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ON A STOCHASTIC INVENTORY MODEL WITH DETERIORATION AND STOCK-DEPENDENT DEMAND ITEMS

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In this article, we propose a new continuous-time stochastic inventory model with deterioration and stock-dependent demand items. We then formulate the problem of finding the optimal impulse control schedule that minimizes the total expected return over an infinite horizon, as a quasivariational inequality (QVI) problem. The QVI is shown to lead to an (s, S) policy, where s and S are determined uniquely as a solution of some algebraic equations.

1. INTRODUCTION

In this article, we will be concerned with a single-item inventory model in which the product experiences some kind of perishability over time. To name a few for which this phenomenon occurs, there are food products, blood, perfumes, photographic films, and electronic components. For more details about inventory models in the literature treating perishable items, see Raafat [14], Nahmias [12], and Goyal and Giri [7].

In the model of this article, an extra complication is added by assuming that the demand for the product is related to the amount of stock on hand. This is motivated by the fact that it is well known in the marketing literature that demand of certain products is affected by the quantity displayed on the shelf. Levin et al. [9] state "At times, the presence of inventory has a motivational effect on people around it. It is a

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common belief that large piles of goods displayed in a supermarket will lead the customer to buy more." A number of deterministics mathematical inventory models dealing with this phenomenon were suggested (see, e.g., Corstjens and Doyle [5] and Mandal and Phaujdar [11]).

The model of this article combines perishability, stock-dependent demand rate, and randomness. We do this by modeling the plant equation governing the changes of the inventory level as a stochastic differential equation. Some of the previously treated models are obtained as a special case of the proposed model.

Let $x(t)$ denote the level of stock at time t and assume that we are allowed to intervene at any time to increase stock to any level that we wish with setup cost $k > 0$, unit cost $c > 0$, and holding costs given by

$$f(x) = \begin{cases} -px & \text{for } x \leq 0 \text{ (shortage cost)} \\ qx & \text{for } x > 0 \text{ (holding cost),} \end{cases}$$

with $p > 0$ and $q > 0$.

We also assume that costs are exponentially discounted at a rate $\alpha > 0$.

In the model treated in this article, unmet demand is backlogged. The case of lost sale of partially backlogged demand is not treated.

A replenishment policy consists of a sequence (t_i, Q_i), $i = 1, \ldots,$ where t_i represents the ith time of ordering and Q_i represents the quantity ordered at time t_i, where $t_1 < t_2 < \cdots$.

Let

$$V_n = \{(t_i, Q_i)\}_{i=1,\ldots,n}. \tag{1}$$

Policies described by (1) are called impulse-control policies. We shall assume that plant equation of our inventory model is given by

$$dx(t) = -(g + \{ax(t)^\beta + \theta x(t)\} I(x(t) > 0))\, dt + \sigma\, dw_t + \sum_{i \geq 0} Q_i \delta(t - t_i), \tag{2}$$

where $I(A)$ is the indicator function of the set A and δ is the Dirac function. The quantities g, a, θ, and σ are all strictly positive real-valued parameters, $0 < \beta < 1$, and $\{w_t\}$ is a standard Brownian motion. Let

$$\mathcal{F}_n = \sigma\{x(s), s \leq t_n\} \tag{3}$$

be the σ-algebra generated by the history of the inventory level $x(t)$ up to time t_n.

Note that if $a = \theta = 0$, then the model represented by (3) reduces to the standard model where demand is driven by a Brownian motion with drift g and variance σ^2. This model has been examined by Bather [1], Sulem [15], and Harrison et al. [8]. The model can be thought of as the usual deterministic demand model in which demand is perturbed by a Brownian motion noise. If $\sigma = 0$, $a = 0$, and $\theta > 0$, then (3) reduces to the model examined by Benkherouf and Mahmoud [2], in which the parameter θ is interpreted as the deterioration rate. These models were first suggested by Ghare and Schrader [6] and are known in the literature as exponential decay models [12]. Also, the model implicitly assumes that when $x(t) < 0$, shortages have no effect on the demand as well as deterioration. If $\sigma = 0$, $a > 0$, and $\theta > 0$, then the term reflecting the stock-demand phenomena is captured by $ax(t)^{\beta}$. The parameter a interpreted as a scale parameter, whereas β is interpreted as a shape parameter (see Goyal and Giri [7]). The form $ax(t)^{\beta}$ may be found in [11]. For our case, this form is convenient for technical reasons, as will become apparent later.

