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DECIDABLE MODELS OF ω-STABLE THEORIES

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Abstract. We characterize the ω-stable theories all of whose countable models admit decidable presentations. In particular, we show that for a countable ω-stable T, every countable model of T admits a decidable presentation if and only if all n-types in T are recursive and T has only countably many countable models. We further characterize the decidable models of ω-stable theories with countably many countable models as those which realize only recursive types.

§1. Introduction.

The following is a fundamental question of recursive model theory:

QUESTION 1.1. *For which theories T is it true that every countable model of T admits a decidable presentation*?

There are two clearly necessary conditions for a theory to have this property. First, for all n, every n-type (a complete type in n variables) in T must be recursive, as it is realized in a decidable countable model. Second, T must have (up to isomorphism) only countably many countable models. Millar [8] showed that these conditions are not sufficient to guarantee that every countable model of T admits a decidable presentation. We show that these conditions suffice for the class of countable ω-stable theories. In particular,

THEOREM 1.2. *Let T be an ω-stable theory. Then every countable model of T is decidably presentable if and only if all n-types consistent with T are recursive and T has only countably many countable models.*

In fact, we will prove the following stronger theorem.

THEOREM 1.3. *Let T be a recursive ω-stable theory with countably many countable models. Let M be a model of T. Then M has a decidable presentation if and only if all types realized in M are recursive.*

This theorem is stronger than Theorem 1.2, as it implies the nontrivial implication in Theorem 1.2 and characterizes the decidable models in a more general class of theories. Also, note that Goncharov and Nurtazin [4] constructed a theory which witnesses that ω-stability does not suffice for Theorem 1.3 without the assumption on the number of countable models.

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Theorems 1.2 and 1.3 can be seen as a generalization of the result of Harrington [5] and Khisamiev [6] that every countable model of a countable uncountably categorical theory admits a decidable presentation if and only if the theory is recursive. At the same time, the result is an effectivization of Vaught's conjecture for ω-stable theories. For the model theory, we will rely heavily on the analysis in Shelah, Harrington and Makkai [10] and Bouscaren [1] where Vaught's conjecture and Martin's conjecture for ω-stable theories are proved.

The key to the analysis in Shelah, Harrington and Makkai [10] and Bouscaren [1] is to find dimension invariants which characterize a model. In other words, there is some $n \in \omega$ so that each model M contains an extended basis whose type can be characterized via an invariant from $(\omega + 1)^n$. Further, M is prime over this basis, thus giving a surjection from $(\omega + 1)^n$ onto the set of models of T. There are three obstacles to effectivizing the analysis in Shelah, Harrington and Makkai [10] and Bouscaren [1]. First, only assuming recursiveness of n-types does not immediately guarantee the recursiveness of the type of an extended basis, which has infinitely many elements. This obstacle will be overcome by using a theorem from Buechler [2] showing that the type of Morley sequences in the appropriate types are recursive. Second, we have to build a prime model over the extended basis. This is done by showing that we only need to omit a recursively enumerable list of types, namely those types which would witness that the basis itself is not complete. We then use the recursive omitting types theorem to omit this list.

A third, and insurmountable, obstacle to effectivizing the analysis in Shelah, Harrington and Makkai [10], is in the inherent noneffectiveness of choosing the finite tree of types which are used in the analysis. We would need this to get a uniform version of our result. That is an algorithm which, given a recursive complete theory T as above, will output an enumeration of decidable presentations of all of the countable models of T. In Sections 2 and 3, we show that for each theory as above, there is an effective enumeration of decidable presentations of all models of T, and we will show in Section 4 that the uniform version of the result is false.

Due to the relationship between Question 1.1 and Vaught's conjecture, the full answer to Question 1.1 is likely to be difficult. On the other hand, Vaught's conjecture has been solved by Buechler [3] for superstable theories of finite rank. Thus we pose the immediate version of Question 1.1 for these theories:

QUESTION 1.4. *Suppose T is superstable of finite rank. When is every countable model of T decidably presentable? Do the conditions above suffice?*

§2. Building the extended basis.

