (1991)).

In this paper we show that, for important categories of domains with stable embeddings as arrows, universal homogeneous objects exist as a consequence of general results in the theory of graphs. In particular, we start from the observation that Rado's universal homogeneous countable graph (Rado 1967) can be used directly to build the universal homogeneous countable coherence space (Girard 1987) (that is, a universal coherent atomic dI-domain). We then extend Rado's construction to hypercoherences, introduced in Ehrhard (1993) to build a model of classical linear logic (Girard 1987) from an extensional notion of sequential function. The same construction is carried out for hypercoherences for which consistency is downward closed. These are essentially the qualitative domains of Girard (1986) (that is, the atomic dI-domains), so we obtain a direct proof of the existence of a universal homogeneous qualitative domain. As a matter of fact, the straightforward use of graph-theoretic techniques is made possible exactly because we consider domains with strong atomicity properties; for constructions of universal domains without atomicity property see Droste and Göbel (1993). The present universal homogeneous objects can also be shown to exist using the more abstract results proved in Droste and Göbel (1993). But we would like to stress that the universal homogeneous structures presented in this paper all have the set of natural numbers as carrier, and their construction makes use only of very basic number-theoretic notions. We also briefly discuss probabilistic versions of these constructions. These proceed as follows: as underlying set we again take the set ω of natural numbers. Let $p \in (0,1)$ be fixed: for any finite subset of ω having at least two elements, decide with probability p to put it into the hypercoherence. We then show that with probability 1 we obtain the uniquely determined (up to isomorphism) universal homogeneous hypercoherence. This shows that, instead of pursuing more or less complicated constructions, in the present categories it suffices to perform our construction in a purely random manner and we obtain (somehow surprisingly) with probability 1 the required universal homogeneous object.

2. An introduction to universality

In this section, we give an introduction to universality using categorical ideas, and present some basic facts that we shall use in the following; we refer the reader to Droste and Göbel (1993) for some of the intuitions behind the following definitions. The underlying idea is that objects in our categories are relational structures and arrows are embeddings. Let $\mathscr{C}$ be a category where all arrows are monic, and let $\mathscr{C}^*$ be a full subcategory of $\mathscr{C}$. An arrow $f : A \to B$ in $\mathscr{C}$ is an *increment* iff f is not an isomorphism but $f = g \circ h$ implies that either g or h is an isomorphism. An object $U \in \mathrm{Obj}(\mathscr{C})$ is:

— $\mathscr{C}^*$*-universal* iff for every object $A \in \mathrm{Obj}(\mathscr{C}^*)$ there is an arrow $f : A \to U$; for example, when $\mathscr{C}$ is a preordered set, a $\mathscr{C}^*$-universal object is the same as an upper bound of $\mathscr{C}^*$. For relational structures this means that any object $A \in \mathrm{Obj}(\mathscr{C}^*)$ can be embedded into U;

— $\mathscr{C}^*$*-homogeneous* iff for any $A \in \mathrm{Obj}(\mathscr{C}^*)$ and for any two arrows $f, g : A \to U$ there exists an automorphism h of U (that is, an arrow $h : U \to U$ of $\mathscr{C}$ that is an isomorphism) such that $h \circ g = f$ (in other words, every time an object of $\mathscr{C}^*$ can be 'mapped' into U using two arrows, these arrows just differ for the composition by an automorphism of U). For relational structures this means that any isomorphism between two $\mathscr{C}^*$-substructures of U can be extended to an automorphism of U;

— $\mathscr{C}^*$*-saturated* iff for any $A, B \in \mathrm{Obj}(\mathscr{C}^*)$ and for any two arrows $f : A \to U$ and $g : A \to B$ there is an arrow $h : B \to U$ such that $h \circ g = f$ (this can be interpreted as follows: if an object A of $\mathscr{C}^*$ can be mapped to U via f, and if it can also be mapped to some other object B, then f can be naturally extended to a map from B to U);

— $\mathscr{C}^*$*-stepwise-saturated* iff for any two objects $A, B \in \mathrm{Obj}(\mathscr{C}^*)$ and any two arrows $f : A \to U$ and $g : A \to B$ such that g is an increment, there exists an arrow $h : B \to U$ with $h \circ g = f$. (Notice that every $\mathscr{C}^*$-saturated object is also $\mathscr{C}^*$-stepwise-saturated).

In analogy with the definition of a finite (or compact, or isolated) element of a partially ordered set, we say that an object B of a category $\mathscr{C}$ is *finite* (in $\mathscr{C}$) if, for every ω-chain $(A_i, f_i : A_i \to A_{i+1})$ in $\mathscr{C}$ with colimit A, and every arrow $g : B \to A$, there is an $i \in \omega$ such that there is a unique $h : B \to A_i$ satisfying $g = f_{i\infty} \circ h$, where $f_{i\infty} : A_i \to A$ is the i-th component of the universal cone with vertex A. A category is *semi-algebroidal* if every ω-chain of finite objects has a colimit, and, moreover, every object is the colimit of an ω-chain of finite objects.

We say that $\mathscr{C}$ is *incremental* iff it contains a weakly initial object and for any arrow $f : A \to B$ between two finite objects $A, B \in \mathrm{Obj}(\mathscr{C}_f)$ there exists a finite chain $(A_i, f_i)_{i=0,\ldots,n-1}$ such that $A = A_0$, $B = A_n$, $f = f_{n-1} \circ \cdots \circ f_1 \circ f_0$ and each $f_i : A_i \to A_{i+1}$ is an increment.

We shall only be interested in the case in which $\mathscr{C}$ is a semi-algebroidal category with initial object and $\mathscr{C}^*$ is the subcategory of all finite objects. We then have the following proposition.

Proposition 2.1. Let $\mathscr{C}$ be a semi-algebroidal category with initial object, U be an object of $\mathscr{C}$ and $\mathscr{C}_f$ be the full subcategory of its finite objects.

1 U is a $\mathscr{C}$-universal $\mathscr{C}_f$-homogeneous object if and only if U is a $\mathscr{C}_f$-saturated object. Moreover, in this case U is unique up to isomorphism.

2 If $\mathscr{C}$ is incremental and U is $\mathscr{C}_f$-stepwise-saturated, then U is $\mathscr{C}_f$-saturated as well.

Proof. (1) was proved in Droste and Göbel (1993). For (2), suppose that $f : A \to U$ and $g : A \to B$ are arrows, where A and B are two finite objects. Since the category

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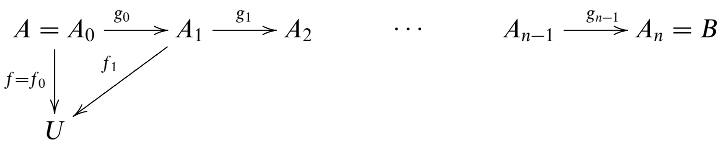
is incremental, there exists a finite chain $A = A_0 \stackrel{g_0}{\to} A_1 \stackrel{g_1}{\to} \cdots A_{n-1} \stackrel{g_{n-1}}{\to} A_n = B$ where $g = g_{n-1} \circ \cdots \circ g_1 \circ g_0$ and each g_i is an increment. Now let $f_0 = f$. By definition of stepwise-saturation, we obtain an arrow $f_1 : A_1 \to U$ such that $f_1 \circ g_0 = f_0$, as in the commutative diagram



Proceeding in this way, we finally obtain an arrow $f_n : B \to U$ with $f_n \circ g_{n-1} \circ \cdots g_1 \circ g_0 = f$. Letting $h = f_n$ we have $h \circ g = f$, as required. □

According to this result, in order to prove that a certain object U is universal and homogeneous we just have to show that, whenever we have an embedding $f : A \to U$ of a finite object A, and whenever A is embeddable into B, there is an embedding $g : B \to U$ through which f factors. In particular, if the category is incremental, an inductive proof of saturation is possible.