Note that between interventions, (2) gives

$$dx(t) = -(g + \{ax(t)^{\beta} + \theta x(t)\}I(x(t) > 0))\,dt + \sigma\,dw_t. \tag{4}$$

The plant equation (4) can be thought of as the usual plant equation of the deterministic model ($\sigma = 0$, $a > 0$, and $\theta > 0$) perturbed by Brownian motion. Also, note that the form $(g + \{ax(t)^{\beta} + \theta x(t)\})$ and the fact that σ is constant guarantee the existence of a unique nonexploding solution of (4).

Assume that V_n is $\mathcal{F}_n$-measurable. Then, the optimal replenishment schedule that minimizes the total discounted costs over an infinite horizon may be stated as that of finding the sequence V^* that solves

$$y(x) = \inf_{V}\left(E\left[\int_0^{\infty} f(x(t))e^{-\alpha t}\,dt + \sum_{i\ge 0}(k + cQ_i)e^{-\alpha t_i}\,|\,x(0) = x\right]\right), \tag{5}$$

where the expectation is taken over all possible realizations of the process $x(t)$ under policy V. Also, let

$$V_{\infty} = \lim_{n\to\infty} V_n = V.$$

The main contributions of the article are twofold:

1. It formulates the problem stated in (5) as a quasivariational inequality (QVI) problem following Bensoussan and Lions [3].
2. It shows that the QVI has a unique optimal solution if and only if $(-p + \alpha c) < 0$. This solution is of (s, S) type. The values of s and S are obtained as a unique solution of some algebraic equations. This generalizes an earlier work of Sulem [15] to the current setup.

Before we embark on dealing with objectives (1) and (2), we note that although our results are similar in nature to those obtained by Constantinides and Richard [4] and Sulem [15], our approach is different in a number of places and is very general and could form a basis for a general approach for tackling inventory control models whose dynamics are more general than those in (3).

In the next section, we formulate the problem addressed in (5) as a QVI problem. Section 3 is concerned with the issue of existence and uniqueness of the solution of the QVI problem. The article concludes with some remarks on the problem of finding a replenishment schedule that minimizes the total cost per unit time.

2. FORMULATION OF THE QVI PROBLEM

This section is concerned with formulating the problem of finding the optimal replenishment schedule described in Section 1 that minimizes the total expected discounted costs over an infinite horizon addressed in (5) as a QVI problem.

Fix t and observe the inventory level in a short time interval of length τ. Thus, we have two cases:

1. If $x(t) > 0$ and no order is made in the interval $(t, t+\tau)$, then (3) and (4) imply that

$$y(x) \leq E\left[\int_t^{t+\tau} f(x(t))e^{-\alpha(s-t)}\,ds + y(x(t+\tau))e^{-\alpha\tau}\,|x(t) = x\right]. \qquad \textbf{(6)}$$

   Write

$$x(t+\tau) = x(t) + \triangle x_\tau$$

   and use the fact that for a standard Brownian motion $w_t, E[w_t] = 0$ and $E[w_t^2] = t$. The Taylor expansion of the right-hand side of (6) gives

$$\begin{aligned} y(x) &\leq \tau f(x(t)) + y(x(t)) + E[\triangle x_\tau]y'(x(t)) \\ &\quad + \frac{1}{2}E[\triangle x_\tau]^2 y''(x(t)) - \alpha\tau y(x(t)) - \alpha\tau E(\triangle x_\tau)y'(x(t)) \\ &\quad - \frac{1}{2}\alpha\tau E((\triangle x_\tau)^2)y''(x(t)) + O(\tau^2), \end{aligned}$$

   leading to

$$\begin{aligned} 0 &\leq \tau f(x(t)) - (g + ax(t)^\beta + \theta x(t))\tau y'(x(t)) + \frac{1}{2}\sigma^2\tau y''(x(t)) \\ &\quad - \alpha\tau y(x(t)) + O(\tau^2). \end{aligned}$$