Throughout the remainder of this section as well as Section 3, we assume T is a countable ω-stable theory with countably many countable models. We will also assume M is a countable model of T realizing only recursive types. We will use the analysis presented in Bouscaren [1] of the models of countable ω-stable theories with countably (equivalently $< 2^\omega$) many countable models to show that we can give a decidable presentation for M.

The analysis in Shelah, Harrington and Makkai [10] and Bouscaren [1] describes the models of T in terms of dimensions of a finite tree of types. We say that a type p *needs* a tuple $\bar{a}$ over a set A if p is orthogonal to A but not to $A \cup \bar{a}$, p is stationary, strongly regular, eventually nonisolated (eni), and the type of $\bar{a}$ over A is stationary and weight 1. If p needs $\bar{a}$ over A, then we say p needs $q := \text{tp}(\bar{a}/A)$. In this case, we say q *supports* p, and q is a supportive type. Using this notion, Shelah, Harrington and Makkai show that there is a tree of depth 2 of types, whose realizations entirely control a model of T. In our following presentation of this tree of types, we will use notation exactly following Bouscaren [1].

Shelah, Harrington and Makkai describe, for any model M of an ω-stable theory T with fewer than continuum many countable models, a subset X of the model built as maximal independent realizations of a finite tree of types of depth ≤ 2. This subset X, which we will call the extended basis of M, then determines the model M as the prime model over X. To describe X, we will need the following sets, which are shown to exist in Shelah, Harrington and Makkai [10]:

- a^0 is a finite tuple from M which has isolated type over the empty set.
- R is a finite set of pairwise orthogonal types r_i over a^0 such that each r_i is stationary, trivial, and has weight 1. Furthermore, if p is any supportive type, p is nonorthogonal to one of the r_i.
- For each $r_i \in R$, we fix an $\tilde{r}_i \in S(a^0)$ such that if $\bar{b}$ realizes r_i, then $\bar{b} \supset b'$ with $tp(b') = \tilde{r}_i$. Furthermore, $\bar{b}$ is dominated by b' over a^0, and $\bar{b}$ is isolated over $a^0 \cup b'$. We let $\tilde{R}$ be the finite set of these $\tilde{r}_i$.
- For any $\bar{b}$ realizing some $r_i \in R$, we define a finite set $P_{\bar{b}} = \{p^j_{\bar{b}} | j < l\}$ of pairwise orthogonal strongly regular types over $a^0 \cup \bar{b}$ such that:
  - Each $p^j_{\bar{b}} \in P_{\bar{b}}$ is eni and, in fact, nonisolated over $a^0 \cup \bar{b}$.
  - If $b' \subset \bar{b}$ realizes $\tilde{r}_i$, then $p^j_{\bar{b}}$ needs b'/a^0.
  - If q is any eni-type needing b'/a^0, then q is nonorthogonal to a member of $P_{\bar{b}}$.

  Note that if $\bar{b}_1, \bar{b}_2$ realize r_i, then $P_{\bar{b}_1}$ and $P_{\bar{b}_2}$ have the same number of types, and we take them to be automorphic types over a^0. Thus, we may equivalently write P_{r_i}.
- A finite set $d^0 \supset a^0$ such that $tp(d^0/a^0)$ is isolated and stationary.
- A finite set Q of pairwise orthogonal strongly regular types over d^0.