In Droste and Göbel (1993), necessary and sufficient conditions for the finite objects of an *algebroidal*† category $\mathscr{C}$ were given so that $\mathscr{C}$ contains a $\mathscr{C}$-universal $\mathscr{C}_f$-homogeneous object U. These abstract conditions (the 'amalgamation property') can be shown to be satisfied in the categories studied in this paper and thus could be used to ensure the existence and uniqueness, and even to construct such an object U. However, that construction would be of very high complexity. The present proofs will provide explicit and easy definitions of objects U that are $\mathscr{C}_f$-saturated. Then, from Proposition 2.1 (which itself has an easy proof) it follows that U is universal and homogeneous.

In this paper, we will provide, for several categories $\mathscr{C}$, explicit descriptions of universal homogeneous objects U in $\mathscr{C}$. Here and in the following, the words 'universal' and 'homogeneous' will always mean $\mathscr{C}$-universal and $\mathscr{C}_f$-homogeneous, respectively.

3. Universal domain constructions

3.1. *Graphs and coherence spaces*

In this subsection we shall recall Rado's result on the existence of a universal homogeneous (countable) graph, and see how this yields a construction for a universal homogenous coherence space.

In the present paper, a *graph* is given by a pair $\boldsymbol{G} = (G, \sim_G)$, where G is a non-empty set of vertices and $\sim_G$ is a reflexive and symmetric adjacency relation‡, the adjacency relation. A *graph homomorphism* f from the graph $\boldsymbol{G}$ to the graph $\boldsymbol{G}'$ is a map $f : G \to G'$ such that $f(x) \sim_{G'} f(y)$ whenever $x \sim_G y$. We shall say that $\boldsymbol{G}$ is an induced subgraph (or, simply, a *subgraph*) of $\boldsymbol{G}'$ if $G \subseteq G'$ and $\sim_G = \sim_{G'} \cap (G \times G)$.

† A category is algebroidal if it is semi-algebroidal, the subcategory of finite objects has a countable skeleton and the Hom-set between any two finite objects is countable.

‡ Observe that we are dealing here with reflexive graphs, that is, graphs with a self-loop at each vertex.

A *(graph) embedding*† is a homomorphism $f : \boldsymbol{G} \to \boldsymbol{G}'$ of graphs that is injective and such that

$$\forall x, y \in G.\, x \sim_G y \text{ iff } f(x) \sim_{G'} f(y).$$

Of course, a graph can be embedded into another one iff the former is isomorphic to a subgraph of the latter.

We let ω-**Graph** be the category of countable graphs with embeddings as arrows; the main properties of this category are listed in the following proposition.

Proposition 3.1. In the category ω-**Graph** every arrow is monic, and an object is finite iff it has a finite underlying set. Moreover, ω-**Graph** is an algebroidal, incremental category, where $f : \boldsymbol{G} \to \boldsymbol{G}'$ is an increment iff $G' = f[G] \cup \{g\}$, with $g \notin f[G]$.

The important property of the category ω-**Graph** we are interested in was studied in Rado (1967), where it is proved by means of an explicit construction that a universal homogeneous graph exists. We shall give essentially the same proof, exploiting the fact that ω-**Graph** is an incremental category and using Proposition 2.1 (for more on Rado's graph, see Cameron (1990)). In the following, let $\wp_{\text{fin}}(\omega)$ be the collection of finite subsets of ω, and let $\Phi : \wp_{\text{fin}}(\omega) \to \omega$ be the bijection that assigns to $u \in \wp_{\text{fin}}(\omega)$ the natural number

$$\Phi(u) =_{\text{def}} \sum_{n \in u} 2^n.$$

Theorem 3.2. (Rado 1967) Let $\boldsymbol{G}_U$ be the graph having $U = \omega$ as underlying set, and with compatibility relation $\sim_U$ defined as follows:

$$n \sim_U m \text{ iff } n = m \text{ or } \min\{n, m\} \in \Phi^{-1}(\max\{n, m\}).$$

Then $\boldsymbol{G}_U$ is the universal homogeneous object of the category ω-**Graph**.

Proof. Note that, for $n, m \in U$ with $n < m$, we have $n \sim_U m$ if and only if 2^n occurs in the unique expansion of m as a sum of distinct powers of 2. We show that $\boldsymbol{G}_U$ is stepwise saturated. The conclusion then follows from Proposition 3.1, using Proposition 2.1. Let $\boldsymbol{G}$ be finite, $f : \boldsymbol{G} \to \boldsymbol{G}_U$ be an embedding and $g : \boldsymbol{G} \to \boldsymbol{G}'$ be an increment. We may assume that f is the identity, that is, that $\boldsymbol{G}$ is an induced subgraph of $\boldsymbol{G}_U$. Since g is an increment, we can write G' as $g[G]\dot{\cup}\{y\}$, and put $A = \{x \in G \mid g(x) \sim_{G'} y\}$. Let $r = \max f[G] + 1$ and $z = \sum_{x \in A} 2^x + 2^r$; clearly, $z \in G_U \setminus G$ and for any $x \in G$ we have

$$x \sim_U z \text{ iff } x \in A \text{ iff } g(x) \sim_{G'} y. \tag{3}$$

Now define $h : \boldsymbol{G}' \to \boldsymbol{G}_U$ by

$$h(w) = \begin{cases} g^{-1}(w) & \text{if } w \in g[G] \\ z & \text{if } w = y. \end{cases}$$

Clearly, by (3), h is a graph embedding, and $h \circ g = f$. □

† These homomorphisms are called 'strong embeddings' in Rado (1967) and 'rigid embeddings' in Boldi *et al.* (1993).

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Rado's construction of a universal homogeneous graph can be used in a rather straightforward way to obtain a universal homogeneous coherence space. We can do this by exploiting the equivalence between the category ω-**Graph** and that of coherent atomic dI-domains, that is, coherence spaces: this equivalence associates to each graph the partial order of its cliques (with respect to inclusion), and, conversely, to each coherence space the countable graph (called the *web* of the coherence space in Girard (1987)) whose vertices are the atoms, two atoms being coherent iff they are compatible.