   Dividing by τ and letting $\tau \to 0$ gives

$$-\frac{1}{2}\sigma^2 y''(x(t)) + (g + ax(t)^\beta + \theta x(t))y'(x(t)) + \alpha y(x) \leq f.$$

2. If $x(t) \leq 0$ and no order is made in the interval $(t, t+\tau)$, then a similar argument to that used in case 1 gives

$$-\frac{1}{2}\sigma^2 y''(x(t)) + gy'(x(t)) + \alpha y(x(t)) \leq f.$$

3. If an order of size Q is placed at time t, then the inventory level jumps from $x(t)$ to $x(t) + Q$. This means

$$y(x(t)) \leq k + \inf_Q[CQ + y(x(t) + Q)].$$

Now, define two operators A and M as follows:

$$Ay(x) = \begin{cases} -\frac{1}{2}\sigma^2 y''(x) + (g + ax^\beta + \theta x)y'(x) + \alpha y(x) & \text{if } x > 0 \\ -\frac{1}{2}\sigma^2 y''(x) + gy'(x) + \alpha y(x) & \text{if } x \leq 0. \end{cases} \quad \textbf{(7)}$$

$$My(x) = k + \inf_Q [CQ + y(x(t) + Q)]. \quad \textbf{(8)}$$

Then, it follows from (7) and (8) that the optimal solution of (5) is given as a solution of the QVI problem:

$$\begin{aligned} Ay &\leq f, \\ y &\leq My, \\ (Ay - f)(y - My) &= 0. \end{aligned} \quad \textbf{(9)}$$

In the next section, we shall show that QVI (9) has a unique solution if and only if $(-p + \alpha c) < 0$. This solution is characterized by a pair (s, S) which is obtained from a unique solution of some algebraic equations.

3. SOLUTION OF THE QVI PROBLEM

Our approach in solving (9) will initially follow that of Sulem [15]. We *postulate* that the optimal solution to (9) is characterized by two values s and S, where $S > s$. These values divide the inventory space into two regions: the continuation region

$$C = \{x \in \mathbb{R}; y(x) < My(x)\} = \{x \in \mathbb{R}; x > s\},$$

where no order is made, and

$$Ay = f,$$

where A is defined in (7). The complement

$$\bar{C} = \{x \in \mathbb{R}; y(x) = My(x)\} = \{x \in \mathbb{R}; x \leq s\},$$

where M is given by (8), corresponds to the states in which an order is made.

In $\bar{C}$, we have

$$y(x) = k + \inf_Q [cQ + y(x + Q)] \quad \textbf{(10)}$$

$$= k + c(S - x) + y(S). \quad \textbf{(11)}$$

To find the values of s and S, we argue as follows:

1. The solution to the QVI problem (9) is continuously differentiable and continuity at the boundary point s gives, from (11), that

$$y'(s) = -c. \quad \textbf{(12)}$$

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2. The infinimum in (10) is attained at S. Hence,

$$y'(S) = -c. \tag{13}$$

3. Also, y is continuous at s, given from (11) that

$$y(S) = y(s) - k - c(S - s). \tag{14}$$

4. Also, we require for some technical reasons (see Bensoussan and Lions [3]) that

$$\lim_{x \to +\infty} \frac{y(x)}{f(x)} < \infty. \tag{15}$$

Condition (15) will be called the growth condition, as it is easily seen that it means that the expected return $y(s)$ has a growth that is at most linear at ∞.

THEOREM 1: *There exists a unique solution to the (QVI) problem (9) if and only if* $(-p + \alpha c) < 0$.