Simply in order to further follow Bouscaren [1], we will append constants for a^0 to the language so that we may assume $a^0 = \emptyset$. As the type of a^0 is isolated, it is recursive in T, so adding constants along with the complete type of those constants to the theory maintains our hypotheses. The purpose of all these definitions becomes quite clear with Proposition 1 from Bouscaren [1]:

Proposition 2.1. *Let M be a model of T. Let $R(M)$ be a maximal independent set of realizations in M of types in R. For each $\bar{b} \in R(M)$, let $P_{\bar{b}}(M)$ be a maximal independent set of realizations in M of types in $P_{\bar{b}}$. Let $\bar{d} \subset M$ realize $tp(d^0)$ such that $\bar{d}$ is isolated over $R(M)$, and let $Q_{\bar{d}}(M)$ be a maximal independent set of realizations in M of the types in $Q_{\bar{d}}$.*

Then M is prime over $X = R(M) \cup \bar{d} \cup Q_{\bar{d}}(M) \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}(M)$.

We call this X an extended basis for M. From this alone, it is not apparent that the type of this extended basis can be characterized by an element of $(\omega + 1)^n$. It appears as though there might be infinitely many "dimensions" that each realization of a supportive type in R might support. To remedy this, Bouscaren introduces the following notion of dimension and equivalence relation on these dimensions.

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DEFINITION 2.2. *For $\bar{b} \in M$ realizing r_i and $\bar{s} \in (\omega+1)^l$, we write $D(\bar{b}, M) = \bar{s}$ if $\dim(p^j_{\bar{b}}, M) = \bar{s}(j)$ for each $j < l$.*

For $\bar{s}, \bar{t} \in (\omega+1)^l$, we say these dimensions are r_i-equivalent, $\bar{s} \sim_{r_i} \bar{t}$, if for all models N of T, if there exists $\bar{c} \in N$ satisfying r_i such that $D(\bar{c}, N) = \bar{s}$, then there exists $\bar{e} \in N$ satisfying r_i where $\bar{c}$ and $\bar{e}$ are dependent and $D(\bar{e}, N) = \bar{t}$.

Bouscaren showed [1] that $\sim_{r_i}$ is an equivalence relation with finitely many classes. Thus, if we fix representatives $\bar{s}_{i,0}, \dots \bar{s}_{i,k(i)}$ of these classes, we may assume in our extended basis that each realization of r_i has one of these finitely many dimensions. Thus, for a given model M, once we fix representatives to be our extended basis, the model is determined by the following two pieces of information:

- For each i and each $j \leq k(i)$, the number of realizations of r_i in the basis whose dimension is $\bar{s}_{i,j}$.
- For each of the finitely many $q \in Q$, the number of elements in $Q_{\bar{d}}(M)$ realizing q.

Note that the type of $\bar{d}$ over $R(M) \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}$ does not matter by [10, Lemma 1.4]. We will choose a $\bar{d}$ whose type is isolated over $R(M) \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}$. This information is then coded by an element of $(\omega+1)^{\sum_i k(i)+|Q|}$. We will show that for each element of $(\omega+1)^{\sum_i k(i)+|Q|}$, there exists a recursive type of an extended basis with precisely those dimensions. The main obstruction to building the extended basis is in building the types of Morley sequences. The remaining difficulties come from putting together the various pieces of the extended basis. We will recursively put together the various pieces of the extended basis using the fact that the various parts of the extended basis are orthogonal. The key to building Morley sequences is the following result by Buechler.

THEOREM 2.3 (Follows from Lemmas 4 and 5 in Buechler [2]). *Let T be ω-stable and $p \in S(\bar{c})$ be stationary strongly regular. Let $\bar{a}_0$ and $\bar{a}_1$ be two independent realizations of p, and let q be the type of $\bar{a}_0\bar{a}_1\bar{c}$. Let I be an infinite independent set of realizations of p. Then the type of $\bar{c}I$ is recursive in q.*

We now show that the extended basis of M has a recursive type. First, we will build the type of the $R(M) \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}$ part of the basis. We fix a tuple $\tau \in (\omega+1)^l$ which tells us the sizes of the sets contained in the basis. For each r_i, we dedicate a collection of tuples from ω, and for each such tuple $\bar{b}$, we dedicate a collection of tuples for each $p^j_{\bar{b}} \in P_{\bar{b}}$, all to match the assigned sizes from τ. Thus, each such tuple has a definite role; we have specifed which type it is supposed to realize over which tuple.