Let us first recall some domain-theoretic terminology that we shall use below. For more standard definitions we refer the reader to Plotkin (1978b) and Droste (1991). A Scott domain $(D, \sqsubseteq)$ is an algebraic cpo in which any upper bounded subset has a supremum. An *atom* of D is a minimal element of D strictly above $\bot$. Furthermore, a Scott domain is:

— *coherent* if every pairwise compatible set has a least upper bound; that is, if $A \subseteq D$ is such that every pair $x, y \in A$ is compatible (that is, has an upper bound), then $\sqcup A$ exists;
— *distributive* if, for every $x, y, z \in D$, the equality

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$$

holds whenever y and z are compatible;
— *finitary* if every compact element dominates at most finitely many elements;
— *atomic* if every element is the least upper bound of the set of atoms it dominates.

A distributive finitary Scott domain is often called a *dI-domain*. Observe that a Scott domain D is an atomic dI- domain if and only if, for each $x \in D$, the set $\{d \in D \mid d \sqsubseteq x\}$, with the induced partial order, is a powerset Boolean algebra.

Scott-continuous functions are not the appropriate maps on dI-domains, as they do not preserve the finitary character in general. Rather, continuity has to be strengthened to *stability* (Berry 1978). Accordingly, the order relation among stable functions is not the pointwise order but a stronger one: if D, E are dI-domains, a continuous function $f : D \to E$ is *stable* if, for all $x, x' \in D$, if x and x' are compatible, then

$$f(x \sqcap x') = f(x) \sqcap f(x').$$

If $f, g : D \to E$ are stable functions, then f is *stably less than* g (written $f \sqsubseteq_s g$) if and only if, whenever $x, x' \in D$ are such that $x \sqsubseteq x'$, we have

$$f(x) = f(x') \sqcap g(x).$$

Now, coherence spaces (in the sense of Girard (1987)) are precisely the coherent atomic dI-domains. To explain this, for each graph $\boldsymbol{G}$ we let Clique($\boldsymbol{G}$) be the set of cliques of $\boldsymbol{G}$, ordered by inclusion (a clique is a set of vertices that is complete with respect to the adjacency relation). It is easy to prove that Clique($\boldsymbol{G}$) is indeed a coherent atomic dI-domain, for each graph $\boldsymbol{G}$, and conversely every such domain D can be seen in that way: just take a graph $\boldsymbol{G}$ that has one vertex for each atom, and where adjacent vertices are compatible atoms. Then, clearly, Clique($\boldsymbol{G}$) $\cong D$.

We can go further in this analogy, and turn it into a categorical equivalence. In order to do this, we recall the well-known notion of stable embedding–projection pair, or SEPP for short (Kahn and Plotkin 1978). A pair $\langle e, p\rangle$ of continuous functions between domains $e : (D_0, \sqsubseteq_0) \to (D_1, \sqsubseteq_1)$ and $p : (D_1, \sqsubseteq_1) \to (D_0, \sqsubseteq_0)$ is called an *embedding–projection pair* iff $p \circ e = \mathbf{1}_{D_0}$ and $e \circ p \sqsubseteq \mathbf{1}_{D_1}$ (here $\sqsubseteq$ denotes the pointwise ordering among continuous maps). It is a *stable* embedding–projection pair iff, moreover, for all $x_0 \in D_0$ and $x_1 \in D_1$, if $x_1 \sqsubseteq_1 e(x_0)$, then $x_1 = e(p(x_1))$. It is not difficult to see that $\langle e, p\rangle$ is a stable embedding–projection pair if and only if

$$p \circ e = \mathbf{1}_{D_0} \quad e \circ p \sqsubseteq_s \mathbf{1}_{D_1}.$$

Moreover, stable embeddings map atoms to atoms and preserve compatibility.

The category of coherent atomic dI-domains with countably many atoms, and with stable embedding–projection pairs as arrows, is denoted by ω-**CAdIDom**. Therefore, if **Coh** denotes the category of coherence spaces with countably many atoms and stable functions as arrows, the category ω-**CAdIDom** is simply the (non-full) subcategory $\mathbf{Coh}^e$ of **Coh** that has the same objects as **Coh** and whose arrows are embeddings.

We can now turn Clique$(-)$ into a functor from ω-**Graph** to ω-**CAdIDom** as follows: if $f : \boldsymbol{G} \to \boldsymbol{G}'$ is a graph embedding, we let $\mathrm{Clique}(f) = \langle e_f, p_f\rangle : \mathrm{Clique}(\boldsymbol{G}) \to \mathrm{Clique}(\boldsymbol{G}')$ be defined as follows: $e_f(C) = \{f(x) \mid x \in C\} = f[C]$ for each clique C of $\boldsymbol{G}$, and $p_f(C') = f^{-1}[C']$ for each clique C' of $\boldsymbol{G}$ (note that this definition is well-given, because of the properties of graph embeddings). The following result is a formal statement of an observation that is basically due to Girard (1987).

Theorem 3.3. The functor Clique gives a categorical equivalence between ω-**Graph** and ω-**CAdIDom**.

Proof. It is routine to check that Clique is indeed a functor – the only non-trivial part is that Clique(f) is in fact stable: suppose that $f : \boldsymbol{G} \to \boldsymbol{G}'$ is an embedding, and let $C' \subseteq e_f(C) = f[C]$ (where C is a clique of $\boldsymbol{G}$). Then clearly $f[f^{-1}[C']] = C'$, so $e_f(p_f(C')) = C'$.

As noted before, for every object D of ω-**CAdIDom**, there is a graph $\boldsymbol{G}$ such that $\mathrm{Clique}(\boldsymbol{G}) \cong D$. So, we are left to prove that Clique is full and faithful. Faithfulness is easy: if $\mathrm{Clique}(f) = \mathrm{Clique}(g)$, then, in particular, for each vertex x one has $e_f(\{x\}) = e_g(\{x\})$, that is, $\{f(x)\} = \{g(x)\}$, hence $f = g$. For fullness, let $\boldsymbol{G}, \boldsymbol{G}'$ be two graphs and $\langle e, p\rangle : \mathrm{Clique}(\boldsymbol{G}) \to \mathrm{Clique}(\boldsymbol{G}')$ be a SEPP; since stable embeddings map atoms to atoms, for each $x \in G$ we let $f(x)$ be the only element of $e(\{x\})$. Now, for every clique C of $\boldsymbol{G}$ we have $e_f(C) = f[C] = \{f(x) \mid x \in C\} = \cup_{x\in C} e(\{x\}) = e(C)$ and thus also $p_f = p$, since a stable embedding determines the corresponding projection in a unique way. So $\mathrm{Clique}(f) = \langle e, p\rangle$. □

As a consequence, we have the following corollary.

Corollary 3.4. Clique$(\boldsymbol{G}_U)$ is the universal homogeneous object of ω-**CAdIDom**.

A more restricted universality result for the same class of domains is proved in Asperti and Longo (1991, §2.4.2), using Plotkin's domain $\mathbf{T}^\omega$. There, it is shown that $\mathbf{T}^\omega$ is universal for coherence spaces with a denumerable web.

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3.2. *Hypercoherences and hereditary hypercoherences*

In this section, we aim to generalise the notion of graph by replacing the adjacency relation (which is a binary one) with a finitary predicate, along the lines of Girard (1987).

Recall that one possible way to interpret the adjacency relation of a graph is to treat it as a binary consistency predicate; reflexivity of this relation is thus quite a natural requirement, since it means that every point is self-consistent. We shall describe two ways of generalizing such a notion.