The proof of Theorem 1 is lengthy and technical and proceeds in two main stages. In the first stage, we show that (10)–(15) lead to a unique pair (s, S), where $S > s$. The technical machinery needed to prove this result relies heavily on asymptotic analysis of differential equations—in particular, the WKB method (see Olver [13] for more details). In the second stage, we show that this pair characterizes uniquely the solution of the QVI problem (9).

Before we proceed to the proof, note that when the state variable $x(t)$ lies between (s, S), then the system is left to move freely (without intervention); in this case, the dynamics of the system evolves like the differential equation $Ay = f$. Now, if s is strictly positive, we get $s = S$ from (12), (13), and (15). This means that $k = 0$ by (14), contradicting the assumption that $k > 0$. Hence, s must be strictly negative.

Let

$$L(x) = y(x) + cx. \tag{16}$$

Also, write

$$y(x) := \begin{cases} y_-(x) & \text{if } x < 0 \\ y_+(x) & \text{if } x \geq 0 \end{cases}$$

for the solution

$$Ay(x) := \begin{cases} -px & \text{if } x < 0 \\ qx & \text{if } x \geq 0, \end{cases} \tag{17}$$

where A is given by (7).

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THEOREM 2: *If* $(-p + \alpha c) < 0$, *then*

(i) $\lim_{x\to+\infty} L(x) = +\infty$
(ii) $\lim_{s\to-\infty} L(s) = +\infty$.

PROOF: Note from (7) that for $x > 0$ we have

$$-\frac{1}{2}\sigma^2 y''(x) + (g + ax^\beta + \theta x)y'(x) + \alpha y(x) = qx. \tag{18}$$

Rewrite (18) as

$$-y''(x) + 2\,\frac{(g + ax^\beta + \theta x)}{\sigma^2}\,y'(x) + 2\,\frac{\alpha}{\sigma^2}\,y(x) = 2\,\frac{q}{\sigma^2}\,x.$$

Let

$$P(x) = \frac{2}{\sigma^2}(g + ax^\beta + \theta x),$$

$$\bar{\alpha} = 2\,\frac{\alpha}{\sigma^2}, \qquad \bar{q} = 2\,\frac{q}{\sigma^2}.$$

Then, (18) reduces to

$$-y''(x) + P(x)y(x) + \bar{\alpha}y(x) = \bar{q}x.$$

Let

$$y(x) = z(x)\exp\left\{\frac{1}{2}\int_0^x P(t)dt\right\}.$$

Then, it can be shown that (18) gives

$$-z''(x) + z(x)Q(x) = \bar{q}x\exp\left\{-\frac{1}{2}\int_0^x P(t)\,dt\right\}, \tag{19}$$

with

$$\begin{aligned} Q(x) &= \frac{1}{4}P^2(x) - \frac{1}{2}P'(x) + \bar{\alpha} \\ &= \frac{1}{\sigma^4}(g + ax^\beta + \theta x)^2 - \frac{1}{\sigma^2}(a\beta x^{\beta-1} + \theta) + \bar{\alpha}. \end{aligned} \tag{20}$$

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Using the WKB method (see Olver [13]), the complementary solutions of (19) are asymptotic, as $x \to +\infty$:

$$z_1(x) \approx Q^{-1/4}(x)\exp\left\{+\int_0^x \sqrt{Q(t)}\,dt\right\},$$

$$z_2(x) \approx Q^{-1/4}(x)\exp\left\{-\int_0^x \sqrt{Q(t)}\,dt\right\},$$

from which we get

$$\begin{aligned} y_1(x) &\approx z_1(x)\exp\left\{+\frac{1}{2}\int_0^x P(t)\,dt\right\} \\ &\approx Q^{-1/4}(x)\exp\left\{\int_0^x \left(\sqrt{Q(t)} + \frac{1}{2}P(t)\right)dt\right\}, \end{aligned} \tag{21}$$

$$\begin{aligned} y_2(x) &\approx z_2(x)\exp\left\{+\frac{1}{2}\int_0^x P(t)\,dt\right\} \\ &\approx Q^{-1/4}(x)\exp\left\{\int_0^x \left(-\sqrt{Q(t)} + \frac{1}{2}P(t)\right)dt\right\}. \end{aligned} \tag{22}$$