We now give a recursive enumeration of a partial type. If $\bar{b}, \bar{x}_1, \dots, \bar{x}_n$ are tuples dedicated for each $\bar{x}_i$ to realize the type $p^j_{\bar{b}}$, we enumerate all formulae to make this true. Here, either $n = 1$, in which case we know that the type of $\bar{b}\bar{x}_1$ is recursive, as it appears in M, or Theorem 2.3 gives us a recursive type over $\bar{b}$ of an infinite independent set of realizations of $p^j_{\bar{b}}$. If $\bar{x}_1, \bar{x}_2$ are dedicated to be independent realizations of r_i, we enumerate all formulae to make this true. By triviality of r_i, the enumerated types of the intended independent realizations of r_i are complete types. By orthogonality of the various pieces of the extended basis, this partial type is complete (see, e.g., [10, Lemma 1.5]), thus this yields an enumeration of the entire type.

We have shown that there is an infinite recursive type S_0 where S_0 is the type of $R(M) \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}$. Now, we find a realization of $tp(d^0)$ in M which is isolated over $R(M) \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}$ and we add a formula to S_0 isolating the type of $\bar{d}$. Now, we show that there is a recursive type S which is the type of $R(M) \cup \bar{d} \cup Q_{\bar{d}} \cup \bigcup_{\bar{b} \in R(M)} P_{\bar{b}}$. Again, we dedicate tuples from ω to be realizations of the types q. Again, if $\bar{x}_1, \ldots, \bar{x}_n$ are dedicated to be independent realizations of the type q over $\bar{d}$, then we enumerate formulae to make this true. Again, we are either using the type of an infinite set of realizations of q given by Theorem 2.3 to uniformly enumerate these formulae or we are using the type of a finite maximal set of realizations in M. By [1, Lemma 2] this gives the complete type of $R(M) \cup \bar{d} \cup Q_{\bar{d}}(M)$, and by [10, Lemma 1.5], this gives the complete type of the entire extended basis.

§3. Omitting types to build M.

We will use the following omitting types theorem, which is a weak version of the result in Millar [7]:

THEOREM 3.1. *Let T be a complete decidable theory, and let Ψ be a recursively enumerable set of nonprincipal (not necessarily complete) types in T. Then there exists a decidable model of T omitting each type in Ψ. Furthermore, the index of the decidable model is uniform in the indices of Ψ and T.*

Now we use the recursive ω-type S of an extended basis with the specified roles as above. Note that this S is a complete theory, and we will apply Theorem 3.1 to S. Let $S' = \{c_0, c_1, \ldots\}$ be the constants specified in S. We give a recursively enumerable list of recursive types as follows:

- If recursive, for each $r_i \in R$, the type of a new realization of r_i independent from the realizations of r_i in S'.
- If recursive, for each $\bar{b}$ from S' realizing r_i and each $p^j_{\bar{b}} \in P_{\bar{b}}$, the type of a new realization of $p^j_{\bar{b}}$ independent from the realizations of $p^j_{\bar{b}}$ in S' over $\bar{b}$.
- If recursive, for each $q \in Q$, the type of a new realization of q independent from the realizations of q in S'.

Note that these types are recursive unless the related part of the extended basis has finite size n and the type of n independent realizations of the given type is recursive, but the type of $n+1$ independent realizations is not recursive. In any case, we simply do not include the nonrecursive types in the list Ψ of types to omit. Our list is a recursively enumerable list of recursive types, and each type in our list is nonprincipal since it is not realized in M, where S' names the chosen extended basis of M.

By Theorem 3.1, there is a decidable model of $T \cup S$ which omits each type in Ψ. We claim that this model N is, in fact, the prime model over the extended basis realizing S. This follows from Proposition 2.1, the fact that N is a model of T, and the fact that the realization of S is a maximal extended basis contained in N. This is either because we explicitly omitted the type of a new realization or because the type of a new realization, being nonrecursive, is automatically omitted in any decidable model, thus in N. Thus by Proposition 2.1, N is prime over the realization of S. Thus, we have built *the* model with the given dimensions, showing that it is decidable. As M was chosen to be an arbitrary model of T realizing only recursive types, we have proved Theorem 1.3 and thus also Theorem 1.2.