3.2.1. *Hypercoherences* Hypercoherences were introduced in Ehrhard (1993) on the basis of previous work he did with Bucciarelli (Bucciarelli and Ehrhard 1994) on the notion of strong stability as an extensional theory of sequential functions. In the following, let $A \subseteq^+_{\text{fin}} B$ denote the fact that A is a finite non-empty subset of B.

A *hypercoherence* $\boldsymbol{X} = (X, \Gamma(X))$ consists of a countable set X (the *web* of $\boldsymbol{X}$) together with a family $\Gamma(X)$ of finite, non-empty subsets of X defining what is called the *atomic coherence* of $\boldsymbol{X}$, such that for all $a \in X$ we have that $\{a\} \in \Gamma(X)$. Given hypercoherences $\boldsymbol{X}, \boldsymbol{Y}$, define an *embedding* of $\boldsymbol{X}$ into $\boldsymbol{Y}$ as an injective mapping $f : X \to Y$ such that for all $u \subseteq^+_{\text{fin}} X$

$$u \in \Gamma(X) \Leftrightarrow f[u] \in \Gamma(Y).$$

It is routine to check that the following property holds for the category ω-**HCoh** whose objects are hypercoherences and whose arrows are embeddings.

Proposition 3.5. In the category ω-**HCoh** every arrow is monic, and an object is finite iff it has a finite underlying set. Moreover, ω-**HCoh** is an algebroidal, incremental category, where $f : \boldsymbol{X} \to \boldsymbol{Y}$ is an increment iff $Y = f[X] \cup \{y\}$ for some $y \notin f[X]$.

Using the coding apparatus of Theorem 3.2, we can show that ω-**HCoh** has a universal homogeneous object $\boldsymbol{U}$, defined as follows:

— $U = \omega$;
— $A \in \Gamma(U)$ if and only if:
  - either $A = \{n\}$ for some $n \in \omega$, or
  - the $\Phi(A \setminus \{\max A\})$-th bit in the formal infinite binary expansion of $\max A$ is 1; in other words, if $k = \Phi(A \setminus \{\max A\})$, then 2^k occurs in the unique expansion of $\max A$ as a sum of distinct powers of two.

Theorem 3.6. $\boldsymbol{U}$ is the universal homogeneous object in the category ω-**HCoh**.

Proof. We follow the same strategy as for Theorem 3.2 and prove that $\boldsymbol{U}$ is stepwise saturated. The result follows by an appeal to Proposition 3.5 and Proposition 2.1. Let $\boldsymbol{X}$ be a finite hypercoherence, let $f : \boldsymbol{X} \to \boldsymbol{U}$ be an embedding and $g : \boldsymbol{X} \to \boldsymbol{Y}$ be an increment. We may assume that f is an inclusion, so that $\boldsymbol{X}$ is a substructure of $\boldsymbol{U}$. Write $Y = g[X]\dot{\cup}\{y\}$. Let $\mathscr{X} = \{C \subseteq^+_{\text{fin}} X \mid g[C] \cup \{y\} \in \Gamma(Y)\}$ and put

$$z = \sum_{C \in \mathscr{X}} 2^{\Phi(C)} + 2^{\Phi(X)+1}.$$

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Clearly, $z \in U \setminus X$ and for any $C \subseteq^+_{\text{fin}} X$ we have $\Phi(C) < z$, so

$$C \cup \{z\} \in \Gamma(U) \Leftrightarrow C \in \mathscr{X} \Leftrightarrow g[C] \cup \{y\} \in \Gamma(Y). \tag{4}$$

Now define $h : \boldsymbol{Y} \to \boldsymbol{U}$ by

$$h(w) = \begin{cases} g^{-1}(w) & \text{if } w \in g[X] \\ z & \text{if } w = y. \end{cases}$$

Obviously, h is an embedding by (4), and $h \circ g : \boldsymbol{X} \to \boldsymbol{X}$ is the identity, showing that $\boldsymbol{U}$ is stepwise saturated. □

3.2.2. *Hereditary hypercoherences* A natural condition that can be imposed on a hypercoherence $\boldsymbol{X}$ is that the atomic coherence $\Gamma(X)$ of $\boldsymbol{X}$ be downward closed, in the sense that $u \subseteq^+_{\text{fin}} v \in \Gamma(X)$ implies $u \in \Gamma(X)$. The hypercoherences in which this condition is satisfied are the *hereditary hypercoherences.*

We let ω-**HCoh**$_h$ be the full subcategory of ω-**HCoh** consisting of hereditary hypercoherences. Also in this case, we have the following proposition.

Proposition 3.7. In the category ω-**HCoh**$_h$ every arrow is monic, and an object is finite iff it has a finite underlying set. Moreover, ω-**HCoh**$_h$ is an algebroidal, incremental category.

We are now ready to prove a generalization of Rado's theorem to hereditary hypercoherences. Let $\boldsymbol{U} = (U, \Gamma(U))$ be the universal homogeneous object in the category ω-**HCoh** from Theorem 3.6. Let $\boldsymbol{U}_h$ be the hereditary hypercoherence $(U_h, \Gamma(U_h))$ where:

— $U_h = U$;
— $A \in \Gamma(U_h)$ if and only if, for all $u \subseteq^+_{\text{fin}} A$, $u \in \Gamma(U)$.

This is clearly hereditary. We prove that $\boldsymbol{U}_h$ is indeed the universal homogeneous hereditary hypercoherence.

Theorem 3.8. $\boldsymbol{U}_h$ is the universal homogeneous object of the category ω-**HCoh**$_h$.

Proof. We follow exactly the proof of Theorem 3.6, starting, of course, with finite hereditary hypercoherences $\boldsymbol{X}$, $\boldsymbol{Y}$ and an embedding f into $\boldsymbol{U}_h$. Observe that in this case $\mathscr{X}$ is closed downwards and $\mathscr{X} \subseteq \Gamma(X)$. We have to show that, for any $C \subseteq^+_{\text{fin}} X$ and z as defined above, we have

$$C \cup \{z\} \in \Gamma(U_h) \Leftrightarrow C \in \mathscr{X}. \tag{5}$$

The implication from left to right is clear. So assume $C \in \mathscr{X}$. Then $C \cup \{z\} \in \Gamma(U)$. Choose any $u \subseteq C$. Clearly, $u \in \mathscr{X}$, so $u \cup \{z\} \in \Gamma(U)$. Also, $u \in \Gamma(X)$, and hence $u = f(u) \in \Gamma(U_h)$. Thus $C \cup \{z\} \in \Gamma(U_h)$, proving equation (5). The rest of the proof proceeds as before. □

Remark. It would be interesting if one could derive Theorem 3.8 directly from Theorem 3.6. However, note that a hereditary hypercoherence which is a substructure of $\boldsymbol{U}_h$ via an embedding f in general is not a substructure of $\boldsymbol{U}$ via the same embedding. An important point to keep in mind here is that coherence in $\boldsymbol{U}$ is a consequence of the coding, whereas in $\boldsymbol{U}_h$ it is forced *a posteriori*.