Note that, in general, when $x \to 0, \sqrt{1+x} \approx 1 + \frac{1}{2}x$, and recall (20) and the fact that

$$P(x) = 2\,\frac{g + ax^{\beta} + \theta x}{\sigma^2}$$

and

$$P'(x) = 2\,\frac{a\beta x^{\beta-1} + \theta}{\sigma^2}.$$

This means that both $P'(x)/P^2(x)$ and $\bar{\alpha}/P^2(x)$ go to zero as $x \to \infty$, leading to

$$\begin{aligned} \sqrt{Q(x)} &= \sqrt{\frac{1}{4}P^2(x) - \frac{1}{2}P'(x) + \bar{\alpha}} = \sqrt{\frac{1}{4}P^2(x)\left\{1 - 2\,\frac{P'(x)}{P^2(x)} + 4\,\frac{\bar{\alpha}}{P^2(x)}\right\}}, \\ &\approx \frac{1}{2}P(x)\left\{1 - \frac{P'(x)}{P^2(x)} + 2\,\frac{\bar{\alpha}}{P^2(x)}\right\}. \end{aligned}$$

Thus,

$$\sqrt{Q(x)} + \frac{1}{2}P(x) \approx P(x) - \frac{P'(x)}{2P(x)} + \frac{\bar{\alpha}}{P(x)} \tag{23}$$

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and

$$\sqrt{Q(x)} - \frac{1}{2}P(x) \approx \frac{P'(x)}{2P(x)} - \frac{\bar{\alpha}}{P(x)}. \tag{24}$$

It follows from (21) and (23), after some algebra, that

$$y_1(x) \approx \exp\left\{\frac{\theta}{\sigma^2}x^2\right\}.$$

Also, it can be shown, using (22) and (24), that $y_2(x) \to 0$ as $x \to \infty$.

Now, note that asymptotically the general solution has the form

$$y(x) \approx c_1 y_1(x) + c_2 y_2(x) + \tilde{y}_p, \tag{25}$$

where $y_1(x)$ and $y_2(x)$ are given by (21) and (22), respectively, and $\tilde{y}_p$ is the asymptotic particular solution to be found later. The symbols c_1 and c_2 refer to some coefficients.

Now, the growth condition (15) implies that $c_1 = 0$. This means that in order to show that part (i) is true, we only need to check that $\tilde{y}_p$ is well behaved. To this end, we only need to look for a formal solution $\tilde{y}_p \approx \sum x^\tau \sum_{n\geq 0}(a_n/x^n)$, which is an asymptotic series (see Olver [13]). Also, keep in mind that an asymptotic series may not converge. There are several ways of finding the coefficients a_n. We shall use an iterative method. Rewrite (19) as

$$y(x) = \frac{q}{\alpha}x + \frac{\sigma^2}{2\alpha}y'' - \frac{g + ax^\beta + \theta x}{\alpha}y'$$

and set

$$y_{n+1}(x) = \frac{\sigma^2}{2\alpha}y_n''(x) - \frac{g + ax^\beta + \theta x}{\alpha}y_n'(x),$$

with

$$y_0 = \frac{q}{\alpha}x.$$

It follows that

$$y_1(x) = -\frac{g + ax^\beta + \theta x}{\alpha}\frac{q}{\alpha} \approx -\frac{\theta}{\alpha}\frac{q}{\alpha}x$$

and

$$y_{n+1}(x) \approx -\left(\frac{\theta}{\alpha}\right)xy_n'(x) \approx (-1)^{n+1}\left(\frac{\theta}{\alpha}\right)^{n+1}\frac{q}{\alpha}x.$$

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It follows that the particular solution is asymptotic such that

$$\tilde{y}_p \approx \frac{q}{\alpha} x.$$

It follows from (25) that $\lim_{x \to +\infty} y(x) + cx = +\infty$.