§4. Nonuniformity. In this section, we show that the result cannot hold uniformly. That is, there is no algorithm which, when given an ω-stable theory T all of whose types are recursive with only countably many countable models, will enumerate decidable presentations of all models of T. Were such an algorithm to exist, we could, in particular, enumerate all n-types realized in any model of T. Thus, there would be an algorithm which, when given an ω-stable theory T all of whose types are recursive with only countably many countable models, outputs an enumeration of all types consistent with T. We will show that no such algorithm exists.

To do so, we employ a standard construction coding a recursive tree $E \subseteq 2^{<\omega}$ with no terminal nodes into a recursive theory. Given $E \subseteq 2^{<\omega}$ a tree with no terminal nodes, we let T_E be the theory in the signature $L = \{U_\sigma(x) \mid \sigma \in 2^{<\omega}\}$ that is axiomatized by the following statements:

- T_E contains the sentence $\forall x U_\lambda(x)$, where λ is the empty string.
- For each σ and τ which are incomparable, T_E contains the sentence $\neg\exists x(U_\sigma(x)\wedge U_\tau(x))$.
- For each σ, T_E contains the sentence $\forall x \left(U_\sigma(x) \leftrightarrow (U_{\sigma\frown 0}(x) \vee U_{\sigma\frown 1}(x))\right)$.
- For each $\sigma \in E$, T_E contains the schema $\{\exists^n x U_\sigma(x) \mid n \in \omega\}$.
- For each $\sigma \notin E$, T_E contains the sentence $\neg\exists x U_\sigma(x)$.

This describes a complete theory and $E \equiv_T T_E$. Last, we note that the n-types in T_E are determined entirely by 1-types and the 1-types correspond to paths through E. Thus, T_E is ω-stable if and only if E has only countably many paths, and T_E has countably many countable models if and only if E has only finitely many nonisolated paths.

Suppose towards a contradiction that there is an algorithm which, when given an ω-stable theory T all of whose types are recursive with only countably many countable models, outputs an enumeration of all types consistent with T. We will, via a standard diagonalization, construct a computable tree E so that the algorithm fails to enumerate all 1-types for the theory T_E. In fact, we will ensure that E has exactly one nonisolated path, and that this path is recursive. By the recursion theorem, we may use information about the enumeration of the 1-types that the algorithm provides: $p_0, p_1, p_2, \dots$. Until the algorithm enumerates either $U_0(x)$ or $U_1(x)$ into p_0, at stage n we put the strings 0^n and $1 \frown 0^{n-1}$ into E. If the algorithm never enumerates either $U_0(x)$ or $U_1(x)$ into p_0, then we have constructed an ω-stable theory with countably many countable models all of whose types are recursive, and the algorithm, being partial, has failed. If, at stage n, $U_0(x)$ is enumerated into p_0, we decide that the nonisolated path in E will extend 1, we will never again split the node 0^n, and we will always extend it by 0's. We then split the $1 \frown 0^{n-1}$ node into two and repeat the process. In this way, we have diagonalized so that the one nonisolated path in E does not correspond to any of the types p_i. Thus we have found an ω-stable theory T_E all of whose types are recursive with only countably many countable models, but the algorithm has failed to enumerate one of its types.

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§5. A related problem. A longstanding open problem of recursive model theory (see the introduction of Millar [9]) is the following question:

QUESTION 5.1. *Does every decidable complete theory with countably many countable models have a decidable prime model*?

We note that the answer to Question 5.1 is positive for the class of ω-stable theories. This is an immediate application of Theorem 1.3, as every type realized in the prime model is isolated in T, and thus, recursive.

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