We now prove the equivalence between the category of hereditary hypercoherences with embeddings, and the category of atomic dI-domains, with stable embedding–projection pairs as arrows. A consequence of this result is an explicit construction of the universal homogeneous atomic dI-domain, which exactly mimics what we did for the coherent case.

We let ω-**AdIDom** be the category of atomic dI-domains with countably many atoms, and stable embedding–projection pairs as arrows. We first define a functor

$$\mathrm{qD} : \omega\text{-}\mathbf{HCoh} \to \omega\text{-}\mathbf{AdIDom}$$

as follows:

— for each hypercoherence $\boldsymbol{X}$,

$$\mathrm{qD}(\boldsymbol{X}) = \{x \subseteq X \mid \forall u \subseteq^{+}_{\text{fin}} x.u \in \Gamma(X)\}$$

— if $f : \boldsymbol{X} \to \boldsymbol{Y}$ is an embedding of hypercoherences, we let $\mathrm{qD}(f) = \langle e_f, p_f \rangle$ where, for any $A \in \mathrm{qD}(\boldsymbol{X})$ and $B \in \mathrm{qD}(\boldsymbol{Y})$, we define $e_f(A) = \{f(x) \mid a \in A\} = f[A]$ and $p_f(B) = \{x \in X \mid f(x) \in B\} = f^{-1}[B]$.

We shall prove that qD gives an equivalence between ω-**AdIDom** and the subcategory ω-$\mathbf{HCoh}_h$ of ω-**HCoh**. The fact that it is indeed a functor is routine: just note that if $B \subseteq e_f(A) = f[A]$, then $e_f(p_f(B)) = f[f^{-1}[B]] = B$, as required.

Furthermore, we also have the following property, which was noted originally in Girard (1986).

Theorem 3.9. qD gives a categorical equivalence between ω-$\mathbf{HCoh}_h$ and ω-**AdIDom**.

Proof. If D is an atomic dI-domain, define $\boldsymbol{X}$ to be the hereditary hypercoherence having as web the set of atoms of D, and with atomic coherence defined by letting $A \in \Gamma(X)$ if and only if A is a compatible subset of D. Now define the map $\varphi : \mathrm{qD}(\boldsymbol{X}) \to D$ as follows

$$\begin{array}{rccc} \varphi : & \mathrm{qD}(\boldsymbol{X}) & \to & D \\ & A & \mapsto & \sqcup A. \end{array}$$

Observe that, if A is a consistent subset of X, then it is a compatible subset of D, and so $\sqcup A$ exists. We must prove that φ is an order-isomorphism. Surjectivity of φ directly follows from atomicity. For injectivity, suppose $\varphi(A) = \varphi(B)$, that is, $\sqcup A = \sqcup B$. This implies that $\sqcup A$ and $\sqcup B$ dominate the same set of atoms, and thus necessarily $A = B$. It is then easy to show that φ is actually an order-isomorphism, and thus $\mathrm{qD}(\boldsymbol{X}) \cong D$.

The proof of fullness and faithfullness for qD is analogous to the case of graphs, and is thus omitted. □

Finally, we obtain the following corollary.

Corollary 3.10. $\mathrm{qD}(\boldsymbol{U}_h)$ is the universal homogeneous atomic dI-domain.

Now observe that atomic dI-domains are essentially the same as the *qualitative domains* in the sense of Girard (1986).

Definition 3.11 (Qualitative domains). A *qualitative domain* $\mathfrak{Q} = (|Q|, Q)$ consists of a countable set $|Q|$ and a collection Q of subsets of $|Q|$ satisfying the conditions:

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— if $a \in |Q|$, then $\{a\} \in Q$;
— if $x \in Q$ and $y \subseteq x$, then $y \in Q$;
— if $D \subseteq Q$ is $\subseteq$-directed, then $\bigcup D \in Q$.

If $\mathfrak{Q}$ and $\mathfrak{R}$ are qualitative domains, then a stable function $f : \mathfrak{Q} \to \mathfrak{R}$ is a stable function from $(Q, \subseteq)$ to $(R, \subseteq)$.

Therefore, we obtain from the above corollary the existence of a universal homogeneous object in the category of qualitative domains with stable embedding–projection pairs.

3.2.3. *A note on probabilistic constructions* In this section, we discuss briefly a probabilistic technique that can be used as an alternative way to show the existence of universal homogeneous representations; in particular, our starting point will be the result of Erdős and Rényi (1963) (discussed in more detail in Cameron (1990); see also Erdős and Spencer (1974)), where a probabilistic construction of Rado's universal graph is given.

For a fixed probability $p \in (0,1)$, consider the graph built by using the following probabilistic procedure: take ω as vertex set, and choose, for every pair of distinct vertices, independently with probability p whether to join them or not. It turns out that the resulting graph is isomorphic to $\boldsymbol{U}$ with probability 1, whatever value was chosen for p.

Before proceeding, a few remarks are in order. First, this heuristic construction can be phrased in a mathematically precise way. One considers the space of all binary graphs on ω and defines on it a probability measure. The result stated above means that in this space the subset of all graphs that are universal and homogeneous has measure 1. The details are technically involved, we just refer the reader to Boldi (1997), Cameron (1990) and Erdős and Spencer (1974).

Intuitively, the construction is clear but surprising. Homogeneity of U means that each isomorphism between two finite substructures of U can be extended to an automorphism of U. Hence, U bears a high amount of symmetry. The above result says that as long as we construct U in a purely but consistently random (hence, 'chaotic') fashion, we will obtain this high degree of symmetry with probability 1.

We can rephrase the argument of Erdős and Rényi (1963) as follows. First observe that, by virtue of Proposition 2.1, if the random graph is universal and homogeneous, it must be isomorphic to $\boldsymbol{U}$. Thus, we can limit ourselves to proving that the random graph is *not* universal homogeneous (or, equivalently, *not* stepwise-saturated) with probability 0.

Suppose that the random graph $\boldsymbol{G}_R = (\omega, \sim_R)$ obtained by the previous procedure is not stepwise-saturated. This means that there is a finite subgraph $\boldsymbol{G}$ of $\boldsymbol{G}_R$ and an increment $g : \boldsymbol{G} \to \boldsymbol{G}'$ such that there does not exist any embedding $h : \boldsymbol{G}' \to \boldsymbol{G}_R$ for which $h \circ g$ is the identity on G. Write $G' = G \dot{\cup} \{y\}$, and let $A = \{x \in G \mid g(x) \sim_{G'} y\}$. Thus, by the argument in the proof of Theorem 3.2, there is no $z \in \omega \setminus G$ such that,

$$\forall x \in G. \quad x \sim_R z \Leftrightarrow x \in A.$$

But since the set $\omega \setminus G$ is infinite whereas G is finite, the probability of this happening is 0.

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The above construction can be turned into a similar probabilistic procedure for building the universal homogeneous hypercoherence, as well as the universal homogeneous hereditary one. We will only present the construction for the latter, which is just slightly more complex than the former. Consider a fixed enumeration $A_0, A_1, A_2, \dots$ of the finite non-empty subsets of ω satisfying the following constraints:

1 The enumeration is injective; that is, $A_i \neq A_j$ whenever $i \neq j$.
2 If $B \subset A_i$, then there exists $j < i$ such that $B = A_j$.