Next, we proceed to the proof of part (ii). The form (25) indicates that the solution $y_+(x)$ is of the type

$$y_+(x) = \epsilon\phi_1(x) + y_{p+}(x), \tag{26}$$

where $\phi_1(x)$ is the complementary solution and $y_{p+}(x)$ is a particular solution.

Note that $\phi_1(x) \to 0$ as $x \to \infty$ as shown in part (i). Also, note that when $x < 0$, $y_-(x)$ (the solution to $Ay = px$ when $x < 0$) has an explicit form:

$$y_-(x) = ae^{\lambda_1(x-s)} + be^{\lambda_2(x-s)} + k_1 x + k_2, \tag{27}$$

where

$$\lambda_1 = \frac{1}{\sigma^2}(g + \sqrt{g^2 + 2\alpha\sigma^2}), \qquad \lambda_2 = \frac{1}{\sigma^2}(g - \sqrt{g^2 + 2\alpha\sigma^2}),$$

with $k_1 = -(p/\alpha)$, $k_2 = gp/\alpha^2$, and a and b are real coefficients to be determined.

The solution $y(x)$ of $Ay = f$ is continuously differentiable at the point 0. By matching $y_-(0) = y_+(0)$ and $y'_-(0) = y'_+(0)$, we get

$$ae^{-\lambda_1 s} + be^{-\lambda_2 s} + k_2 = \epsilon\phi_1(0) + y_{p+}(0),$$

$$a\lambda_1 e^{-\lambda_1 s} + b\lambda_2 e^{-\lambda_2 s} + k_1 = \epsilon\phi'_1(0) + y'_{p+}(0).$$

Using condition (12), $y'_-(s) = -c$, leads, after some tedious but direct algebra, to $y(s) \approx k_1 s$ as $s \to -\infty$. It follows that if $(c + k_1) < 0$, we get

$$L(s) = y(s) + cs \approx (c + k_1)s \to +\infty \quad \text{as } s \to -\infty.$$

This completes the proof. ∎

The next two lemmas pave the way for the main result of the article.

Lemma 3: *If $(-p + \alpha c) < 0$, then the solution (s, S) satisfying (12)–(15) exists.*

Proof: Write the solution of (17) as $y(x, s)$. Let

$$L(x, s) = y(x, s) + cx.$$

Then, it follows from (11), (13), and (14) that the problem of finding (s, S) reduces to the problem of solving the system of nonlinear equations given by

$$L'(S, s) = 0,$$

$$L(s, s) = k + L(S, s).$$

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First, we show that $S \to 0$ as $s \to 0$. Assume that $L''(s) > 0$ and $S > 0$; this implies that there exists some $x^* \in (0, S)$, such that $L'(x^*) = 0$ and $L(x^*) > L(S)$ with $L''(x^*) < 0$ and $L''(S) > 0$. It follows from (17) that

$$\alpha L(x^*) = (q + \alpha c)x^* + (g + ax^{*\beta} + \theta x^*) + \frac{1}{2}\sigma^2 L''(x^*)$$
$$< (q + \alpha c)S + (g + aS^\beta + \theta S) + \frac{1}{2}\sigma^2 L''(S) = \alpha L(S),$$

contradicting the fact that $L(x^*) > L(S)$. Hence, $S \to 0$ as $s \to 0$.

Now, if $L''(s) \leq 0$, then the same argument used above shows that $S \to 0$ as $s \to 0$. In other words, as $s \to 0$, we have

$$L(s, s) < k + L(S, s).$$

Also, we know by Theorem 2 that $L(s, s) \to +\infty$ as $s \to -\infty$ and $k + L(x, s) \to +\infty$ as $x \to +\infty$. This means there exists some $S(s^*) \in (-\infty, +\infty)$ such that $L'(S^*(s), s) = 0$ and $L(S^*(s), s) < \infty$, implying that as $s \to -\infty$, $L(s, s) > k + L(S^*(s), s)$. The lemma is then immediate. ■

LEMMA 4: *Assume that* $(-p + \alpha c) < 0$ *and* (s, S) *found from solving (12)–(15). Then,* $L''(s) \leq 0$.

PROOF: Assume that $L''(s) > 0$. Note that $L(S) < L(s)$ from (10) implies that there exists some $x^* \in (s, S)$ such that $L'(x^*) = 0$. However, $L'(s) = 0$; hence, there exists some $Z \in (s, x^*)$ such that $L''(Z) = 0$ and $L'(Z) > 0$ and Z is a local maximum of the function L'.

Suppose first that $x^* < 0$, then (17) with $x < 0$ implies that $L'''(Z) > 0$, contradicting the assertion that Z is a local maximum.

Now, if $x^* > 0$, then (18) gives

$$\alpha L(x^*) = (q + \alpha c)x^* + c(g + ax^{*\beta} + \theta x) + \frac{1}{2}\sigma^2 L''(x^*)$$

with $L''(x^*) \leq 0$. However, $L'(S) = 0$. This implies that there exists a turning point $Z^* \in (x^*, S)$ such that $L'(Z^*) = 0$ and $L''(Z^*) > 0$, meaning that

$$(q + \alpha c)x^* + c(g + ax^{*\beta} + \theta x) + \frac{1}{2}\sigma^2 L''(x^*)$$
$$< (q + \alpha c)Z^* + c(g + aZ^{\beta_*} + \theta x) + \frac{1}{2}\sigma^2 L''(Z^*),$$

which contradicts the fact that $L(x^*) > L(Z^*)$. This completes the proof. ■

LEMMA 5: *Under the assumptions of Lemma 3, we have*

(i) $L'(x) \leq 0, \qquad s \leq x \leq S$
(ii) $L'(x) \geq 0, \qquad x \geq S.$

The proof is similar to that of Lemma 1.

The following corollary is immediate from Lemma 4.

COROLLARY 6: *If* $(-p+\alpha c)<0$, *then if s is known, S is uniquely determined.*

THEOREM 7: *Under the assumptions of Lemma 2, the* (s,S) *policy is optimal for the QVI problem (9).*

PROOF: We need to check that the inequalities $y<My$ for $x>s$ and $Ay\le f$ for $x\le s$ are satisfied.

Now, Lemma 4 implies that the infinimum of the expression $My(x)$ in (8) is achieved at the point $\xi=S-x$, $s\le x\le S$, and at $\xi=0$ for $x\ge S$. It follows that $My(x)=K+c(S-x)+y(S)$ for $s\le x\le S$ and $My(x)=K+y(x)$ for $x\ge S$.

If $s\le x\le S$, we thus have

$$y(x)-My(x)=y(x)-K-c(S-x)-y(S). \qquad \textbf{(28)}$$

It follows that

$$(y(x)-My(x))'=y'(x)-c=L'(x)\le 0,$$

by Lemma 4. This means that $y(x)-My(x)\le y(S)-My(S)$. Also, $y(x)-My(x)=-k$ for $x\ge S$. Hence, $y(x)-My(x)<0$ for $x>s$.

Next, we show that $Ay\le f$ for $x\le s$. Note that for $x\le s$, we have

$$y(x)=K+c(S-x)+y(S)=y(s)+c(s-x).$$

However, $Ay\le f$, when $x\le s$, leads to

$$-gc+\alpha y(s)+\alpha cs\le(-p+\alpha c)x.$$

Since $(-p+\alpha c)<0$ and $x\le s\le 0$, it is enough to show that

$$-gc+\alpha y(s)+\alpha cs\le(-p+\alpha c)s$$

or, equivalently,

$$-gc+\alpha y(s)\le -ps.$$

Now, (7) with Lemma 3 gives the result. ■

LEMMA 8: *If* $(-p+\alpha c)<0$ *and* (s,S) *is a solution obtained from solving (12)–(15). Then,* (s,S) *is the unique solution of the QVI problem (9).*