Such an enumeration can be built step by step by the following recursive procedure:

— **First step** Set $A_{-1} = \emptyset$ and put $S = \omega$;
— **Inductive step** Suppose that you have already built the subsequence $A_{-1}, \dots, A_k$, and let $n = \min S$. Then, for each $i = -1, \dots, k$, define $A_{k+i+2} = A_i \cup \{n\}$; moreover, delete n from the set S.

This procedure produces the sequence $A_0 = \{0\}$, $A_1 = \{1\}$, $A_2 = \{0, 1\}$, $A_3 = \{2\}$, $A_4 = \{0, 2\}$ *etc.*

Now, build a random hereditary hypercoherence $\boldsymbol{X}_R = (\omega, \Gamma(X_R))$ as follows. For each $i \in \omega$, decide whether $A_i \in \Gamma(X_R)$ using the following randomized algorithm:

1 If $|A_i| = 1$, then put A_i in the consistency predicate.
2 Otherwise, consider all the indices $j < i$ such that $A_j \subset A_i$. If there is an index for which $A_j \notin \Gamma(X_R)$, then do not put A_i in $\Gamma(X_R)$, otherwise, put it in with probability p.

The procedure is defined in such a way that the resulting structure is necessarily hereditary. Now we only have to prove that $\boldsymbol{X}_R$ is stepwise saturated with probability 1.

Suppose that $\boldsymbol{Y}$ is a finite substructure of $\boldsymbol{X}_R$, and that $g : \boldsymbol{Y} \to \boldsymbol{Y}'$ is an increment (write $Y' = Y \dot{\cup} \{y\}$). Also let

$$\mathscr{X} = \{C \subseteq^+_{\text{fin}} Y \mid g[C] \cup \{y\} \in \Gamma(Y')\}.$$

For each $z \in \omega \setminus Y$ such that

$$\forall C \subseteq^+_{\text{fin}} Y. \quad C \cup \{z\} \in \Gamma(X_R) \Leftrightarrow C \in \mathscr{X}$$

the map $h_z : \boldsymbol{Y}' \to \boldsymbol{X}_R$ defined by

$$h_z(x) = \begin{cases} g^{-1}(x) & \text{if } x \in g[Y] \\ z & \text{if } x = y \end{cases}$$

is such that $h_z \circ g = \text{id}_Y$. So, if $\boldsymbol{X}_R$ is not stepwise saturated, then no such z exists. But, since $\mathscr{X}$ is finite, there are infinitely many one-element extensions of the sets of $\mathscr{X}$ (and also of the sets $C \subseteq^+_{\text{fin}} Y$ such that $C \notin X$ and $|C| > 1$) appearing in the enumeration. Hence the probability that no such z exist is 0. Consequently, with probability 1, $\boldsymbol{X}_R$ is stepwise saturated.

3.3. *Categories of hypercoherences*

3.3.1. *Linear morphisms* There are two interesting categories that have hypercoherences as objects. We shall prove that the universal hypercoherence $\boldsymbol{U}$ is the universal homogeneous object in both of them. In the following, if $u \subseteq X \times Y$, for any two sets X, Y, we let $u_1 =_{\mathrm{def}} \{a \in X \mid \exists b \in Y.(a,b) \in u\}$ and, similarly, $u_2 =_{\mathrm{def}} \{b \in Y \mid \exists a \in X.(a,b) \in u\}$.

Definition 3.12 (Linear function space). The hypercoherence $\boldsymbol{X} \multimap \boldsymbol{Y}$ is defined as follows:

– $X \multimap Y =_{\mathrm{def}} X \times Y$.
– $w \in \Gamma(X \multimap Y)$ if and only if w is a non-empty and finite subset of $X \multimap Y$ such that $w_1 \in \Gamma(X)$ implies both that $w_2 \in \Gamma(Y)$ and that if w_2 is a singleton, then w_1 is also.

Now define a *linear morphism* between hypercoherences $\boldsymbol{X}, \boldsymbol{Y}$ to be an element of $\mathrm{qD}(\boldsymbol{X} \multimap \boldsymbol{Y})$. We write $f : \boldsymbol{X} \multimap \boldsymbol{Y}$ to mean that f is a linear morphism from $\boldsymbol{X}$ to $\boldsymbol{Y}$. Furthermore, defining id_X as the set $\{(a,a) \mid a \in X\}$ and using $\circ$ to denote the composition of relations (that is, $R \circ S = \{(a,c) \mid \exists b.(a,b) \in S, (b,c) \in R\}$), we get a (monoidal) category ω-**HCohL** of hypercoherences with linear morphisms, which is a model of classical linear logic (Girard 1987).

If $f : \boldsymbol{X} \to \boldsymbol{Y}$ is an embedding, we define $f^+ \subseteq X \times Y$ as the graph of f, and $f^- \subseteq Y \times X$ as the graph of the inverse of f. (Observe that the sets f^+, f^- are the traces (Girard 1986) of stable (and linear) functions from $\mathrm{qD}(\boldsymbol{X})$ to $\mathrm{qD}(\boldsymbol{Y})$ and *vice-versa*, respectively.) The definition of embedding–projection pair in the category ω-**HCohL** is obvious.

Definition 3.13. Given linear morphisms $e : \boldsymbol{X} \multimap \boldsymbol{Y}$ and $p : \boldsymbol{Y} \multimap \boldsymbol{X}$, we say that $\langle e, p \rangle$ form an *embedding–projection pair* from $\boldsymbol{X}$ to $\boldsymbol{Y}$ if $p \circ e = \mathrm{id}_X$ and $e \circ p \subseteq \mathrm{id}_Y$.

We obtain then a category ω-**HCohL**^{e} whose objects are hypercoherences and whose arrows are (linear) embedding–projection pairs.

It is easy to see that given two arbitrary relations $R \subseteq A \times B$ and $S \subseteq B \times A$, the following two properties are equivalent:

— $S \circ R = \mathrm{id}_A$ and $R \circ S \subseteq \mathrm{id}_B$.
— R is the graph of an injective function, and S is the graph of its inverse.

Using this fact, the following lemma is immediate.

Lemma 3.14. For any two hypercoherences $\boldsymbol{X}, \boldsymbol{Y}$ and any embedding $f : \boldsymbol{X} \to \boldsymbol{Y}$, we have that $f^+ : \boldsymbol{X} \multimap \boldsymbol{Y}$, $f^- : \boldsymbol{Y} \multimap \boldsymbol{X}$ and $f^- \circ f^+ = \mathrm{id}_X$, $f^+ \circ f^- \subseteq \mathrm{id}_Y$. Furthermore, the construction of f^+, f^- from f defines a functor $L : \omega$-**HCoh** $\to \omega$-**HCohL**^{e}, which is the identity on objects.

In order to define an equivalence between the categories ω-**HCoh** and ω-**HCohL**^{e} we still need the following lemma.

Lemma 3.15. If $\langle e, p \rangle$ is an embedding–projection pair from $\boldsymbol{X}$ to $\boldsymbol{Y}$, then there is a unique embedding of hypercoherences $f_{e,p} : \boldsymbol{X} \to \boldsymbol{Y}$ such that $e = f^+_{e,p}$ and $p = f^-_{e,p}$. Furthermore, this construction defines a functor $E : \omega$-**HCohL**$^e \to \omega$-**HCoh**, which is the identity on objects.

Proof. Given an embedding–projection pair $\langle e : \boldsymbol{X} \multimap \boldsymbol{Y}, p : \boldsymbol{Y} \multimap \boldsymbol{X} \rangle$ in ω-**HCohL**^{e}, by the above remark we have an injection $f_{e,p} : X \to Y$ (whose graph is e). We have to show that $u \in \Gamma(X)$ if and only if $f_{e,p}[u] \in \Gamma(Y)$. For the direct implication, let $w = \{(a, f_{e,p}(a)) \mid a \in u\}$. Then $w \subseteq^+_{\mathrm{fin}} e$ and $u = w_1 \in \Gamma(X)$, so $f_{e,p}[u] = w_2 \in \Gamma(Y)$. Conversely, take $v = \{(f_{e,p}(a), a) \mid a \in u\}$ and observe that $v \subseteq^+_{\text{fin}} p$. Therefore $f_{e,p}[u] = v_1 \in \Gamma(Y)$ implies $u = v_2 \in \Gamma(X)$. Functoriality of this construction is straightforward. □

We have thus proved the following result.

Theorem 3.16. The categories ω-**HCoh** and ω-**HCohL**^{e} are equivalent. Moreover, $\boldsymbol{U}$ is the universal homogeneous object in the category ω-**HCohL**^{e}.

3.3.2. *Strongly stable maps* A different category of hypercoherences can be defined by passing from linear morphisms to *strongly stable maps* (Bucciarelli and Ehrhard 1994; Ehrhard 1993). In order to define these, we need to extend our analysis of the qualitative domain qD($\boldsymbol{X}$) canonically associated with a hypercoherence X.

In the following, we shall make use of the following auxiliary notation: if C is a set and $A, B \subseteq \wp(C)$, we write $B \sqsubseteq A$ if and only if

- for all $x \in A$ there exists $y \in B$ such that $x \subseteq y$,
- for all $y \in B$ there exists $x \in A$ such that $x \subseteq y$.

It should be noted that $\sqsubseteq$ is the reverse Egli–Milner ordering (see Plotkin (1976), for example).

Definition 3.17 (Qualitative domain with coherence (Ehrhard 1993)). A *qualitative domain with coherence* (qDC) is a pair

$$\boldsymbol{Q} = (\mathfrak{Q}, \mathscr{C}(Q))$$

consisting of a qualitative domain $\mathfrak{Q} = (|Q|, Q)$ and a family $\mathscr{C}(Q)$ of finite, non-empty subsets of Q satisfying the following constraints:

1 If $x \in Q$, then $\{x\} \in \mathscr{C}(Q)$.
2 If $A \in \mathscr{C}(Q)$ and $B \subseteq^+_{\text{fin}} Q$ is such that $B \sqsubseteq A$, then $B \in \mathscr{C}(Q)$.
3 If $\Delta_1, \ldots, \Delta_n$ are directed subsets of Q such that, for all $x_1 \in \Delta_1, \ldots, x_n \in \Delta_n$ we have $\{x_1, \ldots, x_n\} \in \mathscr{C}(Q)$, then $\{\bigcup \Delta_1, \ldots, \bigcup \Delta_n\} \in \mathscr{C}(Q)$.

Definition 3.18 (Strongly stable map). For qDC $\boldsymbol{Q}, \boldsymbol{R}$, a strongly stable map from $\boldsymbol{Q}$ to $\boldsymbol{R}$ is a (Scott) continuous function $f : Q \to R$ such that, if $A \in \mathscr{C}(Q)$, then:

- $f[A] \in \mathscr{C}(R)$,
- $f(\bigcap A) = \bigcap f[A]$.

A hypercoherence $\boldsymbol{X}$ determines a qDC, as follows.

Definition 3.19 (qDC induced by a hypercoherence). Let $\boldsymbol{X}$ be a hypercoherence. Then we can build a qualitative domain with coherence

$$\text{qDC}(\boldsymbol{X}) = ((X, \text{qD}(\boldsymbol{X})), \mathscr{C}(\boldsymbol{X}))$$

where

$$\mathscr{C}(\boldsymbol{X}) =_{\text{def}} \{A \subseteq^+_{\text{fin}} \text{qD}(\boldsymbol{X}) \mid \forall u \subseteq^+_{\text{fin}} X. u \lhd A \Rightarrow u \in \Gamma(X)\},$$

where $u \lhd A$ if and only if $\forall x \in u. \exists d \in A. x \in d$ and $\forall d \in A. \exists x \in u. x \in d$.

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We can now define a strongly stable map f from a hypercoherence $\boldsymbol{X}$ to a hypercoherence $\boldsymbol{Y}$ to be a strongly stable map $f : \mathrm{qDC}(\boldsymbol{X}) \to \mathrm{qDC}(\boldsymbol{Y})$. The resulting category will be called ω-**HCohFS**. The proof of the following proposition is straightforward.

Proposition 3.20. A strongly stable map $f : \boldsymbol{X} \to \boldsymbol{Y}$ is a stable map from $\mathrm{qD}(\boldsymbol{X})$ to $\mathrm{qD}(\boldsymbol{Y})$. In particular, it is possible to define its trace

$$\mathrm{tr(f)} =_{\mathrm{def}} \{(x, b) \in \mathrm{qD}(\boldsymbol{X}) \times Y \mid b \in f(x) \text{ and } x \text{ is minimal with this property}\}.$$

We can therefore consider the stable ordering among strongly stable maps, and define the category ω-**HCohFS**^{e} whose objects are hypercoherences and whose arrows are pairs of strongly stable maps $\langle e : \boldsymbol{X} \to \boldsymbol{Y}, p : \boldsymbol{Y} \to \boldsymbol{X}\rangle$ that form stable embedding–projection pairs, that is, they are such that $p \circ e = \mathrm{id}_X$ and $e \circ p \sqsubseteq_s \mathrm{id}_Y$.

We now want to show that there is an equivalence between the categories ω-**HCoh** and ω-**HCohFS**. To this end, we shall define two functors

$$S : \omega\text{-}\mathbf{HCoh} \to \omega\text{-}\mathbf{HCohFS}^e \qquad T : \omega\text{-}\mathbf{HCohFS}^e \to \omega\text{-}\mathbf{HCoh}.$$

Lemma 3.21. The mapping $S : \omega$-**HCoh** $\to \omega$-**HCohFS**^{e} that is the identity on objects and transforms an embedding $f : \boldsymbol{X} \to \boldsymbol{Y}$ into

$$S(f) =_{\mathrm{def}} \langle f^+ : \boldsymbol{X} \to \boldsymbol{Y}, f^- : \boldsymbol{Y} \to \boldsymbol{X}\rangle,$$

where

$$\begin{array}{rcrcl} f^+ & : & \mathrm{qD}(\boldsymbol{X}) & \to & \mathrm{qD}(\boldsymbol{Y}) \\ & & x & \mapsto & \{f(a) \mid a \in x\} = f[x] \\ f^- & : & \mathrm{qD}(\boldsymbol{Y}) & \to & \mathrm{qD}(\boldsymbol{X}) \\ & & y & \mapsto & \{a \in X \mid f(a) \in y\} = f^{-1}[y] \end{array}$$

defines a functor.