PROOF: Assume that we have two solutions $(s_1,S(s_1))$ and $(s_2,S(s_2))$, with $s_1<s_2$. This means that $L'(s_1)=L'(s_2)=0$, and $L''(s_1)\le 0$ and $L''(s_2)\le 0$ by (12) and Lemma 3, respectively. This implies that there exists $x^*\in(s_1,s_2)$ such that $L''(x^*)=0$ and $L'(x^*)<0$. Also, we have $Ay(x^*)\le f(x^*)$, giving

$$\alpha L(x^*)<(-p+\alpha c)x^*+gc.$$

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Also, we have $Ay(x^*) = f(x^*)$ since $s_1 < s_2$, giving

$$-\frac{1}{2}\sigma^2 L''(x^*) + gL'(x^*) + \alpha L(x^*) = (-p + \alpha c)x^* + gc.$$

This is in contradiction with the assertion that $L'(x^*) < 0$ and $L''(x^*) = 0$. This completes the proof. ■

LEMMA 9: *If* $(-p + \alpha c) \geq 0$*, then the QVI problem (9) admits no solution.*

PROOF: Assume that there is a solution (s, S) to the (QVI) problem (9). Then,

$$Ay \leq f \quad \text{for } x \leq s$$

or, equivalently,

$$L(x) \leq \frac{1}{\alpha}(-p + \alpha c)x + \frac{gc}{\alpha}. \tag{29}$$

If $(-p + \alpha c) > 0$, then, clearly, (29) is violated when $x \to -\infty$. Therefore, (s, S) cannot be a solution to the QVI problem (9) in this case.

Now, assume that $(-p + \alpha c) = 0$; then, $Ay \leq f$ when $x \leq s$ gives

$$L(S) \leq \frac{gc}{\alpha},$$

which, in turn, leads by (14) and (16) to

$$L(S) \leq \frac{gc}{\alpha} - K.$$

Using the fact that $Ay(S) = f(S)$ and $L'(S) = 0$, we get

$$-\frac{1}{2}\sigma^2 L''(S) - \alpha K \geq \begin{cases} (-p + \alpha c)S & \text{if } S < 0 \\ (q + \alpha c)S + (aS^\beta + \theta S)c & \text{if } S > 0. \end{cases}$$

The left-hand side of the above inequality is strictly negative and the right-hand side is strictly positive, which leads to a contradiction. Theorem 1 follows from Theorem 2 and Lemmas 1–6. ■

Assume that we are interested in impulse-control policies of the form $V = \{(t_i, Q_i)\}_{i=1,\ldots}$, where the t_i's represent the ordering times and the Q_i's represent the quantity ordered, with total cost per unit time given by

$$y_V(x) = \lim_{T\to\infty} \frac{E_x\left[\int_0^T f(x(t))\,dt + \sum_{t_i \leq T}(K + cQ_i)\right]}{T},$$

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where the dynamics of the process are given by (3) and the expectation is taken with respect to all realizations of the process. Then, we say that V^* is average cost optimal if

$$y_{V^*}(x) = \inf_V y_V(x).$$

Let $\mu = y_{V^*}(x)$. Then, it is known (see Lions and Perthame [10]) that the optimal cost y in (2) behaves like $((\mu/\alpha) + y_0)$, where y_0 satisfies some QVI problem that can be obtained from (7). Also, it is known that the optimal (s, S) policy obtained from (5) converges to the optimal policy minimizing the expected average future costs.

In this article, we proposed a new continuous-time stochastic inventory model for stock-dependent demand items. We then formulated the problem of finding the optimal impulse-control schedule that minimizes the total expected return over an infinite horizon, as a quasivariational inequality problem. The QVI was shown to lead to an (s, S) policy, where s and S are determined uniquely as a solution of some algebraic equations.

We would like to point out two interesting questions that could form the basis for some nice research problems:

1. How much can we modify the plant equation (4) while retaining the nice structure of the solution?
2. What happens if the cost function is modified to be general (convex or otherwise)?

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