Proof. Let us first show that f^+, f^- are well-defined. If $x \in \mathrm{qD}(\boldsymbol{X})$, then $f^+(x) \in \mathrm{qD}(\boldsymbol{Y})$: observe that $u \subseteq^+_{\mathrm{fin}} f^+(x)$ implies $u = f^+(v)$, where $v = \{a \in X \mid f(a) \in u\}$. As $v \in \Gamma(X)$, we have $f[v] = f^+(v) \in \Gamma(Y)$ because f is an embedding of hypercoherences. Also, if $y \in \mathrm{qD}(\boldsymbol{Y})$, then $f^-(y) \in \mathrm{qD}(\boldsymbol{X})$: let $v \subseteq^+_{\mathrm{fin}} f^-(y)$, then $f[v] \subseteq^+_{\mathrm{fin}} y$, so $f[v] \in \Gamma(Y)$ and therefore $v \in \Gamma(X)$.

Assume now that $A \in \mathscr{C}(\boldsymbol{X})$. We have to show first that $f^+[A] = \{f^+(x) \mid x \in A\} = \{f[x] \mid x \in A\} \subseteq^+_{\mathrm{fin}} \mathrm{qD}(\boldsymbol{Y})$ is an element of $\mathscr{C}(\boldsymbol{Y})$. Let $v \subseteq^+_{\mathrm{fin}} Y$ be such that $v \lhd f^+[A]$, that is, $\forall b \in v.\exists d \in f^+[A].b \in d$ and $\forall d \in f^+[A].\exists b \in v.b \in d$. We show that $v \in \Gamma(Y)$. Define $u = f^{-1}[v]$. Then, clearly $v = f[u]$, and it is straightforward to show that

(1) $\forall a \in u.\exists x \in A.a \in x$.
(2) $\forall x \in A.\exists a \in u.a \in u$.

Therefore $u \in \Gamma(X)$, because $u \lhd A$. Hence $v = f[u] \in \Gamma(Y)$.

Finally, we have to show that, for all $A \in \mathscr{C}(\boldsymbol{X})$,

$$f^+(\bigcap A) = \bigcap f^+[A].$$

The left to right inclusion is easy. Conversely, $b \in \bigcap f^+[A]$ means that $b \in f[x]$ for all

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$x \in A$. Then $b = f(a)$ for a unique $a \in X$, as f is injective, and this implies that $a \in \bigcap A$, so $f(a) = b \in f[\bigcap A] = f^+(\bigcap A)$. This concludes the proof that f^+ is strongly stable.

Let us now show that $f^-[B] \in \mathscr{C}(\boldsymbol{X})$ whenever $B \in \mathscr{C}(\boldsymbol{Y})$. For this, we have to show that for all $u \subseteq^+_{\text{fin}} X$, if $u \lhd f^-[B]$, then $u \in \Gamma(X)$. But this is easy, as it can readily be checked that $u \lhd f^-[B]$ implies $f[u] \lhd B$. Now $f[u] \lhd B$ implies that $f[u] \in \Gamma(Y)$, so $u \in \Gamma(X)$ because f is an embedding, and this shows that $f^-[B] \in \mathscr{C}(\boldsymbol{X})$. Finally, it is also easy to show that $f^-(\bigcap B) = \bigcap f^-[B]$ for all $B \in \mathscr{C}(\boldsymbol{Y})$.

Clearly, $f^-(f^+(x)) = f^{-1}[f[x]] = x$ for all $x \in \mathrm{qD}(\boldsymbol{X})$, so $f^- \circ f^+ = \mathrm{id}_X$, and, in order to prove that $f^+ \circ f^- \sqsubseteq_s \mathrm{id}_Y$, we have to show that

$$\forall y, y' \in \mathrm{qD}(\boldsymbol{Y}).y \subseteq y' \Rightarrow f^+(f^-(y)) = y \cap f^+(f^-(y')),$$

but this is obvious, as is the functoriality of the construction. □

The construction of a functor in the reverse direction $T : \omega$-**HCohFS**$^e \to \omega$-**HCoh** closely follows the pattern of the analogous constructions for coherence spaces or qualitative domains (Girard 1987; Girard 1986), and uses the fact that the trace of a stable embedding e of $\mathrm{qD}(\boldsymbol{X})$ into $\mathrm{qD}(\boldsymbol{Y})$ is determined by a set of pairs of the form (a, b) where $a \in X, b \in Y$ and $e[\{a\}] = \{b\}$. We will just state the result in the following lemma.

Lemma 3.22. Let $T : \omega$-**HCohFS**$^e \to \omega$-**HCoh** be the identity on objects and assign to a (strongly) stable embedding–projection pair $\langle e : \boldsymbol{X} \to \boldsymbol{Y}, p : \boldsymbol{Y} \to \boldsymbol{X}\rangle$ the embedding of hypercoherences $f_{e,p} : \boldsymbol{X} \to \boldsymbol{Y}$ defined by $f_{e,p}(a) = b$ if $(a, b) \in \mathrm{tr(e)}$. Then T is a functor.

Finally, we have also proved the following results.

Corollary 3.23. The categories ω-**HCoh** and ω-**HCohFS**^{e} are equivalent.

Theorem 3.24. $\boldsymbol{U}$ is the universal homogeneous object in the category ω-**HCohFS**^{e}.

4. Conclusions

We have built three universal objects in categories of domains arising at the confluence of important areas of research in the semantics of programming languages and the logic of computation, namely the study of sequentiality and linear logic. Our results exploit a direct correspondence between the objects of these categories and graph-like structures, and are all based on Rado's construction of a universal graph (Rado 1967) and its generalizations, which also seem to have an interest independently of their application to domain theory. (For this reason, they are somewhat in between the two approaches to the construction of universal domains mentioned in the introduction). As shown in Boldi (1997), the same technique can be applied to other categories, notably a suitable class of event structures with minimal enabling (Winskel 1980): in this case, though, the natural functor associating the domain of configurations to a given event structure is not an equivalence, and thus homogeneity of the universal domain cannot be proved by the methods used in this paper.

We have not dealt in this paper with the applications of our results, which are expected to be found in the solution of recursive domain equations, as is usual for universal

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domains (see, for example, Scott (1976), Plotkin (1978a) and Scott (1982)). Rather, we have limited ourselves to the study of their basic mathematical aspects, focussing on the use of elementary combinatorial techniques in the construction of domains. However, one point should at least be mentioned as an open problem: homogeneity is a desirable property of a universal domain because it ensures its uniqueness, and it means that the domain has a very high degree of symmetry, and hence a large automorphism group. However, it would be quite interesting to find out a different motivation for homogeneity relevant to the computational features of the universal domain. Finally, it is known that in cartesian closed categories, a universal object gives rise to a model of the untyped λ-calculus. Hence, what does the homogeneity of the present universal objects mean for the associated model of the untyped λ-calculus?

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